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The limiting normal cone to pointwise defined sets in Lebesgue spaces

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We consider subsets of Lebesgue spaces which are defined by pointwise constraints. We provide formulas for corresponding variational objects (tangent and normal cones). Our main result shows that the limiting normal cone is always dense in the Clarke normal cone and contains the convex hull of the pointwise limiting normal cone. A crucial assumption for this result is that the underlying measure is non-atomic, and this is satisfied in many important applications (Lebesgue measure on subsets of $\mathbb{R}^d$ or the surface measure on hypersurfaces in $\mathbb{R}^d$). Finally, we apply our findings to an optimization problem with complementarity constraints in Lebesgue spaces.

Keywords: decomposable set, Lebesgue spaces, limiting normal cone, mathematical program with complementarity constraint, measurability
MSC: 49J53, 90C30

1 Introduction

It is standard in optimal control of (ordinary or partial) differential equations to formulate pointwise control constraints, i.e., one requires that the control function u belongs to a set $\mathbb{K} \subset L^p(\mathfrak{m};\mathbb{R}^q)$, $p \in (1,\infty)$, of the type

$$\mathbb{K} := \{u \in L^p(\mathfrak{m};\mathbb{R}^q) \mid u(\omega) \in K(\omega) \text{ f.a.a. } \omega \in \Omega\}.$$

Here, $K\colon \Omega \rightrightarrows \mathbb{R}^q$ is a set-valued map with certain measurability properties, and $(\Omega, \Sigma, \mathfrak{m})$ is a complete, σ-finite, and non-atomic measure space. For simplicity, one may think of a

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measurable subset of $\mathbb{R}^d$ equipped with the Lebesgue measure. In this paper, we do not postulate any convexity properties on the images of K. Hence, the set $\mathbb{K}$ is, in general, not convex.

In order to state optimality conditions for optimization problems comprising constraints of the type $u \in \mathbb{K}$, one needs to study the variational geometry of $\mathbb{K}$ in more detail. Particularly, different tangent and normal cones to $\mathbb{K}$ are of major interest for this issue. Our main result is that the limiting normal cone to $\mathbb{K}$ is always dense in the Clarke normal cone. Moreover, we show that it contains the convex hull of the pointwise limiting normal cone. Additionally, we provide formulas for other variational objects associated with $\mathbb{K}$.

We organized this article as follows: In Section 2, we mention the notation used throughout the paper and some preliminary results from variational analysis and measure theory. Afterwards, we derive formulas for different tangent and normal cones to the set $\mathbb{K}$ in Section 3. Finally, in Section 4, we study mathematical programs comprising complementarity constraints in Lebesgue spaces, which can be stated as optimization problems with a geometric constraint involving a non-convex set $\mathbb{K}$. It will be demonstrated that Mordukhovich's concept of stationarity, which possesses a certain strength for finite-dimensional mathematical programs with complementarity constraints, see Flegel and Kanzow [2006], is unpleasantly weak for these problems. In particular, it coincides with weak stationarity.

2 Notation and preliminary results

2.1 Notation

Basic notation

Here, we subsume the notation used throughout the article. Let $\mathbb{N}$, $\mathbb{Q}^+$, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{R}^+_0$, and $\mathbb{R}^q$ denote the positive natural numbers, the positive rational numbers, the real numbers, the positive real numbers, the non-negative real numbers, and the set of all real vectors with q components, respectively. For arbitrary $x \in \mathbb{R}^q$ and $\delta > 0$, the sets $\mathbb{U}_\delta(x)$ and $\mathbb{B}_\delta(x)$ represent the open and closed ball with radius δ around x with respect to (w.r.t.) the Euclidean norm $|\cdot|$ in $\mathbb{R}^q$. The Euclidean inner product of two vectors $x, y \in \mathbb{R}^q$ is denoted by $x \cdot y$.

Now, suppose that X is a (real) Banach space with norm $\|\cdot\|_X$, and let $A \subset X$ be a non-empty set. We denote by $\operatorname{cl} A$, $\operatorname{cl}_w A$, $\operatorname{cl}_w^{\mathrm{seq}} A$, and $\overline{\operatorname{conv}} A$ the closure, the weak closure (i.e., the closure w.r.t. the weak topology in X), the sequential weak closure (weak accumulation points of weakly convergent sequences), and the closed convex hull of A, respectively. By $X^\star$ we denote the (topological) dual space of X, whereas $\langle \cdot, \cdot \rangle \colon X^\star \times X \to \mathbb{R}$ is the corresponding dual pairing. We define the polar cone and the annihilator of A by

$$A^\circ := \{x^\star \in X^\star \mid \forall x \in A : \langle x^\star, x\rangle \le 0\}, \qquad A^\perp := \{x^\star \in X^\star \mid \forall x \in A : \langle x^\star, x\rangle = 0\}.$$

Recall that X is reflexive if the canonical embedding from X into $X^{\star\star}$ is surjective, in particular, $X \cong X^{\star\star}$ holds. Hence, in this reflexive case, it is consistent to use

$$B^\circ := \{x \in X \mid \forall x^\star \in B : \langle x^\star, x\rangle \le 0\}, \qquad B^\perp := \{x \in X \mid \forall x^\star \in B : \langle x^\star, x\rangle = 0\}$$

for any non-empty set $B \subset X^\star$.

Variational analysis: tangent cones

Let A be a closed subset of the Banach space X and let $\bar{x} \in A$ be given. Then the radial cone to A at $\bar{x}$ is given by

$$\mathcal{R}_A(\bar{x}) := \{d \in X \mid \exists t_0 > 0 : \bar{x} + td \in A \,\forall t \in (0, t_0)\}.$$

The tangent (or Bouligand) cone, the weak tangent cone, the adjacent tangent cone, and the Clarke tangent cone to A at $\bar{x}$ are defined as

$$\begin{aligned}
\mathcal{T}_A(\bar{x}) &:= \{d \in X \mid \exists\{t_n\} \subset \mathbb{R}^+ \,\exists\{d_n\} \subset X : t_n \searrow 0,\, d_n \to d,\, \bar{x} + t_n\, d_n \in A \,\forall n \in \mathbb{N}\},\\
\mathcal{T}^w_A(\bar{x}) &:= \{d \in X \mid \exists\{t_n\} \subset \mathbb{R}^+ \,\exists\{d_n\} \subset X : t_n \searrow 0,\, d_n \rightharpoonup d,\, \bar{x} + t_n\, d_n \in A \,\forall n \in \mathbb{N}\},\\
\mathcal{T}^\flat_A(\bar{x}) &:= \left\{ d \in X \;\middle|\; \begin{array}{l} \forall\{t_n\} \subset \mathbb{R}^+ \text{ such that } t_n \searrow 0 : \\ \qquad \exists\{d_n\} \subset X : d_n \to d,\, \bar{x} + t_n\, d_n \in A \,\forall n \in \mathbb{N} \end{array} \right\},\\
\mathcal{T}^C_A(\bar{x}) &:= \left\{ d \in X \;\middle|\; \begin{array}{l} \forall\{x_n\} \subset A \,\forall\{t_n\} \subset \mathbb{R}^+ \text{ such that } x_n \to \bar{x},\, t_n \searrow 0 : \\ \qquad \exists\{d_n\} \subset X : d_n \to d,\, x_n + t_n\, d_n \in A \,\forall n \in \mathbb{N} \end{array} \right\},
\end{aligned}$$

respectively, see, e.g., [Aubin and Frankowska, 2009, Section 4.1]. It is easily seen that $\mathcal{T}_A(\bar{x})$, $\mathcal{T}^\flat_A(\bar{x})$, and $\mathcal{T}^C_A(\bar{x})$ are closed, while the relations

$$\mathcal{T}^C_A(\bar{x}) \subset \mathcal{T}^\flat_A(\bar{x}) \subset \mathcal{T}_A(\bar{x}) \subset \mathcal{T}^w_A(\bar{x})$$

hold. Moreover, we have $\mathcal{R}_A(\bar{x}) \subset \mathcal{T}^\flat_A(\bar{x})$. The set A is called derivable at the point $\bar{x}$ if $\mathcal{T}_A(\bar{x}) = \mathcal{T}^\flat_A(\bar{x})$ is satisfied, and A is said to be derivable if it is derivable at all of its points, cf. [Aubin and Frankowska, 2009, Definition 4.1.5]. For closed, convex sets, all the introduced tangent cones coincide and, consequently, any such set is derivable. Furthermore, $\operatorname{cl} \mathcal{R}_A(\bar{x}) = \mathcal{T}_A(\bar{x})$ holds true for closed, convex sets A.

Suppose that, in addition to the above assumptions, the Banach space X is reflexive. For an arbitrary vector $\bar{\eta} \in \mathcal{T}_A(\bar{x})^\circ$, we define the critical cone to A w.r.t. $(\bar{x}, \bar{\eta})$ by

$$\mathcal{K}_A(\bar{x}, \bar{\eta}) := \mathcal{T}_A(\bar{x}) \cap \{\bar{\eta}\}^\perp.$$

Variational analysis: normal cones

Let A be a closed subset of the reflexive Banach space X and let $\bar{x} \in A$ be given. Let us define the Clarke (or convexified) normal cone, the Fréchet (or regular) normal cone, the limiting (or basic, Mordukhovich) normal cone, and the strong limiting (or norm-limiting) normal cone to A at $\bar{x}$ by

$$\begin{aligned}
&\mathcal{N}_A^C(\bar{x}) := \mathcal{T}_A^C(\bar{x})^\circ, \\
&\widehat{\mathcal{N}}_A(\bar{x}) := \left\{\eta \in X^\star \;\middle|\; \limsup_{x\to\bar{x},\, x\in A} \frac{\langle \eta, x-\bar{x}\rangle}{\|x-\bar{x}\|_X} \le 0\right\}, \\
&\mathcal{N}_A(\bar{x}) := \{\eta \in X^\star \mid \exists\{x_n\} \subset A\, \exists\{\eta_n\} \subset X^\star : x_n \to \bar{x},\, \eta_n \rightharpoonup \eta,\, \eta_n \in \widehat{\mathcal{N}}_A(x_n)\ \forall n \in \mathbb{N}\}, \\
&\mathcal{N}_A^S(\bar{x}) := \{\eta \in X^\star \mid \exists\{x_n\} \subset A\, \exists\{\eta_n\} \subset X^\star : x_n \to \bar{x},\, \eta_n \to \eta,\, \eta_n \in \widehat{\mathcal{N}}_A(x_n)\ \forall n \in \mathbb{N}\}.
\end{aligned}$$

Due to the reflexivity of X, the Fréchet normal cone satisfies $\widehat{\mathcal{N}}_A(\bar{x}) = \mathcal{T}_A^w(\bar{x})^\circ$, see [Mordukhovich, 2006, Corollary 1.11]. From [Mordukhovich, 2006, Theorem 3.57], we have $\mathcal{N}_A^C(\bar{x}) = \overline{\operatorname{conv}}\, \mathcal{N}_A(\bar{x})$, and this yields the inclusions

$$\widehat{\mathcal{N}}_A(\bar{x}) \subset \mathcal{N}_A^S(\bar{x}) \subset \mathcal{N}_A(\bar{x}) \subset \mathcal{N}_A^C(\bar{x}).$$

It is well known that all these cones coincide whenever A is convex. If the space X is finite-dimensional, we immediately obtain $\widehat{\mathcal{N}}_A(\bar{x}) = \mathcal{T}_A(\bar{x})^\circ$ and $\mathcal{N}_A^S(\bar{x}) = \mathcal{N}_A(\bar{x})$. To our knowledge, strong limiting normals were introduced in Geremew et al. [2009] first. In our analysis in Section 3, they appear naturally as a pointwise equivalent to the finite-dimensional limiting normal cone and it has to be investigated whether this can be exploited analytically.

A closed subset A of the reflexive Banach space X is called sequentially normally compact (SNC for short) at $\bar{x} \in A$ if for any sequences $\{x_n\} \subset A$ and $\{\eta_n\} \subset X^\star$ which satisfy $x_n \to \bar{x}$, $\eta_n \rightharpoonup 0$, and $\eta_n \in \widehat{\mathcal{N}}_A(x_n)$ for all $n \in \mathbb{N}$, we have $\eta_n \to 0$. Clearly, any subset of a finite-dimensional Banach space is SNC at all of its points. On the other hand, a singleton in X is SNC at its point if and only if X is finite-dimensional, see [Mordukhovich, 2006, Theorem 1.21].

Measurability of set-valued mappings

Let (Ω, Σ) be a measurable space, (Y, d) be a separable, complete metric space, and $F\colon \Omega \rightrightarrows Y$ be a set-valued mapping. By $\operatorname{gph} F := \{(\omega, y) \in \Omega \times Y \mid y \in F(\omega)\}$ we denote the graph of F.

The mapping F is called measurable if for any open set $O \subset Y$, the preimage of O under F, i.e., $F^{-1}(O) := \{\omega \in \Omega \mid F(\omega) \cap O \neq \varnothing\}$, is measurable. Note that there exist equivalent characterizations of the measurability of F if F is closed-valued, see [Papageorgiou and Kyritsi-Yiallourou, 2009, Theorem 6.2.20]. In particular, the closed-valued mapping F is measurable if and only if there is a sequence of measurable functions $f_n\colon \Omega \to Y$, $n \in \mathbb{N}$, such that $F(\omega) = \operatorname{cl}\{f_n(\omega)\}_{n\in\mathbb{N}}$ is valid for all $\omega \in \Omega$, see [Papageorgiou and Kyritsi-Yiallourou, 2009, Theorem 6.3.18].

Moreover, it is easily seen that F is measurable if and only if $\operatorname{cl} F$ is measurable, see [Papageorgiou and Kyritsi-Yiallourou, 2009, Proposition 6.2.12], where $\operatorname{cl} F$ is the set-valued mapping defined via $(\operatorname{cl} F)(\omega) := \operatorname{cl}(F(\omega))$. Hence, this concept is not suited for mappings with non-closed images.

We call F graph-measurable whenever the set $\operatorname{gph} F$ is measurable w.r.t. the measurable space $(\Omega \times Y, \Sigma \otimes \mathcal{B}(Y))$. Here, $\mathcal{B}(Y)$ denotes the Borel σ-algebra of Y, i.e., the smallest σ-algebra which contains all open sets of the metric space (Y, d), and $\Sigma \otimes \mathcal{B}(Y)$ represents the smallest σ-algebra which contains the Cartesian product $\Sigma \times \mathcal{B}(Y)$. Note that a measurable map F with closed images is graph-measurable as well, see [Papageorgiou and Kyritsi-Yiallourou, 2009, Proposition 6.2.10].

A function $f\colon \Omega \times Y \to \mathbb{R}$ is called a Carathéodory function if for any $\omega \in \Omega$, the function $f(\omega, \cdot)$ is continuous, whereas $f(\cdot, y)$ is measurable for any $y \in Y$. Consequently, f is $\Sigma \otimes \mathcal{B}(Y)$-measurable, see [Papageorgiou and Kyritsi-Yiallourou, 2009, Theorem 6.2.6].

In this paper, the set Y will frequently be $\mathbb{R}$ or $\mathbb{R}^q$, and we use $\mathcal{B} := \mathcal{B}(\mathbb{R})$ and $\mathcal{B}^q := \mathcal{B}(\mathbb{R}^q)$ for convenience.

Measure space and Lebesgue spaces

Now, assume that $(\Omega, \Sigma, \mathfrak{m})$ is a complete and σ-finite measure space. In order to exclude trivial situations, we always suppose $\mathfrak{m}(\Omega) > 0$. Then for any $p \in [1, \infty]$, we denote by $L^p(\mathfrak{m}; \mathbb{R}^q)$ the usual Lebesgue space of (equivalence classes of) measurable functions from Ω to $\mathbb{R}^q$ equipped with the usual norm. Recall that the dual of $L^p(\mathfrak{m}; \mathbb{R}^q)$ is isometric to $L^{p'}(\mathfrak{m}; \mathbb{R}^q)$, where $p \in [1, \infty)$, $1/p + 1/p' = 1$. The duality pairing is given by

$$\langle v, u \rangle := \int_\Omega u(\omega) \cdot v(\omega) \, \mathrm{d}\omega, \qquad u \in L^p(\mathfrak{m}; \mathbb{R}^q), v \in L^{p'}(\mathfrak{m}; \mathbb{R}^q). \tag{1}$$

Note that we will use $\mathrm{d}\omega$ instead of $\mathrm{d}\mathfrak{m}$ since the measure $\mathfrak{m}$ is fixed throughout the paper. Moreover, $L^p(\mathfrak{m}; \mathbb{R}^q)$ is reflexive for $p \in (1, \infty)$.

We say that the measure space $(\Omega, \Sigma, \mathfrak{m})$ is non-atomic if for all $M \in \Sigma$ with $\mathfrak{m}(M) > 0$, there exists $\hat{M} \in \Sigma$ with $0 < \mathfrak{m}(\hat{M}) < \mathfrak{m}(M)$.

Note that an open subset $\Omega \subset \mathbb{R}^d$, $d \geq 1$, equipped with the σ-algebra of Lebesgue-measurable subsets and the Lebesgue measure is complete, σ-finite, and non-atomic.

We use $\chi_A\colon \Omega \to \mathbb{R}$ to represent the characteristic function of the set A, which has value 1 for all $\omega \in A$ and equals zero otherwise.

2.2 Preliminary results

Criteria for derivability

In this section, we want to present some conditions which ensure that a given set is derivable at a certain point of interest.

Our first result is a simple consequence of some calculus rules for tangent cones to a finite union of sets.

Lemma 2.1. *Let $D_1, \ldots D_k \subset X$ be derivable subsets of a Banach space X. Then $D := \bigcup_{i=1}^k D_i$ is derivable as well.*

Proof. We choose $d \in D$ arbitrarily and set $I(d) := \{i \in \{1, \ldots, k\} \mid d \in D_i\}$. From [Aubin and Frankowska, 2009, Chapter 4.1] we get the inclusions

$$\mathcal{T}_D(d) = \bigcup_{i \in I(d)} \mathcal{T}_{D_i}(d) = \bigcup_{i \in I(d)} \mathcal{T}^\flat_{D_i}(d) \subset \mathcal{T}^\flat_D(d) \subset \mathcal{T}_D(d),$$

see in particular [Aubin and Frankowska, 2009, Tables 4.1 and 4.2]. Hence, we have $\mathcal{T}_D(d) = \mathcal{T}^\flat_D(d)$. □

Due to the above lemma, the union of finitely many closed, convex sets of a Banach space is derivable at all of its points.

The following lemma addresses the derivability of preimages of closed, convex sets under differentiable maps. Its proof follows from [Bonnans and Shapiro, 2000, Corollary 2.91].

Lemma 2.2. *Let $D \subset Y$ be a nonempty, closed, convex subset of a Banach space Y and let $H \colon X \to Y$ be continuously Fréchet differentiable where X is another Banach space. We consider the set $S := \{x \in X \mid H(x) \in D\}$ and assume that Robinson's constraint qualification is satisfied at $\bar{x} \in S$, i.e. the condition*

$$H'(\bar{x})[X] - \mathcal{R}_D(H(\bar{x})) = Y$$

is valid. Then S is derivable at $\bar{x}$.

Measurable sets and maps

First, we give two measurability results.

Lemma 2.3. *Let $(\Omega, \Sigma, \mathfrak{m})$ be a complete and σ-finite measure space and (Y, d) be a complete, separable metric space. Furthermore, let $F \colon \Omega \rightrightarrows Y$ be a measurable set-valued mapping with compact images and let $f \colon \Omega \times Y \to \mathbb{R}$ be a Carathéodory map. Then the set*

$$M := \{\omega \in \Omega \mid \forall y \in F(\omega) : f(\omega, y) \leq 0\}$$

is measurable.

Proof. Let us define a set-valued map $\mathcal{F} \colon \Omega \rightrightarrows \mathbb{R}$ by $\mathcal{F}(\omega) := \{f(\omega, y) \mid y \in F(\omega)\}$ for any $\omega \in \Omega$. Observe that $M = \{\omega \in \Omega \mid \mathcal{F}(\omega) \subset (-\infty, 0]\}$ is satisfied. Due to the fact that f is continuous w.r.t. its second component, whereas $F(\omega)$ is compact for any $\omega \in \Omega$, the images of $\mathcal{F}$ are closed. Applying [Aubin and Frankowska, 2009, Theorem 8.2.8] yields the measurability of $\mathcal{F}$. The statement of the lemma follows from $M = \Omega \setminus \mathcal{F}^{-1}((0, \infty))$. □

Lemma 2.4. *Let $(\Omega, \Sigma, \mathfrak{m})$ be a complete and σ-finite measure space and let $\{a_k\}_{k \in \mathbb{N}}$ be a sequence of measurable functions from Ω to $\mathbb{R}$. We assume that for a.a. $\omega \in \Omega$, there exists $N(\omega) \in \mathbb{N}$, such that $a_k(\omega) \geq 0$ is satisfied for all $k \geq N(\omega)$. Then the function $f \colon \Omega \to \mathbb{N}$ defined via*

$$f(\omega) := \min\{n \in \mathbb{N} \mid \forall k \geq n : a_k(\omega) \geq 0\}$$

is measurable.

Proof. Since $\mathbb{N}$ is countable, it is sufficient to show that the sets $f^{-1}(n)$ are measurable for all $n \in \mathbb{N}$. It is easy to check that

$$f^{-1}(n) = \{\omega \in \Omega \mid a_{n-1}(\omega) < 0 \text{ and } \forall i \geq n : a_i(\omega) \geq 0\}$$

holds. Here, we used the convention $a_0 \equiv -1$. Since the functions a_i are supposed to be measurable, every $f^{-1}(n)$ is the intersection of countably many measurable sets, thus, measurable. □

For a closed set $C \subset \mathbb{R}^q$ and $v \in \operatorname{conv} C$, we introduce the notation

$$r_C(v) := \min \left\{ \max_{i=1,\ldots,q+1} |v_i| \;\middle|\; \begin{array}{c} \exists \{v_i\}_{i=1}^{q+1} \subset C \, \exists \{\lambda_i\}_{i=1}^{q+1} \subset [0,1] : \\ \sum_{i=1}^{q+1} \lambda_i = 1 \text{ and } \sum_{i=1}^{q+1} \lambda_i \, v_i = v \end{array} \right\}.$$

Carathéodory's theorem implies $r_C(v) < \infty$. It is clear that the minimum is attained due to the coercivity of the maximum norm. Observe that $r_C(v)$ is the smallest radius $r \geq 0$ such that $v \in \operatorname{conv}(C \cap \mathbb{B}_r(0))$ is valid. One can exploit Carathéodory's theorem once more in order to see

$$r_C(v) = \min \left\{ \max_{i=1,\ldots,m} |v_i| \;\middle|\; \begin{array}{c} \exists m \in \mathbb{N} \, \exists \{v_i\}_{i=1}^{m} \subset C \, \exists \{\lambda_i\}_{i=1}^{m} \subset [0,1] : \\ \sum_{i=1}^{m} \lambda_i = 1 \text{ and } \sum_{i=1}^{m} \lambda_i \, v_i = v \end{array} \right\}. \tag{2}$$

For the sake of completeness, we set $r_C(v) = +\infty$ for any $v \notin \operatorname{conv} C$.

Now, let $C\colon \Omega \rightrightarrows \mathbb{R}^q$ be a measurable set-valued mapping with closed images and let $v\colon \Omega \to \mathbb{R}^q$ be a measurable function satisfying $v(\omega) \in \operatorname{conv} C(\omega)$ for a.a. $\omega \in \Omega$. Applying [Aubin and Frankowska, 2009, Theorem 8.2.11] yields that the marginal function $r_{C(\cdot)}(v(\cdot))$ is measurable as well.

The following classical result shows that there are arbitrarily small sets of positive measure in a non-atomic measure space.

Lemma 2.5. *Let $(\Omega, \Sigma, \mathfrak{m})$ be a non-atomic measure space, and let $A \in \Sigma$ be a set of positive measure. Then there exists a sequence $\{M_n\} \subset \Sigma$ of measurable subsets of A such that M_n is a set of positive measure for any $n \in \mathbb{N}$ and $\mathfrak{m}(M_n) \to 0$ is satisfied.*

The final result of this section shows the absolute continuity of the Lebesgue integral. It is a straightforward consequence of, e.g., the dominated convergence theorem.

Lemma 2.6. *Let $(\Omega, \Sigma, \mathfrak{m})$ be a measure space and $\{M_n\} \subset \Sigma$ be a sequence with the property $\mathfrak{m}(M_n) \to 0$, and choose $\xi \in L^1(\mathfrak{m})$ arbitrarily. Then we have*

$$\int_{M_n} \xi(\omega) \, \mathrm{d}\omega = \int_{\Omega} \chi_{M_n}(\omega) \, \xi(\omega) \, \mathrm{d}\omega \to 0.$$

3 Variational geometry of decomposable sets in Lebesgue spaces

Throughout this section, we use the following standing assumption.

Assumption 3.1. *Let $(\Omega, \Sigma, \mathfrak{m})$ be a complete, σ-finite, and non-atomic measure space.*

We fix a measurable set-valued map $K\colon \Omega \rightrightarrows \mathbb{R}^q$, such that its images are closed and derivable a.e. on Ω.

Moreover, we fix $p \in (1, \infty)$ as well as the conjugate exponent p' satisfying the relation $1/p + 1/p' = 1$. We further suppose that $L^p(\mathfrak{m})$ and $L^{p'}(\mathfrak{m})$ are separable.

Using the set-valued mapping K, we define

$$\mathbb{K} := \{u \in L^p(\mathfrak{m}; \mathbb{R}^q) \mid u(\omega) \in K(\omega) \text{ f.a.a. } \omega \in \Omega\}. \tag{3}$$

It is easy to check that this set is decomposable in the following sense.

Definition 3.2. *A set $\mathbb{K} \subset L^p(\mathfrak{m}; \mathbb{R}^q)$ is said to be decomposable if for every triple $(A, f_1, f_2) \in \Sigma \times \mathbb{K} \times \mathbb{K}$, we have $\chi_A f_1 + (1 - \chi_A) f_2 \in \mathbb{K}$.*

This notion of decomposability can be retraced to Rockafellar [1968]. In Hiai and Umegaki [1977], the authors present properties, characterizations, and calculus rules for decomposable sets. A convenient overview of the corresponding theory can be found in the recent monograph Papageorgiou and Kyritsi-Yiallourou [2009].

It is well known, see [Papageorgiou and Kyritsi-Yiallourou, 2009, Theorem 6.4.6], that a non-empty set $\mathbb{K} \subset L^p(\mathfrak{m}; \mathbb{R}^q)$ is decomposable and closed if and only if there exists a measurable set-valued map $K\colon \Omega \rightrightarrows \mathbb{R}^q$ with closed images, such that $\mathbb{K}$ possesses the representation (3).

Due to the derivability of $K(\omega)$, we have a convenient expression for the tangent cone to the set $\mathbb{K}$.

Lemma 3.3 ([Aubin and Frankowska, 2009, Corollary 8.5.2])**.** *For all $\bar{u} \in \mathbb{K}$, we have*

$$\mathcal{T}_{\mathbb{K}}(\bar{u}) = \{v \in L^p(\mathfrak{m}; \mathbb{R}^q) \mid v(\omega) \in \mathcal{T}_{K(\omega)}(\bar{u}(\omega)) \text{ f.a.a. } \omega \in \Omega\}.$$

Now, we state a lemma which characterizes three basic operations on decomposable sets which will be exploited in our analysis.

Lemma 3.4. *Let $\mathbb{C} \subset L^p(\mathfrak{m}; \mathbb{R}^q)$ be closed and decomposable, and denote the associated set-valued mapping by C. Then we have*

$$\begin{aligned} \operatorname{cl}_w \mathbb{C} = \overline{\operatorname{conv}}\, \mathbb{C} &= \{v \in L^p(\mathfrak{m}; \mathbb{R}^q) \mid v(\omega) \in \overline{\operatorname{conv}}\, C(\omega) \text{ f.a.a. } \omega \in \Omega\}, \\ \operatorname{conv} \mathbb{C} &\subset \left\{v \in L^p(\mathfrak{m}; \mathbb{R}^q) \,\middle|\, r_{C(\cdot)}(v(\cdot)) \in L^p(\mathfrak{m})\right\} \subset \operatorname{cl}_w^{\text{seq}} \mathbb{C}, \\ \mathbb{C}^\circ &= \{\eta \in L^{p'}(\mathfrak{m}; \mathbb{R}^q) \mid \eta(\omega) \in C(\omega)^\circ \text{ f.a.a. } \omega \in \Omega\}. \end{aligned}$$

In particular, the set-valued maps $\omega \mapsto \overline{\operatorname{conv}}(C(\omega))$ and $\omega \mapsto (C(\omega))^\circ$ are closed-valued and measurable.

Proof. Let us prove the first statement. We always have the inclusion $\operatorname{cl}_w \mathbb{C} \subset \overline{\operatorname{conv}}\, \mathbb{C}$ since $\overline{\operatorname{conv}}\, \mathbb{C}$ is weakly closed. From [Papageorgiou and Kyritsi-Yiallourou, 2009, Proposition 6.4.14, Remark 6.4.15] we find that a decomposable set is weakly closed if and only if it is strongly closed and convex (for this assertion it is essential that $\mathfrak{m}$ is non-atomic). Since $\operatorname{cl}_w \mathbb{C}$ is decomposable, the inclusion $\overline{\operatorname{conv}}\, \mathbb{C} \subset \operatorname{cl}_w \mathbb{C}$ follows. From [Papageorgiou and Kyritsi-Yiallourou, 2009, Proposition 6.4.19] we derive the second equality.

Next, we verify the first inclusion of the second statement. Therefore, we set

$$S := \left\{ v \in L^p(\mathfrak{m}; \mathbb{R}^q) \,\middle|\, r_{C(\cdot)}(v(\cdot)) \in L^p(\mathfrak{m}) \right\}.$$

First, we show $\mathbb{C} \subset S$. For an arbitrary $v \in \mathbb{C}$, we easily see $r_{C(\omega)}(v(\omega)) = |v(\omega)|$ for a.a. $\omega \in \Omega$. By definition of $L^p(\mathfrak{m}; \mathbb{R}^q)$, $|v(\cdot)|$ belongs to $L^p(\mathfrak{m})$, i.e. $v \in S$ holds.

Secondly, choose $v, v' \in S$ and $\kappa \in [0, 1]$ arbitrarily. Since we have $v(\omega), v'(\omega) \in \operatorname{conv} C(\omega)$ a.e. on Ω, $\kappa v(\omega) + (1 - \kappa)v'(\omega) \in \operatorname{conv} C(\omega)$ is obtained a.e. on Ω. Thus, the function $r_{C(\cdot)}(\kappa v(\cdot) + (1 - \kappa)v'(\cdot))$ is well defined and measurable. A straightforward calculation using (2) yields

$$0 \le r_{C(\omega)}(\kappa v(\omega) + (1 - \kappa)v'(\omega)) \le \max \left\{ r_{C(\omega)}(v(\omega)); r_{C(\omega)}(v'(\omega)) \right\}.$$

Thus, $r_{C(\cdot)}(v(\cdot)), r_{C(\cdot)}(v'(\cdot)) \in L^p(\mathfrak{m})$ implies $r_{C(\cdot)}(\kappa v(\cdot) + (1 - \kappa)v'(\cdot)) \in L^p(\mathfrak{m})$. Consequently, $\kappa v + (1 - \kappa)v' \in S$ holds, and this shows the convexity of S. Combining the above results, S is a convex set containing $\mathbb{C}$, i.e. $\operatorname{conv} \mathbb{C} \subset S$ is valid.

Next, we show the second inclusion of the lemma's second assertion. First, we consider the case of a finite measure, i.e., $\mathfrak{m}(\Omega) < \infty$. Let $v \in L^p(\mathfrak{m}; \mathbb{R}^q)$ be given such that $r_{C(\cdot)}(v(\cdot)) \in L^p(\mathfrak{m})$. In particular, this implies $v(\omega) \in \operatorname{conv} C(\omega)$ for a.a. $\omega \in \Omega$. We introduce a set-valued mapping $\Upsilon : \Omega \rightrightarrows \mathbb{R}^{q+1} \times \mathbb{R}^{q(q+1)}$ by

$$\Upsilon(\omega) := \left\{ ((\lambda_1, \dots, \lambda_{q+1}), v_1, \dots, v_{q+1}) \,\middle|\, \begin{array}{l} \lambda_i \ge 0,\ i = 1, \dots, q+1, \\ v_i \in C(\omega) \cap \mathbb{B}_{r_{C(\omega)}(v(\omega))}(0),\ i = 1, \dots, q+1, \\ \sum_{i=1}^{q+1} \lambda_i = 1,\ \sum_{i=1}^{q+1} \lambda_i\, v_i = v(\omega) \end{array} \right\}$$

for all $\omega \in \Omega$. It is easy to see from [Aubin and Frankowska, 2009, Theorem 8.2.9] that Υ is measurable. Applying the measurable selection theorem, see [Aubin and Frankowska, 2009, Theorem 8.1.3], yields the existence of measurable functions $\lambda_i \colon \Omega \to [0, 1]$ and $v_i \colon \Omega \to \mathbb{R}^q$, $i = 1, \dots, q+1$, which satisfy $v_i(\omega) \in C(\omega)$ for all $i = 1, \dots, q+1$, $\sum_{i=1}^{q+1} \lambda_i(\omega) = 1$, and $v(\omega) = \sum_{i=1}^{q+1} \lambda_i(\omega)\, v_i(\omega)$ for a.a. $\omega \in \Omega$. Recalling that $r_{C(\cdot)}(v(\cdot))$ is an element of $L^p(\mathfrak{m})$, we additionally obtain $v_1, \dots, v_{q+1} \in L^p(\mathfrak{m}; \mathbb{R}^q)$.

Now, we define the set-valued mapping $E \colon \Omega \rightrightarrows \mathbb{R}^{q+1}$ via $E(\omega) = \{e_i\}_{i=1}^{q+1}$. Therein, $e_1, \dots, e_{q+1} \in \mathbb{R}^{q+1}$ denote the $q+1$ unit vectors in $\mathbb{R}^{q+1}$. We denote by $\mathbb{E} \subset L^p(\mathfrak{m}; \mathbb{R}^{q+1})$ the decomposable set associated to E. By owing to the first assertion, we observe that $(\lambda_1, \dots, \lambda_{q+1})$ belongs to $\operatorname{cl}_w \mathbb{E} = \overline{\operatorname{conv}}\, \mathbb{E}$. Since $L^{p'}(\mathfrak{m})$ is assumed to be separable, the weak topology is metrizable on bounded subsets of $L^p(\mathfrak{m}; \mathbb{R}^{q+1})$. Thus, we can employ that $\mathbb{E}$ lies in the ball with radius $\mathfrak{m}(\Omega)^{1/p}$, and we get a sequence $\{w_k\}_{k \in \mathbb{N}} \subset \mathbb{E}$, with $w_k \rightharpoonup (\lambda_1, \dots, \lambda_{q+1})$ in $L^p(\mathfrak{m}; \mathbb{R}^{q+1})$. Since $\{w_k\}_{k \in \mathbb{N}}$ is even bounded in $L^\infty(\mathfrak{m}; \mathbb{R}^{q+1})$, we obtain $w_k \overset{\star}{\rightharpoonup} (\lambda_1, \ldots, \lambda_{q+1})$ in $L^\infty(\mathfrak{m}; \mathbb{R}^{q+1})$. This yields

$$\sum_{i=1}^{q+1} (w_k)_i \, v_i \rightharpoonup \sum_{i=1}^{q+1} \lambda_i \, v_i = v$$

and by the property $\{w_k\}_{k\in\mathbb{N}} \subset \mathbb{E}$ we have $\sum_{i=1}^{q+1} (w_k)_i \, v_i \in \mathbb{C}$ for all $k \in \mathbb{N}$. Hence, $v \in \operatorname{cl}_w^{\mathrm{seq}} \mathbb{C}$ is satisfied. This shows the second assertion in the case that $\mathfrak{m}(\Omega) < \infty$.

In the general case, the second assertion can be proved by working on the decomposition of Ω into countably many parts of finite measure and by employing the estimate $|v_i(\omega)| \le r_{C(\omega)}(v(\omega))$ for a.a. $\omega \in \Omega$.

Finally, the proof for the last assertion is straightforward and, hence, omitted. $\square$

Combining Lemmas 3.3 and 3.4, we obtain a pointwise characterization of the critical cone.

Corollary 3.5. *For any $\bar\eta \in \mathcal{T}_{\mathbb{K}}(\bar u)^\circ$, we have*

$$\mathcal{K}_{\mathbb{K}}(\bar u, \bar\eta) = \{v \in L^p(\mathfrak{m}; \mathbb{R}^q) \mid v(\omega) \in \mathcal{K}_{K(\omega)}(\bar u(\omega), \bar\eta(\omega)) \text{ f.a.a. } \omega \in \Omega\}.$$

Now, we can estimate the weak tangent cone to $\mathbb{K}$.

Lemma 3.6. *We have*

$$\mathcal{T}_{\mathbb{K}}(\bar u) \subset \mathcal{T}_{\mathbb{K}}^w(\bar u) \subset \overline{\operatorname{conv}}\, \mathcal{T}_{\mathbb{K}}(\bar u).$$

Proof. The first inclusion follows from the definition of the involved cones.

It remains to show the second inclusion.

Let $v \in \mathcal{T}_{\mathbb{K}}^w(\bar u)$ be given. We proceed by contradiction and assume $v \notin \overline{\operatorname{conv}}\, \mathcal{T}_{\mathbb{K}}(\bar u)$. Hence, there exists $\eta \in \mathcal{T}_{\mathbb{K}}(\bar u)^\circ$ with $\langle \eta, v\rangle > 0$. Therefore, there are $E_1 \in \Sigma$, $\mathfrak{m}(E_1) > 0$, and $\alpha > 0$ with $\eta(\omega) \cdot v(\omega) \ge \alpha$ for almost all $\omega \in E_1$.

Since v belongs to the weak tangent cone, there are $v_n \rightharpoonup v$ in $L^p(\mathfrak{m}; \mathbb{R}^q)$ and $t_n \searrow 0$ with $u_n := \bar u + t_n v_n \in \mathbb{K}$. In particular, we have $u_n \to \bar u$ in $L^p(\mathfrak{m}; \mathbb{R}^q)$. By passing to a subsequence, we find $E_2 \in \Sigma$ with $\mathfrak{m}(E_2) > \frac{2}{3}\, \mathfrak{m}(E_1)$ and $E_2 \subset E_1$ such that $\|u_n - \bar u\|_{L^\infty(\mathfrak{m}|_{E_2}; \mathbb{R}^q)} \to 0$.

We set $M := \|\eta\|_{L^{p'}(\mathfrak{m};\mathbb{R}^q)} \sup_{n\in\mathbb{N}} \|v_n\|_{L^p(\mathfrak{m};\mathbb{R}^q)}$ and $\varepsilon := \alpha\, \mathfrak{m}(E_1)/(4\, M)$. Observe that for any $\delta > 0$ the set

$$E_\delta := E_2 \cap \big\{\omega \in \Omega \;\big|\; \forall u \in K(\omega) \cap \mathbb{B}_\delta(\bar u(\omega)) : \eta(\omega) \cdot (u - \bar u(\omega)) \le \varepsilon\, |\eta(\omega)|\, |u - \bar u(\omega)|\big\}$$

is measurable by Lemma 2.3. Since $\eta(\omega) \in \mathcal{T}_{K(\omega)}(\bar u(\omega))^\circ$ holds for almost all $\omega \in \Omega$, see Lemma 3.4, we find a sufficiently small $\bar\delta$ such that $\mathfrak{m}(E_{\bar\delta}) > \mathfrak{m}(E_1)/2$ is satisfied.

Now, for n large enough, we have $\|u_n - \bar u\|_{L^\infty(\mathfrak{m}|_{E_{\bar\delta}}; \mathbb{R}^q)} \le \bar\delta$. Thus, since $v_n \rightharpoonup v$, we have for n large enough

$$\begin{aligned}
\frac{2}{3}\, \alpha\, \mathfrak{m}(E_{\bar\delta}) &\le \int_{E_{\bar\delta}} \eta(\omega) \cdot v_n(\omega) \,\mathrm{d}\omega = \frac{1}{t_n} \int_{E_{\bar\delta}} \eta(\omega) \cdot (u_n(\omega) - \bar u(\omega)) \,\mathrm{d}\omega \\
&\le \frac{\varepsilon}{t_n} \int_{E_{\bar\delta}} |\eta(\omega)|\, |u_n(\omega) - \bar u(\omega)| \,\mathrm{d}\omega \le \frac{\varepsilon}{t_n} \|\eta\|_{L^{p'}(\mathfrak{m};\mathbb{R}^q)} \|u_n - \bar u\|_{L^p(\mathfrak{m};\mathbb{R}^q)} \\
&= \varepsilon\, \|\eta\|_{L^{p'}(\mathfrak{m};\mathbb{R}^q)} \|v_n\|_{L^p(\mathfrak{m};\mathbb{R}^q)} \le \varepsilon\, M = \frac{\alpha}{4}\, \mathfrak{m}(E_1) \le \frac{\alpha}{2}\, \mathfrak{m}(E_{\bar\delta}).
\end{aligned}$$

This is a contradiction. $\square$

Now, we can compute the Fréchet normal cone as the dual of $\mathcal{T}^w_{\mathbb{K}}(\bar{u})$ in a pointwise fashion.

Corollary 3.7. *We have*

$$\widehat{\mathcal{N}}_{\mathbb{K}}(\bar{u}) = \{\eta \in L^{p'}(\mathfrak{m}; \mathbb{R}^q) \mid \eta(\omega) \in \widehat{\mathcal{N}}_{K(\omega)}(\bar{u}(\omega)) \text{ } f.a.a. \text{ } \omega \in \Omega\}.$$

Proof. Using Lemma 3.6 as well as $A^\circ = (\overline{\operatorname{conv}} A)^\circ$ for arbitrary sets in a Banach space, we obtain

$$\mathcal{T}_{\mathbb{K}}(\bar{u})^\circ \supset \mathcal{T}^w_{\mathbb{K}}(\bar{u})^\circ \supset (\overline{\operatorname{conv}}\, \mathcal{T}_{\mathbb{K}}(\bar{u}))^\circ = \mathcal{T}_{\mathbb{K}}(\bar{u})^\circ.$$

Together with Lemma 3.4, this yields

$$\begin{aligned}\widehat{\mathcal{N}}_{\mathbb{K}}(\bar{u}) = \mathcal{T}^w_{\mathbb{K}}(\bar{u})^\circ = \mathcal{T}_{\mathbb{K}}(\bar{u})^\circ &= \{\eta \in L^{p'}(\mathfrak{m}; \mathbb{R}^q) \mid \eta(\omega) \in \mathcal{T}_{K(\omega)}(\bar{u}(\omega))^\circ \text{ f.a.a. } \omega \in \Omega\} \\ &= \{\eta \in L^{p'}(\mathfrak{m}; \mathbb{R}^q) \mid \eta(\omega) \in \widehat{\mathcal{N}}_{K(\omega)}(\bar{u}(\omega)) \text{ f.a.a. } \omega \in \Omega\}.\end{aligned}$$

This completes the proof. $\square$

Note that we even have $\widehat{\mathcal{N}}_{\mathbb{K}}(\bar{u}) = \mathcal{T}_{\mathbb{K}}(\bar{u})^\circ$.

In the upcoming theorem, we consider the graphical Fréchet normal cone mapping of $\mathbb{K}$, i.e., the mapping $\omega \mapsto \operatorname{gph} \widehat{\mathcal{N}}_{K(\omega)}$. Therein, for any $\omega \in \Omega$, $\widehat{\mathcal{N}}_{K(\omega)} \colon \mathbb{R}^q \rightrightarrows \mathbb{R}^q$ denotes the Fréchet normal cone map induced by $K(\omega)$, i.e., it maps any $u \in K(\omega)$ to the Fréchet normal cone $\widehat{\mathcal{N}}_{K(\omega)}(u)$ and any $u \notin K(\omega)$ is mapped to $\varnothing$.

Theorem 3.8. *The set-valued map $\omega \mapsto \operatorname{gph} \widehat{\mathcal{N}}_{K(\omega)}$ is graph-measurable, that is, the set*

$$\begin{aligned}\operatorname{gph} \operatorname{gph} \widehat{\mathcal{N}}_{K(\cdot)} &= \Big\{(\omega, u, \nu) \in \Omega \times \mathbb{R}^q \times \mathbb{R}^q \;\Big|\; (u, \nu) \in \operatorname{gph} \widehat{\mathcal{N}}_{K(\omega)}\Big\} \\ &= \Big\{(\omega, u, \nu) \in \Omega \times \mathbb{R}^q \times \mathbb{R}^q \;\Big|\; u \in K(\omega),\ \nu \in \widehat{\mathcal{N}}_{K(\omega)}(u)\Big\}\end{aligned}$$

is measurable w.r.t. the σ-algebra $\Sigma \otimes \mathcal{B}^q \otimes \mathcal{B}^q$.

Proof. We start by giving a convenient expression for $\operatorname{gph} \widehat{\mathcal{N}}_C$ for a closed set $C \subset \mathbb{R}^q$. By elementary calculations, we have

$$\begin{aligned}\operatorname{gph} \widehat{\mathcal{N}}_C &= \{(u, \nu) \in \mathbb{R}^q \times \mathbb{R}^q \mid \nu \in \widehat{\mathcal{N}}_C(u)\} \\ &= \{(u, \nu) \in C \times \mathbb{R}^q \mid \nu \cdot (c - u) \le o(|c - u|) \text{ as } C \ni c \to u\} \\ &= \big\{(u, \nu) \in C \times \mathbb{R}^q \;\big|\; \forall \varepsilon \in \mathbb{Q}^+ \; \exists \delta \in \mathbb{Q}^+ \; \forall c \in \mathbb{U}_\delta(u) \cap C : \nu \cdot (c - u) \le \varepsilon\, |c - u|\big\}.\end{aligned}$$

Since we are going to prove measurability, it will be beneficial to write $\operatorname{gph} \widehat{\mathcal{N}}_C$ via countable intersections and unions. We find

$$\operatorname{gph} \widehat{\mathcal{N}}_C = \bigcap_{\varepsilon \in \mathbb{Q}^+} \bigcup_{\delta \in \mathbb{Q}^+} \big\{(u, \nu) \in C \times \mathbb{R}^q \;\big|\; \forall c \in \mathbb{U}_\delta(u) \cap C : \nu \cdot (c - u) \le \varepsilon\, |c - u|\big\}$$

from the above representation.

Now, we consider a countable set $\{c_m\}_{m\in\mathbb{N}}$ which is dense in C. Immediately, we find that $\{c_m\}_{m\in\mathbb{N}} \cap \mathbb{U}_\delta(u)$ is dense in $C \cap \mathbb{U}_\delta(u)$. Hence,

$$\begin{aligned}
&\{(u,\nu) \in C \times \mathbb{R}^q \mid \forall c_m \in \mathbb{U}_\delta(u) : \nu \cdot (c_m - u) \le \varepsilon\, |c_m - u|\} \\
&\quad = \bigcap_{m\in\mathbb{N}} \Big(\big[(C \setminus \mathbb{U}_\delta(c_m)) \times \mathbb{R}^q\big] \cup \big\{(u,\nu) \in C \times \mathbb{R}^q \;\big|\; \nu \cdot (c_m - u) \le \varepsilon\, |c_m - u|\big\} \Big) \\
&\quad = \bigcap_{m\in\mathbb{N}} \Big((C \times \mathbb{R}^q) \cap \Big[\big[\big(\mathbb{R}^q \setminus \mathbb{U}_\delta(c_m)\big) \times \mathbb{R}^q\big] \\
&\qquad\qquad\qquad \cup \big\{(u,\nu) \in \mathbb{R}^q \times \mathbb{R}^q \;\big|\; \nu \cdot (c_m - u) \le \varepsilon\, |c_m - u|\big\} \Big] \Big).
\end{aligned}$$

Now, we are in position to verify the measurability of the graph of $\omega \mapsto \operatorname{gph} \widehat{\mathcal{N}}_{K(\omega)}$. Since $\omega \mapsto K(\omega)$ is non-empty, closed-valued, and measurable, there exist measurable functions $\{k_m\}_{m\in\mathbb{N}}$ such that $K(\omega) = \operatorname{cl}\{k_m(\omega)\}_{m\in\mathbb{N}}$ for all $\omega \in \Omega$, see [Papageorgiou and Kyritsi-Yiallourou, 2009, Theorem 6.3.18]. We define the sets

$$\begin{aligned}
I_1 &:= \operatorname{gph} K \times \mathbb{R}^q, \\
I_2(\delta, m) &:= \big\{(\omega, u, \nu) \in \Omega \times \mathbb{R}^q \times \mathbb{R}^q \;\big|\; \delta - |k_m(\omega) - u| \le 0\big\}, \\
I_3(\varepsilon, m) &:= \big\{(\omega, u, \nu) \in \Omega \times \mathbb{R}^q \times \mathbb{R}^q \;\big|\; \nu \cdot (k_m(\omega) - u) - \varepsilon\, |k_m(\omega) - u| \le 0\big\}.
\end{aligned}$$

From the preparation above, we have

$$\operatorname{gph} \operatorname{gph} \widehat{\mathcal{N}}_{K(\cdot)} = \bigcap_{\varepsilon\in\mathbb{Q}^+} \bigcup_{\delta\in\mathbb{Q}^+} \bigcap_{m\in\mathbb{N}} \Big(I_1 \cap \big[I_2(\delta, m) \cup I_3(\varepsilon, m)\big] \Big). \tag{4}$$

Hence, it remains to show the measurability of the sets I_1, $I_2(\delta, m)$, and $I_3(\varepsilon, m)$. Since $\omega \mapsto K(\omega)$ is closed-valued and measurable, it is graph-measurable, [Papageorgiou and Kyritsi-Yiallourou, 2009, Proposition 6.2.10], and hence, the measurability of I_1 follows. In order to show the measurability of $I_2(\delta, m)$ and $I_3(\varepsilon, m)$, we introduce the functions

$$\begin{aligned}
\varphi_{\delta,m}(\omega, (u, \nu)) &:= \delta - |k_m(\omega) - u|, \\
\psi_{\varepsilon,m}(\omega, (u, \nu)) &:= \nu \cdot (k_m(\omega) - u) - \varepsilon\, |k_m(\omega) - u|.
\end{aligned}$$

Obviously, these functions are measurable in ω for fixed (u, ν) and continuous in (u, ν) for fixed ω. Hence, these functions are Carathéodory functions, and, consequently, measurable w.r.t. $\Sigma \otimes \mathcal{B}^q \otimes \mathcal{B}^q$. The measurability of $I_2(\delta, m)$ and $I_3(\varepsilon, m)$ follows from

$$I_2(\delta, m) = \varphi_{\delta,m}^{-1}((-\infty, 0]) \quad \text{and} \quad I_3(\varepsilon, m) = \psi_{\varepsilon,m}^{-1}((-\infty, 0]).$$

Hence, the assertion follows from the representation (4). □

The next step is the characterization of the limiting normal cone. We first give an expression for the strong limiting normal cone.

Lemma 3.9. *We have*

$$\mathcal{N}^S_{\mathbb{K}}(\bar u) = \{\eta \in L^{p'}(\mathfrak{m};\mathbb{R}^q) \mid \eta(\omega) \in \mathcal{N}_{K(\omega)}(\bar u(\omega)) \text{ f.a.a. } \omega \in \Omega\}.$$

Proof. Clearly, we have $\operatorname{gph}\mathcal{N}^S_{\mathbb{K}} = \operatorname{cl}\operatorname{gph}\widehat{\mathcal{N}}_{\mathbb{K}}$ from the definition. Hence,

$$\begin{aligned}\operatorname{gph}\mathcal{N}^S_{\mathbb{K}} &= \operatorname{cl}\big\{(u,\eta) \in L^p(\mathfrak{m},\mathbb{R}^q) \times L^{p'}(\mathfrak{m};\mathbb{R}^q) \bigm| (u(\omega),\eta(\omega)) \in \operatorname{gph}\widehat{\mathcal{N}}_{K(\omega)} \text{ f.a.a. } \omega \in \Omega\big\}\\ &= \big\{(u,\eta) \in L^p(\mathfrak{m},\mathbb{R}^q) \times L^{p'}(\mathfrak{m};\mathbb{R}^q) \bigm| (u(\omega),\eta(\omega)) \in \operatorname{cl}\operatorname{gph}\widehat{\mathcal{N}}_{K(\omega)} \text{ f.a.a. } \omega \in \Omega\big\}\\ &= \big\{(u,\eta) \in L^p(\mathfrak{m},\mathbb{R}^q) \times L^{p'}(\mathfrak{m};\mathbb{R}^q) \bigm| (u(\omega),\eta(\omega)) \in \operatorname{gph}\mathcal{N}_{K(\omega)} \text{ f.a.a. } \omega \in \Omega\big\}\end{aligned}$$

is obtained from Lemma 3.7, Theorem 3.8, and [Papageorgiou and Kyritsi-Yiallourou, 2009, Proposition 6.4.20]. This yields the claim. □

Now, it is possible to derive lower and upper estimates for the limiting normal cone.

Lemma 3.10. *We have*

$$\operatorname{cl}^{\mathrm{seq}}_w \mathcal{N}^S_{\mathbb{K}}(\bar u) \subset \mathcal{N}_{\mathbb{K}}(\bar u) \subset \overline{\operatorname{conv}}\,\mathcal{N}^S_{\mathbb{K}}(\bar u).$$

Proof. We begin with the first inclusion. Let $\eta \in \operatorname{cl}^{\mathrm{seq}}_w \mathcal{N}^S_{\mathbb{K}}(\bar u)$ be given. Hence, there is a sequence $\{\eta_k\}_{k\in\mathbb{N}} \subset \mathcal{N}^S_{\mathbb{K}}(\bar u)$ with $\eta_k \rightharpoonup \eta$. By definition of the strong limiting normal cone, we find $u_k \in L^p(\mathfrak{m};\mathbb{R}^q)$ and $\hat\eta_k \in \widehat{\mathcal{N}}_{\mathbb{K}}(u_k)$ with $\|u_k - \bar u\|_{L^p(\mathfrak{m};\mathbb{R}^q)} \le 1/k$ and $\|\hat\eta_k - \eta_k\|_{L^{p'}(\mathfrak{m};\mathbb{R}^q)} \le 1/k$. This readily implies that $\hat\eta_k \rightharpoonup \eta$ and together with $u_k \to \bar u$ we obtain $\eta \in \mathcal{N}_{\mathbb{K}}(\bar u)$.

Now, we show the inclusion $\mathcal{N}_{\mathbb{K}}(\bar u) \subset \overline{\operatorname{conv}}\,\mathcal{N}^S_{\mathbb{K}}(\bar u)$ by contradiction. Assume there is $\eta \in \mathcal{N}_{\mathbb{K}}(\bar u) \setminus \overline{\operatorname{conv}}\,\mathcal{N}^S_{\mathbb{K}}(\bar u)$. Then, there is $v \in \mathcal{N}^S_{\mathbb{K}}(\bar u)^\circ$ with $\langle \eta, v\rangle > 0$. Hence, there exist $E_1 \in \Sigma$, $\mathfrak{m}(E_1) > 0$, and $\alpha > 0$ with $\eta(\omega)\cdot v(\omega) \ge \alpha$ for almost all $\omega \in E_1$.

Since $\eta \in \mathcal{N}_{\mathbb{K}}(\bar u)$ is satisfied, there exist sequences $\{u_n\} \subset \mathbb{K}$ and $\{\eta_n\} \subset L^{p'}(\mathfrak{m};\mathbb{R}^q)$, $\eta_n \in \widehat{\mathcal{N}}_{\mathbb{K}}(u_n)$, such that

$$u_n \to u \quad \text{in } L^p(\mathfrak{m};\mathbb{R}^q) \quad \text{and} \quad \eta_n \rightharpoonup \eta \quad \text{in } L^{p'}(\mathfrak{m};\mathbb{R}^q).$$

By passing to a subsequence, we find $E_2 \in \Sigma$ with $\mathfrak{m}(E_2) > \frac{2}{3}\,\mathfrak{m}(E_1)$ and $E_2 \subset E_1$ such that $\|u_n - u\|_{L^\infty(\mathfrak{m}|_{E_2};\mathbb{R}^q)} \to 0$.

We set $M := \|v\|_{L^p(\mathfrak{m};\mathbb{R}^q)} \sup_n \|\eta_n\|_{L^{p'}(\mathfrak{m};\mathbb{R}^q)}$ and $\varepsilon := \alpha\,\mathfrak{m}(E_1)/(4\,M) > 0$.

Now, we fix $\omega \in E_2$. By the calculations above, we have

$$v(\omega) \in \mathcal{N}_{K(\omega)}(\bar u(\omega))^\circ, \qquad \eta_n(\omega) \in \widehat{\mathcal{N}}_{K(\omega)}(u_n(\omega)).$$

Next, we show that there is $N(\omega) \in \mathbb{N}$ such that

$$\eta_n(\omega)\cdot v(\omega) \le \varepsilon\,|\eta_n(\omega)|\,|v(\omega)| \quad \forall n \ge N(\omega). \tag{5}$$

If this would not be the case, we would have $v(\omega) \ne 0$ and there would be a subsequence η_{n_k} with

$$\eta_{n_k}(\omega)\cdot v(\omega) > \varepsilon\,|\eta_{n_k}(\omega)|\,|v(\omega)|$$

for all k. W.l.o.g. we can assume $\eta_{n_k}(\omega)/|\eta_{n_k}(\omega)| \to \bar\eta(\omega) \in \mathcal{N}_{K(\omega)}(\bar u(\omega))$ (since we have $u_n(\omega) \to \bar u(\omega)$). Hence,

$$0 \ge \bar\eta(\omega) \cdot v(\omega) \leftarrow \frac{\eta_{n_k}(\omega)}{|\eta_{n_k}(\omega)|} \cdot v(\omega) > \varepsilon\, |v(\omega)|$$

and this is a contradiction, since $v(\omega) \ne 0$ holds. Thus, we have shown that (5) holds for some $N(\omega) \in \mathbb{N}$.

Let us define the function $P \colon \Omega \to \mathbb{N}$ by

$$P(\omega) := \min\bigl\{n \in \mathbb{N} \bigm| \forall k \ge n : \varepsilon\, |\eta_k(\omega)|\, |v(\omega)| - \eta_k(\omega) \cdot v(\omega) \ge 0\bigr\}.$$

Since (5) holds for some $N(\omega)$, $P(\omega)$ is finite for all $\omega \in \Omega$. Applying Lemma 2.4 yields the measurability of P. Consequently, $E_3 := \{\omega \in E_2 \mid \hat N \ge P(\omega)\}$ is measurable and satisfies $\mathfrak{m}(E_3) \ge \mathfrak{m}(E_1)/2$ for some large enough $\hat N \in \mathbb{N}$. Hence, we have for large enough n

$$\begin{aligned}\frac{2}{3}\,\alpha\, \mathfrak{m}(E_3) &\le \int_{E_3} v(\omega) \cdot \eta_n(\omega)\, \mathrm{d}\omega \le \varepsilon \int_{E_3} |v(\omega)|\, |\eta_n(\omega)|\, \mathrm{d}\omega \\ &\le \varepsilon\, \|v\|_{L^p(\mathfrak{m};\mathbb{R}^q)}\, \|\eta_n\|_{L^{p'}(\mathfrak{m};\mathbb{R}^q)} \le \varepsilon\, M = \frac{\alpha}{4}\,\mathfrak{m}(E_1) \le \frac{\alpha}{2}\,\mathfrak{m}(E_3).\end{aligned}$$

This is a contradiction. □

Now, we are in the position to prove our main result.

Theorem 3.11. *We have*

$$\operatorname{conv} \mathcal{N}^S_{\mathbb{K}}(\bar u) \subset \Bigl\{\eta \in L^{p'}(\mathfrak{m};\mathbb{R}^q) \Bigm| r_{\mathcal{N}_{K(\cdot)}(\bar u(\cdot))}(\eta(\cdot)) \in L^{p'}(\mathfrak{m})\Bigr\} \subset \mathcal{N}_{\mathbb{K}}(\bar u) \subset \mathcal{N}^C_{\mathbb{K}}(\bar u).$$

Moreover, we obtain

$$\operatorname{cl} \mathcal{N}_{\mathbb{K}}(\bar u) = \mathcal{N}^C_{\mathbb{K}}(\bar u) = \Bigl\{\eta \in L^{p'}(\mathfrak{m};\mathbb{R}^q) \Bigm| \eta(\omega) \in \mathcal{N}^C_{K(\omega)}(\bar u(\omega)) \text{ f.a.a. } \omega \in \Omega\Bigr\},$$

i.e. $\mathcal{N}_{\mathbb{K}}(\bar u)$ is dense in $\mathcal{N}^C_{\mathbb{K}}(\bar u)$.

Proof. The first assertion follows from Lemmas 3.4, 3.9, and 3.10. Combining this result with Lemma 3.10, we find

$$\operatorname{conv} \mathcal{N}^S_{\mathbb{K}}(\bar u) \subset \mathcal{N}_{\mathbb{K}}(\bar u) \subset \overline{\operatorname{conv}}\, \mathcal{N}^S_{\mathbb{K}}(\bar u).$$

This yields $\operatorname{cl} \mathcal{N}_{\mathbb{K}}(\bar u) = \overline{\operatorname{conv}}\, \mathcal{N}_{\mathbb{K}}(\bar u) = \mathcal{N}^C_{\mathbb{K}}(\bar u)$, and applying Lemmas 3.4 and 3.9 implies the second assertion. □

The above theorem shows that the limiting normal cone to any closed, decomposable set in a Lebesgue space contains the convex hull of the strong limiting normal cone and is dense in the Clarke normal cone, i.e., the limiting normal cone might be unpleasantly large. In particular, if it is closed, then it equals the Clarke normal cone. We again emphasize that it is crucial for Theorem 3.11 that $\mathfrak{m}$ is non-atomic. Observe that our result is related to the Lyapunov convexity theorem, see [Aubin and Frankowska, 2009, Theorem 8.7.3], which implies the convexity of certain integral functions whenever the underlying measure is non-atomic, see [Aubin and Frankowska, 2009, Theorem 8.6.4] as well as [Mordukhovich and Sagara, 2016, Theorem 2.8, Proposition 2.10] and the references therein. On the other hand, we need to emphasize that Theorem 3.11 does not show the convexity of the limiting normal cone to closed, decomposable sets in general. A connection between (an extension of) Lyapunov's convexity theorem and decomposable sets, see Definition 3.2, was already depicted in Olech [1990]. More precisely, the author shows that the image of a decomposable set under integration is convex.

4 Complementarity constraints in Lebesgue spaces

Let $(\Omega, \Sigma, \mathfrak{m})$ be a complete, σ-finite, and non-atomic measure space such that $L^2(\mathfrak{m})$ is separable. In this section, we consider the optimization problem

$$\begin{aligned} \text{Minimize} \quad & f(x) \\ \text{s.t.} \quad & g(x) \in C, \quad G(x) \in \mathcal{C}, \quad H(x) \in \mathcal{C}^\circ, \quad \langle H(x), G(x) \rangle = 0, \end{aligned} \tag{MPCC}$$

where the abbreviation MPCC stands for *mathematical program with complementarity constraint*, in more detail. Here, $f\colon X \to \mathbb{R}$ is Fréchet differentiable, $g\colon X \to Y$, $G\colon X \to L^2(\mathfrak{m}, \mathbb{R}^q)$, and $H\colon X \to L^2(\mathfrak{m}, \mathbb{R}^q)$ are continuously Fréchet differentiable, $q \geq 1$, X and Y are Banach spaces, $C \subset Y$ is a closed, convex set, and $\mathcal{C} \subset L^2(\mathfrak{m}; \mathbb{R}^q)$ is the closed, convex cone defined below:

$$\mathcal{C} := \{u \in L^2(\mathfrak{m}; \mathbb{R}^q) \mid u(\omega) \geq 0 \text{ f.a.a. } \omega \in \Omega\}.$$

From Lemma 3.4 we immediately obtain

$$\mathcal{C}^\circ = \{\eta \in L^2(\mathfrak{m}; \mathbb{R}^q) \mid \eta(\omega) \leq 0 \text{ f.a.a. } \omega \in \Omega\}.$$

The problem (MPCC) can be stated equivalently as

$$\begin{aligned} \text{Minimize} \quad & f(x) \\ \text{s.t.} \quad & (g(x), G(x), H(x)) \in C \times \mathbb{C}, \end{aligned} \tag{6}$$

where $\mathbb{C} \subset L^2(\mathfrak{m}; \mathbb{R}^q)^2$ is the pointwise defined set

$$\mathbb{C} := \left\{(u, \eta) \in L^2(\mathfrak{m}; \mathbb{R}^q)^2 \;\middle|\; 0 \leq u(\omega) \perp \eta(\omega) \leq 0 \text{ f.a.a. } \omega \in \Omega\right\}.$$

Defining $\Xi := \{(a, b) \in \mathbb{R}^2 \mid a \geq 0,\, b \leq 0,\, a \cdot b = 0\}$, we obtain

$$\mathbb{C} = \left\{(u, \eta) \in L^2(\mathfrak{m}; \mathbb{R}^q)^2 \;\middle|\; (u(\omega), \eta(\omega)) \in \Xi^q \text{ f.a.a. } \omega \in \Omega\right\},$$

where Ξ^q can be represented as the finite union of polyhedral sets and, therefore, is closed and derivable by Lemma 2.1.

One may think of (MPCC) as an optimal control problem with pointwise complementarity constraints on the control which appears, e.g., when among two possible control variables at most one can be non-zero at any time. Observe that (MPCC) is an MPCC in Banach spaces as studied in a more general way in Wachsmuth [2015] and, afterwards, in Mehlitz and Wachsmuth [2016], Wachsmuth [2016]. As it was shown in [Mehlitz and Wachsmuth, 2016, Lemma 3.1], Robinson's constraint qualification fails to hold at any feasible point of (MPCC) which is why the KKT conditions of this problem may turn out to be too strong to hold at local optimal solutions. Hence, the concepts of weak and strong stationarity were introduced in the aforementioned papers which are less restrictive than the KKT conditions. Below we apply these concepts to (MPCC). One may check [Mehlitz and Wachsmuth, 2016, Lemma 5.1] for a detailed validation.

Definition 4.1. *Let $\bar{x} \in X$ be a feasible point of* (MPCC)*.*

1. *We call the point $\bar{x}$ a weakly stationary point if there exist multipliers $\lambda \in Y^\star$ and $\mu, \nu \in L^2(\mathfrak{m}; \mathbb{R}^q)$ which solve the system*

$$0 = f'(\bar{x}) + g'(\bar{x})^\star \lambda + G'(\bar{x})^\star \mu + H'(\bar{x})^\star \nu, \tag{7a}$$

$$\lambda \in \mathcal{T}_C(g(\bar{x}))^\circ, \tag{7b}$$

$$\forall i \in \{1, \ldots, q\}: \quad \mu_i = 0 \quad \text{a.e. on } I^{+0}(\bar{x}, i), \tag{7c}$$

$$\forall i \in \{1, \ldots, q\}: \quad \nu_i = 0 \quad \text{a.e. on } I^{0-}(\bar{x}, i). \tag{7d}$$

   Here and in what follows, the maps $g'(\bar{x})^\star : Y^\star \to X^\star$, $G'(\bar{x})^\star : L^2(\mathfrak{m}; \mathbb{R}^q) \to X^\star$, and $H'(\bar{x})^\star : L^2(\mathfrak{m}; \mathbb{R}^q) \to X^\star$ are the adjoint operators of $g'(\bar{x})$, $G'(\bar{x})$, and $H'(\bar{x})$, respectively.

2. *We call the point $\bar{x}$ a strongly stationary point if there exist multipliers $\lambda \in Y^\star$ and $\mu, \nu \in L^2(\mathfrak{m}; \mathbb{R}^q)$ which satisfy* (7) *and, additionally,*

$$\forall i \in \{1, \ldots, q\}: \quad \mu_i \le 0,\ \nu_i \ge 0 \quad \text{a.e. on } I^{00}(\bar{x}, i). \tag{8}$$

Therein, for any $i \in \{1, \ldots, q\}$, the sets $I^{+0}(\bar{x}, i)$, $I^{0-}(\bar{x}, i)$, and $I^{00}(\bar{x}, i)$ are given by

$$\begin{aligned} I^{+0}(\bar{x}, i) &= \{\omega \in \Omega \mid G(\bar{x})(\omega)_i > 0,\ H(\bar{x})(\omega)_i = 0\}, \\ I^{0-}(\bar{x}, i) &= \{\omega \in \Omega \mid G(\bar{x})(\omega)_i = 0,\ H(\bar{x})(\omega)_i < 0\}, \\ I^{00}(\bar{x}, i) &= \{\omega \in \Omega \mid G(\bar{x})(\omega)_i = 0,\ H(\bar{x})(\omega)_i = 0\}. \end{aligned}$$

Observe that these concepts equal pointwise the concept of weak and strong stationarity for common finite-dimensional MPCCs, see Ye [2005].

The following result is taken from [Mehlitz and Wachsmuth, 2016, Theorem 3.1] and [Wachsmuth, 2015, Proposition 5.2].

Proposition 4.2. *Let $\bar{x} \in X$ be a local minimizer of* (MPCC)*. Suppose that the constraint qualification*

$$(g'(\bar{x}), G'(\bar{x}), H'(\bar{x}))[X] - \mathcal{R}_C(g(\bar{x})) \times S = Y \times L^2(\mathfrak{m}; \mathbb{R}^q)^2, \tag{9}$$

where $S \subset L^2(\mathfrak{m};\mathbb{R}^q)^2$ *is given by*

$$S := \left\{ (v,\eta) \in L^2(\mathfrak{m};\mathbb{R}^q)^2 \;\middle|\; \begin{array}{ll} v_i = 0 \text{ a.e. on } I^{0-}(\bar{x},i) \cup I^{00}(\bar{x},i), & \forall i \in \{1,\ldots,q\} \\ \eta_i = 0 \text{ a.e. on } I^{+0}(\bar{x},i) \cup I^{00}(\bar{x},i) & \end{array} \right\},$$

holds. Then $\bar{x}$ *is weakly stationary.*

Further, suppose that $(g'(\bar{x}), H'(\bar{x}), G'(\bar{x}))$ *is surjective. Then* $\bar{x}$ *is strongly stationary.*

From the theory of finite-dimensional or semidefinite MPCCs, other stationarity notions between weak and stationarity are known, see Ye [2005], Ding et al. [2014]. One of these concepts is Mordukhovich stationarity (M-stationarity for short). Here, the multipliers (μ,ν) corresponding to the complementarity constraints are assumed to be elements of the limiting normal cone to the complementarity set. Considering the more general complementarity condition

$$\tilde{G}(x) \in K, \quad \tilde{H}(x) \in K^\circ, \quad \langle \tilde{H}(x), \tilde{G}(x) \rangle = 0$$

for continuously Fréchet differentiable mappings $\tilde{G}\colon X \to W$ and $\tilde{H}\colon X \to W^\star$, a reflexive Banach space W, and a closed, convex cone $K \subset W$, this would lead to

$$(\mu,\nu) \in \mathcal{N}_{\mathcal{K}(K)}(\tilde{G}(\bar{x}), \tilde{H}(\bar{x})),$$

where $\mathcal{K}(K) := \{(u,\eta) \in K \times K^\circ \mid \langle \eta, u \rangle = 0\}$ defines the complementarity set induced by K. For $K := (\mathbb{R}^+_0)^q$, this condition takes the form

$$\begin{aligned} \forall i \in I^{+0}(\bar{x}): &\quad \mu_i = 0, \\ \forall i \in I^{0-}(\bar{x}): &\quad \nu_i = 0, \\ \forall i \in I^{00}(\bar{x}): &\quad \mu_i \cdot \nu_i = 0 \text{ or } (\mu_i < 0 \text{ and } \nu_i > 0), \end{aligned}$$

where the index sets $I^{+0}(\bar{x})$, $I^{0-}(\bar{x})$, and $I^{00}(\bar{x})$ are defined as stated below:

$$\begin{aligned} I^{+0}(\bar{x}) &:= \{i \in \{1,\ldots,q\} \mid \tilde{G}(\bar{x})_i > 0,\ \tilde{H}(\bar{x})_i = 0\}, \\ I^{0-}(\bar{x}) &:= \{i \in \{1,\ldots,q\} \mid \tilde{G}(\bar{x})_i = 0,\ \tilde{H}(\bar{x})_i < 0\}, \\ I^{00}(\bar{x}) &:= \{i \in \{1,\ldots,q\} \mid \tilde{G}(\bar{x})_i = 0,\ \tilde{H}(\bar{x})_i = 0\}. \end{aligned}$$

It is well known from Flegel and Kanzow [2006] that any local minimizer of a standard finite-dimensional MPCC is M-stationary if an MPCC-tailored MFCQ-type condition is satisfied.

Applying this idea to (MPCC), this yields the following definition. Note that we have $\mathcal{K}(\mathcal{C}) = \mathbb{C}$.

Definition 4.3. *Let* $\bar{x} \in X$ *be a feasible point of* (MPCC)*. We call* $\bar{x}$ *an M-stationary point if there exist multipliers* $\lambda \in Y^\star$ *and* $\mu,\nu \in L^2(\mathfrak{m};\mathbb{R}^q)$*, which satisfy* (7a), (7b)*, as well as*

$$(\mu,\nu) \in \mathcal{N}_{\mathbb{C}}(G(\bar{x}), H(\bar{x})). \tag{10}$$

Following the variational calculus of Mordukhovich, see Mordukhovich [2006], we obtain the following result.

Proposition 4.4. *Let $\bar{x} \in X$ be a local minimizer of (MPCC). Then any of the following constraint qualifications is sufficient for M-stationarity:*

1. *the operator $(g'(\bar{x}), H'(\bar{x}), G'(\bar{x}))$ is surjective,*
2. *the set $C \times \mathbb{C}$ is SNC at $(g(\bar{x}), G(\bar{x}), H(\bar{x}))$, the spaces X as well as Y are Asplund spaces, and*

$$\left.\begin{aligned} 0 &= g'(\bar{x})^\star\lambda + G'(\bar{x})^\star\mu + H'(\bar{x})^\star\nu, \\ \lambda &\in \mathcal{T}_C(g(\bar{x}))^\circ, \\ (\mu,\nu) &\in \mathcal{N}_{\mathbb{C}}(G(\bar{x}), H(\bar{x})) \end{aligned}\right\} \Longrightarrow \lambda = 0,\ \mu = 0,\ \nu = 0 \tag{11}$$

is satisfied.

Proof. Defining a continuously Fréchet differentiable map $F\colon X \to Y \times L^2(\mathfrak{m};\mathbb{R}^q)^2$ by $F(x) := (g(x), G(x), H(x))$ for any $x \in X$ and $\Theta := C \times \mathbb{C}$, (MPCC) is equivalent to

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{s.t.} \quad & F(x) \in \Theta. \end{aligned}$$

Let $M \subset X$ denote the feasible set of (MPCC). Since $\bar{x}$ is a local minimizer, the relation $-f'(\bar{x}) \in \mathcal{N}_M(\bar{x})$ is obtained, see [Mordukhovich, 2006, Proposition 5.1]. In order to verify the M-stationarity conditions, it is sufficient to show $\mathcal{N}_M(\bar{x}) \subset F'(\bar{x})^\star[\mathcal{N}_\Theta(F(\bar{x}))]$. The first postulated constraint qualification leads to the surjectivity of $F'(\bar{x})$ and [Mordukhovich, 2006, Theorem 1.17] applies. On the other hand, this inclusion also follows from [Mordukhovich, 2006, Theorem 3.8] when the second constraint qualification holds. □

A surprising observation regarding M-stationarity is mentioned in the following theorem.

Theorem 4.5. *Let $\bar{x} \in X$ be a feasible point of (MPCC). Then it is weakly stationary if and only if it is M-stationary.*

Proof. For brevity, we put $z := G(\bar{x})$ and $z^\star := H(\bar{x})$. In order to verify the theorem's assertion, we need to show

$$\mathcal{N}_{\mathbb{C}}(z, z^\star) = \left\{(\mu,\nu) \in L^2(\mathfrak{m};\mathbb{R}^q) \,\middle|\, (\mu,\nu) \text{ satisfies (7c) and (7d)}\right\}.$$

Therefore, we invoke Theorem 3.11. First, we obtain

$$\begin{aligned} \mathcal{N}_{\mathbb{C}}(z, z^\star) &\subset \mathcal{N}_{\mathbb{C}}^C(z, z^\star) \\ &= \left\{(\mu,\nu) \in L^2(\mathfrak{m};\mathbb{R}^q)^2 \,\middle|\, (\mu(\omega),\nu(\omega)) \in \overline{\operatorname{conv}}\, \mathcal{N}_{\Xi^q}(z(\omega), z^\star(\omega)) \text{ f.a.a } \omega \in \Omega\right\}. \end{aligned}$$

For fixed $i \in \{1, \ldots, q\}$, we have

$$\mathcal{N}_\Xi(z(\omega)_i, z^\star(\omega)_i) = \begin{cases} \{0\} \times \mathbb{R} & \text{if } \omega \in I^{+0}(\bar{x}, i), \\ \mathbb{R} \times \{0\} & \text{if } \omega \in I^{0-}(\bar{x}, i), \\ \big((-\mathbb{R}_0^+) \times \mathbb{R}_0^+\big) \cup \Xi & \text{if } \omega \in I^{00}(\bar{x}, i). \end{cases} \tag{12}$$

Thus, a pair $(\mu, \nu) \in \mathcal{N}_\mathbb{C}(z, z^\star)$ satisfies (7c) and (7d).

On the other hand, Theorem 3.11 yields

$$\operatorname{conv} \mathcal{N}_\mathbb{C}^S(z, z^\star) \subset \mathcal{N}_\mathbb{C}(z, z^\star).$$

Take a pair $(\mu, \nu) \in L^2(\mathfrak{m}; \mathbb{R}^q)^2$ which satisfies the conditions (7c) and (7d) and define $\mu^1 := 2\mu$, $\mu^2 := 0$, $\nu^1 := 0$, and $\nu^2 := 2\nu$. Using Lemma 3.9 and (12), $(\mu^1, \nu^1), (\mu^2, \nu^2) \in \mathcal{N}_\mathbb{C}^S(z, z^\star)$ follows. Thus, we have

$$(\mu, \nu) = \tfrac{1}{2}(\mu^1, \nu^1) + \tfrac{1}{2}(\mu^2, \nu^2) \in \operatorname{conv} \mathcal{N}_\mathbb{C}^S(z, z^\star) \subset \mathcal{N}_\mathbb{C}(z, z^\star).$$

This completes the proof. □

From Corollary 3.7 and the proof of the above theorem, we obtain the following result.

Corollary 4.6. *Let $\bar{x} \in X$ be a feasible point of (MPCC). Then we have*

$$\widehat{\mathcal{N}}_\mathbb{C}(G(\bar{x}), H(\bar{x})) = \left\{ (\mu, \nu) \in L^2(\mathfrak{m}; \mathbb{R}^q)^2 \;\middle|\; \begin{array}{l} \mu_i = 0 \text{ a.e. on } I^{+0}(\bar{x}, i), \\ \nu_i = 0 \text{ a.e. on } I^{0-}(\bar{x}, i), \\ \mu_i \le 0 \text{ a.e. on } I^{00}(\bar{x}, i), \\ \nu_i \ge 0 \text{ a.e. on } I^{00}(\bar{x}, i) \end{array} \quad \forall i = 1, \ldots, q \right\},$$

$$\mathcal{N}_\mathbb{C}(G(\bar{x}), H(\bar{x})) = \left\{ (\mu, \nu) \in L^2(\mathfrak{m}; \mathbb{R}^q)^2 \;\middle|\; \begin{array}{l} \mu_i = 0 \text{ a.e. on } I^{+0}(\bar{x}, i), \\ \nu_i = 0 \text{ a.e. on } I^{0-}(\bar{x}, i) \end{array} \quad \forall i = 1, \ldots, q \right\}.$$

Hence, the conditions (7c), (7d), and (8) are equivalent to $(\mu, \nu) \in \widehat{\mathcal{N}}_\mathbb{C}(G(\bar{x}), H(\bar{x}))$. Recalling the convexity of C and the product rule for Fréchet and limiting normals, see [Mordukhovich, 2006, Proposition 1.2], a feasible point $\bar{x} \in X$ of (MPCC) is strongly stationary if and only if

$$-f'(\bar{x}) \in (g'(\bar{x}), G'(\bar{x}), H'(\bar{x}))^\star \big[\widehat{\mathcal{N}}_{C \times \mathbb{C}}(g(\bar{x}), G(\bar{x}), H(\bar{x}))\big]$$

is satisfied. It is M-stationary if and only if

$$-f'(\bar{x}) \in (g'(\bar{x}), G'(\bar{x}), H'(\bar{x}))^\star \big[\mathcal{N}_{C \times \mathbb{C}}(g(\bar{x}), G(\bar{x}), H(\bar{x}))\big]$$

holds.

Remark 4.7. *Suppose that $\bar{x} \in X$ is a feasible point of (MPCC) where the constraint qualification (9) is satisfied. Polarizing both sides of this equation, it is easily seen that the constraint qualification (11) holds as well. The converse does not hold true in general.*

Clearly, the first constraint qualification presented in Proposition 4.4 already implies strong stationarity, see Proposition 4.2, which is stronger than M-stationarity by means of Theorem 4.5. However, the second constraint qualification fails to be applicable as the following lemma shows.

Lemma 4.8. *Suppose that $\bar{x} \in X$ is a feasible point of (MPCC). Then the set $\mathbb{C}$ is not SNC at $(G(\bar{x}), H(\bar{x}))$.*

Proof. First, suppose that $I^{00}(\bar{x}, 1)$ is a set of positive measure. Let $A := I^{00}(\bar{x}, 1)$ hold in the setting of Lemma 2.5 and let $\{M_n\} \subset \Sigma$ be the sequence defined therein. For any $n \in \mathbb{N}$, we define

$$\forall \omega \in \Omega: \quad \mu_n(\omega)_1 := -\mathfrak{m}(M_n)^{-\frac{1}{2}} \chi_{M_n}(\omega), \quad \nu_n(\omega)_1 := \mathfrak{m}(M_n)^{-\frac{1}{2}} \chi_{M_n}(\omega)$$

and

$$\forall \omega \in \Omega \, \forall i \in \{2, \ldots, q\}: \quad \mu_n(\omega)_i := \nu_n(\omega)_i := 0.$$

Clearly, we have $(\mu_n, \nu_n) \in \widehat{\mathcal{N}}_{\mathbb{C}}(G(\bar{x}), H(\bar{x}))$ from Corollary 4.6. Choose an arbitrary function $\xi \in L^2(\mathfrak{m}; \mathbb{R}^q)$. Then we obviously have $\xi \in L^2(\mathfrak{m}|_{M_n}; \mathbb{R}^q)$, i.e., Hölder's inequality yields

$$\begin{aligned} |\langle \xi_1, \mu_{n,1} \rangle| &= \mathfrak{m}(M_n)^{-\frac{1}{2}} \left| \int_{M_n} \xi_1(\omega) \, \mathrm{d}\omega \right| \\ &\leq \mathfrak{m}(M_n)^{-\frac{1}{2}} \left(\int_{M_n} 1 \, \mathrm{d}\omega \right)^{\frac{1}{2}} \left(\int_{M_n} \xi_1^2(\omega) \, \mathrm{d}\omega \right)^{\frac{1}{2}} = \left(\int_{M_n} \xi_1^2(\omega) \, \mathrm{d}\omega \right)^{\frac{1}{2}}. \end{aligned}$$

The latter integral tends to zero as n goes to infinity, see Lemma 2.6. Hence, we have $\mu_n \rightharpoonup 0$. On the other hand, we easily calculate

$$\forall n \in \mathbb{N}: \quad \|\mu_n\|_{L^2(\mathfrak{m};\mathbb{R}^q)} = \mathfrak{m}(M_n)^{-\frac{1}{2}} \left(\int_{M_n} 1 \, \mathrm{d}\omega \right)^{\frac{1}{2}} = 1,$$

i.e., $\mu_n \not\to 0$. Similarly, we can show $\nu_n \rightharpoonup 0$ and $\nu_n \not\to 0$. Hence, $\mathbb{C}$ is not SNC at the point $(G(\bar{x}), H(\bar{x}))$.

Suppose that $I^{00}(\bar{x}, 1)$ is a set of measure zero (w.l.o.g. we assume $I^{00}(\bar{x}, 1) = \varnothing$). Then either $I^{+0}(\bar{x}, 1)$ or $I^{0-}(\bar{x}, 1)$ possesses positive measure. We assume first that $\mathfrak{m}(I^{+0}(\bar{x}, 1)) > 0$ holds. Again, we use Lemma 2.5 with $A := I^{+0}(\bar{x}, 1)$ and consider the corresponding sequence $\{M_n\}$. Let us define

$$\forall n \in \mathbb{N} \, \forall \omega \in \Omega: \quad u_n(\omega)_1 := \begin{cases} (1 - \chi_{M_n}(\omega)) \, G(\bar{x})(\omega)_1 & \text{if } \omega \in I^{+0}(\bar{x}, 1), \\ 0 & \text{if } \omega \in I^{0-}(\bar{x}, 1). \end{cases}$$

as well as

$$\forall i \in \{2, \ldots, q\} \, \forall \omega \in \Omega: \quad u_n(\omega)_i := G(\bar{x})(\omega)_i.$$

From $\chi_{M_n} \to 0$ we have $u_n \to G(\bar{x})$. We set for any $n \in \mathbb{N}$

$$\forall \omega \in \Omega: \quad \mu_n(\omega)_1 := -\mathfrak{m}(M_n)^{-\frac{1}{2}} \chi_{M_n}(\omega), \quad \nu_n(\omega)_1 := 0$$

and

$$\forall i \in \{2, \ldots, q\} \, \forall \omega \in \Omega: \quad \mu_n(\omega)_i := \nu_n(\omega)_i := 0.$$

Thus, we obtain $(\mu_n, \nu_n) \in \widehat{\mathcal{N}}_{\mathbb{C}}(u_n, H(\bar{x}))$ for all $n \in \mathbb{N}$ from Corollary 4.6. As above, we have $\mu_n \rightharpoonup 0$ but $\mu_n \not\to 0$. Consequently, $\mathbb{C}$ is not SNC at $(G(\bar{x}), H(\bar{x}))$. Similarly, we can proceed in the case where $I^{0-}(\bar{x}, 1)$ is a set of positive measure. □

The above argumentation depicts that Mordukhovich's stationarity concept seems not to be appropriate for (MPCC): it is equivalent to weak stationarity and corresponding constraint qualifications are either too strong or not applicable.

On the other hand, one easily sees from Lemma 3.9 that the condition

$$-f'(\bar{x}) \in (g'(\bar{x}), G'(\bar{x}), H'(\bar{x}))^\star \big[\mathcal{N}^S_{C \times \mathbb{C}}(g(\bar{x}), G(\bar{x}), H(\bar{x}))\big] \tag{13}$$

coincides with the pointwise concept of M-stationarity which is known from the theory of finite-dimensional MPCCs. It is a question of future research under which constraint qualifications a local minimizer $\bar{x} \in X$ of (MPCC) satisfies this condition. Particularly, it is necessary to find conditions implying

$$\mathcal{N}^S_M(\bar{x}) \subset (g'(\bar{x}), G'(\bar{x}), H'(\bar{x}))^\star \big[\mathcal{N}^S_{C \times \mathbb{C}}(g(\bar{x}), G(\bar{x}), H(\bar{x}))\big],$$

where M denotes the feasible set of (MPCC), since we already have $-f'(\bar{x}) \in \mathcal{N}^S_M(\bar{x})$ from [Mordukhovich, 2006, Proposition 5.1]. Although there are only a few calculus rules for strong limiting normals available yet, it might be possible to exploit the product structure of the set $\mathbb{C}$ and, e.g., [Mordukhovich, 2006, Theorem 3.4, Lemma 5.58] to derive (13) using different techniques. This, however, is clearly beyond the scope of this paper.

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