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Andreas Hehl, Ira Neitzel

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SPP1962 at Weierstrass Institute for Applied Analysis and Stochastics (WIAS)
Leibniz Institute in the Forschungsverbund Berlin e.V.
Mohrenstraße 39, 10117 Berlin, Germany
E-Mail: spp1962@wias-berlin.de

World Wide Web: http://spp1962.wias-berlin.de/

Local quadratic convergence of the SQP method for an optimal control problem governed by a regularized fracture propagation model

Andreas Hehl and Ira Neitzel

Institut für Numerische Simulation
Rheinische Friedrich-Wilhelms-Universität Bonn
Friedrich-Hirzebruch-Allee 7, 53115 Bonn (Germany)

Abstract. We prove local quadratic convergence of the sequential quadratic programming (SQP) method for an optimal control problem of tracking type governed by the Euler-Lagrange equation of a time-discrete regularized fracture or damage energy minimization problem. This lower-level energy minimization describing the fracture process contains a penalization term for violation of the irreversibility condition in the fracture growth process, as well as a viscous regularization (corresponding to a time-step restriction) to obtain convexity. Nonetheless, due to the quasilinear structure of the Euler-Lagrange equations, the control problem is nonconvex. For the convergence proof, we follow the approach from [**53**], utilizing strong regularity of generalized equations.

1. Introduction

In this work, we analyze the convergence of the sequential quadratic programming (SQP) method applied to an optimal control problem for regularized fracture propagation including control constraints. The model problem is the same as in e.g. [**30**], and closely related to [**46, 47**]. It is of tracking-type, with a control q in a control set Q_{ad} acting as a boundary force, with associated state pair $\mathbf{u} = (u, \varphi)$ in a state space V consisting of a displacement u and a phase-field φ. It reads:

$$
(\mathrm{NLP}^{\gamma,\eta}) \quad \begin{cases} \min\limits_{q \in Q_{\mathrm{ad}},\, \mathbf{u} \in V} & J(q, \mathbf{u}) := \dfrac{1}{2}\|u - u_d\|^2_{L^2(\Omega,\mathbb{R}^2)} + \dfrac{\alpha}{2}\|q\|^2_{L^2(\Gamma)}, \\ \text{subject to :} & A(\mathbf{u}) + R(\varphi;\gamma) = B(q). \qquad\qquad (\mathrm{EL}^{\gamma,\eta}) \end{cases}
$$

We will precisely define the mathematical setting, including the operators A, R, and B in Section 2 below. The problem stems from a bi-level optimization problem with an upper-level tracking type functional, and a lower-level energy minimization problem for variational fracture propagation. In fact, we consider one time step of a time discrete and spatially continuous problem, where a regularized version of an energy minimization problem describing the lower-level fracture propagation is eventually replaced by its Euler-Lagrange equations. A nonregularized version of the fracture propagation has originally been considered in [**12, 13, 21**]. To avoid the irregular fracture set, an Ambrosio-Tortorelli regularization cf. [**5**] is used; i.e. an additional phase-field variable φ is introduced to replace the irregular Hausdorff

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measure. The phase-field variable φ has values in $[0,1]$, and describes the condition of the material at every point in the domain, with $\varphi = 1$ where the material is completely sound, and $\varphi = 0$ where the material is fully broken, guaranteeing a smooth transition between those two states. We apply a viscous regularization to guarantee strict convexity of the lower-level minimization problem and eventually unique solvability of its Euler-Lagrange equation. Conditions on the viscous regularization parameter η correspond to a time step restriction in the temporal discretization of the problem, see [**41**]. Nevertheless, the Euler-Lagrange equations are of quasilinear type, making the overall control problem nonconvex. Finally, a violation of the irreversibility condition in the fracture growth process is penalized using a regularization with parameter γ in form of a (higher-order) penalization, as in [**42**]. The corresponding terms appear in the operator R in the differential equation, whereas the differential operator A stems from the actual (regularized) fracture propagation process. For a more detailed description of the mathematical and physical background of $(\mathrm{NLP}^{\gamma,\eta})$, we refer to the introductions of [**30**, **46**].

Further approaches in the field of the optimization involving a fracture setting are given in [**19**, **20**], where the control of a viscous damage model was considered in a continuous setting, and [**1**, **45**], where shape optimization was used. An approach where the propagation of a crack was limited through controlling the release of the associated energy was used in [**17**, **34**]. An optimal control problem of a two-field damage model, and a nonsmooth (viscous damage) coupled system, was analyzed in [**6**, **51**]. For results concerning the lower-level fracture problem, we refer the interested reader to [**59**], where modelling and numerical analysis of multiphysics phase-field fracture models are addressed. Phase-field models are also applicable in other fields, like material science, medical applications and image segmentation. For the former, we refer to [**7–9**]. In the context of tumor growth, phase-field models have been used in e.g. [**22**, **23**]. For the latter, the analysis of the Mumford-Shah image segmentation functional [**44**] through phase-field methods [**5**, **10**, **11**] is still an active field of research, see e.g. [**50**]. For an overview about numerical implementation of phase-field models, we refer to [**16**].

Let us give a brief summary of the current state of research for the model problem $(\mathrm{NLP}^{\gamma,\eta})$: Existence of solutions and first-order necessary optimality conditions for the model problem, under an additional trivial kernel assumption, but without a viscous approximation and control constraints, have been proven in [**46**]. In [**47**], convergence of regularized solutions with respect to γ was proven. Subsequently, convergence (w.r.t γ) of the dual variables was established in [**29**]. Finite element discretization error estimates have been derived for a linearized fracture control problem in [**43**], and algorithmic concepts, respectively the space-time formulation and time discretization, were studied in [**39**, **40**]. Further, in [**29**] the sequential quadratic programming (SQP) method for $(\mathrm{NLP}^{\gamma,\eta})$ was described, and a preliminary analysis of the underlying quadratic subproblem was made, under an additional rather strong local coercivity condition, cf. Section 4. This is the starting point of our analysis. Utilizing second-order sufficient conditions we carry out a rigorous convergence analysis. We can rely on the results from [**30**], where we investigated second-order necessary and second-order sufficient conditions (SSC) with minimal gap and without two-norm discrepancy. It is well known that SSCs are commonly the basis for convergence proofs of the SQP method. For an introduction we refer the interested reader to e.g. the introduction of [**24**]. For SQP of control constrained problems governed by semilinear elliptic and parabolic equations we refer to [**24, 52, 53, 56, 57**], for semilinear problems with mixed control-state-constraints to [**25, 26**], and for the Navier-Stokes equation to [**32, 35, 36, 58**]. In [**31, 33**], the SQP method for the optimal control of a (semilinear) phase-field equation was considered. Only recently, convergence of the SQP method for quasilinear parabolic optimal optimal control in function space setting was proven in [**37**].

We will continue the work established in [**29, 30**], and analyze the SQP method applied to the quasilinear fracture control problem ($\mathrm{NLP}^{\gamma,\eta}$), utilizing the typical procedure of proving convergence of SQP methods in infinite dimensional spaces that goes back to [**3**]. We follow the ideas of e.g. [**26, 53**] and apply Newton's method to a generalized equation that corresponds to the necessary optimality conditions of the model problem. A Newton-Kantorovich like convergence theorem, cf. [**3, 4, 38**] will then ensure local quadratic convergence of the generated sequence. This theorem relies in particular on the so-called strong regularity property, cf. [**49**], which allows to generalize the implicit function theorem to generalized equations. It was later used to show convergence of Newton's method in the context of (unconstrained) optimal control in Banach spaces in [**18**]. Ensuring this strong regularity property and additional e.g. Lipschitz results for our model problem requires a careful, nontrivial analysis. We benefit from results that we have proven in the context of SSC in [**30**].

Strong regularity is closely related to second-order sufficient conditions (SSCs). Let us therefore briefly comment on different types of second-order optimality conditions. On the one hand, it is preferable to keep the gap between the necessary and sufficient conditions as close as possible, which leads to SSC incorporating so-called strongly active constraints, cf. [**14, 15**]. We have established such a result for ($\mathrm{NLP}^{\gamma,\eta}$) in [**30**]. Yet, this only ensures coercivity on some subspace of Q_{ad} and it is not clear that the directions generated by the SQP method belong to this subset. On the other hand, choosing an SSC on the whole control space Q is a very strong assumption. In this work, we will use so called σ-strongly active constraints, see e.g. [**53**]. Following the ideas of [**53**], we will establish convergence of the SQP method for certain auxiliary quadratic subproblems. In a second step, we will show that the solutions of the auxiliary problems in fact correspond to the solution of the SQP method for ($\mathrm{NLP}^{\gamma,\eta}$), restricted to a neighborhood of the optimal solution.

The outline of the present work is as follows: We start with a detailed description of the problem setting including all assumptions on the model problem as well as the notation used, in Section 2. In Section 3, we collect regularity and existence result for ($\mathrm{EL}^{\gamma,\eta}$) and solvability as well as necessary and sufficient optimality conditions for ($\mathrm{NLP}^{\gamma,\eta}$), respectively. In Section 4, we describe the SQP method for ($\mathrm{NLP}^{\gamma,\eta}$), and start with some preliminary considerations about the quadratic SQP subproblem. In Section 5 we develop convergence results for auxiliary problems via the strong regularity property, which we transfer to the SQP method for ($\mathrm{NLP}^{\gamma,\eta}$) from Section 4.

2. Problem setting and assumptions

In this section, we state the precise setting of the model problem, which is the same as in [**30**]. Let us recall the problem formulation

$$(\mathrm{NLP}^{\gamma,\eta}) \qquad \begin{cases} \min\limits_{q\in Q_{\mathrm{ad}},\mathbf{u}\in V} & J(\mathbf{u},q) := \dfrac{1}{2}\|u-u_d\|^2_{L^2(\Omega,\mathbb{R}^2)} + \dfrac{\alpha}{2}\|q\|^2_{L^2(\Gamma)}, \\ \text{subject to}: & A(\mathbf{u}) + R(\varphi;\gamma) = B(q), \end{cases} \qquad (\mathrm{EL}^{\gamma,\eta})$$

from the introduction.

We assume $\Omega \subset \mathbb{R}^2$ to be a polygonal Gröger regular domain, cf. [**27**], with boundary $\delta\Omega = \Gamma\dot{\cup}\Gamma_D$, where Γ is the Neumann part of the boundary on which q acts as a boundary force. The remaining part of $\delta\Omega$ is denoted by Γ_D, on which homogeneous Dirichlet boundary conditions are prescribed. As in [**47**, Section 2], Ω is also assumed to be $W^{2,q}$-regular for the homogeneous Neumann-problem $-\varepsilon\nabla\varphi + \frac{1}{\varepsilon}\varphi = f$.

The given function $u_d \in L^2(\Omega,\mathbb{R}^2)$ denotes a desired displacement, and the Tikhonov cost parameter α is a fixed positive real number. The control space Q is given by $Q = L^2(\Gamma)$, and for $q_a, q_b \in L^\infty(\Gamma)$ with $q_a < q_b$ a.e. on Γ, the set of admissible controls is denoted by

$$Q_{\mathrm{ad}} := \{q \in Q \quad | \quad q_a \le q \le q_b \quad \text{a.e. on } \Gamma\}.$$

The state $\mathbf{u} = (u,\varphi) \in V$ consists of a pair of functions, with displacement u and phase-field φ. We fix some general notation for function spaces, along with the definition of the state space V. For $p > 2$, $q := p/2 > 1$, we define the spaces

$$\begin{aligned} V_u &:= H^1_D(\Omega;\mathbb{R}^2) := \{v \in H^1(\Omega;\mathbb{R}^2) \,|\, v = 0 \text{ on } \Gamma_D\}, & V_\varphi &:= H^1(\Omega), \\ W_u &:= W^{1,p}_D(\Omega;\mathbb{R}^2), & W_\varphi &:= W^{2,q}(\Omega), \\ V &:= V_u \times V_\varphi, & W &:= W_u \times W_\varphi, \\ W^\times &:= W^{-1,p}(\Omega;\mathbb{R}^2) \times L^q(\Omega), \end{aligned}$$

and for better readability we introduce the short notation

$$Y := W \times Q \times W \qquad \text{and} \qquad Z := W^\times \times Q \times W^\times. \tag{2.1}$$

We will frequently use the notation $y = (\mathbf{u}, q, \mathbf{z}) \in Y$ for functions triples consisting of a state $\mathbf{u}$, control q, and an adjoint state $\mathbf{z}$ (to be introduced later).

We understand that all spaces are defined on the domain Ω unless otherwise stated, and often omit the dependency on Ω for the sake of readability. For norms and inner products, we agree that $(\cdot,\cdot)$ denotes the usual L^2-inner product with corresponding norm $\|\cdot\|$, and $(\cdot,\cdot)_Q$ corresponds to the inner product of Q i.e. $L^2(\Gamma)$. For functions $\mathbf{v} = (v^u, v^\varphi) \in W$, the norm in the space W is given by

$$\|\mathbf{v}\|_W = \|(v^u, v^\varphi)\|_W = \|v^u\|_{1,p} + \|v^\varphi\|_{2,q}.$$

We will denote dual spaces with a superscript *, e.g., V^*, and agree that $\langle\cdot,\cdot\rangle$ stands for a duality pairing where the spaces are omitted if obvious from the context, otherwise denoted by a subscript. Note that for our choice of p, q and spatial dimensions $N = 2$, we have $W_u \hookrightarrow V_u$ and $W_\varphi \hookrightarrow V_\varphi$ by the Sobolev-Kondrachov theorem. Further, $W_\varphi \hookrightarrow L^\infty$ holds, which will often be used without further notice. Lastly, let us introduce $B^{\mathbb{V}}_r(v)$ as the open ball of radius r centered at $v \in \mathbb{V}$ w.r.t the norm of $\mathbb{V}$, where $\mathbb{V}$ can be any Banach space.

As explained in the introduction, the equation ($\mathrm{EL}^{\gamma,\eta}$) is in fact a necessary optimality condition of an energy minimization problem. The operators involved are the so called nonlinear phase-field operator operator $A\colon V \supset W \to V^*$, the penalization operator $R\colon V_\varphi \to V_\varphi^*$, and the control-action operator on the Neumann boundary Γ, $B\colon Q \to V^*$, which for $\mathbf{u} = (u,\varphi) \in W$, with $0 \le \varphi \le 1$, are

$$\begin{aligned}
\langle A(\mathbf{u}), \mathbf{v}\rangle :&= \Big(g(\varphi)\mathbb{C}e(u), e(v^u)\Big) + \epsilon(\nabla\varphi, \nabla v^\varphi) - \frac{1}{\epsilon}(1-\varphi, v^\varphi) \\
&\quad + \eta(\varphi - \varphi^-, v^\varphi) + (1-\kappa)(\varphi\mathbb{C}e(u) : e(u), v^\varphi), \\
\langle R(\varphi;\gamma), v^\varphi\rangle :&= \gamma([(\varphi-\varphi^-)^+]^3, v^\varphi), \\
\langle Bq, (v^u, v^\varphi)\rangle :&= (q, v^u)_Q,
\end{aligned}$$

for all $\mathbf{v} = (v^u, v^\varphi) \in V$ and given phase-field $\varphi^- \in W_\varphi$ with $0 \le \varphi^- \le 1$.

The operators A and R will also be used as mappings into the more regular spaces $W^\times$ and L^q, where we will use test functions $\mathbf{v} \in V$, since $W^\times \hookrightarrow V^*$. The parameter $\varepsilon > 0$ is a fixed phase-field parameter. Further, let $\kappa > 0$ and $g(x) := (1-\kappa)x^2 + \kappa$. Both κ and g appear in the problem due to an Ambrosio-Tortorelli regularization [**5**] and an additional regularization for the elastic energy degeneracy. For more details, we point to [**46**, Section 2]. Moreover, $\mathbb{C}$ denotes the rank-4 elasticity tensor with usual properties, cf. [**48**, Section 3] and a sufficiently large $\eta \ge 0$ will be referred to as the (viscosity) regularization parameter, cf. [**41**] and also [**29, 30, 47**] in the context of ($\mathrm{NLP}^{\gamma,\eta}$). For sufficiently large η, unique solvability of ($\mathrm{EL}^{\gamma,\eta}$) as well as differentiability of the corresponding control-to-state-operator are known. We will frequently make use of results from [**30, 47**], that hold under such a condition, we therefore tacitly assume:

Assumption 2.1 (Viscous approximation). Let $\eta \ge 0$ be chosen large enough for all results and calculations of the following sections that depend on such a condition on η.

Finally, the given parameter $\gamma > 0$ is called the penalization parameter. It stems from the regularization of the irreversibility condition of the fracture problem. Originally of 4th-order in the energy minimization problem, i.e. $\frac{\gamma}{2}\|(\varphi-\varphi^-)^+\|^4_{L^4(\Omega)}$, this penalization leads to the R-term in the Euler-Lagrange equations after differentiation, cf. [**42**]. The high order is needed to ensure the second-order differentiability of ($\mathrm{EL}^{\gamma,\eta}$) needed for the SQP method. Note that $\varphi^- \in W_\varphi$ is actually the phase-field of the previous time step if more than one time-step is considered.

3. Preliminary results for ($\mathrm{EL}^{\gamma,\eta}$) and ($\mathrm{NLP}^{\gamma,\eta}$)

In this section we give a quick overview about available theoretical results, cf. e.g. [**29, 30, 47**].

3.1. The Euler-Lagrange equation ($\mathrm{EL}^{\gamma,\eta}$). We briefly summarize results concerning solvability of ($\mathrm{EL}^{\gamma,\eta}$), its linearization, and the associated solution operators. The proofs can be found in [**30**] and the references therein, cf. [**28, 46, 47**]. By [**30**, Lemma 3.1], for $\eta \ge 0$ being sufficiently large and $0 \le \varphi^- \in W_\varphi$, we know that ($\mathrm{EL}^{\gamma,\eta}$) has a unique weak solution $\mathbf{u} \in W$ for every $q \in Q$, i.e. $\mathbf{u} \in V$ satisfies

$$\langle A(\mathbf{u}), \mathbf{v}\rangle + \langle R(\varphi;\gamma), v^\varphi\rangle = \langle B(q), \mathbf{v}\rangle \qquad \forall\, \mathbf{v} \in V.$$

We denote the associated solution operator, also frequently called the control-to-state-operator, by

$$G\colon Q \to W, \quad G(q) = \mathbf{u} = (u, \varphi). \tag{3.1}$$

By [**30**, Proposition 3.3], we know that again for $\eta \geq 0$ sufficiently large, G, respectively A and R, are twice continuously Fréchet-differentiable from Q into W, respectively from W into $W^\times$ and from W_φ into L^q. The first derivative of G, $\tilde{\mathbf{u}} := G'(q)\tilde{q}$, $\tilde{\mathbf{u}} = (\tilde{u}, \tilde{\varphi}) \in W$, is the solution of

$$A'(\mathbf{u})\tilde{\mathbf{u}} + R'(\varphi;\gamma)\tilde{\varphi} = B(\tilde{q}), \tag{3.2}$$

for $\mathbf{u} = G(q)$ and $(\tilde{q}, 0) \in W^\times$, cf. also Lemma 3.1 in the following. The second derivative $\mathbf{y} := G''(q)[\tilde{q}_1, \tilde{q}_2]$, $\mathbf{y} = (\mathrm{y}^u, \mathrm{y}^\varphi) \in W$ is the solution of

$$A'(\mathbf{u})\mathbf{y} + R'(\varphi;\gamma)\mathrm{y}^\varphi = -A''(\mathbf{u})[\tilde{\mathbf{u}}_1, \tilde{\mathbf{u}}_2] - R''(\varphi;\gamma)[\tilde{\varphi}_1, \tilde{\varphi}_2], \tag{3.3}$$

for $\mathbf{u} = G(q)$, and $\tilde{\mathbf{u}}_i = G'(q)\tilde{q}_i$, $i = 1, 2$.

In the above, the operators $A'(\mathbf{u})\colon V \to V^*$, $A''(\mathbf{u})\colon W \times W \to V^*$, $R'(\varphi,\gamma)\colon V_\varphi \to V_\varphi^*$, and $R''(\varphi;\gamma)\colon W_\varphi \times W_\varphi \to V_\varphi^*$ are given by

$$\begin{aligned}
\langle A'(\mathbf{u})\tilde{\mathbf{u}}, \mathbf{v}\rangle &:= \Big(g(\varphi)\mathbb{C}e(\tilde{u}), e(v^u)\Big) + 2(1-\kappa)(\varphi\mathbb{C}e(u) : e(\tilde{u}), v^\varphi) \\
&\quad + 2(1-\kappa)(\varphi\mathbb{C}e(u)\tilde{\varphi}, e(v^u)) + \epsilon(\nabla\tilde{\varphi}, \nabla v^\varphi) + \frac{1}{\epsilon}(\tilde{\varphi}, v^\varphi) \\
&\quad + \eta(\tilde{\varphi}, v^\varphi) + (1-\kappa)(\tilde{\varphi}\mathbb{C}e(u) : e(u), v^\varphi), \\
\langle A''(\mathbf{u})[\tilde{\mathbf{u}}_1, \tilde{\mathbf{u}}_2], \mathbf{v}\rangle &:= 2(1-\kappa)\Big[(\tilde{\varphi}_2\mathbb{C}e(u)\tilde{\varphi}_1, e(v^u)) + (\tilde{\varphi}_2\mathbb{C}e(\tilde{u}_1)\varphi, e(v^u)) \\
&\quad + (\tilde{\varphi}_2\mathbb{C}e(u) : e(\tilde{u}_1), v^\varphi) + (\varphi\mathbb{C}e(\tilde{u}_2)\tilde{\varphi}_1, e(v^u)) \\
&\quad + (\tilde{\varphi}_1\mathbb{C}e(\tilde{u}_2) : e(u), v^\varphi) + (\varphi\mathbb{C}e(\tilde{u}_2) : e(\tilde{u}_1), v^\varphi)\Big], \\
\langle R'(\varphi;\gamma)\tilde{\varphi}, v^\varphi\rangle &:= 3\gamma([(\varphi - \varphi^-)^+]^2\tilde{\varphi}, v^\varphi), \\
\langle R''(\varphi;\gamma)[\tilde{\varphi}_1, \tilde{\varphi}_2], v^\varphi\rangle &:= 6\gamma([(\varphi - \varphi^-)^+]\tilde{\varphi}_1\tilde{\varphi}_2, v^\varphi),
\end{aligned}$$

for all $\mathbf{v} \in V$. Note that the operators $A'(\mathbf{u})$ and $R'(\varphi;\gamma)$ are self-adjoint, cf. [**30**, Subsection 3.3], and as for A and R, we can also use A' and R' as mappings from W into $W^\times$ and from W_φ into L^q, respectively. We will therefore also use test functions $\mathbf{v} \in V$ when working with the operators A', A'', R', and R''.

Let us also collect some boundedness results for A', R', A'' and R'' for later use. For that, let $\mathbf{u}, \tilde{\mathbf{u}} \in W$, then an easy calculation similar to [**30**, Lemma 3.9] from the definition of A' and R' ensures the existence of a constant $c > 0$, such that

$$\|A'(\mathbf{u})\tilde{\mathbf{u}}\|_{W^\times} \leq c\|\mathbf{u}\|_W^2\|\tilde{\mathbf{u}}\|_W, \tag{3.4}$$

$$\|R'(\varphi;\gamma)\tilde{\varphi}\|_q \leq c\|\varphi\|_\infty^2\|\tilde{\varphi}\|_\infty \leq c\|\mathbf{u}\|_W^2\|\tilde{\mathbf{u}}\|_W. \tag{3.5}$$

Let additionally $\mathbf{z} \in W$. Then, from the calculations in the proof of [**30**, Lemma 3.9], the linearity and symmetry of the terms $\tilde{\mathbf{u}}_1, \tilde{\mathbf{u}}_2, \mathbf{z}$ in the definition of $\langle A''(\mathbf{u})[\tilde{\mathbf{u}}_1, \tilde{\mathbf{u}}_2], \mathbf{z}\rangle$, as well as $\tilde{\varphi}_1, \tilde{\varphi}_2, z^\varphi$ in the definition of $\langle R''(\varphi;\gamma)[\tilde{\varphi}_1, \tilde{\varphi}_2], v^\varphi\rangle$, respectively, we find

$$\|A''(\mathbf{u})[\tilde{\mathbf{u}}, \cdot]^*\mathbf{z}\|_{W^\times} \leq c\|\mathbf{u}\|_W\|\tilde{\mathbf{u}}\|_W\|\mathbf{z}\|_W, \tag{3.6}$$

$$\|R''(\varphi;\gamma)[\tilde{\varphi}, \cdot]^*z^\varphi\|_q \leq c\|\varphi\|_\infty\|\tilde{\varphi}\|_\infty\|z^\varphi\|_\infty \leq c\|\mathbf{u}\|_W\|\tilde{\mathbf{u}}\|_W\|\tilde{\mathbf{z}}\|_W, \tag{3.7}$$

for a constant $c > 0$. Note in particular that there exists a $c > 0$, such that for $\mathbf{u}, \tilde{\mathbf{u}}_1, \tilde{\mathbf{u}}_2, \mathbf{z} \in W$, the boundedness results

$$|\langle A''(\mathbf{u})[\tilde{\mathbf{u}}_1, \tilde{\mathbf{u}}_2], \mathbf{v}\rangle| \le c\|\mathbf{u}\|_W \|\tilde{\mathbf{u}}_1\|_W \|\tilde{\mathbf{u}}_2\|_W \|\mathbf{v}\|_W, \tag{3.8}$$

$$|\langle R''(\varphi;\gamma)[\tilde{\varphi}_1, \tilde{\varphi}_2], v^\varphi\rangle| \le c\|\mathbf{u}\|_W \|\tilde{\mathbf{u}}_1\|_W \|\tilde{\mathbf{u}}_2\|_W \|\mathbf{v}\|_W \tag{3.9}$$

hold. Moreover, analogously to the estimations made in the proof of [**30**, Lemma 3.9], we obtain local Lipschitz continuity results for A' and R', i.e. for all $\tilde{\mathbf{u}}, \mathbf{u}_1, \mathbf{u}_2 \in W$, there exists a constant $c > 0$, such that

$$\|\big(A'(\mathbf{u}_1) - A'(\mathbf{u}_2)\big)\tilde{\mathbf{u}}\|_{W^\times} \le c\big[\|\mathbf{u}_1\|_W + \|\mathbf{u}_2\|_W\big]\|\mathbf{u}_1 - \mathbf{u}_2\|_W \|\tilde{\mathbf{u}}\|_W, \tag{3.10}$$

$$\|\big(R'(\varphi_1;\gamma) - R'(\varphi_2;\gamma)\big)\tilde{\varphi}\|_q \le c\big[\|\mathbf{u}_1\|_W + \|\mathbf{u}_2\|_W\big]\|\mathbf{u}_1 - \mathbf{u}_2\|_W \|\tilde{\mathbf{u}}\|_W. \tag{3.11}$$

Furthermore, for all $\mathbf{u}_1, \mathbf{u}_2, \tilde{\mathbf{u}}, \mathbf{z} \in W$, there exists a $c > 0$, such that

$$\|(A''(\mathbf{u}_1) - A''(\mathbf{u}_2))[\tilde{\mathbf{u}}, \cdot]^*\mathbf{z}\|_{W^\times} \le c\|\mathbf{u}_1 - \mathbf{u}_2\|_W \|\tilde{\mathbf{u}}\|_W \|\mathbf{z}\|_W, \tag{3.12}$$

$$\|\big(R''(\varphi_1;\gamma) - R''(\varphi_2;\gamma)\big)[\tilde{\varphi}, \cdot]^* z^\varphi\|_q \le c\|\mathbf{u}_1 - \mathbf{u}_2\|_W \|\tilde{\mathbf{u}}\|_W \|\mathbf{z}\|_W. \tag{3.13}$$

For further explicit reference, we state a result from [**30**, Lemma 3.3], for linear equations as in (3.2) and (3.3), with arbitrary right-hand side data.

Lemma 3.1. *Let $\eta \ge 0$ sufficiently large and let $\mathbf{u} \in W$ be given, then for every $\mathbf{f} = (f^u, f^\varphi) \in V^*$, the linearized partial differential equation*

$$A'(\mathbf{u})\tilde{\mathbf{u}} + R'(\varphi;\gamma)\tilde{\varphi} = \mathbf{f} \tag{$\mathrm{EL}^{\gamma,\eta}_{\mathrm{lin}}$}$$

has a unique weak solution $\tilde{\mathbf{u}} \in V$, that fulfills the estimate

$$\|\tilde{\mathbf{u}}\|_V \le c\|\mathbf{f}\|_{V}, \tag{3.14}$$

for a $c > 0$. If a fortiori $\mathbf{f} \in W^\times$, then $\tilde{\mathbf{u}} \in W$ and it holds

$$\|\tilde{u}\|_{1,p} + \|\tilde{\varphi}\|_{2,q} \le c\max(\|\mathbf{u}\|_W^1, \|\mathbf{u}\|_W^2, \|\mathbf{u}\|_W^3, \|\mathbf{u}\|_W^4)\|\mathbf{f}\|_{W^\times}, \tag{3.15}$$

for a $c > 0$.

We can therefore introduce the solution operator $\mathcal{G}_\mathbf{u}$ corresponding to ($\mathrm{EL}^{\gamma,\eta}_{\mathrm{lin}}$), for arbitrary right-hand sides $\mathbf{f} \in V^*$ and given $\mathbf{u} \in W$, by

$$\mathcal{G}_\mathbf{u} \colon V^* \to V, \quad \mathcal{G}_\mathbf{u}(\mathbf{f}) =: \tilde{\mathbf{u}}. \tag{3.16}$$

Note that the first and second derivatives G' and G'' of the control-to state operator G can be expressed as

$$\begin{aligned} G'(q)\tilde{q} &= \mathcal{G}_\mathbf{u}\big(B(\tilde{q})\big),\\ G''(q)[\tilde{q}_1, \tilde{q}_2] &= \mathcal{G}_\mathbf{u}\big(-A''(\mathbf{u})[\tilde{\mathbf{u}}_1, \tilde{\mathbf{u}}_2] - R''(\varphi;\gamma)[\tilde{\varphi}_1, \tilde{\varphi}_2]\big), \end{aligned}$$

where again $\mathbf{u} = G(q)$, and $\tilde{\mathbf{u}}_i = \mathcal{G}_\mathbf{u}\big(B(\tilde{q}_i)\big) = G'(q)\tilde{q}_i$. We will use this in the description of the SQP subproblems below.

3.2. The optimization problem ($\mathrm{NLP}^{\gamma,\eta}$).

We can now gather some known results for the control problem ($\mathrm{NLP}^{\gamma,\eta}$), see [**29, 30, 46**]. We first write ($\mathrm{NLP}^{\gamma,\eta}$) in a usual reduced form. Utilizing the control-to-state operator G, and implicitly using the embedding $W \hookrightarrow L^2(\Omega, \mathbb{R}^2) \times L^2(\Omega)$, the reduced functional $f \colon Q \to \mathbb{R}$ is defined by $f(q) := J(G(q), q) = J(\mathbf{u}, q)$. Then ($\mathrm{NLP}^{\gamma,\eta}$) is equivalent to

$$\min f(q), \quad \text{subject to} \quad q \in Q_{\mathrm{ad}}. \tag{$\mathrm{NLP}^{\gamma,\eta}_{\mathrm{red}}$}$$

Existence of at least one global minimizer $\bar{q}$ of ($\mathrm{NLP}^{\gamma,\eta}_{\mathrm{red}}$) with associated state $\bar{\mathbf{u}} \in W$ has been shown in [**30**, Proposition 4.1] and in [**46**, Theorem 4.3] for a model problem without control constraints. Due to the nonconvex structure of $(\mathrm{NLP}^{\gamma,\eta})$, we call $\bar{q} \in Q_{\mathrm{ad}}$ a local minimizer of $(\mathrm{NLP}^{\gamma,\eta})$ in the sense of $L^2(\Gamma)$, if there exists an $r > 0$ such that

$$f(\bar{q}) \leq f(q) \quad \forall\, q \in Q_{\mathrm{ad}} \quad \text{with} \quad \|\bar{q} - q\|_Q \leq r. \tag{3.17}$$

First-order necessary optimality conditions for $(\mathrm{NLP}^{\gamma,\eta})$ have been derived in [**30**, Lemma 4.3], by adapting the results of [**46**, **47**]. They read as follows:

Lemma 3.2. *Let $\bar{q} \in Q_{ad}$ be a local minimizer of $(\mathrm{NLP}^{\gamma,\eta}_{\mathrm{red}})$ with associated state $\bar{\mathbf{u}} \in W$. Then there exists an adjoint state $\bar{\mathbf{z}} = (\bar{z}^u, \bar{z}^\varphi) \in W$, such that*

$$A(\bar{\mathbf{u}}) + R(\bar{\varphi};\gamma) = B\bar{q} \in V^*, \tag{$\mathrm{EL}^{\gamma,\eta}$}$$

$$(A'(\bar{\mathbf{u}}))^*\bar{\mathbf{z}} + R'(\bar{\varphi};\gamma)^*\bar{z}^\varphi = \bar{u} - u_d \in V^*, \tag{$\mathrm{AE}^{\gamma,\eta}$}$$

$$(B^*\bar{\mathbf{z}} + \alpha\bar{q}, q - \bar{q})_Q \geq 0 \qquad \forall q \in Q_{ad} \tag{$\mathrm{VI}^{\gamma,\eta}$}$$

holds.

Note that Lemma 3.2 already makes use of differentiability of f. In fact, for $\eta \geq 0$ being sufficiently large, the reduced functional f is twice Fréchet-differentiable from Q into $\mathbb{R}$ by [**30**, Corollary 3.5 and Section 3.3], with derivatives

$$\begin{aligned} &f'(q)\tilde{q} = \big(B^*\mathbf{z} + \alpha q, \tilde{q}\big)_Q, \\ &f''(q)[\tilde{q}_1,\tilde{q}_2] = (\tilde{u}_1,\tilde{u}_2) + \alpha(\tilde{q}_1,\tilde{q}_2)_Q - \langle A''(\mathbf{u})[\tilde{\mathbf{u}}_1,\tilde{\mathbf{u}}_2],\mathbf{z}\rangle - \langle R''(\varphi;\gamma)[\tilde{\varphi}_1,\tilde{\varphi}_2], z^\varphi\rangle, \end{aligned} \tag{3.18}$$

where $\mathbf{u} = G(q)$, $\mathbf{z} = G'(q)^*(u - u_d)$, $\tilde{\mathbf{u}}_i = \mathcal{G}_{\mathbf{u}}\big(B(\tilde{q}_i)\big)$, and $\tilde{q}, \tilde{q}_i \in Q$, for $i = 1, 2$. Moreover, by [**30**, Lemma 3.10] and for $\eta \geq 0$ sufficiently large, f is second-order locally Lipschitz continuous in Q, i.e. for every $\rho > 0$ there exists a constant $c_L = c_L(\rho) > 0$ such that for all $q_1, q_2, \in Q$ with $\|q_1 - q_2\|_Q \leq \rho$, it holds

$$\Big|\Big(f''(q_1) - f''(q_2)\Big)[\tilde{q}_1,\tilde{q}_2]\Big| \leq c_L\|q_1 - q_2\|_Q\|\tilde{q}_1\|_Q\|\tilde{q}_2\|_Q. \tag{3.19}$$

Let us also point out that the existence, uniqueness, and regularity result from Lemma 3.1 is applicable to the adjoint equation $(\mathrm{AE}^{\gamma,\eta})$, due to the self-adjointness of A' and R', and the regularities $\bar{u} \in W_u$ and $u_d \in L^2(\Omega,\mathbb{R}^2)$.

To close this section, we point out that in [**30**, Theorem 4.6 and Remark 4.7], we have established second-order sufficient conditions optimality conditions. If $\bar{q} \in Q_{\mathrm{ad}}$, with associated state $\bar{\mathbf{u}} \in W$ and adjoint state $\bar{\mathbf{z}} \in W$, satisfies the first-order necessary conditions from Lemma 3.2 as well as the coercivity condition

$$\exists\, \delta_{\mathrm{SSC}} > 0 \quad \text{such that} \quad f''(\bar{q})(d^q)^2 \geq \delta_{\mathrm{SSC}}\|d^q\|_Q^2 \quad \forall\, d^q \in \mathcal{C}(\bar{q}), \tag{3.20}$$

with a cone of critical directions $\mathcal{C}(\bar{q})$ defined by

$$\mathcal{C}(\bar{q}) := \{d^q \in Q_{\mathrm{ad}} \mid d^q(x) = 0 \quad \text{if} \quad B^*\bar{\mathbf{z}}(x) + \alpha\bar{q}(x) \neq 0\}, \tag{3.21}$$

then there exist constants $\epsilon > 0$ and $c > 0$ such that the quadratic growth condition

$$f(q) \geq f(\bar{q}) + c\|q - \bar{q}\|_Q^2 \quad \forall\, q \in Q_{\mathrm{ad}} \quad \text{that satisfy} \quad \|q - \bar{q}\|_Q \leq \epsilon$$

holds. However, for proving convergence of the SQP method under a second-order sufficiency condition, we will have to ensure that the descent directions determined by the quadratic subproblem stay within the cone of critical directions. If this cone of critical directions is too small, we cannot guarantee this. Following the ideas of [**53**], we therefore use slightly stronger SSC that involve so-called σ-strongly active constraints, for an arbitrarily small but fixed parameter $\sigma > 0$. Let us therefore define the set

$$\mathcal{I}(\sigma) := \{x \in \Omega \quad \Big| \quad |B^*\bar{\mathbf{z}} + \alpha\bar{q}| \geq \sigma\}, \quad \text{for} \quad \sigma > 0 \tag{3.22}$$

and the cone of (σ-)critical directions

$$\mathcal{C}_\sigma(\bar{q}) := \{d^q \in Q_{\mathrm{ad}} \mid d^q(x) = 0 \text{ on } \mathcal{I}(\sigma)\}. \tag{3.21'}$$

Note that for the cone $\mathcal{C}(\bar{q})$ defined in (3.21) we have the inclusion $\mathcal{C}(\bar{q}) \subset \mathcal{C}_\sigma(\bar{q})$ for $\sigma > 0$. We therefore impose the following assumption, slightly stronger than (3.20):

Assumption 3.3. Let $\bar{q} \in Q_{\mathrm{ad}}$ and the associated function triple $(\bar{\mathbf{u}}, \bar{q}, \bar{\mathbf{z}}) \in Y$ fulfill the first-order necessary conditions given in Lemma 3.2. There exist constants $\sigma > 0$ and $\delta_{\mathrm{SSC}} > 0$ such that

$$f''(\bar{q})(d^q)^2 \geq \delta_{\mathrm{SSC}} \|d^q\|_Q^2 \quad \forall\, d^q \in \mathcal{C}_\sigma(\bar{q}). \tag{3.20'}$$

Then, [**30**, Theorem 4.6] obviously also results in a quadratic growth condition under Assumption 3.3.

Corollary 3.4. *Let $\bar{q} \in Q_{ad}$, with associated state $\bar{\mathbf{u}} \in W$ and adjoint state $\bar{\mathbf{z}} \in W$, fulfill Assumption 3.3. There exist constants $\epsilon > 0$ and $c > 0$ such that the quadratic growth condition*

$$f(q) \geq f(\bar{q}) + c\|q - \bar{q}\|_Q^2$$

holds for every $q \in Q_{ad}$ with $\|q - \bar{q}\|_Q \leq \epsilon$. In particular, $\bar{q}$ is a strict locally optimal control in the sense of L^2.

In the following, we tacitly assume that $\bar{q}$ is a fixed local minimizer of $(\mathrm{NLP}^{\gamma,\eta})$ and $\bar{y} = (\bar{\mathbf{u}}, \bar{q}, \bar{\mathbf{z}}) \in Y$ will always denote a fixed triple that satisfies the first-order necessary conditions of Lemma 3.2 and the second-order sufficient conditions of Assumption 3.3.

4. The SQP method

Let us now describe the SQP method for $(\mathrm{NLP}^{\gamma,\eta})$. To do so, we first define the Lagrangian functional

$$\mathcal{L}\colon Y \to \mathbb{R}, \quad \mathcal{L}(y) := J(\mathbf{u}, q) + \langle A(\mathbf{u}), \mathbf{z}\rangle + \langle R(\varphi;\gamma), z^\varphi\rangle - \langle B(q), \mathbf{z}\rangle, \tag{4.1}$$

with $y = (\mathbf{u}, q, \mathbf{z})$ as introduced in Section 2. Applying the differentiability results for A and R, it is clear that second-order Fréchet-differentiability of $\mathcal{L}$ from Y into $\mathbb{R}$ holds. Using (3.18) we see

$$\begin{aligned} \mathcal{L}''(y)[(\tilde{\mathbf{u}}_1, \tilde{q}_1), (\tilde{\mathbf{u}}_2, \tilde{q}_2)] &= f''(q)[\tilde{q}_1, \tilde{q}_2], \\ &= (\tilde{u}_1, \tilde{u}_2) + \alpha(\tilde{q}_1, \tilde{q}_2)_Q \\ &\quad - \langle A''(\mathbf{u})[\tilde{\mathbf{u}}_1, \tilde{\mathbf{u}}_2], \mathbf{z}\rangle - \langle R''(\varphi;\gamma)[\tilde{\varphi}_1, \tilde{\varphi}_2], z^\varphi\rangle. \end{aligned} \tag{4.2}$$

Note that therefore $\mathcal{L}''$ is also locally Lipschitz continuous, with Lipschitz estimate induced directly from (3.19).

As is well-known, the SQP method involves the iterative solution of quadratic subproblems, which we will denote by (QP_k). Given a current iterate

$y^k = (\mathbf{u}^k, q^k, \mathbf{z}^k) \in Y$, find a pair $\mathbf{d}^k = (\mathbf{d}^{\mathbf{u},k}, d^{q,k}) = (\mathbf{u}^{k+1} - \mathbf{u}^k, q^{k+1} - q^k) \in W \times Q$ that solves
(QP_k)

$$\begin{aligned}
\min_{\mathbf{d}^k} J_k(\mathbf{d}^k) &:= J'(\mathbf{u}^k, q^k)\mathbf{d}^k + \frac{1}{2}\mathcal{L}''(y^k)[\mathbf{d}^k, \mathbf{d}^k], \\
\text{s.t.} \quad & A'(\mathbf{u}^k)\mathbf{d}^{\mathbf{u},k} + R'(\varphi^k;\gamma)d^{\varphi,k} = B(d^{q,k}) + B(q^k) - A(\mathbf{u}^k) - R(\varphi^k;\gamma) \\
\text{and} \quad & q^{k+1} \in Q_{\mathrm{ad}}.
\end{aligned}$$

Let us point out that we can equivalently express the equation for $(\mathbf{d}^{\mathbf{u},k}, d^{q,k})$ by

$$\mathbf{d}^{\mathbf{u},k} = \mathcal{G}_{\mathbf{u}^k}\big(B(d^{q,k}) + B(q^k) - A(\mathbf{u}^k) - R(\varphi^k;\gamma)\big),$$

with the operator $\mathcal{G}_{\mathbf{u}^k}$ defined in (3.16). Further, the derivatives J' and $\mathcal{L}''$ are always taken w.r.t to the function pair $(\mathbf{u}, q)$. We also explicitly recall the functionals J' and $\mathcal{L}''$ as

$$J'(\mathbf{u}^k, q^k)\mathbf{d}^k = (u^k - u_d, d^{u,k}) + \alpha(q^k, d^{q,k}), \tag{4.3}$$

$$\begin{aligned}
\mathcal{L}''(y^k)[\mathbf{d}^k, \mathbf{d}^k] = \|d^{u,k}\|^2 + \alpha\|d^{q,k}\|^2 &- 6\gamma([(\varphi^k - \varphi^-)^+]d^{\varphi,k}d^{\varphi,k}, z^{\varphi,k}) \\
&- 2(1-\kappa)\Big[(d^{\varphi,k}\mathbb{C}e(u^k)d^{\varphi,k}, e(z^{u,k})) \\
&\qquad + 2(d^{\varphi,k}\mathbb{C}e(d^{u,k})\varphi^k, e(z^{u,k})) \\
&\qquad + 2(d^{\varphi,k}\mathbb{C}e(u^k) : e(d^{u,k}), z^{\varphi,k}) \\
&\qquad + (\varphi^k\mathbb{C}e(d^{u,k}) : e(d^{u,k}), z^{\varphi,k})\Big].
\end{aligned} \tag{4.4}$$

For now, assume that $\mathbf{d}^k = (\mathbf{d}^{\mathbf{u},k}, d^{q,k})$ is a solution of (QP_k). First-order necessary optimality conditions for (QP_k) are then a straightforward adaption of [**29**, Section 4.2].

Lemma 4.1. *Let $y^k \in Y$ be given and $\mathbf{d}^k$ be a minimizer of (QP_k). Then, there exists an adjoint state $\mathbf{z}^{k+1} = (z^{u,k+1}, z^{\varphi,k+1}) \in W$ such that*

$$A'(\mathbf{u}^k)\mathbf{d}^{\mathbf{u},k} + R'(\varphi^k;\gamma)d^{\varphi,k} = B(q^{k+1}) - A(\mathbf{u}^k) - R(\varphi^k;\gamma), \tag{4.5a}$$

$$\begin{aligned}
(A'(\mathbf{u}^k))^*\mathbf{z}^{k+1} + R'(\varphi^k;\gamma)^*z^{\varphi,k+1} &= u^{k+1} - u_d - A''(\mathbf{u}^k)[\mathbf{d}^{\mathbf{u},k}, \cdot]^*\mathbf{z}^k \\
&\quad - R''(\varphi^k;\gamma)[d^{\varphi,k}, \cdot]^*z^{\varphi,k},
\end{aligned} \tag{4.5b}$$

$$\big(B^*\mathbf{z}^{k+1} + \alpha q^{k+1}, q - q^{k+1}\big)_Q \geq 0 \qquad \forall q \in Q_{ad} \tag{4.5c}$$

holds.

We want to point out that convexity of the objective functional $J_k(\mathbf{d}^k)$ at every iterate y^k is still an open question at this point. Solvability of (QP_k) has been addressed in [**29**, Lemma 4.1], under the condition that a strong coercivity property is fulfilled in the current iterate $y^k \in Y$, i.e. that

$$\exists\, c > 0 \quad \text{such that} \quad \mathcal{L}''(y^k)[\mathbf{d}^k, \mathbf{d}^k] \geq c\|d^{q,k}\|_Q^2,$$

for all $\mathbf{d}^k \in W \times Q_{\mathrm{ad}}$ that satisfy $\mathbf{d}^{\mathbf{u},k} = \mathcal{G}_{\mathbf{u}^k}(d^{q,k})$. Transferring such a condition from one iterate to the next would be rather straight forward using the Lipschitz property of $\mathcal{L}''$, if the algorithm is initialized close to a local minimum fulfilling the strong condition

$$\mathcal{L}''(\bar{y})[\mathbf{d}^k, \mathbf{d}^k] \geq c\|d^{q,k}\|_Q^2,$$

for all $\mathbf{d}^k \in W \times Q$ that satisfy $\mathbf{d}^{\mathbf{u},k} = \mathcal{G}_{\bar{\mathbf{u}}}(d^{q,k})$ with a constant $c > 0$, but is more involved for $d^{q,k} \in \mathcal{C}_\sigma(\bar{q})$.

We end this section by stating the SQP algorithm, as already used in [**29**, Algorithm 4.1].

Algorithm 4.2 (SQP algorithm for $(\mathrm{NLP}^{\gamma,\eta})$).

(0) Choose $y^0 = (\mathbf{u}^0, q^0, \mathbf{z}^0) \in Y$, and set $k = 0$.
(1) STOP, if $y^k = (\mathbf{u}^k, q^k, \mathbf{z}^k)$ satisfies the first-order necessary optimality conditions of $(\mathrm{NLP}^{\gamma,\eta})$ from Lemma 3.2.
(2) Solve (QP_k) to obtain $\mathbf{d}^k$ with associated adjoint $\mathbf{z}^{k+1}$.
(3) Set $(\mathbf{u}^{k+1}, q^{k+1}) = (\mathbf{u}^k, q^k) + \mathbf{d}^k$, with associated adjoint $\mathbf{z}^{k+1}$, set $k = k+1$ and go to step 1.

To prove convergence under the weaker condition involving the cone of (σ-)critical directions $\mathcal{C}_\sigma(\bar{q})$ from (3.21') requires the discussion of several auxiliary results. We want to point out that we can follow a meanwhile classical approach in the convergence of SQP methods, see e.g. [**3**, **4**, **53**], also [**25**, **26**], yet the application for our specific problem requires the careful application of appropriate regularity results.

5. Convergence analysis for an auxiliary sequence

Following the ideas of [**4**, **53**], we will first show that the SQP method of Algorithm 4.2 corresponds to iteratively applying Newton's method to a generalized equation. In order to do this, we will transform the optimality conditions from Lemma 3.2 and Lemma 4.1 into generalized equations and identify the associated Newton steps. We can then investigate convergence results for Newton's method with auxiliary subproblems.

5.1. Optimality conditions as generalized equation. Let us start by looking at the first-order optimality conditions of $(\mathrm{NLP}^{\gamma,\eta})$ from Lemma 3.2. Following [**4**, **53**], we define the generalized equation

$$0 \in F(\bar{y}) + N(\bar{y}), \tag{GE}$$

where the mapping $F\colon Y \longrightarrow Z$ and the set-valued map $N\colon Y \rightrightarrows Z$ are given by

$$F(y) := \begin{pmatrix} (A'(\mathbf{u}))^*\mathbf{z} + R'(\varphi;\gamma)^* z^\varphi - u + u_d \\ B^*\mathbf{z} + \alpha q \\ A(\mathbf{u}) + R(\varphi;\gamma) - Bq \end{pmatrix}, \quad N(y) := \begin{pmatrix} 0 \\ N_{\mathrm{nc}}(q) \\ 0 \end{pmatrix}. \tag{5.1}$$

Here, $N_{\mathrm{nc}}(q)$ denotes the normal cone of Q_{ad} at a $q \in Q$, i.e.

$$N_{\mathrm{nc}}(q) = \{d^q \in Q \,|\, (d^q, \tilde{q} - q)_Q \leq 0 \quad \text{for all} \quad \tilde{q} \in Q_{\mathrm{ad}}\}.$$

Note that due to the nonlinearity of A and A' with respect to $\mathbf{u}$, (GE) is also nonlinear. The operator F is Fréchet differentiable from Y into Z, with derivative

$$F'(y)\tilde{y} = \begin{pmatrix} A''(\mathbf{u})[\tilde{\mathbf{u}}, \cdot]^*\mathbf{z} + A'(\mathbf{u})^*\tilde{\mathbf{z}} + R''(\varphi;\gamma)[\tilde{\varphi}, \cdot]^* z^\varphi + R'(\varphi;\gamma)^*\tilde{z}^\varphi - \tilde{u} \\ B^*\tilde{\mathbf{z}} + \alpha\tilde{q} \\ A'(\mathbf{u})\tilde{\mathbf{u}} + R'(\varphi;\gamma)\tilde{\varphi} - B(\tilde{q}) \end{pmatrix}, \tag{5.2}$$

where $\tilde{y} = (\tilde{\mathbf{u}}, \tilde{q}, \tilde{\mathbf{z}})$. This directly follows from the second-order continuous Fréchet differentiability of A, R, and the linearity of B. We can thus apply Newton's method to (GE). Given the function triple $y^k = (\mathbf{u}^k, q^k, \mathbf{z}^k) \in Y$, the next iterate $y^{k+1} = (\mathbf{u}^{k+1}, q^{k+1}, \mathbf{z}^{k+1}) \in Y$ is determined by solving the generalized equation

$$0 \in F(y^k) + F'(y^k)\big(y^{k+1} - y^k\big) + N(y^{k+1}). \tag{NM}$$

Writing out the definitions of F, F', and N, we see that (NM) is equivalent to

$$\begin{aligned} A(\mathbf{u}^k) + R(\varphi^k;\gamma) - Bq^{k+1} \qquad\qquad& \\ +A'(\mathbf{u}^k)(\mathbf{u}^{k+1} - \mathbf{u}^k) + R'(\varphi^k;\gamma)(\varphi^{k+1} - \varphi^k) &= 0, \\ (A'(\mathbf{u}^k))^*\mathbf{z}^{k+1} + R'(\varphi^k;\gamma)^* z^{k+1,\varphi} + A''(\mathbf{u}^k)[\mathbf{u}^{k+1} - \mathbf{u}^k, \cdot\,]^*\mathbf{z}^k \qquad& \\ +R''(\varphi^k;\gamma)[\varphi^{k+1} - \varphi^k, \cdot\,]^* z^{\varphi,k} - u^{k+1} + u_d &= 0, \\ \big(B^*\mathbf{z}^{k+1} + \alpha q^{k+1}, q - q^{k+1}\big)_Q &\geq 0 \quad \forall q \in Q_{\mathrm{ad}}, \end{aligned}$$

which is precisely the formulation of (4.5a)-(4.5c), recalling $\mathbf{d}^{\mathbf{u},k} = \mathbf{u}^{k+1} - \mathbf{u}^k$.

5.2. An auxiliary subproblem $(\widehat{\mathrm{QP}}_k)$. We already pointed out that a priori it is not clear whether the directions $\mathbf{d}^k$ produced by Algorithm 4.2 lie in the cone of critical directions from (3.21') that Assumption 3.3 uses. This suggests to look at auxiliary subproblems. We follow the approach of [**53**] and change the definition of the admissible set in a first step. The auxiliary subproblem, for $\hat{\mathbf{d}}^k = (\hat{\mathbf{d}}^{\mathbf{u},k}, \hat{d}^{q,k}) = (\hat{\mathbf{u}}^{k+1} - \mathbf{u}^k, \hat{q}^{k+1} - q^k) \in W \times Q$, reads

$(\widehat{\mathrm{QP}}_k)$

$$\begin{aligned} &\min_{\hat{\mathbf{d}}^k} J_k(\hat{\mathbf{d}}^k) = J'(\mathbf{u}^k, q^k)\hat{\mathbf{d}}^k + \frac{1}{2}\mathcal{L}''(y^k)[\hat{\mathbf{d}}^k, \hat{\mathbf{d}}^k], \\ &\qquad \text{s. t. } A'(\mathbf{u}^k)\hat{\mathbf{d}}^{\mathbf{u},k} + R'(\varphi^k;\gamma)\hat{d}^{\varphi,k} = B(\hat{d}^{q,k}) + B(q^k) - A(\mathbf{u}^k) - R(\varphi^k;\gamma), \\ &\qquad \text{and } q^{k+1} \in \widehat{Q_{\mathrm{ad}}} := \{q \in Q_{\mathrm{ad}} \mid q = \bar{q} \text{ on } \mathcal{I}(\sigma)\}, \end{aligned}$$

for which we recall $\mathcal{I}(\sigma) := \{x \in \Omega \,\big|\, |B^*\bar{\mathbf{z}} + \alpha\bar{q}| \geq \sigma\}$, cf. (3.22). To establish unique solvability of $(\widehat{\mathrm{QP}}_k)$, we prove a coercivity result, following [**55**, Lemma 6.2].

Lemma 5.1. *Let $\delta_{SSC} > 0$ be as in Assumption 3.3. Then, there exists a constant $\omega_1 > 0$ such that for all $y^k \in Y$ with $\|y^k - \bar{y}\|_Y \leq \omega_1$,*

$$\mathcal{L}''(y^k)[(\tilde{\mathbf{u}}_{\mathbf{u}^k}, \tilde{q}), (\tilde{\mathbf{u}}_{\mathbf{u}^k}, \tilde{q})] \geq \frac{\delta_{SSC}}{2}\|\tilde{q}\|_Q^2$$

holds for all $(\tilde{\mathbf{u}}_{\mathbf{u}^k}, \tilde{q}) \in W \times Q_{ad}$ that satisfy $\tilde{\mathbf{u}}_{\mathbf{u}^k} = \mathcal{G}_{\mathbf{u}^k}\big(B(\tilde{q})\big)$ and $\tilde{q} = 0$ on $\mathcal{I}(\sigma)$.

Proof. Let $\tilde{q}, \tilde{\mathbf{u}}_{\mathbf{u}^k}, \tilde{\mathbf{u}}_{\bar{\mathbf{u}}}$ be given as assumed and observe that the difference $\mathbf{d}^{\mathbf{u}} := \tilde{\mathbf{u}}_{\mathbf{u}^k} - \tilde{\mathbf{u}}_{\bar{\mathbf{u}}}$, $\mathbf{d}^{\mathbf{u}} = (d^u, d^\varphi)$ fulfills

$$\mathbf{d}^{\mathbf{u}} = \mathcal{G}_{\bar{\mathbf{u}}}\big([A'(\bar{\mathbf{u}}) - A'(\mathbf{u}^k)]\tilde{\mathbf{u}}_{\mathbf{u}^k} + [R'(\bar{\varphi};\gamma) - R'(\varphi^k;\gamma)]\big), \tag{5.3}$$

where we can apply Lemma 3.1 to obtain:

$$\|\mathbf{d}^{\mathbf{u}}\|_W \leq c\|[A'(\bar{\mathbf{u}}) - A'(\mathbf{u}^k)]\tilde{\mathbf{u}}_{\mathbf{u}^k}\|_{W^\times} + c\|[R'(\bar{\varphi};\gamma) - R'(\varphi^k;\gamma)]\tilde{\varphi}_{\mathbf{u}^k}\|_q. \tag{5.4}$$

Now, applying the Lipschitz results (3.10) and (3.11) to (5.4) we obtain

$$\|\mathbf{d}^{\mathbf{u}}\|_W \leq 2c[\|\bar{\mathbf{u}}\|_W + \|\mathbf{u}^k\|_W]\|\bar{\mathbf{u}} - \mathbf{u}^k\|_W\|\tilde{\mathbf{u}}_{\mathbf{u}^k}\|_W,$$

which by the triangle equation $\|\mathbf{u}^k\|_W \leq \|\bar{\mathbf{u}}\|_W + \|\mathbf{u}^k - \bar{\mathbf{u}}\|_W$, and setting $M := 2\|\bar{\mathbf{u}}\|_W + \|\mathbf{u}^k - \bar{\mathbf{u}}\|_W$, leads to

$$\|\mathbf{d}^{\mathbf{u}}\|_W \leq cM\|\mathbf{u}^k - \bar{\mathbf{u}}\|_W \|\tilde{\mathbf{u}}_{\mathbf{u}^k}\|_W \leq cM\|\bar{\mathbf{u}} - \mathbf{u}^k\|_W \|\tilde{q}\|_Q, \tag{5.5}$$

observing again from Lemma 3.1 that $\|\tilde{\mathbf{u}}_{\mathbf{u}^k}\|_V \leq c\|\tilde{q}\|_Q$.

Writing $\tilde{\mathbf{u}}_{\mathbf{u}^k} = \tilde{\mathbf{u}}_{\bar{\mathbf{u}}} + (\tilde{\mathbf{u}}_{\mathbf{u}^k} - \tilde{\mathbf{u}}_{\bar{\mathbf{u}}})$, a short calculation shows

$$\begin{aligned}
&\mathcal{L}''(y^k)[(\tilde{\mathbf{u}}_{\mathbf{u}^k}, \tilde{q}), (\tilde{\mathbf{u}}_{\mathbf{u}^k}, \tilde{q})] \\
&\quad = \mathcal{L}''(y^k)[(\tilde{\mathbf{u}}_{\bar{\mathbf{u}}} + (\tilde{\mathbf{u}}_{\mathbf{u}^k} - \tilde{\mathbf{u}}_{\bar{\mathbf{u}}}), \tilde{q}), (\tilde{\mathbf{u}}_{\bar{\mathbf{u}}} + (\tilde{\mathbf{u}}_{\mathbf{u}^k} - \tilde{\mathbf{u}}_{\bar{\mathbf{u}}}), \tilde{q})] \\
&\quad = \mathcal{L}''(\bar{y})[(\tilde{\mathbf{u}}_{\bar{\mathbf{u}}}, \tilde{q}), (\tilde{\mathbf{u}}_{\bar{\mathbf{u}}}, \tilde{q})] + \Big[\mathcal{L}''(y^k) - \mathcal{L}''(\bar{y})\Big][(\tilde{\mathbf{u}}_{\bar{\mathbf{u}}}, \tilde{q}), (\tilde{\mathbf{u}}_{\bar{\mathbf{u}}}, \tilde{q})] \\
&\qquad + 2(\tilde{u}_{\bar{\mathbf{u}}}, \tilde{u}_{\mathbf{u}^k} - \tilde{u}_{\bar{\mathbf{u}}}) + \|\tilde{u}_{\mathbf{u}^k} - \tilde{u}_{\bar{\mathbf{u}}}\|^2 \\
&\qquad - 2\langle A''(\mathbf{u}^k)[\tilde{\mathbf{u}}_{\bar{\mathbf{u}}}, \tilde{\mathbf{u}}_{\mathbf{u}^k} - \tilde{\mathbf{u}}^{\bar{\mathbf{u}}}], \mathbf{z}^k\rangle - \langle A''(\mathbf{u}^k)[\tilde{\mathbf{u}}^{\mathbf{u}^k} - \tilde{\mathbf{u}}_{\bar{\mathbf{u}}}, \tilde{\mathbf{u}}_{\mathbf{u}^k} - \tilde{\mathbf{u}}_{\bar{\mathbf{u}}}], \mathbf{z}^k\rangle \\
&\qquad - 2\langle R''(\varphi^k;\gamma)[\tilde{\varphi}_{\bar{\mathbf{u}}}, \tilde{\varphi}_{\mathbf{u}^k} - \tilde{\varphi}^{\bar{\mathbf{u}}}], z^{\varphi,k}\rangle - \langle R''(\varphi^k;\gamma)[\tilde{\varphi}_{\mathbf{u}^k} - \tilde{\varphi}_{\bar{\mathbf{u}}}, \tilde{\varphi}^{\mathbf{u}^k} - \tilde{\varphi}^{\bar{\mathbf{u}}}], z^{\varphi,k}\rangle.
\end{aligned} \tag{5.6}$$

We now estimate all terms from the right-hand side of (5.6) from below. We recognize that $\tilde{q}$ lies in the cone of critical directions $\mathcal{C}_\sigma(\bar{q})$ from (3.21'), thus we can use Assumption 3.3 in the first term on the right-hand side of (5.6) to obtain

$$\mathcal{L}''(\bar{y})[(\tilde{\mathbf{u}}_{\bar{\mathbf{u}}}, \tilde{q}), (\tilde{\mathbf{u}}_{\bar{\mathbf{u}}}, \tilde{q})] \geq \delta_{\mathrm{SSC}}\|\tilde{q}\|_Q^2.$$

For the second term, the Lipschitz result (3.19) and (4.2), with constant $c_L \leq c\omega_1$ lead to

$$\Big[\mathcal{L}''(y^k) - \mathcal{L}''(\bar{y})\Big][(\tilde{\mathbf{u}}_{\bar{\mathbf{u}}}, \tilde{q}), (\tilde{\mathbf{u}}_{\bar{\mathbf{u}}}, \tilde{q})] \geq -\, c\omega_1\|q^k - \bar{q}\|_Q\|\tilde{q}\|_Q^2.$$

Next, for the third term, the Cauchy-Schwarz inequality and the classical embedding result $\|u\|_2 \leq \|u\|_{1,2} \leq \|\mathbf{u}\|_V$ yield

$$\begin{aligned}
2(\tilde{u}_{\bar{\mathbf{u}}}, \tilde{u}_{\mathbf{u}^k} - \tilde{u}_{\bar{\mathbf{u}}}) + \|\tilde{u}_{\mathbf{u}^k} - \tilde{u}_{\bar{\mathbf{u}}}\|^2 &\geq -\, 2\|\tilde{u}_{\bar{\mathbf{u}}}\|_2\|\tilde{u}_{\mathbf{u}^k} - \tilde{u}_{\bar{\mathbf{u}}}\|_2 + 0 \\
&\geq -\, 2\|\tilde{\mathbf{u}}_{\bar{\mathbf{u}}}\|_V\|\tilde{\mathbf{u}}_{\mathbf{u}^k} - \tilde{\mathbf{u}}_{\bar{\mathbf{u}}}\|_V \\
&\geq -\, 2cM\|\bar{\mathbf{u}} - \mathbf{u}^k\|_W\|\tilde{q}\|_Q^2,
\end{aligned}$$

where we used $\|\tilde{\mathbf{u}}_{\bar{\mathbf{u}}}\|_W \leq c\|\tilde{q}\|_Q$, again by Lemma 3.1, and (5.5) to obtain the last inequality.

Finally note that by Lemma 3.1, it also holds $\|\tilde{\mathbf{u}}_{\bar{\mathbf{u}}}\|_W \leq c\|\tilde{q}\|_Q$. Thus, using the estimates for A'' and R'' from (3.8) and (3.9), and (5.5), we further obtain

$$\begin{aligned}
-2\langle A''(\mathbf{u}^k)[\mathbf{d}^{\bar{\mathbf{u}}}, \mathbf{d}^{\mathbf{u}^k} - \mathbf{d}^{\bar{\mathbf{u}}}], \mathbf{z}^k\rangle &\geq -\, 2cM\,\|\tilde{\mathbf{u}}_{\bar{\mathbf{u}}}\|_W\|\tilde{\mathbf{u}}_{\mathbf{u}^k} - \tilde{\mathbf{u}}_{\bar{\mathbf{u}}}\|_W \\
&\geq -\, 2cM\,\|\bar{\mathbf{u}} - \mathbf{u}^k\|_W\|\tilde{q}\|_Q^2, \\
-\langle A''(\mathbf{u}^k)[\tilde{\mathbf{u}}_{\mathbf{u}^k} - \tilde{\mathbf{u}}_{\bar{\mathbf{u}}}, \tilde{\mathbf{u}}_{\mathbf{u}^k} - \tilde{\mathbf{u}}_{\bar{\mathbf{u}}}], \mathbf{z}^k\rangle &\geq -\, cM\,\|\bar{\mathbf{u}} - \mathbf{u}^k\|_W^2\|\tilde{q}\|_Q^2, \\
-2\langle R''(\varphi^k;\gamma)[\tilde{\varphi}_{\bar{\mathbf{u}}}, \tilde{\varphi}_{\mathbf{u}^k} - \tilde{\varphi}_{\bar{\mathbf{u}}}], z^{\varphi,k}\rangle &\geq -\, 2cM\,\|\bar{\mathbf{u}} - \mathbf{u}^k\|_W\|\tilde{q}\|_Q^2, \\
-\langle R''(\varphi^k;\gamma)[\tilde{\varphi}_{\mathbf{u}^k} - \tilde{\varphi}_{\bar{\mathbf{u}}}, \tilde{\varphi}^{\mathbf{u}^k} - \tilde{\varphi}_{\bar{\mathbf{u}}}], z^{\varphi,k}\rangle &\geq -\, cM\,\|\bar{\mathbf{u}} - \mathbf{u}^k\|_W^2\|\tilde{q}\|_Q^2.
\end{aligned}$$

Assume now $\|\bar{\mathbf{u}} - \mathbf{u}^k\|_W \leq \omega_1$ and $\|\bar{q} - q^k\|_W \leq \omega_1$, and note that therefore $M \leq c + \omega_1$, for an $\omega_1 > 0$ to be determined. Collecting all estimates and inserting them into (5.6) eventually leads to

$$\begin{aligned}&\mathcal{L}''(y^k)[(\tilde{\mathbf{u}}_{\mathbf{u}^k}, \tilde{q}), (\tilde{\mathbf{u}}_{\mathbf{u}^k}, \tilde{q})]\\ &\quad\geq \Big(\underbrace{\delta_{\text{SSC}} - c\omega_1^2 - c\omega_1(2+\omega_1)(\omega_1+c) - c\omega_1(2+\omega_1)(\omega_1+c)}_{:=I}\Big)\|\tilde{q}\|_Q^2.\end{aligned}$$

The constant $\omega_1 > 0$ can be chosen small enough that $I =: \frac{\delta_{\text{SSC}}}{2} \geq 0$. This concludes the proof. $\square$

An existence and uniqueness result for solutions to $(\widehat{\mathrm{QP}}_k)$ now follows from the compactness of $\widehat{Q}_{\text{ad}}$, and the strict convexity of the objective functional of $(\widehat{\mathrm{QP}}_k)$ that is immediately implied by the coercivity condition shown in Lemma 5.1, for ω_1 sufficiently small.

Corollary 5.2. *Let $y^k \in Y$ fulfill $\|y^k - \bar{y}\|_Y \leq \omega_1$ for a sufficiently small $\omega_1 > 0$. Then $(\widehat{\mathrm{QP}}_k)$ has a unique solution $\hat{d}^{q,k} \in \widehat{Q_{ad}}$ with associated $\hat{\mathbf{d}}^{\mathbf{u},k} \in W$.*

Proof. By Lemma 5.1, the objective functional $J_k(\hat{\mathbf{d}}^k)$ of $(\widehat{\mathrm{QP}}_k)$ is the sum of a linear functional and a uniformly convex functional, for ω_1 sufficiently small. The set $\widehat{Q_{\text{ad}}}$ is uniformly compact in Q, the claim now follows from standard arguments, cf. [**55**, Theorem 2.14]. $\square$

Again, first-order optimality conditions for $(\widehat{\mathrm{QP}}_k)$ follow in a standard way.

Corollary 5.3. *Let $y^k \in Y$ be given. A control $\hat{\mathbf{d}}^{q,k} = \hat{q}^{k+1} - q^k$ with $\hat{q}^{k+1} \in \widehat{Q_{ad}}$, with associated optimal state $\hat{\mathbf{d}}^{\mathbf{u},k}$ and adjoint state $\hat{\mathbf{z}}^{k+1} = (\hat{z}^{u,k+1}, \hat{z}^{\varphi,k+1})$, is optimal for the subproblem $(\widehat{\mathrm{QP}}_k)$ if and only if $(\hat{\mathbf{d}}^{\mathbf{u},k}, \hat{\mathbf{d}}^{q,k}, \hat{\mathbf{z}}^{k+1}) \in Y$ satisfies the optimality system*

$$A'(\mathbf{u}^k)\hat{\mathbf{d}}^{\mathbf{u},k} + R'(\varphi^k;\gamma)\hat{d}^{\varphi,k} = B(\hat{q}^{k+1}) - A(\mathbf{u}^k) - R(\varphi^k;\gamma), \tag{5.7a}$$

$$\begin{aligned}(A'(\mathbf{u}^k))^*\hat{\mathbf{z}}^{k+1} + R'(\varphi^k;\gamma)^*\hat{z}^{\varphi,k+1} &= \hat{u}^{k+1} - u_d - A''(\mathbf{u}^k)[\hat{\mathbf{d}}^{\mathbf{u},k},\cdot]^*\mathbf{z}^k\\ &\quad - R''(\varphi^k;\gamma)[\hat{d}^{\varphi,k},\cdot]^* z^{\varphi,k},\end{aligned} \tag{5.7b}$$

$$\left(B^*\hat{\mathbf{z}}^{k+1} + \alpha\hat{q}^{k+1}, q - \hat{q}^{k+1}\right)_Q \geq 0 \qquad \forall q \in \widehat{Q_{ad}}. \tag{5.7c}$$

Note that for $\|y^k - \bar{y}\|_Y \leq \omega_1$, with ω_1 as in Lemma 5.1 and Corollary 5.2, the first-order conditions are also sufficient, since $(\widehat{\mathrm{QP}}_k)$ is (strictly) convex. The strict convexity means in particular that solutions of the optimality system from Corollary 5.3 are also locally unique in an ω_1-neighborhood of $\bar{y}$.

Analogously to the course of action at the end of Section 5.1, we observe the equivalence of the optimality system of Corollary 5.3 with a generalized equation, which is introduced as

$$0 \in F(y) + \hat{N}(y), \tag{$\widehat{\mathrm{GE}}$}$$

where F is given as in (5.1) and $\hat{N}(y) := (0, \hat{N}_{\text{nc}}(q), 0)^T$, with $\hat{N}_{\text{nc}}(q)$ defined by

$$\hat{N}_{\text{nc}}(q) = \{d^q \in Q \mid (d^q, \tilde{q} - q)_Q \leq 0 \quad \text{for all} \quad \tilde{q} \in \widehat{Q_{\text{ad}}}\}.$$

Formally, the Newton subproblem associated to $(\widehat{\mathrm{GE}})$ reads: Given the function triple $y^k \in Y$, the next iterate $\hat{y}^{k+1}$ is determined by solving the generalized equation

$$0 \in F(y^k) + F'(y^k)(\hat{y}^{k+1} - y^k) + \hat{N}(\hat{y}^{k+1}). \tag{$\widehat{\mathrm{NM}}$}$$

5.3. The strong regularity property. We now prove local quadratic convergence of the sequence of solutions generated by ($\widehat{\mathrm{NM}}$), or equivalently by ($\widehat{\mathrm{QP}}_k$). As in [**26**, **53**], we use a Newton-Kantorovich like convergence theorem, following the approach of [**2**]. It ensures that the generated sequence $\{\hat{q}^k\} \subset \widehat{Q_{\mathrm{ad}}}$ of global solutions of ($\widehat{\mathrm{QP}}_k$) is well-defined and stays in the convergence radius of Newton's method, if started from a good initial guess, and ensures local quadratic convergence to $\bar{q}$.

Let us continue with some notation. Firstly, let $y_\delta = (\mathbf{u}_\delta, q_\delta, \mathbf{z}_\delta)$ and $y_{\delta'} = (\mathbf{u}_{\delta'}, q_{\delta'}, \mathbf{z}_{\delta'})$, and let $\delta := (\delta_1, \delta_2, \delta_3)$ and $\tilde{\delta} := (\tilde{\delta}_1, \tilde{\delta}_2, \tilde{\delta}_3)$ denote triples of perturbations that lie in the space Z, cf. (2.1). Next, we write $B_r^Z(\tilde{\delta})$ and $B_r^Y(\tilde{y})$ for the open balls

$$B_r^Z(\tilde{\delta}) := \{\delta \in Z \quad | \quad \|\delta - \tilde{\delta}\|_Z < r\}, \tag{5.8}$$

$$B_r^Y(\tilde{y}) := \{y \in Y \quad | \quad \|y - \tilde{y}\|_Y < r\}. \tag{5.9}$$

We will often set $\tilde{\delta} = 0$. With this, we can state the definition of the strong regularity property, see [**18**, **49**].

Definition 5.4. We say that the generalized equation ($\widehat{\mathrm{GE}}$) has the strong regularity property at $\bar{y}$ if there exist radii $r_1, r_2 > 0$ and a constant $L_{sr} > 0$, such that for all perturbations $\delta \in B_{r_1}^Z(0)$ the perturbed generalized equation

$$\delta \in F(\bar{y}) + F'(\bar{y})(y_\delta - \bar{y}) + \hat{N}(y_\delta) \tag{5.10}$$

suffices to the following properties:

(1) The perturbed generalized equation (5.10) has a solution $y_\delta \in B_{r_2}^Y(\bar{y})$.
(2) y_δ is the only solution of (5.10) in $B_{r_2}^Y(\bar{y})$.
(3) Let $y_\delta, y_{\delta'}$ be the unique solutions to (5.10) in $B_{r_2}^Y(\bar{y})$ for $\delta, \delta' \in B_{r_1}^Z(0)$. Then the Lipschitz condition

$$\|y_\delta - y_{\delta'}\|_Y \leq L_{sr} \|\delta - \delta'\|_Z$$

holds.

Let us put on record that $\delta \in Z$ only enters (5.10) linearly and further that $\bar{y}$ is a solution to both ($\widehat{\mathrm{GE}}$) and (5.10) for $\delta = 0$. Closely following [**53**], we point out that (5.10) is exactly the perturbation of the linearization of ($\widehat{\mathrm{GE}}$) in $\bar{y}$, i.e. a perturbation of ($\widehat{\mathrm{NM}}$) in this function triple. Let us recall this generalized equation for further reference, and from now on also call it

$$\delta \in F(\bar{y}) + F'(\bar{y})(y_\delta - \bar{y}) + \hat{N}(y_\delta), \tag{$\widehat{\mathrm{NM}_\delta}$}$$

for a $\delta \in Z$. It is clear that $(\widehat{\mathrm{NM}}_\delta)$ is the first-order necessary condition for the auxiliary subproblem

$$
\begin{aligned}
(\widehat{\mathrm{QP}}_\delta) \qquad & \min_{\hat{\mathbf{d}}_\delta} J_k(\hat{\mathbf{d}}_\delta) = J'(\bar{\mathbf{u}}, \bar{q})\hat{\mathbf{d}}_\delta + \frac{1}{2}\mathcal{L}''(\bar{y})[\hat{\mathbf{d}}_\delta, \hat{\mathbf{d}}_\delta] \\
& \qquad - \langle \delta_1, \hat{\mathbf{d}}_\delta^{\mathbf{u}} \rangle_{(W^{1,p'} \times L^{q'})^*, W^{1,p} \times L^q} - (\delta_2, \hat{d}_\delta^q)_Q, \\
(\widehat{\mathrm{EL}}_\delta) \qquad & \quad \text{s. t. } A'(\bar{\mathbf{u}})\hat{\mathbf{d}}_\delta^{\mathbf{u}} + R'(\bar{\varphi};\gamma)\hat{d}_\delta^{\varphi} = B(\hat{d}_\delta^q) + B(\bar{q}) + \delta_3 - A(\bar{\mathbf{u}}) - R(\bar{\varphi};\gamma) \\
& \quad \text{and } \hat{q}_\delta \in \widehat{Q_{\mathrm{ad}}},
\end{aligned}
$$

where $\hat{\mathbf{d}}_\delta = (\hat{\mathbf{d}}_\delta^{\mathbf{u}}, \hat{d}_\delta^q) := (\hat{\mathbf{u}}_\delta - \bar{\mathbf{u}}, \hat{q}_\delta - \bar{q})$, and $\delta = (\delta_1, \delta_2, \delta_3) \in Z$.

We start by discussing unique solvability of $(\widehat{\mathrm{QP}}_\delta)$, relying on Assumption 3.3 to guarantee strict convexity of the problem in $\bar{y}$. Note that this follows analogously to [**54**, Lemma 4.1]. Due to $\hat{q}_\delta$ being equal to $\bar{q}$ on $\mathcal{I}(\sigma)$, we have $\hat{d}_\delta^q = 0$ on $\mathcal{I}(\sigma)$. We split $\hat{\mathbf{u}}_\delta = \hat{\mathbf{u}} + \tilde{\mathbf{u}}$, where $\hat{\mathbf{u}} = \mathcal{G}_{\bar{\mathbf{u}}}\big(B(\hat{d}_\delta^q)\big)$ and $\tilde{\mathbf{u}} = \mathcal{G}_{\bar{\mathbf{u}}}\big(B(\bar{q}) - A(\bar{\mathbf{u}}) - R(\bar{\varphi};\gamma) + \delta_3\big)$, i.e. $\tilde{\mathbf{u}}$ does not depend on the control $\hat{d}_\delta^q$. The objective functional of $(\widehat{\mathrm{QP}}_\delta)$ can now be rewritten into

$$
\begin{aligned}
J_k(\hat{\mathbf{d}}_\delta) &= J(\bar{\mathbf{u}}, \bar{q})(\hat{\mathbf{u}} + \tilde{\mathbf{u}}, \hat{d}_\delta^q) + \frac{1}{2}\mathcal{L}(\bar{y})[(\hat{\mathbf{u}} + \bar{\mathbf{u}}_\delta, \hat{d}_\delta^q), (\hat{\mathbf{u}} + \tilde{\mathbf{u}}, \hat{d}_\delta^q)] \\
&\quad - \langle \delta_1, \hat{\mathbf{u}} - \tilde{\mathbf{u}} \rangle_{(W^{1,p'} \times L^{q'})^*, W^{1,p} \times L^q} - (\delta_2, \hat{d}_\delta^q)_Q \\
&= \frac{1}{2}\mathcal{L}''(\bar{y})[(\hat{\mathbf{u}}, \hat{d}_\delta^q), (\hat{\mathbf{u}}, \hat{d}_\delta^q)] \\
&\quad + \mathcal{L}''(\bar{y})[(\hat{\mathbf{u}}, 0), (\tilde{\mathbf{u}}, 0)] + J'(\bar{\mathbf{u}}, \bar{q})(\hat{\mathbf{u}}, \hat{d}_\delta^q) \\
&\quad - \langle \delta_1, \hat{\mathbf{u}} \rangle_{(W^{1,p'} \times L^{q'})^*, W^{1,p} \times L^q} - (\delta_2, \hat{d}_\delta^q)_Q \\
&\quad + \frac{1}{2}\mathcal{L}''(\bar{y})[(\tilde{\mathbf{u}}, 0), (\tilde{\mathbf{u}}, 0)] + J'(\bar{\mathbf{u}}, \bar{q})(\tilde{\mathbf{u}}, 0) \\
&\quad + \langle \delta_1, \tilde{\mathbf{u}} \rangle_{(W^{1,p'} \times L^{q'})^*, W^{1,p} \times L^q}.
\end{aligned}
$$

Note that $(\hat{\mathbf{u}}, \hat{q})$ belongs to the subspace where Assumption 3.3 applies. Thus, the term in the first line of the right-hand side is coercive. The terms in the second and third line constitute a linear functional in $(\hat{\mathbf{u}}, \hat{d}_\delta^q)$, and the terms in the fourth and fifth line are independent of $\hat{d}_\delta^q$. Thus the objective functional $J_k(\hat{\mathbf{d}}_\delta)$ is strictly convex. Existence, and uniqueness of solutions now follow from standard theory, cf. [**26**, Lemma 5.1]. We therefore omit the proof.

Lemma 5.5. *For each $\delta \in Z$, $(\widehat{\mathrm{QP}}_\delta)$ has a unique solution $(\hat{\mathbf{d}}_\delta^{\mathbf{u}}, \hat{d}_\delta^q) \in W \times Q$, with $\hat{q}_\delta \in \widehat{Q_{ad}}$, depending on δ.*

In a canonical way, also first-order necessary conditions for $(\widehat{\mathrm{QP}}_\delta)$ can be shown.

Corollary 5.6. *Let $\delta \in Z$ be given. A control $\hat{d}_\delta^q = \hat{q}_\delta - \bar{q}$ with $\hat{q}_\delta \in \widehat{Q_{ad}}$, with associated optimal state $\hat{\mathbf{d}}_\delta^{\mathbf{u}}$ and adjoint state $\hat{\mathbf{z}}_\delta = (\hat{z}_\delta^u, \hat{z}_\delta^\varphi)$, is optimal for the subproblem $(\widehat{\mathrm{QP}}_\delta)$ if and only if $(\hat{\mathbf{d}}_\delta^{\mathbf{u}}, \hat{d}_\delta^q, \hat{\mathbf{z}}_\delta)$ satisfies the optimality system*

$$A'(\bar{\mathbf{u}})\hat{\mathbf{d}}_\delta^{\mathbf{u}} + R'(\bar\varphi;\gamma)\hat d_\delta^\varphi = B(\hat q_\delta) - A(\bar{\mathbf{u}}) - R(\bar\varphi;\gamma) + \delta_3, \tag{5.11a}$$

$$\begin{aligned}(A'(\bar{\mathbf{u}}))^*\hat{\mathbf{z}}_\delta + R'(\bar\varphi;\gamma)^*\hat z_\delta^\varphi &= \hat u_\delta - u_d - A''(\bar{\mathbf{u}})[\hat{\mathbf{d}}_\delta^{\mathbf{u}},\cdot]^*\bar{\mathbf{z}}\\ &\quad - R''(\bar\varphi;\gamma)[\hat d_\delta^\varphi,\cdot]^*\bar z^\varphi - \delta_1,\end{aligned} \tag{5.11b}$$

$$\Big(B^*\hat{\mathbf{z}}_\delta + \alpha\hat q_\delta - \delta_2, q - \hat q_\delta\Big) \geq 0 \qquad \forall q \in \widehat{Q_{ad}}. \tag{5.11c}$$

As for the optimality conditions of $(\widehat{\mathrm{QP}}_k)$ from Corollary 5.3, the first-order conditions of $(\widehat{\mathrm{QP}}_\delta)$ from (5.6) are also sufficient, due to the strict convexity of $(\widehat{\mathrm{QP}}_\delta)$ in $\bar y$. Further, this again also means that the solutions of the optimality system from Corollary 5.6 are unique.

We will now show the Lipschitz condition from Definition 5.4, for which we closely follow the proof of [**54**, Theorem 4.2].

Lemma 5.7. *Let $\hat q_\delta$ and $\hat q_{\delta'}$ be the the unique solutions of $(\widehat{\mathrm{QP}}_\delta)$ for the perturbations $\delta, \delta' \in Z$, with associated states $\hat{\mathbf{u}}_\delta$, $\hat{\mathbf{u}}_{\delta'}$ and adjoint states $\hat{\mathbf{z}}_{\delta'}, \hat{\mathbf{z}}_\delta$, respectively. Let $\hat y_\delta$ and $\hat y_{\delta'}$ denote the associated function triples. There exists a constant $L > 0$, such that*

$$\|\hat y_\delta - \hat y_{\delta'}\|_Y \leq L\|\delta - \delta'\|_Z.$$

Proof. Let $\mathbf{u} := \hat{\mathbf{u}}_\delta - \hat{\mathbf{u}}_{\delta'}$, $q := \hat q_\delta - \hat q_{\delta'}$, $\mathbf{z} := \hat{\mathbf{z}}_\delta - \hat{\mathbf{z}}_{\delta'}$, and note that (5.11a) and (5.11b) imply

$$A'(\bar{\mathbf{u}})\mathbf{u} + R'_\varphi(\bar\varphi;\gamma)\varphi = B(q) + \delta_3 - \delta_3', \tag{5.12}$$

$$(A'(\bar{\mathbf{u}}))^*\mathbf{z} + R'(\bar\varphi;\gamma)^* z^\varphi = -A''(\bar{\mathbf{u}})[\mathbf{u},\cdot]^*\bar{\mathbf{z}} - R''(\bar\varphi;\gamma)[\varphi,\cdot]^*\bar z^\varphi + u - (\delta_1 - \delta_1'). \tag{5.13}$$

We start by estimating the right-hand side of both equations, using the regularities $q \in Q$, $\delta \in Z$, and the estimates (3.6) and (3.7). Hence, by Lemma 3.1,

$$\|\mathbf{u}\|_W \leq c\|q\|_Q + c\|\delta_3 - \delta_3'\|_{W^\times} \leq c\|q\|_q + c\|\delta - \delta'\|_Z, \tag{5.14}$$

$$\|\mathbf{z}\|_W \leq c\|\mathbf{u}\|_W + c\|\delta_1 - \delta_1'\|_{W^\times} \leq c\|q\|_Q + c\|\delta - \delta'\|_Z. \tag{5.15}$$

In particular, we have $\mathbf{u} \in W \hookrightarrow V$ and $\mathbf{z} \in W \hookrightarrow V$. We can thus test (5.12) with $\mathbf{z}$ and (5.13) with $\mathbf{u}$. Summing up, we obtain

$$\begin{aligned}(q, B^*\mathbf{z}) + \langle \delta_3 - \delta_3', \mathbf{z}\rangle &= -\langle A''(\bar{\mathbf{u}})[\mathbf{u},\mathbf{u}]\bar{\mathbf{z}}\rangle - \langle R''(\bar\varphi;\gamma)[\varphi,\varphi], \bar z^\varphi\rangle\\ &\quad + (u,u) - \langle \delta_1 - \delta_1', \mathbf{u}\rangle.\end{aligned} \tag{5.16}$$

Testing (5.11c) once in $(\hat{\mathbf{z}}_\delta, \hat q_\delta)$ with $\hat q_{\delta'}$ and once in $(\hat{\mathbf{z}}_{\delta'}, \hat q_{\delta'})$ with $\hat q_\delta$, and summing both inequalities, leads to

$$\big(\delta_2 - \delta_2', q\big) - \alpha\big(q, q\big) \geq \big(B^*\mathbf{z}, q\big). \tag{5.17}$$

Inserting (5.16) and (5.17) into the second derivative of the Lagrangian, cf. (4.2), leads to

$$\mathcal{L}''(\bar y)[(\mathbf{u}, q), (\mathbf{u}, q)] \leq \langle \delta_1 - \delta_1', \mathbf{u}\rangle + \big(\delta_2 - \delta_2', q\big) + \langle \delta_3 - \delta_3', \mathbf{z}\rangle. \tag{5.18}$$

We now estimate $\mathcal{L}''$ from below. We split $\mathbf{u}$ into a part that depends on the control q and a part that depends on the perturbations, i.e. $\mathbf{u} = \hat{\mathbf{u}} + \mathbf{u}_\delta$, where

$$A'(\mathbf{u})\hat{\mathbf{u}} + R'(\bar\varphi;\gamma)\hat\varphi = B(q), \qquad A'(\mathbf{u})\mathbf{u}_\delta + R'(\bar\varphi;\gamma)\varphi_\delta = \delta_3 - \delta_3'.$$

Note that Lemma 3.1 ensures

$$\|\hat{\mathbf{u}}\|_V \le c\|q\|_Q,\ \|\hat{\mathbf{u}}\|_W \le c\|q\|_Q,\ \|\mathbf{u}_\delta\|_V \le c\|\delta_3-\delta_3'\|_{W^\times},\ \|\mathbf{u}_\delta\|_W \le c\|\delta_3-\delta_3'\|_{W^\times}. \tag{5.19}$$

An easy calculation, analogously to (5.6) in the proof of Lemma 5.1, yields

$$\begin{aligned}
&\mathcal{L}''(\bar{y})[(\hat{\mathbf{u}}+\mathbf{u}_\delta, q),(\hat{\mathbf{u}}+\mathbf{u}_\delta, q)] \\
&\quad = \mathcal{L}''(\bar{y})[(\hat{\mathbf{u}}, q),(\hat{\mathbf{u}}, q)] + 2(\hat{u}, u_\delta) + \|u_\delta\|^2 - 2\langle A''(\bar{\mathbf{u}})[\mathbf{u}_\delta,\hat{\mathbf{u}}],\bar{\mathbf{z}}\rangle \\
&\quad\quad - \langle A''(\bar{\mathbf{u}})[\mathbf{u}_\delta,\mathbf{u}_\delta],\bar{\mathbf{z}}\rangle - 2\langle R''(\bar{\varphi};\gamma)[\varphi_\delta,\hat{\varphi}],\bar{z}^\varphi\rangle - \langle R''(\bar{\varphi};\gamma)[\varphi_\delta,\varphi_\delta],\bar{z}^\varphi\rangle,
\end{aligned} \tag{5.20}$$

with

$$\begin{aligned}
\mathcal{L}''(\bar{y})[(\hat{\mathbf{u}}, q),(\hat{\mathbf{u}}, q)] &\ge \delta_{\mathrm{SSC}}\|q\|_Q^2, \\
2(\hat{u}, u_\delta) + \|u_\delta\|^2 &\ge -2\|\delta_3-\delta_3'\|_{W^\times}\|q\|_Q, \\
-2\langle A''(\bar{\mathbf{u}})[\mathbf{u}_\delta,\hat{\mathbf{u}}],\bar{\mathbf{z}}\rangle &\ge -2c\|\delta_3-\delta_3'\|_{W^\times}\|\|q\|_Q, \\
-\langle A''(\bar{\mathbf{u}})[\mathbf{u}_\delta,\mathbf{u}_\delta],\bar{\mathbf{z}}\rangle &\ge -c\|\delta_3-\delta_3'\|_{W^\times}^2, \\
-2\langle R''(\bar{\varphi};\gamma)[\varphi_\delta,\hat{\varphi}],\bar{z}^\varphi\rangle &\ge -2c\|\delta_3-\delta_3'\|_{W^\times}\|\|q\|_Q, \\
-\langle R''(\bar{\varphi};\gamma)[\varphi_\delta,\varphi_\delta],\bar{z}^\varphi\rangle &\ge -c\|\delta_3-\delta_3'\|_{W^\times}^2.
\end{aligned}$$

Plugging the above estimates in (5.20) and combining it with (5.18), eventually leads to

$$\begin{aligned}
\langle\delta_1-\delta_1',\mathbf{u}\rangle + \big(\delta_2-\delta_2', q\big) + \langle\delta_3-\delta_3',\mathbf{z}\rangle \ge\ & \delta_{\mathrm{SSC}}\|q\|_Q^2 - 2\|\delta_3-\delta_3'\|_{W^\times}\|q\|_Q \\
& - c\|\delta_3-\delta_3'\|_{W^\times}^2 - 2c\|\delta_3-\delta_3'\|_{W^\times}\|\|q\|_Q \\
& - c\|\delta_3-\delta_3'\|_{W^\times}^2,
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
\delta_{\mathrm{SSC}}\|q\|_Q^2 \le\ & \|\delta_1-\delta_1'\|_{W^\times}\|\mathbf{u}\|_W + c\|\delta_2-\delta_2'\|_Q\|q\|_Q \\
& + \|\delta_3-\delta_3'\|_{W^\times}\|\mathbf{z}\|_W + c\|\delta_3-\delta_3'\|_{W^\times}\|q\|_Q + c\|\delta_3-\delta_3'\|_{W^\times}^2 \\
\le\ & c\|\delta-\delta'\|_Z\big(\|\mathbf{u}\|_W + \|q\|_Q + \|\mathbf{z}\|_W\big) + c\|\delta-\delta'\|_Z^2.
\end{aligned}$$

Inserting the estimates for $\mathbf{u}$ and $\mathbf{z}$ from (5.14) and (5.15) and using Young's inequality, we obtain

$$\|\hat{q}_\delta - \hat{q}_{\delta'}\|_Q \le c\|\delta-\delta'\|_Z$$

for a $c > 0$. Applying (5.14) and (5.15) concludes the proof. □

At this point, we have established all properties of Definition 5.4 for $(\widehat{\mathrm{NM}}_\delta)$. In summary, we have shown the following result in the context of strong regularity.

Theorem 5.8. *The generalized equation* $(\widehat{\mathrm{GE}})$ *is strongly regular at* $\bar{y}$.

5.4. Convergence of $(\widehat{\mathrm{NM}})$. Let us now turn to the proof of convergence for the sequences $\{\hat{q}^k\}$ generated by $(\widehat{\mathrm{NM}})$, or equivalently by solving $(\widehat{\mathrm{QP}}_k)$. Due to the strong regularity of $(\widehat{\mathrm{GE}})$, we can make use of a generalization of the implicit function theorem. The proof relies on standard arguments, see e.g. [**2**, **25**, **26**, **52**]. For completeness, we recapitulate the main arguments of the proof given in [**25**, Theorem 7.1], adapting them to the operator F and its properties, some of them postponed to the Appendix.

Theorem 5.9. *There exists a radius $\omega_2 > 0$ and a constant $C_N > 0$, such that for each starting point $y^0 \in B^Y_{\omega_2}(\bar{y})$ with $q^0 \in \widehat{Q_{ad}}$, the auxiliary subproblem $(\widehat{\mathrm{QP}}_k)$ generates a unique sequence of iterates $\{\hat{y}^k\} \subset B^Y_{\omega_2}(\bar{y})$ that satisfies*

$$\|\hat{y}^{k+1} - \bar{y}\|_Y \leq C_N \|\hat{y}^k - \bar{y}\|_Y^2 \qquad \text{for all } k \in \mathbb{N}. \tag{5.21}$$

Proof. By Theorem 5.8, $(\widehat{\mathrm{GE}})$ is strongly regular. Further, note the auxiliary Lipschitz result for F from Lemma A.1. As a result, we can utilize Dontchev's implicit function theorem for generalized equations, cf. [**18**, Theorem 2.4], which ensures the existence of $r_3, r_4 > 0$, such that for any $\hat{y}^k \in B^Y_{r_3}(\bar{y})$, there exist a unique solution $\hat{y}^{k+1} \in B^Y_{r_4}(\bar{y})$ to $(\widehat{\mathrm{NM}})$. If ω_2 is chosen such that $0 < \omega_2 \leq r_3$, we obtain

$$0 \in F(\bar{y}) + F'(\bar{y})(\bar{y} - \bar{y}) + N(\bar{y}), \tag{5.22}$$

$$0 \in F(\hat{y}^k) + F'(\hat{y}^k)(\hat{y}^{k+1} - \hat{y}^k) + N(\hat{y}^{k+1}). \tag{5.23}$$

Adding and subtracting $F(\bar{y})$ and $F'(\bar{y})(\hat{y}^{k+1} - \bar{y})$ to (5.23), leads to

$$\delta^{k+1} \in F(\bar{y}) + F'(\bar{y})(\hat{y}^{k+1} - \bar{y}) + N(\hat{y}^{k+1}), \tag{5.24}$$

where δ^{k+1} is defined as

$$\delta^{k+1} := F(\bar{y}) - F(\hat{y}^k) + F'(\bar{y})(\hat{y}^{k+1} - \bar{y}) - F'(\hat{y}^k)(\hat{y}^{k+1} - \hat{y}^k). \tag{5.25}$$

Note that we can apply the local Lipschitz result of Lemma A.1 to (5.24), which then yields

$$\|\delta^{k+1}\|_Z \leq L \|\hat{y}^k - \bar{y}\|_Y \leq L\, \omega_2, \tag{5.26}$$

for a constant $L > 0$ depending only on the radii r_3 and r_4.

Next, we recognize that (5.22) and (5.24) are equivalent to the first-order necessary conditions of $(\widehat{\mathrm{QP}}_\delta)$ for $\delta = 0$ and $\delta = \delta^{k+1}$, respectively, Therefore, from Lemma 5.7 we obtain

$$\|\hat{y}^{k+1} - \bar{y}\|_Y \leq L_{sr} \|\delta^{k+1} - 0\|_Z = L_{sr} \|\delta^{k+1}\|_Z. \tag{5.27}$$

Inserting (5.27) and (5.26) shows

$$\|\hat{y}^{k+1} - \bar{y}\|_Y \leq L_{sr} L\, \omega_2. \tag{5.28}$$

To obtain a quadratic convergence result, we estimate $\|\delta^{k+1}\|_Z$ further. Its definition (5.25) yields

$$\|\delta^{k+1}\|_Z \leq \|F(\bar{y}) - F(\hat{y}^k - F'(\hat{y}^k)(\bar{y} - \hat{y}^k)\|_Z + \|(F'(\bar{y}) - F'(\hat{y}^k))(\hat{y}^{k+1} - \bar{y})\|_Z. \tag{5.29}$$

The estimation of the right-hand side is postponed to the appendix. Note that $\|\hat{y}^k\|_Y \leq \|\bar{y}\|_Y + r_3$, thus applying Lemma A.3 and Lemma A.2 yields

$$\|\delta^{k+1}\|_Z \leq c_1 \|\hat{y}^k - \bar{y}\|_Y^2 + c_2 \|\hat{y}^k - \bar{y}\|_Y \|\hat{y}^{k+1} - \bar{y}\|_Y, \tag{5.30}$$

for constants $c_1, c_2 > 0$ depending on the radius r_3. Combining (5.27) with (5.30) now leads to

$$\|\hat{y}^{k+1} - \bar{y}\|_Y \leq L_{sr} c_1 \|\hat{y}^k - \bar{y}\|_Y^2 + L_{sr} c_2 \|\hat{y}^k - \bar{y}\|_Y \|\hat{y}^{k+1} - \bar{y}\|_Y. \tag{5.31}$$

Let us additionally demand $\omega_2 \leq \frac{1}{L_{sr}\, c_1 + L_{sr}^2 c_2\, L}$. Since $\hat{y}^k \in B^Y_{\omega_2}(\bar{y})$, and using (5.28), from (5.31) we obtain

$$\|\hat{y}^{k+1} - \bar{y}\|_Y \leq L_{sr}\, c_1 \omega_2^2 + L_{sr}\, c_2 \omega_2\, L_{sr}\, L\, \omega_2 \leq \omega_2^2 \Big[L_{sr} c_1 + L_{sr}^2 c_2\, L \Big] \leq \omega_2, \tag{5.32}$$

which implies that $\hat{y}^{k+1} \in B^Y_{\omega_2}(\bar{y})$. Finally, let us demand $\omega_2 \leq \frac{1}{2 L_{sr} c_2}$, and choose $C_N = \frac{L_{sr} c_1}{1 - \omega_2 c_2 L_{sr}} > 0$. Again, from (5.30) we obtain

$$\begin{aligned} \|\hat{y}^{k+1} - \bar{y}\|_Y &\leq L_{sr} c_1 \|\hat{y}^k - \bar{y}\|_Y^2 + L_{sr} c_2 \omega_2 \|\hat{y}^{k+1} - \bar{y}\|_Y \\ &\leq C_N \|(\hat{\mathbf{u}}^k, \hat{q}^k, \hat{\mathbf{z}}^k) - (\bar{\mathbf{u}}, \bar{q}, \bar{\mathbf{z}})\|_Y^2. \end{aligned}$$

Overall, setting $\omega_2 =: \min\left(r_3, \frac{1}{L_{sr}\, c_1 + L_{sr}^2 c_2\, L}, \frac{1}{2 L_{sr}\, c_2} \right) > 0$ yields the assertion (5.21). □

6. Local convergence of Algorithm 4.2

In this section, we will show our main result, i.e. that the sequence $\{q^k\}$ of iterates produced by the SQP method from Algorithm 4.2 converges locally quadratically to $\bar{q}$. We have already proven that the auxiliary subproblem $(\widehat{\mathrm{QP}}_k)$ produces feasible iterates, exploiting the strong regularity property, and that these iterates converge quadratically to $\bar{q}$. To carry the results obtained for $(\widehat{\mathrm{QP}}_k)$ over to (QP_k), we introduce yet another auxiliary problem, still following the ideas of [**53**]. We then show equivalence results for the intermediate subproblems and (QP_k) and transfer our convergence result from Theorem 5.9 to Algorithm 4.2.

Let us note that in the following, $y^k = (\mathbf{u}^k, q^k, \mathbf{z}^k) \in Y$ will always refer to a fixed function triple, that lies in a neighborhood of $\bar{y}$, which will be determined in Assumption 6.4 below. At this point, let us also recall that $\bar{y}$ always satisfies Assumption 3.3.

6.1. The intermediate subproblem (QP_k^ω).

As in [**53**] we define the neighborhood Q^ω_{ad} around $\bar{q}$, for an $\omega > 0$, via

$$Q^\omega_{\mathrm{ad}} = \{ q \in Q_{\mathrm{ad}} \,\big|\, \|q - \bar{q}\|_Q \leq \omega \},$$

and introduce, for $\mathbf{d}^k_\omega = (\mathbf{d}^{\mathbf{u},k}_\omega, d^{q,k}_\omega) := (\mathbf{u}^{k+1}_\omega - \mathbf{u}^k, q^{k+1}_\omega - q^k)$:

(QP_k^ω)

$$\begin{aligned} &\min_{\mathbf{d}^k_\omega} J_k(\mathbf{d}^k_\omega) = J'(\mathbf{u}^k, q^k)\mathbf{d}^k_\omega + \frac{1}{2}\mathcal{L}''(y^k)[\mathbf{d}^k_\omega, \mathbf{d}^k_\omega], \\ &\quad \text{s. t. } A'(\mathbf{u}^k)\mathbf{d}^{\mathbf{u},k}_\omega + R'(\varphi^k;\gamma) d^{\varphi,k}_\omega = B(d^{q,k}_\omega) + B(q^k) - A(\mathbf{u}^k) - R(\varphi^k;\gamma) \\ &\quad \text{and } q^{k+1}_\omega \in Q^\omega_{\mathrm{ad}}. \end{aligned}$$

The problem (QP_k^ω) will serve as an intermediate problem between (QP_k) and $(\widehat{\mathrm{QP}}_k)$. The motivation for (QP_k^ω) is the fact that this subproblem is a localization of (QP_k), in the sense that the admissible control set Q_{ad} is restricted to a local neighborhood of $\bar{y}$. Let us start by proving existence of at least one solution.

Lemma 6.1. *Let $\omega > 0$ be sufficiently small, and $\|y^k - \bar{y}\|_Y \leq \omega$. The auxiliary subproblem (QP_k^ω) has at least one solution $d^{q,k}_\omega \in Q^\omega_{ad}$ with associated $\mathbf{d}^{\mathbf{u},k}_\omega \in W$.*

Proof. The objective functional of (QP_k^ω) can be written as

$$J_k(\mathbf{d}_\omega^k) = J_k(\mathbf{u}_\omega^k)_{|\mathbf{u}} + J_k(q_\omega^k)_{|q},$$

where

$$\begin{aligned} J_k(\mathbf{u}_\omega^k)_{|\mathbf{u}} &= (u^k - u_d, d_\omega^{u,k}) + \|d_\omega^{u,k}\|^2 - \langle A''(\mathbf{u}^k)[\mathbf{d}_\omega^{\mathbf{u},k}, \mathbf{d}_\omega^{\mathbf{u},k}], \mathbf{z}^k\rangle \\ &\quad - \langle R''(\varphi^k;\gamma)[d_\omega^{\varphi,k}, d^{\varphi,k}], z^{\varphi,k}\rangle, \\ J_k(q_\omega^k)_{|q} &= \alpha(q^k, d_\omega^{q,k}) + \alpha\|d_\omega^{q,k}\|^2. \end{aligned}$$

Since $q_\omega^{k+1} \in Q_{\mathrm{ad}}^\omega$ and $q^k \in Q$ fixed, $d_\omega^{q,k}$ is bounded in Q. Note that $\mathbf{d}_\omega^{\mathbf{u},k} = \mathcal{G}_{\mathbf{u}^k}\big(B(d_\omega^{q,k} + q^k) - A(\mathbf{u}^k) - R(\varphi^k;\gamma)\big)$ is the unique weak solution to the state equation of (QP_k^ω), in particular this means that $\mathbf{d}^{\mathbf{u},k} \in W$ is bounded by Lemma 3.1, i.e.

$$\|\mathbf{d}^{\mathbf{u},k}\|_W \le c(q^k, \mathbf{u}^k)\|d_\omega^{q,k}\|_Q,$$

where $c = c(q^k, \mathbf{u}^k) > 0$ is a constant only dependent on the Q norm of q^k and the W-norm of $\mathbf{u}^k$, c.f. (3.15). Therefore, we have $\mathbf{u}^k, \mathbf{d}_\omega^{\mathbf{u},k}, \mathbf{z}^k$ bounded in W, and q^k, d_ω^k bounded in Q. Using (3.6), (3.7), and Hölder's inequality, both $J_k(\mathbf{u}_\omega^k)_{|\mathbf{u}}$ and $J_k(q_\omega^k)_{|q}$ are bounded from below. Further, it is easy to verify that $J_k(q_\omega^k)_{|q}$ is convex and continuous w.r.t. $d_\omega^{q,k}$, therefore weakly lower semicontinuous, and that Q_{ad}^ω is weakly sequentially compact. The remainder of the proof now follows standard arguments, cf. e.g. [**55**]. □

We state first-order optimality conditions for (QP_k^ω), which again follow in a standard way.

Corollary 6.2. *Let $\mathbf{d}_\omega^k$ be a local solution to (QP_k^ω) for given $y^k \in Y$. Then there exists an adjoint state pair $\mathbf{z}_\omega^{k+1} = (z_\omega^{u,k+1}, z_\omega^{\varphi,k+1}) \in W$, such that*

$$A'(\mathbf{u}^k)\mathbf{d}_\omega^{\mathbf{u},k+1} + R'_\varphi(\varphi^k;\gamma)d_\omega^{\varphi,k+1} = B(q_\omega^{k+1}) - A(\mathbf{u}^k) - R(\varphi^k;\gamma), \tag{6.1a}$$

$$\begin{aligned} (A'(\mathbf{u}^k))^*\mathbf{z}_\omega^{k+1} + R'(\varphi^k;\gamma)^* z_\omega^{\varphi,k+1} &= u_\omega^{k+1} - u_d - A''(\mathbf{u}^k)[\mathbf{d}_\omega^{\mathbf{u},k+1}, \cdot]^*\mathbf{z}^k \\ &\quad - R''(\varphi^k;\gamma)[d_\omega^{\varphi,k+1}, \cdot]^* z^{\varphi,k}, \end{aligned} \tag{6.1b}$$

$$\Big(B^*\mathbf{z}_\omega^{k+1} + \alpha q_\omega^{k+1}, q - q_\omega^{k+1}\Big) \ge 0 \qquad \forall q \in Q_{ad}^\omega. \tag{6.1c}$$

Note that the optimality conditions of $(\widehat{\mathrm{QP}}_k)$ from Corollary 5.3 and the optimality conditions of (QP_k^ω) from Corollary 6.2 only differ in the variational inequality, in particular only the control sets $\widehat{Q_{\mathrm{ad}}}$, Q_{ad}^ω, respectively, are not identical.

6.2. Equivalence of (QP_k^ω) and $(\widehat{\mathrm{QP}}_k)$. We will show that the (unique) solution $\hat{q}^{k+1}$ of $(\widehat{\mathrm{QP}}_k)$, together with the associated state $\hat{\mathbf{u}}^{k+1}$ and the adjoint state $\hat{\mathbf{z}}^{k+1}$, satisfies the optimality conditions of (QP_k^ω), and that the latter have a unique solution if y^k lies sufficiently close to $\bar{y}$, and $\omega > 0$ sufficiently small. In particular, we will also show that q_ω^{k+1} lies in $\widehat{Q_{\mathrm{ad}}}$, if $\omega > 0$ is sufficiently small.

We start with a technical auxiliary lemma, analogously to [**55**, Lemma 6.5].

Lemma 6.3. *There exists an $\omega_3 > 0$ with the following properties: Suppose $\omega \le \omega_3$, $y^k \in Y$ with $\|y^k - \bar{y}\|_Y \le \omega_3$, and let the triple $y = (\mathbf{u}, q, \mathbf{z})$*

satisfy

$$\begin{aligned}
&q \in Q^{\omega}_{ad},\\
&\mathbf{u} = \mathcal{G}_{\mathbf{u}^k}\big(B(q) + A'(\mathbf{u}^k)\mathbf{u}^k + R'(\varphi^k;\gamma)\varphi^k - A(\mathbf{u}^k) - R(\varphi^k;\gamma)\big),\\
&\mathbf{z} = \mathcal{G}_{\mathbf{u}^k}\big(u - u_d - A''(\mathbf{u}^k)[\mathbf{u}-\mathbf{u}^k,\cdot]^*\mathbf{z}^k - R''(\varphi^k;\gamma)[\varphi-\varphi^k,\cdot]^* z^{\varphi,k}\big).
\end{aligned}$$

Then it holds

$$\begin{aligned}
sign(B^*\mathbf{z} + \alpha q)(x) &= sign(B^*\bar{\mathbf{z}} + \alpha \bar{q})(x) \quad a.e.\ on \quad \mathcal{I}(\sigma),\\
\big|(B^*\mathbf{z} + \alpha q)(x)\big| &\geq \frac{\sigma}{2} \quad a.e.\ on \quad \mathcal{I}(\sigma).
\end{aligned}$$

Before we start the proof, let us point out that the assumptions of the Lemma 6.3 mean that y satisfies the state equation (6.1a) and adjoint equation (6.1b) of $(\mathrm{QP}^{\omega}_{k})$ from Lemma 6.2, but it is not yet clear or required that the variational inequality (6.1c) holds.

Proof. Analogously to [**55**, Lemma 6.5], the function $\mathbf{d}^{\mathbf{u}} = (d^u, d^{\varphi}) := \mathbf{u} - \bar{\mathbf{u}}$ satisfies

$$\begin{aligned}
A'(\mathbf{u}^k)\mathbf{d}^{\mathbf{u}} + R'(\varphi^k;\gamma)d^{\varphi} &= A'(\mathbf{u}^k)(\mathbf{u}^k - \bar{\mathbf{u}}) + R'(\varphi^k)(\varphi^k - \bar{\varphi})\\
&\quad - (A(\mathbf{u}^k) - A(\bar{\mathbf{u}})) - (R(\varphi^k;\gamma) - R(\bar{\varphi};\gamma)) + B(q-\bar{q}).
\end{aligned} \tag{6.2}$$

Taylor expansion of the auxiliary functional $T\colon [0,1] \to W^{\times}$, $T(\theta) := A(\mathbf{u}^k + \theta(\bar{\mathbf{u}} - \mathbf{u}^k))$ yields $T(1) - T(0) = T'(\theta)$, for $\theta \in (0,1)$, hence

$$A(\bar{\mathbf{u}}) - A(\mathbf{u}^k) = A'(\mathbf{u}^k + \theta(\bar{\mathbf{u}} - \mathbf{u}^k))(\bar{\mathbf{u}} - \mathbf{u}^k).$$

The operator R can be handled analogously. Setting $M := c\big[\|\bar{\mathbf{u}}\|_W + \|\mathbf{u}^k - \bar{\mathbf{u}}\|_W\big]$, from (3.10) and (3.11), the right-hand side of (6.2) can thus be estimated, analogously to the proof of Lemma 5.1, in the $W^{\times}$-norm by $cM\|\mathbf{u}^k - \bar{\mathbf{u}}\|_W + c\|q-\bar{q}\|_Q \leq cM(\omega_3 + \omega) \leq cM\omega_3$, using $q \in Q^{\omega}_{\mathrm{ad}}$ in combination with $\omega \leq \omega_3$. Thus, by Lemma 3.1 it holds

$$\|\mathbf{u} - \bar{\mathbf{u}}\|_W = \|\mathbf{d}^{\mathbf{u}}\|_W \leq cM\omega_3.$$

Analogously, we obtain

$$\|\mathbf{z} - \bar{\mathbf{z}}\|_W \leq cM\omega_3,$$

from which we conclude, utilizing again $q \in Q^{\omega}_{\mathrm{ad}}$ in combination with $\omega \leq \omega_3$,

$$|B^*(\mathbf{z} - \bar{\mathbf{z}}) + \alpha(q - \bar{q})| \leq cM(\omega_3 + \omega) \leq cM\omega_3. \tag{6.3}$$

Therefore

$$B^*\mathbf{z} + \alpha q = B^*\bar{\mathbf{z}} + \alpha\bar{q} + B^*(\mathbf{z} - \bar{\mathbf{z}}) + \alpha(q - \bar{q}) \geq \sigma - cM\omega_3 \text{ a.e. on } \mathcal{I}(\sigma),$$

where we used (6.3), and $|B^*\bar{\mathbf{z}} + \alpha\bar{q}| \geq \sigma$ holds due to Assumption 3.3. Since $M \leq c(\|\bar{\mathbf{u}}\|_W + \omega_3)$, choosing $\omega_3 > 0$ sufficiently small completes the proof. □

Let us now summarize all requirements for the different ω_i, $i = 1,2,3$, that allow to apply all previously proven statements.

Assumption 6.4. In all that follows, we chose

$$\omega := \min(\omega_1, \omega_2, \omega_3),$$

and assume that the fixed triple y^k fulfills $\|y^k - \bar{y}\|_Y \leq \omega$.

The next result works similarly to [**55**, Corollary 6.9].

Lemma 6.5. *The unique solution $\hat{q}^{k+1}$ of $(\widehat{\mathrm{QP}}_k)$, with associated state $\hat{\mathbf{u}}^{k+1}$ and adjoint state $\hat{\mathbf{z}}^{k+1}$, satisfies the optimality conditions of (QP_k^ω) from Corollary 6.2.*

Proof. Observe first that $\hat{q}^{k+1} \in Q_{\mathrm{ad}}^\omega$ under Assumption 6.4, as an immediate consequence of Theorem 5.9. To show the remaining properties, observe that

$$\begin{aligned}
\mathbf{d}_\omega^{\mathbf{u},k} &= \mathcal{G}_{\mathbf{u}^k}\big(B(q_\omega^{k+1}) - A(\mathbf{u}^k) - R(\varphi^k;\gamma)\big),\\
\mathbf{z}_\omega^{k+1} &= \mathcal{G}_{\mathbf{u}^k}\big(u_\omega^{k+1} - u_d - A''(\mathbf{u}^k)[\mathbf{d}_\omega^{\mathbf{u},k},\cdot]^*\mathbf{z}^k - R''(\varphi^k;\gamma)[d_\omega^{\varphi,k},\cdot]^* z^{\varphi,k}\big),\\
\hat{\mathbf{d}}^{\mathbf{u},k} &= \mathcal{G}_{\mathbf{u}^k}\big(B(\hat{q}^{k+1}) - A(\mathbf{u}^k) - R(\varphi^k;\gamma)\big),\\
\hat{\mathbf{z}}^{k+1} &= \mathcal{G}_{\mathbf{u}^k}\big(\hat{u}^{k+1} - u_d - A''(\mathbf{u}^k)[\hat{\mathbf{d}}^{\mathbf{u},k},\cdot]^*\mathbf{z}^k - R''(\varphi^k;\gamma)[\hat{d}^{\varphi,k},\cdot]^* z^{\varphi,k}\big),
\end{aligned}$$

i.e. the state equation of $(\widehat{\mathrm{QP}}_k)$ and (QP_k^ω) as well as the adjoint equation of $(\widehat{\mathrm{QP}}_k)$ and (QP_k^ω) are identical. It remains to prove that $\hat{q}^{k+1}$ and $\hat{\mathbf{z}}^{k+1}$ satisfy the variational inequality (6.1c) of (QP_k^ω). We know that $\hat{q}^{k+1}$ and $\hat{\mathbf{z}}^{k+1}$ fulfill the variational inequality (5.7c) from Corollary 5.3, which reads

$$(B^*\hat{\mathbf{z}}^{k+1} + \alpha\hat{q}^{k+1}, q - \hat{q}^{k+1}) \geq 0 \qquad \forall \quad q \in \widehat{Q_{\mathrm{ad}}}.$$

Similarly to [**53**, Corollary 6.9], we recognize that on $\mathcal{I}(\sigma)$, there are two cases: If $B^*\hat{\mathbf{z}}^{k+1} + \alpha\hat{q}^{k+1} \geq \sigma$, we have $q_a = \bar{q} = \hat{q}^{k+1}$ recalling that since $\hat{q}^{k+1} \in \widehat{Q_{\mathrm{ad}}}$, it holds $\hat{q}^{k+1} = \bar{q}$). Likewise, if $B^*\hat{\mathbf{z}}^{k+1} + \alpha\hat{q}^{k+1} \leq -\sigma$, then $q_b = \bar{q} = \hat{q}^{k+1}$. We already know that $\hat{q}^{k+1}$, with associated $\hat{\mathbf{u}}^{k+1}$ and $\hat{\mathbf{z}}^{k+1}$, is feasible for (QP_k^ω), therefore we can utilize Lemma 6.3 for the triple $\hat{y}^{k+1} = (\hat{\mathbf{u}}^{k+1}, \hat{q}^{k+1}, \hat{\mathbf{z}}^{k+1})$ and conclude that either $B^*\hat{\mathbf{z}}^{k+1} + \alpha\hat{q}^{k+1} \geq \frac{\sigma}{2}$ or $B^*\hat{\mathbf{z}}^{k+1} + \alpha\hat{q}^{k+1} \leq -\frac{\sigma}{2}$.

Therefore, $(B^*\hat{\mathbf{z}}^{k+1} + \alpha\hat{q}^{k+1})(q - \hat{q}^{k+1}) \geq 0$ holds on $\mathcal{I}(\sigma)$ for all $q \in [q_a, q_b]$. On $Q\backslash\mathcal{I}(\sigma)$, the controls $q \in \widehat{Q_{\mathrm{ad}}}$ succumb to the constraint $q \in [q_a, q_b]$. Overall, we obtain

$$\begin{aligned}
(B^*\hat{\mathbf{z}}^{k+1} + \alpha\hat{q}^{k+1}, q - \hat{q}^{k+1})_Q &= (B^*\hat{\mathbf{z}}^{k+1} + \alpha\hat{q}^{k+1}, q - \hat{q}^{k+1})_{Q\backslash\mathcal{I}(\sigma)}\\
&\quad + (B^*\hat{\mathbf{z}}^{k+1} + \alpha\hat{q}^{k+1}, q - \hat{q}^{k+1})_{\mathcal{I}(\sigma)} \geq 0 \;\forall\; q \in Q_{\mathrm{ad}}.
\end{aligned} \tag{6.4}$$

Since $Q_{\mathrm{ad}}^\omega \subset Q_{\mathrm{ad}}$, the last inequality in particularly also holds for all $q \in Q_{\mathrm{ad}}^\omega$, which concludes the proof. ☐

Before showing uniqueness of the solution of (QP_k^ω), we need another auxiliary lemma, which shows $q_\omega^{k+1} \in \widehat{Q_{\mathrm{ad}}}$, i.e. feasibility of q_ω^{k+1} for $(\widehat{\mathrm{QP}}_k)$.

Lemma 6.6. *Any locally optimal control $q_\omega^{k+1} \in Q_{ad}^\omega$ of (QP_k^ω), that satisfies the optimality conditions from Corollary 6.2, together with the associated state $\mathbf{u}_\omega^{k+1}$ and adjoint state $\mathbf{z}_\omega^{k+1}$, fulfills*

$$q_\omega^{k+1}(x) = \bar{q}(x) \quad a.e. \;\; on \quad \mathcal{I}(\sigma).$$

Proof. The proof works in the same way as [**53**, Corollary 6.6]. For convenience we will recapitulate it: Let x be on $\mathcal{I}(\sigma)$. We have $\bar{q}(x) = q_b$ where $(B^*\bar{\mathbf{z}} + \alpha\bar{q})(x) \leq -\sigma$, and $\bar{q}(x) = q_a$ where $(B^*\bar{\mathbf{z}} + \alpha\bar{q})(x) \geq \sigma$. For any $q \in Q_{\mathrm{ad}}^\omega$, therefore either $q(x) \in [q_b - \omega, q_b]$ or $q(x) \in [q_a, q_a + \omega]$. By Lemma 6.3, either $B^*\mathbf{z}_\omega^{k+1} + \alpha q_\omega^{k+1} \geq \frac{\sigma}{2}$ or $B^*\mathbf{z}_\omega^{k+1} + \alpha q_\omega^{k+1} \leq -\frac{\sigma}{2}$, thus by (6.1c) it holds either $q_\omega^{k+1} = q_b$ or $q_\omega^{k+1} = q_a$. Thus on $\mathcal{I}(\sigma)$, we have shown $q_\omega^{k+1} = \bar{q}$. ☐

By means of Lemma 6.6, we will now show that solutions of the optimality system from Corollary 6.2 are unique under Assumption 3.3 and Assumption 6.4, using the ideas of [**55**, Theorem 6.12].

Lemma 6.7. *The optimality system from Corollary 6.2 for* (QP_k^ω) *admits a unique KKT triple* $y_\omega^{k+1} \in Y$.

Proof. According to Lemma 6.1, there is at least one solution $q_{\omega,1}^{k+1}$ of (QP_k^ω), hence at least one triple $y_{\omega,1}^{k+1}$ has to satisfy the optimality conditions of (QP_k^ω) from Corollary 6.2. We assume that $y_{\omega,2}^{k+1}$ also satisfies the optimality system from Corollary 6.2. Testing the variational inequality (6.1c) once in $q_{\omega,1}^{k+1}$ with $q_{\omega,2}^{k+1}$, and once in $q_{\omega,2}^{k+1}$ with $q_{\omega,1}^{k+1}$, taking the sum of the resulting inequalities, we obtain

$$0 \le (B^*\mathbf{z}_{\omega,1}^{k+1} + \alpha q_{\omega,1}^{k+1}, q_{\omega,2}^{k+1} - q_{\omega,1}^{k+1}) + (B^* z_{\omega,2}^{k+1} + \alpha q_{\omega,2}^{k+1}, q_{\omega,1}^{k+1} - q_{\omega,2}^{k+1}).$$

Introducing the notation $\mathbf{u} := \mathbf{u}_{\omega,2}^{k+1} - \mathbf{u}_{\omega,1}^{k+1}, q := q_{\omega,2}^{k+1} - q_{\omega,1}^{k+1}$, $\mathbf{z} := \mathbf{z}_{\omega,2}^{k+1} - \mathbf{z}_{\omega,1}^{k+1}$, this leads to

$$\langle \mathbf{z}, Bq\rangle - \alpha(q,q)_Q \ge 0. \tag{6.5}$$

The functions $\mathbf{z}$ and $\mathbf{u}$ satisfy

$$\begin{aligned}(A'(\mathbf{u}^k))^*\mathbf{z} + R'(\varphi^k;\gamma)^* z^\varphi &= -A''(\mathbf{u}^k)[\mathbf{u},\cdot]^*\mathbf{z}^k - R''(\varphi^k;\gamma)[z^\varphi,\cdot]^* z^{\varphi,k} + u,\\ A'(\mathbf{u}^k)\mathbf{u} + R'(\varphi^k;\gamma)\varphi &= Bq.\end{aligned}$$

Testing the weak formulation of the first equation with $\mathbf{u}$ and the weak formulation of the second equation with $\mathbf{z}$ and again taking the sum of both equations, leads to

$$\langle Bq, \mathbf{z}\rangle = -\langle A''(\mathbf{u}^k)[\mathbf{u},\mathbf{u}], \mathbf{z}^k\rangle - \langle R''(\varphi^k;\gamma)[\varphi,\varphi], z^{\varphi,k}\rangle + (u,u). \tag{6.6}$$

Combining (6.5) with (6.6), and using the definition of the Lagrangian function, we obtain

$$\begin{aligned}0 &\ge (u,u) + \alpha(q,q)_Q - \langle A''(\mathbf{u}^k)[\mathbf{u},\mathbf{u}], \mathbf{z}^k\rangle - \langle R''(\varphi^k;\gamma)[\varphi,\varphi], z^{\varphi,k}\rangle\\ &= \mathcal{L}''(y^k)[(\mathbf{u},q),(\mathbf{u},q)].\end{aligned}$$

Due to Assumption 6.4, by Lemma 6.6 we have $q = 0$ on $\mathcal{I}(\sigma)$. Thus, by Lemma 5.1 it holds

$$\frac{\delta}{2}\|q\|_Q^2 \le \mathcal{L}''(y^k)[(\mathbf{u},q),(\mathbf{u},q)] \le 0.$$

However, this means $q = 0$ on Q, thus $q_{\omega_1}^{k+1} = q_{\omega,2}^{k+1}$. ☐

Note that Lemma 6.1 ensures the existence of at least one solution of (QP_k^ω) and Lemma 6.7 implies uniqueness of solutions of the optimality system associated to (QP_k^ω). We immediately conclude:

Corollary 6.8. *The subproblem* (QP_k^ω) *has a unique solution* $q_\omega^{k+1} \in Q_{ad}^\omega$.

Now, on the one hand Corollary 5.2 and Lemma 6.5 guarantee that the unique solution $\hat{q}^{k+1}$ of $(\widehat{\mathrm{QP}}_k)$ with associated state $\hat{\mathbf{u}}^{k+1}$ and adjoint state $\hat{\mathbf{z}}^{k+1}$ is a KKT-triple of (QP_k^ω). On the other hand, Lemma 6.7 guarantees uniqueness of KKT triples of (QP_k^ω). We conclude:

Corollary 6.9. *The unique solution* $\hat{q}^{k+1}$ *for* $(\widehat{\mathrm{QP}}_k)$ *and the unique solution* q_ω^{k+1} *of* (QP_k^ω) *coincide.*

In particular, this means that $\hat{y}^{k+1} = y_\omega^{k+1}$ and that $\hat{q}^{k+1} = q_\omega^{k+1}$ is the unique (global) solution of both the subproblem $(\widehat{\mathrm{QP}}_k)$ and the subproblem (QP_k^ω).

6.3. Equivalence of (QP_k^ω) and (QP_k).

In this section, we show that the unique solution q_ω^{k+1} of (QP_k^ω), with associated state $\mathbf{u}_\omega^{k+1}$ and adjoint state $\mathbf{z}_\omega^{k+1}$, satisfies the optimality conditions from Lemma 4.1 of (QP_k) and that all solutions $y^{k+1} = (\mathbf{u}^{k+1}, q^{k+1}, \mathbf{z}^{k+1})$ of the optimality system from Lemma 4.1 of (QP_k) that lie in a neighborhood of $\bar{y}$ satisfy the optimality conditions from Corollary 6.2 of (QP_k^ω). We can then conclude that the solutions of the problems (QP_k) and (QP_k^ω) coincide.

Let us start with the observation that the proof of Lemma 6.5 already implies that the solution $\hat{q}^{k+1}$ of $(\widehat{\mathrm{QP}}_k)$, with associated state $\hat{\mathbf{u}}^{k+1}$ and adjoint state $\hat{\mathbf{z}}^{k+1}$, not only satisfies the optimality conditions of (QP_k^ω) from Corollary 6.2, but in fact also already the optimality conditions of (QP_k) from Lemma 4.1, cf. in particular (6.4). Since $\hat{q}^{k+1}$ coincides with q_ω^{k+1} due to Corollary 6.9, we conclude:

Corollary 6.10. *The solution q_ω^{k+1} of (QP_k^ω), with associated state $\mathbf{u}_\omega^{k+1}$ and adjoint state $\mathbf{z}_\omega^{k+1}$, satisfies the optimality conditions of (QP_k) from Lemma 4.1.*

Since by definition, q_ω^{k+1} lies in the ω-neighborhood of $\bar{q}$, i.e. $\|q_\omega^{k+1} - \bar{q}\|_Q \le \omega$, we have shown existence of (at least) one stationary point of (QP_k) that lies in the ω-neighborhood of $\bar{q}$. The reverse assertion holds in the following sense.

Lemma 6.11. *Every $y^{k+1} \in Y$ that fulfills the optimality conditions of (QP_k) from Lemma 4.1 and suffices to $\|y^{k+1} - \bar{y}\| \le \omega$ also satisfies the optimality conditions of (QP_k^ω) from Corollary 6.2.*

Proof. We only have to look at the variational inequality (4.5c) which is clearly true due to $Q_{\mathrm{ad}}^\omega \subset Q_{\mathrm{ad}}$. □

Let us summarize: We have shown that the optimality conditions of (QP_k) from Lemma 4.1 have at least one solution, since y_ω^{k+1} satisfies them. Further, since every solution of the optimality conditions of (QP_k) from Lemma 4.1, that suffices to $\|y^{k+1} - \bar{y}\| \le \omega$, also satisfies the optimality conditions of (QP_k^ω) from Corollary 6.2, it holds $y^{k+1} = y_\omega^{k+1}$, since the optimality conditions (QP_k^ω) from Corollary 6.2 admit a unique solution by Lemma 6.7. Thus the unique y_ω^{k+1} from Lemma 6.7 is the unique solution of the optimality conditions of (QP_k) from Lemma 4.1. Moreover, since q_ω^{k+1} is the unique solution to (QP_k^ω), it is in fact a minimizer of (QP_k) in the neighborhood $B_\omega^Q(\bar{q})$, since there holds

$$f(q_\omega^{k+1}) \le f(q) \qquad \forall q \in Q_{ad}^\omega = \{q \in Q_{\mathrm{ad}} \mid \|q - \bar{q}\|_Q \le \omega\}.$$

In short, it holds:

Corollary 6.12. *The subproblem (QP_k) has a unique local minimizer q^{k+1} in the neighborhood $B_\omega^Q(\bar{q})$, which coincides with q_ω^{k+1}, the unique solution of (QP_k^ω).*

6.4. The main result.

Let us finally establish our main result, local quadratic convergence of the SQP method given by Algorithm 4.2 to the local minimizer $\bar{q}$ from Lemma 3.2. This follows by transferring the convergence result for $\{\hat{q}^k\}$ from

Theorem 5.9, via the equivalence of the subproblems $(\widehat{\mathrm{QP}}_k)$ and (QP_k^ω), and the equivalence of the subproblems (QP_k^ω) and (QP_k), to Algorithm 4.2.

Theorem 6.13. *Let $\bar{y} \in Y$ satisfy both the optimality condition from Lemma 3.2 and the second-order sufficient condition from Assumption 3.3 and let Assumption 6.4 hold, i.e. $\omega := \min(\omega_1, \omega_2, \omega_3)$. For all starting triples $y^0 \in Y$ that fulfill $\|y^0 - \bar{y}\|_Y \le \omega$, Algorithm 4.2 generates a sequence $\{y^k\}$ that converges quadratically to $\bar{y}$.*

Proof. Let y^0 be as assumed and let $\{\hat{q}^k\} \subset Q_{\mathrm{ad}}$ denote the sequence of iterates generated by $(\widehat{\mathrm{QP}}_k)$. By Theorem 5.9, we have $\|\hat{y}^k - \bar{y}\|_Y \le \omega$ for all $k > 1$ and $\{\hat{q}^k\}$ converges quadratically to $\bar{q}$. By induction, as well as Corollary 6.9 and Corollary 6.12, $\hat{q}^k$ with associated state $\hat{\mathbf{u}}^k$ and adjoint state $\hat{\mathbf{z}}^k$, is the unique solution of (QP_k), which yields the result. $\square$

To conclude, let us give a remark on possible choices of the neighborhood Q_{ad}^ω. The idea stems from [**53**], where a possibility to eliminate the a priori unknown $\bar{y}$ in the definition of (QP_k^ω) is presented. It relies on the fact that the set Q_{ad}^ω in the previous estimations can be replaced by any convex, closed set $\tilde{Q}_{\mathrm{ad}}$, as long as $\tilde{Q}_{\mathrm{ad}}$ is chosen such that

$$Q_{\mathrm{ad}}^\omega \supset \tilde{Q}_{\mathrm{ad}} \supset Q_{\mathrm{ad}}^{\omega_0}, \tag{6.7}$$

for some $\omega_0 > 0$. A possibility is e.g. using $\omega_0 = \|y^0 - \bar{y}\|_Y$, such that $\omega_0 \le \frac{1}{3}\omega$, then the control set

$$\tilde{Q}_{\mathrm{ad}} = \{q \in Q_{\mathrm{ad}} \mid \|q - q^0\|_Q \le 2\omega_0\}$$

can be used in the SQP method, and we obtain the same solution in $\tilde{Q}_{\mathrm{ad}}$ as in Q_{ad}^ω, which is the solution in $\widehat{Q_{\mathrm{ad}}}$.

Appendix A. Auxiliary results

We establish some auxiliary results that are necessary for the proof of Theorem 5.9. We start with an auxiliary result that is required for the application of Dontchev's implicit function theorem [**18**, Theorem 2.4], and follows along the lines of [**26**, Lemma 6.2].

Lemma A.1. *Let $\bar{y} \in Y$ be given. For any $r_5, r_6 > 0$, there exists a constant $L(r_5, r_6) > 0$ such that for all $y_i = (\mathbf{u}_i, q_i, \mathbf{z}_i) \in Y$ with $\|y_i - \bar{y}\|_Y \le r_5$, $i = 1, 2$, and for all $y \in Y$ with $\|y - \bar{y}\|_Y \le r_6$, the following Lipschitz condition holds:*

$$\|F(y_1) + F'(y_1)(y - y_1) - F(y_2) - F'(y_2)(y - y_2)\|_Z \le L(r_5, r_6)\|y_1 - y_2\|_Y.$$

Proof. Let $r_5, r_6 > 0$, y_i and y be as assumed. We define f_1 and f_2 via

$$\begin{aligned} f_1(\mathbf{u}_i, \mathbf{z}_i) :=& (A'(\mathbf{u}_i))^*\mathbf{z} + R'(\varphi_i;\gamma)^* z^\varphi + A''(\mathbf{u}_i)[\mathbf{u} - \mathbf{u}_i, \cdot\,]^*\mathbf{z}_i \\ &+ R''(\varphi_i;\gamma)[\varphi - \varphi_i, \cdot\,]^* z_i^\varphi, \\ f_2(\mathbf{u}_i) :=& A(\mathbf{u}_i) + R(\varphi_i;\gamma) + A'(\mathbf{u}_i)(\mathbf{u} - \mathbf{u}_i) + R'(\varphi_i;\gamma)(\varphi - \varphi_i), \end{aligned}$$

and obtain

$$\begin{aligned} &F(y_1) + F'(y_1)(y - y_1) - F(y_2) + F'(y_2)(y - y_2) \\ &\quad = \Big(f_1(\mathbf{u}_1, \mathbf{z}_1) - f_1(\mathbf{u}_2, \mathbf{z}_2), 0, f_2(\mathbf{u}_1) - f_2(\mathbf{u}_2) \Big)^T, \end{aligned}$$

and calculate

$$\begin{aligned} &f_1(\mathbf{u}_1, \mathbf{z}_1) - f_1(\mathbf{u}_2, \mathbf{z}_2) \\ &= (A'(\mathbf{u}_1) - A'(\mathbf{u}_2)^* \mathbf{z} + (A''(\mathbf{u}_1) - A''(\mathbf{u}_2))[\mathbf{u} - \mathbf{u}_1, \cdot]^* \mathbf{z}_1 \\ &\quad + A''(\mathbf{u}_2)[\mathbf{u} - \mathbf{u}_1, \cdot]^*(\mathbf{z}_1 - \mathbf{z}_2) + A''(\mathbf{u}_2)[\mathbf{u}_2 - \mathbf{u}_1, \cdot]^* \mathbf{z}_2 \\ &\quad + (R'(\varphi_1; \gamma) - R'(\varphi_2; \gamma))^* z^\varphi + (R''(\varphi_1; \gamma) + R''(\varphi_2; \gamma))[\varphi - \varphi_1, \cdot]^* z_1^\varphi \\ &\quad + R''(\varphi_2; \gamma)[\varphi - \varphi_1, \cdot]^*(z_1^\varphi - z_2^\varphi) + R''(\varphi_2; \gamma)[\varphi_2 - \varphi_1, \cdot]^* z_2^\varphi. \end{aligned}$$

Applying the boundedness and Lipschitz results (3.8)-(3.13), we obtain

$$\|f_1(\mathbf{u}_1, \mathbf{z}_1) - f_1(\mathbf{u}_2, \mathbf{z}_2)\|_{W^\times} \leq L(r_5, r_6)\big(\|\mathbf{u}_1, -\mathbf{u}_2\|_W + \|\mathbf{z}_1 - \mathbf{z}_2\|_W\big),$$

for an $L(r_5, r_6) > 0$. Here, we also used $\|\mathbf{u} - \mathbf{u}_i\|_W \leq \|\mathbf{u} - \bar{\mathbf{u}}\|_W + \|\bar{\mathbf{u}} - \mathbf{u}_i\|_W \leq r_5 + r_6$, for $i = 1, 2$. Estimating the difference for f_2 in a similar way concludes the proof. We omit the details. □

For completeness, we also want to give a Lipschitz result for F'.

Lemma A.2. *The operator F' from* (5.2) *is locally Lipschitz continuous w.r.t to y as a mapping from Y into Z, i.e. there exists a constant $c > 0$, such that for all $y_i = (\mathbf{u}_i, q_i, \mathbf{z}_i) \in Y$ it holds*

$$\|\big(F'(y_1) - F'(y_2)\big)y\|_Z \leq c\big[\|y_1\|_Y + \|y_2\|_Y\big]\|y_1 - y_2\|_Y \|y\|_Y.$$

Proof. Similarly to the proof of Lemma A.1, we define an f_1 and f_2 via

$$\begin{aligned} f_1(\mathbf{u}_i, \mathbf{z}_i) &:= A''(\mathbf{u}_i)[\mathbf{u}, \cdot]^* \mathbf{z}_i + A'(\mathbf{u}_i)^* \mathbf{z} + R''(\varphi_i; \gamma)[\varphi, \cdot]^* z_i^\varphi + R'(\varphi_i; \gamma)^* z^\varphi, \\ f_2(\mathbf{u}_i) &:= A'(\mathbf{u}_i)\mathbf{u} + R'(\varphi_i; \gamma)\varphi, \end{aligned}$$

and obtain

$$\big(F'(y_1) - F'(y_2)\big)y = \big(f_1(\mathbf{u}_1, \mathbf{z}_1) - f_2(\mathbf{u}_2, \mathbf{z}_2), 0, f_2(\mathbf{u}_1) - f_2(\mathbf{u}_2)\big),$$

and calculate

$$\begin{aligned} &f_1(\mathbf{u}_1, \mathbf{z}_1) - f_1(\mathbf{u}_2, \mathbf{z}_2) \\ &\quad = \big(A''(\mathbf{u}_1) - A''(\mathbf{u}_2)\big)[\mathbf{u}, \cdot]^* \mathbf{z}_1 + A''(\mathbf{u}_2)[\mathbf{u}, \cdot]^*(\mathbf{z}_1 - \mathbf{z}_2) \\ &\qquad + \big(R''(\varphi_1; \gamma) - R''(\varphi_2; \gamma)\big)[\varphi, \cdot]^* z_1^\varphi + R''(\varphi_2; \gamma)[\varphi, \cdot]^*(z_1^\varphi - z_2^\varphi) \\ &\qquad + \big(A'(\mathbf{u}_1) - A'(\mathbf{u}_2)\big)^* \mathbf{z} + \big(R'(\varphi_1; \gamma) - R'(\varphi_2; \gamma)\big)^* z^\varphi. \end{aligned}$$

The claim now follows analogously to Lemma A.1. Estimating the difference for f_2 in a similar way concludes the proof. We again omit the details. □

Finally, we need a quadratic bound for the second-order remainder of the derivative of F, that is used in the proof of Theorem 5.9. Note that in [**25**, Theorem 7.1], this bound immediately follows from second-order Fréchet-differentiability of F. However, in our case F is not twice differentiable due to the fact that the operator R is not three times Fréchet-differentiable. This requires some additional calculations.

Lemma A.3. *Let $\bar{y} \in Y$ be given and $y^k \in Y$ with $\|y^k - \bar{y}\|_Y \le r$ for some $r > 0$. There exists a constant $c(r) > 0$ such that*

$$\|F(\bar{y}) - F(y^k) - F'(y^k)(\bar{y} - y^k)\|_Z \le c(r)\|y^k - \bar{y}\|_Y^2.$$

Proof. As mentioned above, the difficulty in obtaining the quadratic term on the right-hand side lies in the fact that F is not twice Fréchet-differentiable, we instead have to exploit the structure of the operator F', and hence R'', to obtain the desired estimate. We recall the definition of F via

$$F(y) = \Bigg(\underbrace{(A'(\mathbf{u}))^*\mathbf{z} + R'(\varphi;\gamma)^* z^\varphi - u - u_d}_{=:I(y)}, \underbrace{B^*\mathbf{z} + \alpha q}_{=:II(y)}, \underbrace{A(\mathbf{u}) + R(\varphi;\gamma) - Bq}_{=:III(y)} \Bigg)^T,$$

and continue by component wise estimation. For the first component of F, using the notation $I(y)$ from above, this means

$$\begin{aligned}
& I(\bar{y}) - I(y^k) - I'(y^k)(\bar{y} - y^k)) \\
&\quad = (A'(\bar{\mathbf{u}}))^*\bar{\mathbf{z}} + R'(\bar{\varphi};\gamma)^*\bar{z}^\varphi - (A'(\mathbf{u}^k))^*\mathbf{z}^k - R'(\varphi^k;\gamma)^* z^{\varphi,k} \\
&\qquad - A''(\mathbf{u}^k)[\bar{\mathbf{u}} - \mathbf{u}^k, \cdot\,]^*\mathbf{z}^k - A'(\mathbf{u}^k)^*(\bar{\mathbf{z}} - \mathbf{z}^k) \\
&\qquad - R''(\varphi^k;\gamma)[\bar{\varphi} - \varphi^k, \cdot\,]^* z^{\varphi,k} - R'(\varphi^k;\gamma)^*(\bar{z}^\varphi - z^{\varphi,k}) \\
&\quad = A''(\mathbf{u}^k)[\bar{\mathbf{u}} - \mathbf{u}^k, \cdot\,]^*(\bar{\mathbf{z}} - \mathbf{z}^k) + rem_{A'}(\cdot, \bar{\mathbf{u}} - \mathbf{u}^k)^*\bar{\mathbf{z}} \\
&\qquad - R''(\varphi^k;\gamma)[\bar{\varphi} - \varphi^k, \cdot\,]^* z^{\varphi,k} + \big(R'(\bar{\varphi};\gamma) - R'(\varphi^k;\gamma)\big)^*\bar{z}^\varphi,
\end{aligned} \tag{A.1}$$

which follows by differentiability of A'. The term involving A'' and $rem_{A'}$ can now be estimated by applying (3.6) and again analogously to [**30**, Proposition 3.3], respectively. Overall, it follows

$$\|A''(\mathbf{u}^k)[\bar{\mathbf{u}} - \mathbf{u}^k, \cdot]^*(\bar{\mathbf{z}} - \mathbf{z}^k)\|_{W^\times} \le c(\mathbf{u}^k)\|y^k - \bar{y}\|_Y^2, \tag{A.2}$$

$$\|rem_{A'}(\cdot, \bar{\mathbf{u}} - \mathbf{u}^k)^*\bar{\mathbf{z}}\|_{W^\times} \le c(\bar{\mathbf{z}})\|\bar{\mathbf{u}} - \mathbf{u}^k\|_W^2. \tag{A.3}$$

For the terms involving R', we introduce the auxiliary functional $T\colon [0,1] \to \mathbb{R}$, $T(\theta) := R'(\varphi^k + \theta(\bar{\varphi} - \varphi^k);\gamma)^*\bar{z}^\varphi$. By Taylor's expansion $T(1) = T(0) + T'(\theta)$, for $\theta \in (0,1)$ and $\bar{\varphi}, \varphi^k, \bar{z}^\varphi \in W_\varphi$, we obtain

$$R'(\bar{\varphi};\gamma)^*\bar{z}^\varphi - R'(\varphi^k;\gamma)^*\bar{z}^\varphi = R''(\varphi^k + \theta(\bar{\varphi} - \varphi^k);\gamma)[\cdot, \bar{\varphi} - \varphi^k])^*\bar{z}^\varphi.$$

Thus for the R' and R'' terms in (A.1), we obtain

$$\begin{aligned}
&\|R''(\varphi^k + \theta(\bar{\varphi} - \varphi^k);\gamma)[\bar{\varphi} - \varphi^k, \cdot]^*\bar{z}^\varphi - R''(\varphi^k;\gamma)[\bar{\varphi} - \varphi^k, \cdot]^* z^{\varphi,k}\|_q \\
&\quad \le 6\gamma\|[(\varphi^k + \theta(\bar{\varphi} - \varphi^k) - \varphi^-)^+](\bar{\varphi} - \varphi^k)(\bar{z}^\varphi - z^{\varphi,k})\|_q \\
&\qquad + 6\gamma\|([(\varphi^k + \theta(\bar{\varphi} - \varphi^k) - \varphi^-)^+ - (\varphi^k - \varphi^-)^+](\bar{\varphi} - \varphi^k) z^{\varphi,k})\|_q \\
&\quad \le 6\gamma\|[(\varphi^k + \theta(\bar{\varphi} - \varphi^k) - \varphi^-)^+]\|_\infty\|\bar{\varphi} - \varphi^k\|_\infty\|\bar{z}^\varphi - z^{\varphi,k}\|_\infty \\
&\qquad + 6\gamma\|[(\varphi^k + \theta(\bar{\varphi} - \varphi^k) - \varphi^-)^+ - (\varphi^k - \varphi^-)^+]\|_\infty\|\bar{\varphi} - \varphi^k\|_\infty\|z^{\varphi,k}\|_\infty \\
&\quad \le c\|\bar{\mathbf{u}}\|_Y\|\bar{y} - y^k\|_Y^2 + c\|z^{\varphi,k}\|_{2,q}\|\varphi^k + \theta(\bar{\varphi} - \varphi^k) - \varphi^- - \varphi^k + \varphi^-\|_\infty\|\bar{z}^\varphi - z^{\varphi,k}\|_\infty \\
&\quad \le c\|\bar{y} - y^k\|_Y^2,
\end{aligned} \tag{A.4}$$

where we used continuity of $(\cdot)^+$ to obtain the second to last inequality. Overall collecting the estimates (A.2), (A.3), and (A.4), we conclude

$$\|I(\bar{y}) - I(y^k) - I'(y^k)(\bar{y} - y^k)\|_{W^\times} \leq c(r)\|\bar{y} - y^k\|_Y^2. \tag{A.5}$$

For the second component in the difference of F, using the notation $II(y)$, it immediately holds

$$\begin{aligned} &II(\bar{y}) - II(y^k) - II(y^k)(\bar{y} - y^k) \\ &\quad = B^*\bar{\mathbf{z}} + \alpha\bar{q} - B^*\mathbf{z}^k - \alpha q^k - B^*(\bar{\mathbf{z}} - \mathbf{z}^k) - \alpha(\bar{q} - q^k) = 0, \end{aligned}$$

which directly leads to

$$\|II(\bar{y}) - II(y^k) - II'(y^k)(\bar{y} - y^k)\|_{W^\times} = 0. \tag{A.6}$$

Finally for the third component in the difference of F, we use the notation $III(\mathbf{u}, q, \mathbf{z})$, and calculate

$$\begin{aligned} &III(\bar{y}) - III(y^k) - III(y^k)(\bar{y} - y^k) \\ &\quad = A(\bar{\mathbf{u}}) + R(\bar{\varphi};\gamma) - B(\bar{q}) - A(\mathbf{u}^k) - R(\varphi^k;\gamma) + B(q^k) \\ &\qquad - A'(\mathbf{u}^k)(\bar{\mathbf{u}} - \mathbf{u}^k) - R'(\varphi^k;\gamma)(\bar{\varphi} - \varphi^k) + B(\bar{q} - q^k) \\ &\quad = rem_A(\mathbf{u}^k, \bar{\mathbf{u}} - \mathbf{u}^k) + R''(\varphi^k + \theta(\bar{\varphi} - \varphi^k))(\bar{\varphi} - \varphi^k)^2, \end{aligned} \tag{A.7}$$

for a $\theta \in (0,1)$. Here, differentiability of A was used, and the term involving R'' follows by using an auxiliary functional $T\colon [0,1] \to \mathbb{R}$, $T(\theta) := R(\varphi^k + \theta(\bar{\varphi} - \varphi^k);\gamma)$, and Taylor's expansion $T(1) = T(0) + T'(0) + \frac{1}{2}T''(\theta)$, for $\theta \in (0,1)$ and $\bar{\varphi}, \varphi^k \in W_\varphi$, i.e.

$$R(\bar{\varphi};\gamma) - R(\varphi^k;\gamma) - R'(\varphi^k;\gamma)(\bar{\varphi} - \varphi^k) = R''(\varphi^k + \theta(\bar{\varphi} - \varphi^k);\gamma)(\bar{\varphi} - \varphi^k)^2.$$

Continuing in (A.7), the estimation of R'' in the L^q norm and of rem_A in $W^\times$ works in the same way as above. We conclude

$$\|III(\bar{y}) - III(y^k) - III'(y^k)(\bar{y} - y^k)\|_{W^\times} \leq c(r)\|\bar{y} - y^k\|_Y^2. \tag{A.8}$$

Combining (A.5), (A.6), and (A.8) concludes the proof. □

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