Deutsche Forschungsgemeinschaft
Priority Programme 1962

On Generalized Nash Equilibrium Problems in Infinite-Dimensional Spaces using Nikaido–Isoda type Functionals

Michael Ulbrich, Julia Wachter

SPP 1962
Non-smooth and Complementarity-based Distributed Parameter Systems: Simulation and Hierarchical Optimization

Preprint Number SPP1962-197

received on October 14, 2022

Edited by
SPP1962 at Weierstrass Institute for Applied Analysis and Stochastics (WIAS)
Leibniz Institute in the Forschungsverbund Berlin e.V.
Mohrenstraße 39, 10117 Berlin, Germany
E-Mail: spp1962@wias-berlin.de

World Wide Web: http://spp1962.wias-berlin.de/

On generalized Nash equilibrium problems in infinite-dimensional spaces using Nikaido–Isoda type functionals

Michael Ulbrich[a] and Julia Wachter[a]

[a]Chair of Mathematical Optimization, Department of Mathematics, School of Computation, Information and Technology, Technical University of Munich, Boltzmannstr. 3, 85748 Garching, Germany

ARTICLE HISTORY
Compiled October 14, 2022

Abstract
In this work, we present an analysis of generalized Nash equilibrium problems in infinite-dimensional spaces with possibly non-convex objective functions of the players. Such settings arise, for instance, in games that involve nonlinear partial differential equation constraints. Due to non-convexity, we work with equilibrium concepts that build on first order optimality conditions, especially Quasi-Nash Equilibria (QNE), i.e., first-order optimality conditions for (Generalized) Nash Equilibria, and Variational Equilibria (VE), i.e., first-order optimality conditions for Normalized Nash Equilibria. We prove existence of these types of equilibria and study characterizations of them via regularized (and localized) Nikaido-Isoda merit functions. We also develop continuity and (continuous) differentiability results for these merit functions under quite weak assumptions, using a generalization of Danskin's theorem. They provide a theoretical foundation for, e.g., using globalized descent methods for computing QNE or VE.

KEYWORDS
generalized Nash equilibrium problems; infinite-dimensional spaces; (regularized) Nikaido–Isoda function; existence of equilibria; merit functions; Danskin's theorem

AMS CLASSIFICATION
91A10; 90C48; 90C26; 49J27; 49J50; 46N10

1. Introduction

In this work, we investigate non-convex generalized Nash equilibrium problems (GNEPs) in infinite-dimensional spaces. Nash equilibrium problems were pioneered by Nash [23] and extended by Arrow and Debreu [1] and by Rosen [27]. Since then, Nash equilibria (NE) have been investigated intensively, e.g., [10, 12–15, 30]. The paper [13] gives a survey on the theory of GNEPs in finite dimensions. In contrast to the finite-dimensional case, the literature on GNEPs in infinite-dimensional spaces is relatively scarce [18, 19, 24, 25, 32, 33]. Most of these available results have somewhat restrictive requirements on the objective function, in particular convexity assumptions. In [18, 19], convex GNEPs with optimal control structure, a linear state equation and control and state constraints are studied, where [19] considers the special case of a quadratic cost function and a linear elliptic state equation.

GNEPs have been applied to many real world scenarios, e.g., in economy, aerodynamics [28], and electricity [9]. Games with infinite-dimensional strategy spaces often involve sys-

CONTACT M. Ulbrich. Email: mulbrich@ma.tum.de

tems described by partial differential equations (PDEs) and have important applications, e.g., in traffic flow [4–6] and natural gas spot markets [18]. Therefore, it is useful to have the same tools at hand as in finite dimensions. We combine the methods of finite-dimensional GNEPs with the optimality theory in Banach spaces, see, e.g., [2, 3, 22, 34], to extend many results for GNEPs to an infinite-dimensional non-convex setting.

Most of the available work on GNEPs studies the case where the objective functions are convex. Frequently used concepts are (regularized) Nikaido–Isoda functions and the corresponding merit functions, see, e.g., [25]. The properties of these merit functions in connection with Nash equilibria (NE), normalized equilibria (NoE), and variational equilibria (VE) are well studied in finite dimensions, cf. [13, 18, 30]. For instance, in the case of convex merit functions, the notion of variational equilibria (VE) is equivalent to normalized equilibria (NoE). Algorithms for computing NE and NoE are often based on (constrained) optimization methods for minimizing these merit functions; this typically makes continuous differentiability of the merit functions a desired property. Concerning theoretical results and numerical methods for Nikaido–Isoda merit function based approaches in the finite-dimensional case, we refer to, e.g., [10, 11, 30, 31]. These merit functions are not necessarily differentiable, but, under appropriate assumptions, their continuous differentiability was shown in, e.g., [30], and, under stronger assumptions involving a constraint qualification, it was shown that their gradient is semismooth in a neighborhood of suitable equilibria, cf. [29]. This opens the door for first and second order methods that minimize regularized Nikaido–Isoda merit functions to compute equilibria [11, 15, 30, 31]. Optimality conditions for GNEPs can be formulated as quasi-variational inequalities (QVIs), but, as noted in [30], numerical methods for quasi-variational inequalities (QVIs) are not yet as developed and as efficient as approaches that build on Nikaido–Isoda merit functions. Recently, promising augmented Lagrangian methods (ALM) for QVIs and GNEPs have been proposed [20, 21]. However, their application to GNEPs still requires to solve subproblems that are NEPs, making the results of the present paper also relevant in an ALM context.

In contrast to the majority of the available literature, we focus on GNEPs posed in infinite-dimensional spaces and the players' objective functions can be non-convex. Due to this non-convexity, our main target are then equilibria that are described by first-order optimality conditions, i.e., Quasi-Nash equilibria (QNE) rather than Nash equilibria (NE) and variational equilibria (VE) rather than normalized Nash equilibria (NoE). These concepts will be introduced in Section 2. In our approach, we reformulate the GNEP in terms of the Nikaido–Isoda function, see [25], and its regularization, see [30], using QVIs, see, e.g., [14, 17, 26]. We consider the corresponding solution map based on the regularized Nikaido–Isoda function and a first order optimality condition, which allows us to prove the existence of a fixed point using a generalized version of the Kakutani fixed point theorem. These fixed points can be interpreted as VE or NoE.

Due to our non-convex setting, the players' solution map (or best response map) can be multivalued and, consequently, the Nikaido–Isoda merit function can be nonsmooth. We thus study regularized and "localized" versions of the Nikaido–Isoda merit function, giving rise to proximal response maps and related concepts. Using a generalized version of Danskin's theorem, we prove continuous differentiability of the regularized and localized Nikaido–Isoda merit function. This enables the use of derivative-based optimization methods, which, when applied for the minimization of these merit functions, can be used to compute QNE and VE. Further, we relate minimizers of the Nikaido–Isoda based merit functions to equilibria of GNEPs, in particular to QNE and VE.

This paper is structured as follows. In Section 2, we introduce the mathematical setting and important definitions for our problem. In Section 3, we consider two approaches for showing existence of equilibria, where the solution maps of specific parametric optimization problems, called proximal best response maps, are shown to have a fixed point using a generalized version of the Kakutani fixed point theorem. These fixed points are related to merit functions, the analytical properties of which are investigated in Section 4. Further, we also consider corresponding "localized" merit functions. We establish connections between these merit functions, (quasi-)VIs, and (local) NoE and (local) NE, respectively. In Section 5, we investigate the differentiability of the two merit functions and their localized versions. To this end, we apply a suitable version of Danskin's theorem, see Appendix A.

2. Preliminaries

We consider the following generalized Nash equilibrium problem, see [13] for reference. Let $N \in \mathbb{N}$ be the number of players. For each $i \in [N] := \{1, \ldots, N\}$, the i-th player's strategy (or control) is denoted by $u^i \in U_i$ and for the strategy set of all players we write $U = \Pi_i U_i$. Here, U_i, $i \in [N]$, are Hilbert spaces. Further, by $u^{-i} \in U_{-i}$ we denote the tuple of all strategies except the one of player i. We use the notation $u = (u^i, u^{-i})$ for emphasizing the i-th player, but without reordering the tuple. For each $i \in [N]$, player i solves the minimization problem

$$\min_{v^i \in F_i(u^{-i})} \theta_i(v^i, u^{-i}), \tag{1}$$

where $F_i(u^{-i}) \subset U_i$ denotes the feasible set of player i, which depends on the strategies u^{-i} chosen by the other players. If this feasible set is independent of u^{-i}, the collection of problems (1), $i \in [N]$, is called a *Nash Equilibrium Problem* (NEP), while in the general case it is called a *Generalized Nash Equilibrium Problem* (GNEP). In the majority of the available literature, the objective functions $\theta_i : U \to \mathbb{R}$ are assumed to be both convex and continuously differentiable in the i-th component. In this paper, we relax this assumption. In particular, we will use proximal response maps, where a regularization term $\frac{\alpha}{2}\|\iota_{H_i}(v^i) - \iota_{H_i}(u^i)\|_{H_i}^2$ is added to the cost function θ_i. Here, $\iota_{H_i} : U_i \to H_i$ is a linear and continuous operator, which maps to some Hilbert space H_i. The parameter $\alpha \geq 0$ is chosen according to Assumption 2.3 below. Detailed requirements on all spaces and problem data will be given in this section in Assumptions 2.1 and 2.2.

We are interested in Nash equilibria and in normalized equilibria, respectively, and in points that satisfy first-order necessary optimality conditions for them. A point $u \in F(u) = \Pi_i F_i(u^{-i})$ is called a *Nash Equilibrium* (NE) for the GNEP (1) if

$$\theta_i(u^i, u^{-i}) \leq \theta_i(v^i, u^{-i})$$

holds for all $v^i \in F_i(u^{-i})$ and all $i \in [N]$.

If $F(u)$ is convex and all θ_i are differentiable with respect to the i-th component at the point u, then the following first-order necessary optimality condition holds (which is sufficient if $\theta_i(\cdot, u^{-i})$ is convex)

$$u^i \in F_i(u^{-i}), \quad \langle (\theta_i)_{u^i}(u^i, u^{-i}), v^i - u^i \rangle_{U_i^*, U_i} \geq 0 \quad \forall\, v^i \in F_i(u^{-i}), \ \forall\, i \in [N]. \tag{2}$$

A point u that satisfies (2) will be called a *Quasi-Nash Equilibrium* (QNE). Later, we will relate QNE of the considered GNEP to NE of a regularized GNEP where a proximal term $\frac{\alpha}{2}\|\iota_{H_i}(v^i) - \iota_{H_i}(u^i)\|_{H_i}^2$ is added to each θ_i.

Now let the feasible set of the i-th player be of the form

$$F_i(u^{-i}) = \{v^i \in U_i : (v^i, u^{-i}) \in \mathcal{X}\}, \tag{3}$$

where $\mathcal{X} \subset U$ is nonempty and closed. Then a point $u \in \mathcal{X}$ is called a *Normalized Equilibrium* (NoE) if for all $v \in \mathcal{X}$ there holds

$$\sum_i \theta_i(u^i, u^{-i}) \leq \sum_i \theta_i(v^i, u^{-i}). \tag{4}$$

For a GNEP with feasible sets $F_i(u^{-i})$ as in (3), every NoE $u \in \mathcal{X}$ is a NE. In fact, if u is a NoE, then for any fixed $i \in [N]$ and for all $v^i \in F_i(u^{-i})$, there hold $v := (v^i, u^{-i}) \in \mathcal{X}$ and

$$\begin{aligned}
\theta_i(v^i, u^{-i}) &= \sum_j \theta_j(v^j, u^{-j}) - \sum_{j \neq i} \theta_j(u^j, u^{-j}) \\
&\geq \sum_j \theta_j(u^j, u^{-j}) - \sum_{j \neq i} \theta_j(u^j, u^{-j}) = \theta_i(u^i, u^{-i}),
\end{aligned}$$

where the inequality follows from u being a NoE.

If $\mathcal{X}$ is convex, then a necessary condition for a NoE is given by the variational inequality (VI)

$$u \in \mathcal{X}, \quad \sum_i \langle (\theta_i)_{u^i}(u^i, u^{-i}), v^i - u^i \rangle_{U_i^*, U_i} \geq 0 \quad \forall\, v \in \mathcal{X}. \tag{5}$$

This follows from the fact that (4) means that $v = u$ solves the problem $\min_{v \in \mathcal{X}} \sum_i \theta_i(v^i, u^{-i})$ and (5) is a first-order necessary optimality condition for this problem. Solutions u to (5) are called *Variational Equilibria* (VE). A very similar argument to the one we used after (4) shows that if u solves (5), then u also solves (2), which means that any VE is a QNE.

There are different ways to characterize and compute the introduced types of equilibria (Nash, Quasi-Nash, normalized, variational). One of them uses the *Nikaido–Isoda function* [25],

$$\Psi(u, v) := \sum_i \big[\theta_i(u) - \theta_i(v^i, u^{-i})\big], \tag{6}$$

or regularized versions of it (see below). It is evident that $u \in \mathcal{X}$ is a NoE if and only if $\Psi(u, v) \leq 0$ for all $v \in \mathcal{X}$. This is equivalent to $u \in \mathcal{X}$ being a solution to $\sup_{v \in \mathcal{X}} \Psi(u, v)$ with optimal value 0, while, if $u \in \mathcal{X}$ is not a NoE, then the optimal value is strictly positive. Thus, the value function $u \mapsto V(u) := \sup_{v \in \mathcal{X}} \Psi(u, v)$, called *Nikaido–Isoda merit function*, is nonnegative on $\mathcal{X}$ and there holds $u \in \mathcal{X}$ and $V(u) = 0$ if and only if u is a NoE. A similar construction, where the supremum is taken over $F(u)$ rather than $\mathcal{X}$, yields a Nikaido–Isoda merit function that characterizes NE.

In order to characterize QNE and VE, we introduce a *regularized Nikaido–Isoda function*, see [30], which is given by

$$\Psi_\alpha(u, v) := \sum_i \Big[\theta_i(u) - \theta_i(v^i, u^{-i}) - \frac{\alpha}{2} \|\iota_{H_i}(v^i) - \iota_{H_i}(u^i)\|_{H_i}^2\Big]. \tag{7}$$

The corresponding *regularized Nikaido–Isoda merit function* is defined as

$$V_\alpha(u) := \sup_{v\in\mathcal{X}} \Psi_\alpha(u,v). \tag{8}$$

Note that there holds $\sup_{v\in\mathcal{X}} \Psi_\alpha(u,v) = \sum_i \theta_i(u) - \inf_{v\in\mathcal{X}} \widetilde{\Psi}_\alpha(u,v)$, where

$$\widetilde{\Psi}_\alpha(u,v) = \sum_i \big[\theta_i(v^i,u^{-i}) + \frac{\alpha}{2}\|\iota_{H_i}(v^i) - \iota_{H_i}(u^i)\|_{H_i}^2\big]. \tag{9}$$

One can proceed in a similar way for generalized Nash equilibria and define the regularized Nikaido–Isoda merit function by

$$\begin{aligned}\widetilde{V}_\alpha(u) &:= \sup_{v\in F(u)} \Psi_\alpha(u,v)\\ &= \sum_i \sup_{v^i\in F_i(u^{-i})} \big[\theta_i(u) - \theta_i(v^i,u^{-i}) - \frac{\alpha}{2}\|\iota_{H_i}(v^i) - \iota_{H_i}(u^i)\|_{H_i}^2\big].\end{aligned} \tag{10}$$

The parameter $\alpha \geq 0$ is chosen according to Assumption 2.3 in order to guarantee suitable convexity properties. In Section 3, we will show under suitable assumptions that, for appropriate $\alpha \geq 0$, the problem $\inf_{v\in\mathcal{X}} \widetilde{\Psi}_\alpha(u,v)$ (which is equivalent to the problem in (8) defining $V_\alpha(u)$) has a well-defined solution map $u \mapsto v_{\widetilde{\Psi}_\alpha}(u)$. We show that the *proximal best response map* $v_{\widetilde{\Psi}_\alpha}$ possesses a fixed point and that any such fixed point is a VE of the GNEP.

In order to avoid that possibly quite large values of α are needed, which might slow down numerical methods that use V_α or $\widetilde{V}_\alpha$ for globalization, we also investigate properties of the GNEP (1) when we localize around some u by a constraint, a penalty term, or a barrier-type term. To this end, let $\widetilde{F}_i(u) \subset F_i(u^{-i})$ be convex with $F_i(u^{-i}) \cap \overline{B}_R(u^i) \subset \widetilde{F}_i(u)$ for some $R > 0$ and set $\widetilde{F}(u) := \Pi_i \widetilde{F}_i(u)$, where $\overline{B}_R(u^i)$ is the closed ball in U_i with center u^i and radius $R > 0$, $i \in [N]$. Further, let $\widetilde{\mathcal{X}}(u) \subset \mathcal{X}$ be convex with $\mathcal{X} \cap \overline{B}_R(u) \subset \widetilde{\mathcal{X}}(u)$. For $i \in [N]$, let $q_i : U_i \to [0,\infty]$ be convex, continuously differentiable at 0 with $q_i(0) = 0$, and let q_i be finite on $\overline{B}_R(0) \subset U_i$. Now, define

$$\begin{aligned}V_\alpha^{\mathrm{loc}}(u) &:= \sup_{v\in\widetilde{\mathcal{X}}(u)} \big[\Psi_\alpha(u,v) - \sum_i q_i(v^i - u^i)\big],\\ \widetilde{V}_\alpha^{\mathrm{loc}}(u) &:= \sup_{v\in\widetilde{F}(u)} \big[\Psi_\alpha(u,v) - \sum_i q_i(v^i - u^i)\big].\end{aligned}$$

By a suitable choice of $\widetilde{F}_i(u)$ or q_i, the relevant points v for the maximization can be localized around u. A typical choice for $\widetilde{F}_i(u)$ would be $\widetilde{F}_i(u) = F_i(u^{-i}) \cap (u^i + M_i)$, where $M_i \subset U_i$ is a closed convex set with 0 in its interior. Equivalent to this would be to choose $\widetilde{F}_i(u) = F_i(u^{-i})$ and to set $q_i(z^i) = \infty$ for all $z^i \notin M_i$. We note that, after introducing q_i, the terms $\frac{\alpha}{2}\|\iota_{H_i}(v^i) - \iota_{H_i}(u^i)\|_{H_i}^2$ could, in principle, be absorbed into the functions q_i, but we will not do this. The main reason of introducing q_i is that it can provide a means of localization without sacrificing differentiability. In fact, the u-dependent constraints $\widetilde{\mathcal{X}}(u)$ and $\widetilde{F}_i(u)$, respectively, affect the smoothness properties of V_α^{loc} and $\widetilde{V}_\alpha^{\mathrm{loc}}$, respectively, in a negative way. If, instead, locality is enforced by suitable barrier or penalty terms $q_i(\cdot - u^i)$, this side effect can be avoided.

We pose the following assumptions on the underlying spaces U, $\widetilde{U}$ and H.

Assumption 2.1.

(A1) $(U_i, (\cdot,\cdot)_{U_i})$ and $(H_i, (\cdot,\cdot)_{H_i})$ are Hilbert spaces with corresponding norms $\|\cdot\|_{U_i}$ and $\|\cdot\|_{H_i}$ respectively, $\iota_{H_i} : U_i \to H_i$ is a linear, compact and injective operator,

(A2) $U := \Pi_i U_i$ and $H := \Pi_i H_i$ are Hilbert spaces with corresponding inner products $(u,v)_U := \sum_i (u_i, v_i)_{U_i}$ and $(u,v)_H := \sum_i (u_i, v_i)_{H_i}$, $\iota_H : U \to H$ is defined by $\iota_H(u)_i := \iota_{H_i}(u^i)$, $i \in [N]$,

(A3) $(\widetilde{U}_i, \|\cdot\|_{\widetilde{U}_i})$ is a normed space and $\widetilde{U} := \Pi_i \widetilde{U}_i$ is a normed space with corresponding norm $\|u\|_{\widetilde{U}} = \left(\sum_i \|u^i\|^2_{\widetilde{U}_i}\right)^{\frac{1}{2}}$, $\iota_{\widetilde{U}_i} : U_i \to \widetilde{U}_i$ is a linear and compact operator and $\iota_{\widetilde{U}} : U \to \widetilde{U}$ is defined by $\iota_{\widetilde{U}}(u)_i := \iota_{\widetilde{U}_i}(u^i)$, $i \in [N]$.

We use the notion of weakly convergent sequences instead of weakly convergent nets. To this end, we use the weak sequential topology, which is induced by weak sequential convergence, whenever results in topological vector spaces are applied that we want to use in the setting of weak sequential convergence. We denote strong convergence and weak convergence by $\to$ and $\rightharpoonup$, respectively. Further, the Banach space and Hilbert space adjoints of a linear operator ι are denoted by ι^* and ι', respectively.

For derivatives, we use the following notation. Let $f : U \to \mathbb{R}$ be a Fréchet differentiable functional. Then $f_x(x) \in U^*$ is the derivative of f at $x \in U$ and we denote the Riesz operator from U to U^* by R_U.

Depending on their nature, our results require different flavors of continuity and differentiability assumptions on the objective functions θ_i. In the following, we collect them and we will require only one of them at a time when we develop specific results.

Assumption 2.2.

(B1) $\theta_i : U \to \mathbb{R}$ is Fréchet differentiable in the i-th component,

(B2) $\theta_i : U \to \mathbb{R}$ is of the form $\theta_i(u) = \tilde{\theta}_i(\iota_{\widetilde{U}}(u)) + \frac{\gamma}{2}\|u^i\|^2_{U_i}$, where $\gamma \geq 0$ and one of the following conditions holds

  (a) $\tilde{\theta}_i : \widetilde{U} \to \mathbb{R}$ is continuous,
  (b) $\tilde{\theta}_i : \widetilde{U} \to \mathbb{R}$ is continuously differentiable in the i-th component,
  (c) $\tilde{\theta}_i : \widetilde{U} \to \mathbb{R}$ is continuously differentiable.

We denote the partial derivative by $(\tilde{\theta}_i)_{\tilde{u}^i} : \widetilde{U}_i \to \widetilde{U}_i^*$. By the chain rule, there holds

$$\left[\tilde{\theta}_i(\iota_{\widetilde{U}_i}(u^i), \iota_{\widetilde{U}_{-i}}(u^{-i}))\right]_{u^i} = \iota^*_{\widetilde{U}_i}(\tilde{\theta}_i)_{\tilde{u}^i}(\iota_{\widetilde{U}_i}(u^i), \iota_{\widetilde{U}_{-i}}(u^{-i})) \in U_i^*. \tag{11}$$

Clearly, under Assumption 2.2, (B2c), the objective function $\theta_i : U \to \mathbb{R}$ is continuously differentiable.

We will refer to the following convexity assumptions on the functions $\widetilde{\Psi}_\alpha$ and on proximally regularized versions of θ_i. Again we collect them here and will require only one of them at a time for obtaining specific results.

Assumption 2.3. Let $\alpha \geq 0$ be such that one of the following conditions holds

(C1) $\widetilde{\Psi}_\alpha(u,\cdot)$ is convex on $\mathcal{X}$,

(C2) $\widetilde{\Psi}_\alpha(u,\cdot)$ is pseudoconvex at u on $\mathcal{X}$,

(C3) $v \mapsto \widetilde{\Psi}_\alpha(u,v) + \sum_i q_i(v^i - u^i)$ is pseudoconvex at u on $\widetilde{\mathcal{X}}(u)$,

(C4) $\theta_i(\cdot, u^{-i}) + \frac{\alpha}{2}\|\iota_{H_i}(\cdot) - \iota_{H_i}(u^i)\|^2_{H_i}$ is pseudoconvex at u^i on $F_i(u^{-i})$ for all i,

(C5) $\theta_i(\cdot, u^{-i}) + \frac{\alpha}{2}\|\iota_{H_i}(\cdot) - \iota_{H_i}(u^i)\|^2_{H_i} + q_i(\cdot - u^i)$ is pseudoconvex at u^i on $\widetilde{F}_i(u)$ for all

i.

Remark 1. Note that (C4) implies (C5). In fact, for all $v^i \in \widetilde{F}_i(u) \subset F_i(u^{-i})$, we get

$$
\begin{aligned}
&\langle (\theta_i)_{u^i}(u), v^i - u^i \rangle_{U_i^*,U_i} \geq 0 \overset{\text{(C4)}}{\Rightarrow} \\
&\qquad \theta_i(u) \leq \theta_i(v^i, u^{-i}) + \tfrac{\alpha}{2} \|\iota_{H_i}(v^i) - \iota_{H_i}(u^i)\|_{H_i}^2 \\
&\qquad\qquad \leq \theta_i(v^i, u^{-i}) + \tfrac{\alpha}{2} \|\iota_{H_i}(v^i) - \iota_{H_i}(u^i)\|_{H_i}^2 + q_i(v^i - u^i).
\end{aligned}
$$

Now, since $\theta_i(u)$ and $(\theta_i)_{u^i}(u)$, respectively, are the value and the derivative of the function in (C5) at u^i, we see that (C5) holds.
Similarly, (C2) implies that $\widetilde{\Psi}_\alpha(u, \cdot)$ is pseudoconvex at u on $\widetilde{\mathcal{X}}(u)$ and due to $q_i(v^i - u^i) \geq q_i(0)$ and $q_i'(0) = 0$ this implies (C3).

3. Fixed points of the proximal best response map

In this section, we consider the two minimization problems $\min_{v \in \mathcal{X}} \widetilde{\Psi}_\alpha(u, v)$ and $\min_{v \in \mathcal{X}} \Phi_\alpha(u, v)$, where

$$
\Phi_\alpha(u, v) = \sum_i \left[\left\langle \left[\tilde{\theta}_i(\iota_{\widetilde{U}}(u)) \right]_{u^i}, v^i \right\rangle_{U_i^*,U_i} + \gamma(u^i, v^i)_{U_i} \right] + \tfrac{\alpha}{2} \|\iota_H(v) - \iota_H(u)\|_H^2.
$$

We prove the existence of a fixed point of the corresponding solution maps. As we will see, these fixed points are closely related to VE of the GNEP. In fact, concerning the first problem, the following holds.

Proposition 3.1. *Let $u \in \mathcal{X}$ and assume that $v = u$ is a global solution to $\min_{v \in \mathcal{X}} \widetilde{\Psi}_\alpha(u, v)$. Then there holds $V_\alpha(u) = 0$. Conversely, if $u \in \mathcal{X}$ satisfies $V_\alpha(u) = 0$, then $v = u$ solves $\min_{v \in \mathcal{X}} \widetilde{\Psi}_\alpha(u, v)$.*

Proof. If $u \in \mathcal{X}$ minimizes $\widetilde{\Psi}_\alpha(u, \cdot)$ on $\mathcal{X}$, it also maximizes $\Psi_\alpha(u, \cdot)$ on $\mathcal{X}$ and thus $V_\alpha(u) = \Psi_\alpha(u, u) = 0$.

If $u \in \mathcal{X}$ and $V_\alpha(u) = 0$ hold, then $\Psi_\alpha(u, u) = 0 = V_\alpha(u) = \sup_{v \in \mathcal{X}} \Psi_\alpha(u, v)$. Thus, u maximizes $\Psi_\alpha(u, \cdot)$ on $\mathcal{X}$, which is equivalent to u minimizing $\widetilde{\Psi}_\alpha(u, \cdot)$ on $\mathcal{X}$. □

This shows that $V_\alpha(u) = 0$ is equivalent to $u \in \mathcal{X}$ being a fixed point of the solution map to $\min_{v \in \mathcal{X}} \widetilde{\Psi}_\alpha(u, v)$. In Section 3.1, we prove that under suitable assumptions this solution map has a fixed point. As will be shown in Theorem 4.3 (parts 4 $\Leftarrow$, 2 $\Leftarrow$, and 3 $\Leftarrow$), $V_\alpha(u) = 0$ yields that u is a VE (and thus also a QNE as we discussed in Section 2). In Section 3.2, we study fixed points of the solution map to $\min_{v \in \mathcal{X}} \Phi_\alpha(u, v)$ and show that they are VE.

3.1. *Proximal best response*

In the following, we consider the optimization problem

$$
\min_{v \in \mathcal{X}} \widetilde{\Psi}_\alpha(u, v) \quad \text{with} \quad \widetilde{\Psi}_\alpha(u, v) = \sum_i \left[\theta_i(v^i, u^{-i}) + \tfrac{\alpha}{2} \|\iota_{H_i}(v^i) - \iota_{H_i}(u^i)\|_{H_i}^2 \right] \tag{12}
$$

and show that the corresponding solution map $u \mapsto v_{\widetilde{\Psi}_\alpha}(u)$ has a fixed point.

Throughout this Section 3.1, we work under the following

Standing Assumption. Let $\theta_i : U \to \mathbb{R}$ fulfill (B2a) and let the feasible set $\mathcal{X} \subset U$ be nonempty, convex, closed, and bounded. Further, let $\alpha \geq 0$ be such that $\widetilde{\Psi}_\alpha : U \times U \to \mathbb{R}$ is convex with respect to the second argument, i.e., assumption (C1) holds.

We investigate the solution map

$$u \in U \mapsto v_{\widetilde{\Psi}_\alpha}(u) := \{v \in U : v \text{ solves } (12)\}.$$

Theorem 3.2. *The solution map $u \mapsto v_{\widetilde{\Psi}_\alpha}(u)$ has a fixed point in $\mathcal{X}$. Any such fixed point $u = v_{\widetilde{\Psi}_\alpha}(u) \in \mathcal{X}$ satisfies $V_\alpha(u) = 0$.*

As already mentioned, we will investigate in Section 4 whether and under which assumptions this condition is equivalent to $u \in \mathcal{X}$ being a VE or a NoE, see Theorem 4.4 below.

In order to prove Theorem 3.2, we want to apply a generalization of the Kakutani fixed point theorem, see [16, Chapter 1, Theorem]. In the following propositions we verify the assumptions of the Kakutani fixed point theorem. To this end, we show that the corresponding solution map $u \mapsto v_{\widetilde{\Psi}_\alpha}(u)$ has nonempty and convex images and we check that $\widetilde{\Psi}_\alpha : U \times U \to \mathbb{R}$ is lower semicontinuous with respect to the weak sequential topology.

Proposition 3.3. *The function $\widetilde{\Psi}_\alpha : U \times U \to \mathbb{R}$ is lower semicontinuous with respect to the weak sequential topology.*

Proof. In order to show the lower semicontinuity of $\widetilde{\Psi}_\alpha : U \times U \to \mathbb{R}$ with respect to the weak sequential topology we consider arbitrary sequences $\{v_k\}_{k \in \mathbb{N}}$ and $\{u_k\}_{k \in \mathbb{N}}$ in U with $v_k \rightharpoonup v$ and $u_k \rightharpoonup u$ in U. By the compactness of the operators $\iota_{\widetilde{U}} : U \to \widetilde{U}$ and $\iota_H : U \to H$, we obtain the strong convergences $\iota_{\widetilde{U}}(u_k) \to \iota_{\widetilde{U}}(u)$, $\iota_{\widetilde{U}}(v_k) \to \iota_{\widetilde{U}}(v)$ in $\widetilde{U}$ and $\iota_H(u_k) \to \iota_H(u)$, $\iota_H(v_k) \to \iota_H(v)$ in H, respectively. For each i we have $\iota_{\widetilde{U}}(v_k^i, u_k^{-i}) \to \iota_{\widetilde{U}}(v^i, u^{-i})$ in $\widetilde{U}$ and $\tilde{\theta}_i(\iota_{\widetilde{U}}(v_k^i, u_k^{-i})) \to \tilde{\theta}_i(\iota_{\widetilde{U}}(v^i, u^{-i}))$ as $k \to \infty$ by the continuity of $\tilde{\theta}_i : \widetilde{U} \to \mathbb{R}$. Further, there holds $\|\iota_H(v_k) - \iota_H(u_k)\|_H \to \|\iota_H(v) - \iota_H(u)\|_H$ as $k \to \infty$. Using the lower semicontinuity of $\|\cdot\|_{U_i}^2$ in the weak sequential topology for all i, we can estimate

$$\begin{aligned}
\widetilde{\Psi}_\alpha(u, v) &= \sum_i \left[\tilde{\theta}_i(\iota_{\widetilde{U}}(v^i, u^{-i})) + \tfrac{\gamma}{2}\|v^i\|_{U_i}^2\right] + \tfrac{\alpha}{2}\|\iota_H(v) - \iota_H(u)\|_H^2 \\
&\leq \lim_{k\to\infty} \left[\sum_i \left[\tilde{\theta}_i(\iota_{\widetilde{U}}(v_k^i, u_k^{-i}))\right] + \tfrac{\alpha}{2}\|\iota_H(v_k) - \iota_H(u_k)\|_H^2\right] + \tfrac{\gamma}{2}\sum_i \left[\liminf_{k\to\infty} \|v_k^i\|_{U_i}^2\right] \\
&= \liminf_{k\to\infty} \widetilde{\Psi}_\alpha(u_k, v_k).
\end{aligned}$$

□

Proposition 3.4. *For all $u \in U$, the solution set $v_{\widetilde{\Psi}_\alpha}(u)$ to (12) is nonempty and convex.*

Proof. We note that U is a reflexive Banach space and $\mathcal{X}$ is nonempty, closed, convex, and bounded by assumption. Further, $\widetilde{\Psi}_\alpha(u, \cdot) : U \to \mathbb{R}$ is lower semicontinuous with respect to the weak sequential topology and with respect to the norm topology by Proposition 3.3. Thus, we can apply [7, Corollary 3.23] to conclude that $\min_{v \in \mathcal{X}} \widetilde{\Psi}_\alpha(u, v)$ possesses at least one solution and hence, $v_{\widetilde{\Psi}_\alpha}(u)$ is nonempty. The convexity of the solution set $v_{\widetilde{\Psi}_\alpha}(u)$ follows from the convexity of $\widetilde{\Psi}_\alpha(u, \cdot)$ and of $\mathcal{X}$. □

Theorem 3.5. *The solution map $u \mapsto v_{\widetilde{\Psi}_\alpha}(u)$ to the optimization problem* $\min_{v \in \mathcal{X}} \widetilde{\Psi}_\alpha(u, v)$ *is closed in the weak sequential topology.*

Proof. Let $\{u_k\}_{k\in\mathbb{N}} \subset U$ and $v_k \in v_{\widetilde{\Psi}_\alpha}(u_k)$, $k \in \mathbb{N}$, be arbitrary sequences such that $u_k \rightharpoonup u$ in U and $v_k \rightharpoonup v$ in U for $k \to \infty$. We have to verify $v \in v_{\widetilde{\Psi}_\alpha}(u)$. To this end, we apply Proposition 3.3 and obtain $\widetilde{\Psi}_\alpha(u,v) \le \liminf_{k\to\infty} \widetilde{\Psi}_\alpha(u_k, v_k)$. For all $z \in \mathcal{X}$ and $y \in v_{\widetilde{\Psi}_\alpha}(x)$ it holds $\widetilde{\Psi}_\alpha(x,y) \le \widetilde{\Psi}_\alpha(x,z)$ and thus, we get, using compactness of $\iota_{\widetilde{U}}$ and ι_H and continuity properties:

$$\begin{aligned} \liminf_{k\to\infty} \widetilde{\Psi}_\alpha(u_k, v_k) &\le \liminf_{k\to\infty} \widetilde{\Psi}_\alpha(u_k, z) \\ &= \sum_i \Big[\lim_{k\to\infty} \tilde{\theta}_i(\iota_{\widetilde{U}}(z^i, u_k^{-i})) + \tfrac{\gamma}{2}\|z^i\|_{U_i}^2 \Big] + \lim_{k\to\infty} \Big[\tfrac{\alpha}{2}\|\iota_H(z) - \iota_H(u_k)\|_H^2 \Big] \\ &= \sum_i \big[\tilde{\theta}_i(\iota_{\widetilde{U}}(z^i, u^{-i})) + \tfrac{\gamma}{2}\|z^i\|_{U_i}^2 \big] + \tfrac{\alpha}{2}\|\iota_H(z) - \iota_H(u)\|_H^2 = \widetilde{\Psi}_\alpha(u,z). \end{aligned} \tag{13}$$

This yields $\widetilde{\Psi}_\alpha(u,v) \le \widetilde{\Psi}_\alpha(u,z)$ for all $z \in \mathcal{X}$. From $v_k \in v_{\widetilde{\Psi}_\alpha}(u_k) \subset \mathcal{X}$ and the closedness of $\mathcal{X} \subset U$ with respect to the weak sequential topology we conclude $v \in \mathcal{X}$ and thus v minimizes $\widetilde{\Psi}_\alpha(u,\cdot)$ on $\mathcal{X}$. This shows $v \in v_{\widetilde{\Psi}_\alpha}(u)$. □

Proof of Theorem 3.2. Since $\mathcal{X}$ is convex and closed, it is closed in the weak topology of U and since $\mathcal{X}$ is also bounded and U is reflexive, $\mathcal{X}$ is weakly compact in U, thus also compact in the weak sequential topology of U by the Eberlein–Šmulian theorem. Theorem 3.5 provides the closedness of the solution map $v_{\widetilde{\Psi}_\alpha}$ in the weak sequential topology and, by Proposition 3.4, this map has nonempty convex images. Thus, Glicksberg's version of the fixed point theorem of Kakutani, cf. [16, Chapter 1, Theorem], is applicable and yields the existence of a fixed point of the solution map.

Thus, there exists a point $u \in \mathcal{X}$ satisfying $v_{\widetilde{\Psi}_\alpha}(u) = u$. For any such fixed point u there holds $\widetilde{\Psi}_\alpha(u,u) = \widetilde{\Psi}_\alpha(u, v_{\widetilde{\Psi}_\alpha}(u)) \le \widetilde{\Psi}_\alpha(u,v)$ for all $v \in \mathcal{X}$. This is equivalent to the property

$$\sum_i \big[\theta_i(u^i, u^{-i}) - \theta_i(v^i, u^{-i}) - \tfrac{\alpha}{2}\|\iota_{H_i}(v^i) - \iota_{H_i}(u^i)\|_{H_i}^2\big] \le 0, \qquad \forall\, v \in \mathcal{X}$$

and hence, it holds $V_\alpha(u) = 0$. □

3.2. *Proximal best response map via optimality condition*

We now consider a proximal best response map based on first order optimality conditions. Therefore, we introduce $\Phi : U \times U \to \mathbb{R}$ and its regularization $\Phi_\alpha : U \times U \to \mathbb{R}$, which are defined by

$$\begin{aligned} \Phi(u,v) &= \sum_i \big[\big\langle \big[\tilde{\theta}_i(\iota_{\widetilde{U}}(u))\big]_{u^i}, v^i \big\rangle_{U_i^*, U_i} + \gamma(u^i, v^i)_{U_i}\big], \\ \Phi_\alpha(u,v) &= \Phi(u,v) + \tfrac{\alpha}{2}\|\iota_H(v) - \iota_H(u)\|_H^2, \end{aligned} \tag{14}$$

where $\alpha \ge 0$ and could be chosen 0. There holds

$$\Phi(u,v) = \sum_i \big[\big\langle \big[\theta_i(u)\big]_{u^i}, v^i - u^i \big\rangle_{U_i^*, U_i}\big] - \gamma\|u\|_U^2. \tag{15}$$

In the following, we study the problem $\min_{v\in\mathcal{X}} \Phi_\alpha(u,v)$ and show the existence of a fixed point of the corresponding solution map $u \mapsto v_{\Phi_\alpha}(u)$.

Throughout this Section 3.2, we work under the following

Standing Assumption. Let $\theta_i : U \to \mathbb{R}$ fulfill (B2b) and let the feasible set $\mathcal{X} \subset U$ be nonempty, convex, closed, and bounded.

Theorem 3.6. *There exists a fixed point in $\mathcal{X}$ of the mapping $u \mapsto v_{\Phi_\alpha}(u)$.*

In order to prove this result, we would like to perform a similar calculation as before, but the terms $(u^i, v^i)_{U_i}$, $i \in [N]$, are not weakly sequentially continuous. As a remedy, we consider $\widetilde{\Phi} : U \times U \to \mathbb{R}$ and the regularized version $\widetilde{\Phi}_\alpha$ given by

$$\begin{aligned} \widetilde{\Phi}(u, v) &= \sum_i \left[\left\langle \left[\tilde{\theta}_i(\iota_{\widetilde{U}}(u))\right]_{u^i}, v^i \right\rangle_{U_i^*, U_i} + \tfrac{\gamma}{2} \|v^i\|_{U_i}^2 \right], \\ \widetilde{\Phi}_\alpha(u, v) &= \widetilde{\Phi}(u, v) + \tfrac{\alpha}{2} \|\iota_H(v) - \iota_H(u)\|_H^2. \end{aligned} \tag{16}$$

We first show that the solution maps of $\min_{v \in \mathcal{X}} \Phi_\alpha(u, v)$ and $\min_{v \in \mathcal{X}} \widetilde{\Phi}_\alpha(u, v)$ coincide.

Proposition 3.7. *Let $u \in \mathcal{X}$. Then the following equivalence holds:*

$$\Phi_\alpha(u, u) \le \Phi_\alpha(u, v) \; \forall\, v \in \mathcal{X} \quad \Longleftrightarrow \quad \widetilde{\Phi}_\alpha(u, u) \le \widetilde{\Phi}_\alpha(u, v) \; \forall\, v \in \mathcal{X}.$$

Proof. $\Rightarrow$: Let $u, v \in \mathcal{X}$ satisfy $\Phi_\alpha(u, u) \le \Phi_\alpha(u, v)$. Using the convexity and differentiability of $\widetilde{\Phi}(u, \cdot)$ in $\mathcal{X}$ yields

$$\begin{aligned} \widetilde{\Phi}_\alpha(u, v) - \widetilde{\Phi}_\alpha(u, u) &\ge \left\langle (\widetilde{\Phi})_v(u, v)_{|_{v=u}}, v - u \right\rangle_{U^*, U} + \tfrac{\alpha}{2} \|\iota_H(v) - \iota_H(u)\|_H^2 \\ &= \sum_i \left[\left\langle \left[\tilde{\theta}_i(\iota_{\widetilde{U}}(u))\right]_{u^i}, v^i - u^i \right\rangle_{U_i^*, U_i} + \gamma (u^i, v^i - u^i)_{U_i} \right] + \tfrac{\alpha}{2} \|\iota_H(v) - \iota_H(u)\|_H^2 \\ &= \Phi_\alpha(u, v) - \Phi_\alpha(u, u) \ge 0. \end{aligned}$$

$\Leftarrow$: Let $u \in \mathcal{X}$ and $\widetilde{\Phi}_\alpha(u, v) \ge \widetilde{\Phi}_\alpha(u, u)$ hold for all $v \in \mathcal{X}$. By convexity of $\mathcal{X}$, for any $v \in \mathcal{X}$ and $t \in (0, 1]$, there then holds $\widetilde{\Phi}_\alpha(u, u) \le \widetilde{\Phi}_\alpha(u, u + t(v - u))$. The convexity of $\Phi_\alpha(u, \cdot)$ yields

$$\begin{aligned} \Phi_\alpha(u, v) - \Phi_\alpha(u, u) &\ge \tfrac{1}{t} \left[\Phi_\alpha(u, u + t(v - u)) - \Phi_\alpha(u, u) \right] \\ &= \tfrac{1}{t} \Big[\widetilde{\Phi}_\alpha(u, u + t(v - u)) - \widetilde{\Phi}_\alpha(u, u) \\ &\qquad + \gamma (u, u + t(v - u))_U - \tfrac{\gamma}{2} \|u + t(v - u)\|_U^2 - \gamma (u, u)_U + \tfrac{\gamma}{2} \|u\|_U^2 \Big] \\ &\ge \tfrac{1}{t} \left[\gamma (u, t(v - u))_U - \tfrac{\gamma}{2} \|u + t(v - u)\|_U^2 + \tfrac{\gamma}{2} \|u\|_U^2 \right] = -\tfrac{\gamma t}{2} \|v - u\|_U^2. \end{aligned}$$

For $t \to 0^+$ the right hand side tends to 0. Thus $\Phi_\alpha(u, v) \ge \Phi_\alpha(u, u)$ holds for all $v \in \mathcal{X}$. $\square$

Since we have shown $\operatorname*{argmin}_{v \in \mathcal{X}} \widetilde{\Phi}_\alpha(u, v) = \operatorname*{argmin}_{v \in \mathcal{X}} \Phi_\alpha(u, v)$, we can study $\widetilde{\Phi}_\alpha$ instead of Φ_α.

Proposition 3.8. *The function $\widetilde{\Phi}_\alpha : U \times U \to \mathbb{R}$ is lower semicontinuous with respect to the weak sequential topology.*

Proof. Let $\{u_k\}_{k \in \mathbb{N}} \subset U$ and $\{v_k\}_{k \in \mathbb{N}} \subset U$ be sequences such that $u_k \rightharpoonup u$ and $v_k \rightharpoonup v$ in U as $k \to \infty$. Further, let i be arbitrary. It holds $(\tilde{\theta}_i)_{\tilde{u}^i}(\iota_{\widetilde{U}}(u_k)) \to (\tilde{\theta}_i)_{\tilde{u}^i}(\iota_{\widetilde{U}}(u))$ in $\widetilde{U}_i^*$

for $k \to \infty$ since we have the strong convergence $\iota_{\widetilde{U}}(u_k) \to \iota_{\widetilde{U}}(u)$. Hence, we arrive at the convergence

$$\begin{aligned}\big\langle \big[\tilde{\theta}_i(\iota_{\widetilde{U}}(u_k))\big]_{u^i}, v_k^i \big\rangle_{U_i^*,U_i} = \big\langle (\tilde{\theta}_i)_{\tilde{u}^i}(\iota_{\widetilde{U}}(u_k)), \iota_{\widetilde{U}_i}(v_k^i) \big\rangle_{\widetilde{U}_i^*,\widetilde{U}_i} &\to \big\langle (\tilde{\theta}_i)_{\tilde{u}^i}(\iota_{\widetilde{U}}(u)), \iota_{\widetilde{U}_i}(v^i) \big\rangle_{\widetilde{U}_i^*,\widetilde{U}_i} \\ &= \big\langle \big[\tilde{\theta}_i(\iota_{\widetilde{U}}(u))\big]_{u^i}, v^i \big\rangle_{U_i^*,U_i}.\end{aligned}$$

Due to the weak sequential lower semicontinuity of $\|\cdot\|_{U_i}^2$, we obtain the weak sequential lower semicontinuity of $\widetilde{\Phi}_\alpha$ as follows

$$\begin{aligned}\widetilde{\Phi}_\alpha(u,v) &= \sum_i \big[\big\langle \big[\tilde{\theta}_i(\iota_{\widetilde{U}}(u))\big]_{u^i}, v^i \big\rangle_{U_i^*,U_i} + \tfrac{\gamma}{2}\|v^i\|_{U_i}^2\big] + \tfrac{\alpha}{2}\|\iota_H(v) - \iota_H(u)\|_H^2 \\ &\leq \lim_{k\to\infty} \Big[\sum_i \big[\big\langle \big[\tilde{\theta}_i(\iota_{\widetilde{U}}(u_k))\big]_{u^i}, v_k^i \big\rangle_{U_i^*,U_i}\big] + \tfrac{\alpha}{2}\|\iota_H(v_k) - \iota_H(u_k)\|_H^2\Big] \\ &\quad + \tfrac{\gamma}{2}\sum_i \big[\liminf_{k\to\infty} \|v_k^i\|_{U_i}^2\big] \\ &= \liminf_{k\to\infty} \Big[\sum_i \big[\big\langle \big[\tilde{\theta}_i(\iota_{\widetilde{U}}(u_k))\big]_{u^i}, v_k^i \big\rangle_{U_i^*,U_i} + \tfrac{\gamma}{2}\|v_k^i\|_{U_i}^2\big] + \tfrac{\alpha}{2}\|\iota_H(v_k) - \iota_H(u_k)\|_H^2\Big] \\ &= \liminf_{k\to\infty} \widetilde{\Phi}_\alpha(u_k,v_k).\end{aligned}$$

□

Proposition 3.9. *The proximal best response map* $u \in \mathcal{X} \mapsto v_{\widetilde{\Phi}_\alpha}(u) = \operatorname{argmin}_{v\in\mathcal{X}} \widetilde{\Phi}_\alpha(u,v) \in \mathcal{X}$ *has nonempty convex images.*

Proof. We get the lower semicontinuity of $\widetilde{\Phi}_\alpha(u,\cdot)$ by Proposition 3.8 and hence, we obtain the existence of a solution to the minimization problem $\min_{v\in\mathcal{X}} \widetilde{\Phi}_\alpha(u,v)$ by [7, Corollary 3.23]. Thus, the set $v_{\widetilde{\Phi}_\alpha}(u)$ is nonempty. The convexity follows as in the proof of Proposition 3.4. □

Proposition 3.10. *The solution map* $u \mapsto v_{\widetilde{\Phi}_\alpha}(u)$ *to the minimization problem* $\min_{v\in\mathcal{X}} \widetilde{\Phi}_\alpha(u,v)$ *is closed in the weak sequential topology.*

Proof. Let $\{u_k\}_{k\in\mathbb{N}} \subset U$ and $v_k \in v_{\widetilde{\Phi}_\alpha}(u_k)$, $k \in \mathbb{N}$, be two weakly convergent sequences in U, i.e., $u_k \rightharpoonup u$ and $v_k \rightharpoonup v$ in U for $k \to \infty$. We prove $v \in v_{\widetilde{\Phi}_\alpha}(u)$. By Proposition 3.8 we know $\widetilde{\Phi}_\alpha(u,v) \leq \liminf_{k\to\infty} \widetilde{\Phi}_\alpha(u_k,v_k)$ and additionally, it holds $\widetilde{\Phi}_\alpha(x,y) \leq \widetilde{\Phi}_\alpha(x,z)$ for all $z \in \mathcal{X}, y \in v_{\widetilde{\Phi}_\alpha}(x)$ and $x \in U$. Thus, for any $z \in \mathcal{X}$, we get

$$\begin{aligned}\widetilde{\Phi}_\alpha(u,v) &\leq \liminf_{k\to\infty} \widetilde{\Phi}_\alpha(u_k,v_k) \leq \liminf_{k\to\infty} \widetilde{\Phi}_\alpha(u_k,z) \\ &= \liminf_{k\to\infty} \Big[\sum_i \big[\big\langle \big[\tilde{\theta}_i(\iota_{\widetilde{U}}(u_k))\big]_{u^i}, z^i \big\rangle_{U_i^*,U_i} + \tfrac{\gamma}{2}\|z^i\|_{U_i}^2\big] + \tfrac{\alpha}{2}\|\iota_H(z) - \iota_H(u_k)\|_H^2\Big] \\ &= \sum_i \big[\big\langle \big[\tilde{\theta}_i(\iota_{\widetilde{U}}(u))\big]_{u^i}, z^i \big\rangle_{U_i^*,U_i} + \tfrac{\gamma}{2}\|z^i\|_{U_i}^2\big] + \tfrac{\alpha}{2}\|\iota_H(z) - \iota_H(u)\|_H^2 = \widetilde{\Phi}_\alpha(u,z).\end{aligned} \tag{17}$$

Thus, $v \in \mathcal{X}$ minimizes $\widetilde{\Phi}_\alpha$ on $\mathcal{X}$, and hence $v \in v_{\widetilde{\Phi}_\alpha}(u)$. □

Proof of Theorem 3.6. As in the proof of Theorem 3.2, we obtain the compactness of $\mathcal{X}$ in the weak sequential topology. By Proposition 3.10 the assumptions of the fixed point theorem of Kakutani, [16, Chapter 1, Theorem], are satisfied and we can apply it to the map

$u \mapsto v_{\widetilde{\Phi}_\alpha}(u)$. Thus, we get the existence of a fixed point of $u \mapsto \operatorname{argmin}_{v\in\mathcal{X}} \widetilde{\Phi}_\alpha(u,v)$. By Proposition 3.7 we obtain a fixed point of $u \mapsto \operatorname{argmin}_{v\in\mathcal{X}} \Phi_\alpha(u,v)$. □

We have shown the existence of a fixed point $u \in \mathcal{X}$ of the solution map to the minimization problem $\min_{v\in\mathcal{X}} \Phi_\alpha(u,v)$, i.e., there holds $\Phi_\alpha(u,u) \le \Phi_\alpha(u,v)$ for all $v \in \mathcal{X}$. By (15), this can be written as

$$\sum_i \left[\langle (\theta_i)_{u^i}(u), u^i\rangle_{U_i^*,U_i}\right] \le \sum_i \left[\langle (\theta_i)_{u^i}(u), v^i\rangle_{U_i^*,U_i}\right] + \tfrac{\alpha}{2}\|\iota_H(v) - \iota_H(u)\|_H^2. \tag{18}$$

For $\alpha = 0$, we obtain the VI

$$\sum_i \langle (\theta_i)_{u^i}(u), v^i - u^i\rangle_{U_i^*,U_i} \ge 0 \quad \forall\, v \in \mathcal{X}. \tag{19}$$

The same VI is also obtained from (18) for arbitrary $\alpha \ge 0$. In fact, for any $v \in \mathcal{X}$ and $t \in (0,1]$, we can insert $u + t(v-u) \in \mathcal{X}$ into (18) to obtain

$$\sum_i \langle (\theta_i)_{u^i}(u), t(v^i - u^i)\rangle_{U*_i,U_i} \ge -\tfrac{\alpha t^2}{2}\|\iota_H(v) - \iota_H(u)\|_H^2.$$

Dividing by t and $t \to 0^+$ yields (19). Next, we study the connection between this VI and a NoE, see Theorem 4.3 and 4.4 below.

4. Nikaido–Isoda type functions

In this section, we study analytical properties of the functionals V_α and $\widetilde{V}_\alpha$ defined in (8) and (10) and their localized modifications V_α^{loc} and $\widetilde{V}_\alpha^{\text{loc}}$. Let $\widetilde{F}_i(u) \subset F_i(u^{-i})$ be convex such that $F_i(u^{-i}) \cap \overline{B}_R(u^i) \subset \widetilde{F}_i(u)$ for some $R > 0$ and set $\widetilde{F}(u) := \Pi_i \widetilde{F}_i(u)$. Further, let $\widetilde{\mathcal{X}}(u) \subset \mathcal{X}$ be convex such that $\mathcal{X} \cap \overline{B}_R(u) \subset \widetilde{\mathcal{X}}(u)$ for some $R > 0$. For $i \in [N]$, let $q_i : U_i \to \overline{\mathbb{R}}_{\ge 0}$ be convex, continuously differentiable at the point 0 with $q_i(0) = 0$, and let q_i be finite on $\overline{B}_R(0)$. Define

$$\begin{aligned} V_\alpha^{\text{loc}}(u) &:= \sup_{v\in\widetilde{\mathcal{X}}(u)} \left[\Psi_\alpha(u,v) - \sum_i q_i(v^i - u^i)\right], \\ \widetilde{V}_\alpha^{\text{loc}}(u) &:= \sum_i \sup_{v^i\in\widetilde{F}_i(u)} \left[\theta_i(u) - \theta_i(v^i, u^{-i}) - \tfrac{\alpha}{2}\|\iota_{H_i}(v^i) - \iota_{H_i}(u^i)\|_{H_i}^2 - q_i(v^i - u^i)\right]. \end{aligned}$$

Especially, we verify that these are merit functions corresponding to a (quasi-)VI. Moreover, we build a connection between these merit functions, (quasi-)VIs, and (local) NoE and (local) NE, respectively. We also prove some relation between the merit functions V_α and V_α^{loc}, respectively.

4.1. Regularized and localized merit functions

In the following, we consider a general feasible set $F_i(u^{-i})$ and study properties of the functionals $\widetilde{V}_\alpha : U \to \mathbb{R}$ and $\widetilde{V}_\alpha^{\text{loc}} : U \to \mathbb{R}$ as defined in (10). Especially, we relate these merit functions to QVIs that express first-order optimality conditions for (local) NE.

Theorem 4.1. *Let $\alpha \geq 0$ and consider $u \in U$ satisfying $u \in F(u)$. For assertions 2–4, let (B1) be satisfied and let $F_i(u^{-i})$ be convex, $i \in [N]$. Also, for "$\Rightarrow$" in 2., let (C5) hold and for "$\Rightarrow$" in 4., let (C4) hold. Then there holds:*

1. $\widetilde{V}_\alpha(u) \geq \widetilde{V}_\alpha^{\mathrm{loc}}(u) \geq 0$.
2. $\langle(\theta_i)_{u^i}(u), v^i - u^i\rangle_{U_i^*,U_i} \geq 0 \;\forall\, v^i \in \widetilde{F}_i(u),\ \forall\, i \iff \widetilde{V}_\alpha^{\mathrm{loc}}(u) = 0$.
3. *For all i:* $\langle(\theta_i)_{u^i}(u), v^i - u^i\rangle_{U_i^*,U_i} \geq 0 \;\forall\, v^i \in F_i(u^{-i}) \iff \langle(\theta_i)_{u^i}(u), v^i - u^i\rangle_{U_i^*,U_i} \geq 0 \;\forall\, v^i \in \widetilde{F}_i(u)$.
4. $\widetilde{V}_\alpha^{\mathrm{loc}}(u) = 0 \iff \widetilde{V}_\alpha(u) = 0$.

Proof. Let $u \in U$ satisfy $u \in F(u)$. Clearly, due to $q_i \geq 0$ and $\widetilde{F}_i(u) \subset F_i(u^{-i})$, there holds $\widetilde{V}_\alpha(u) \geq \widetilde{V}_\alpha^{\mathrm{loc}}(u)$.

1. We set $v^i = u^i \in \widetilde{F}_i(u)$ for all i and conclude

$$\widetilde{V}_\alpha(u) \geq \widetilde{V}_\alpha^{\mathrm{loc}}(u) \geq \sum_i \left[\theta_i(u) - \theta_i(u^i, u^{-i}) - \frac{\alpha}{2}\|\iota_{H_i}(u^i) - \iota_{H_i}(u^i)\|_{H_i}^2 - q_i(0)\right] = 0.$$

2. $\Rightarrow$: Let i be arbitrary. Since $\theta_i(\cdot, u^{-i}) + \frac{\alpha}{2}\|\iota_{H_i}(\cdot) - \iota_{H_i}(u^i)\|_{H_i}^2 + q_i(\cdot - u^i)$ is pseudoconvex at u^i on $\widetilde{F}_i(u)$, we obtain

$$\theta_i(u) \leq \theta_i(v^i, u^{-i}) + \frac{\alpha}{2}\|\iota_{H_i}(v^i) - \iota_{H_i}(u^i)\|_{H_i}^2 + q_i(v^i - u^i) \qquad \forall\, v^i \in \widetilde{F}_i(u),$$

where we applied the inequality

$$\begin{aligned}\Big\langle \big[\theta_i(v^i, u^{-i}) + \tfrac{\alpha}{2}\|\iota_{H_i}(v^i) - \iota_{H_i}(u^i)\|_{H_i}^2 + q_i(v^i - u^i)\big]_{v^i}|_{v^i=u^i}, v^i - u^i\Big\rangle_{U_i^*,U_i} &= \\ = \langle(\theta_i)_{u^i}(u), v^i - u^i\rangle_{U_i^*,U_i} \geq 0.&\end{aligned}$$

This yields

$$\widetilde{V}_\alpha^{\mathrm{loc}}(u) = \sum_i \sup_{v^i \in \widetilde{F}_i(u)} \left[\theta_i(u) - \theta_i(v^i, u^{-i}) - \frac{\alpha}{2}\|\iota_{H_i}(v^i) - \iota_{H_i}(u^i)\|_{H_i}^2 - q_i(v^i - u^i)\right] \leq 0,$$

and together with $\widetilde{V}_\alpha^{\mathrm{loc}}(u) \geq 0$ we conclude $\widetilde{V}_\alpha^{\mathrm{loc}}(u) = 0$.

$\Leftarrow$: Let i be arbitrary. We first show that $\widetilde{V}_\alpha^{\mathrm{loc}}(u) = 0$ implies that for all $w^i \in \widetilde{F}_i(u)$ and all i there holds

$$\theta_i(u) \leq \theta_i(w^i, u^{-i}) + \frac{\alpha}{2}\|\iota_{H_i}(w^i) - \iota_{H_i}(u^i)\|_{H_i}^2 + q_i(w^i - u^i).$$

In fact, otherwise there would exist j and $z^j \in \widetilde{F}_j(u)$ such that

$$\theta_j(u) > \theta_j(z^j, u^{-j}) + \frac{\alpha}{2}\|\iota_{H_j}(z^j) - \iota_{H_j}(u^j)\|_{H_j}^2 + q_j(z^j - u^j)$$

and we would arrive at the contradiction (choose $v^j = z^j$ and $v^i = u^i$, $i \neq j$)

$$\widetilde{V}_\alpha^{\mathrm{loc}}(u) \geq \theta_j(u) - \theta_j(z^j, u^{-j}) - \frac{\alpha}{2}\|\iota_{H_j}(z^j) - \iota_{H_j}(u^j)\|_{H_j}^2 - q_j(z^j - u^j) > 0.$$

Now consider any $v^i \in \widetilde{F}_i(u)$ and $t \in (0,1]$. By convexity of $\widetilde{F}_i(u)$ we have $w^i := u^i + t(v^i - u^i) \in \widetilde{F}_i(u)$ and Taylor's expansion together with $q_i'(0) = 0$ yields

$$\begin{aligned}\theta_i(u) &\le \theta_i(u^i + t(v^i - u^i), u^{-i}) + \frac{\alpha}{2}\|\iota_{H_i}(t(v^i - u^i))\|^2_{H_i} + q_i(t(v^i - u^i))\\ &= \theta_i(u) + \langle(\theta_i)_{u^i}(u), t(v^i - u^i)\rangle_{U_i^*,U_i} + o(t).\end{aligned}$$

Dividing by t and taking the limit $t \downarrow 0$, we arrive at the desired inequality $\langle(\theta_i)_{u^i}(u), v^i - u^i\rangle_{U_i^*,U_i} \ge 0$.

3. Let i be arbitrary fixed.

$\Rightarrow$: Let the VI on the left hold. By the relation $\widetilde{F}_i(u) \subset F_i(u^{-i})$ we obtain $\langle(\theta_i)_{u^i}(u), v^i - u^i\rangle_{U_i^*,U_i} \ge 0$ for all $v^i \in \widetilde{F}_i(u)$.

$\Leftarrow$: Let the inequality $\langle(\theta_i)_{u^i}(u), \tilde{v}^i - u^i\rangle_{U_i^*,U_i} \ge 0$ hold for all $\tilde{v}^i \in \widetilde{F}_i(u)$ and consider any $v^i \in F_i(u^{-i})$. If $v^i \in \widetilde{F}_i(u)$, we are done. In the case $v^i \in F_i(u^{-i})\backslash\widetilde{F}_i(u)$ we construct a suitable $\tilde{v}^i \in \widetilde{F}_i(u)$. To this end, let $t \in (0,1]$ satisfy $t\|v^i - u^i\|_{U_i} \le R$ and set $\tilde{v}^i := u^i + t(v^i - u^i)$. By the convexity of $F_i(u^{-i})$, there holds $\tilde{v}^i \in F_i(u^{-i}) \cap \overline{B}_R(u^i) \subset \widetilde{F}_i(u)$. Further,

$$\langle(\theta_i)_{u^i}(u), v^i - u^i\rangle_{U_i^*,U_i} = \tfrac{1}{t}\langle(\theta_i)_{u^i}(u), \tilde{v}^i - u^i\rangle_{U_i^*,U_i} \ge 0.$$

4. $\Rightarrow$: Let $\widetilde{V}_\alpha^{\mathrm{loc}}(u) = 0$. By applying 2. and 3. we obtain $\langle(\theta_i)_{u^i}(u), v^i - u^i\rangle_{U_i^*,U_i} \ge 0$ for all $v^i \in F_i(u^{-i})$ and all i. Let i be arbitrary. Since $\theta_i(\cdot, u^{-i}) + \frac{\alpha}{2}\|\iota_{H_i}(\cdot) - \iota_{H_i}(u^i)\|^2_{H_i}$ is pseudoconvex at u^i on $F_i(u^{-i})$, it holds

$$\theta_i(u) \le \theta_i(v^i, u^{-i}) + \frac{\alpha}{2}\|\iota_{H_i}(v^i) - \iota_{H_i}(u^i)\|^2_{H_i} \quad \forall\, v^i \in F_i(u^{-i}).$$

Hence, we arrive at

$$0 \le \widetilde{V}_\alpha(u) = \sum_i \sup_{v^i \in F_i(u^{-i})} [\theta_i(u) - \theta_i(v^i, u^{-i}) - \frac{\alpha}{2}\|\iota_{H_i}(v^i) - \iota_{H_i}(u^i)\|^2_{H_i}] \le 0.$$

$\Leftarrow$: This direction is an immediate consequence of $\widetilde{V}_\alpha(u) = 0$ and $0 \le \widetilde{V}_\alpha^{\mathrm{loc}}(u) \le \widetilde{V}_\alpha(u)$, cf. part 1. $\square$

From parts 1–3 of Theorem 4.1 we conclude that, if $F(u)$ is convex and (B1), (C5) hold, then $\widetilde{V}_\alpha^{\mathrm{loc}}$ can be used as a merit function for any of the following two systems of QVIs

$$u^i \in \widetilde{F}_i(u), \quad \langle(\theta_i)_{u^i}(u), v^i - u^i\rangle_{U_i^*,U_i} \ge 0 \quad \forall\, v^i \in \widetilde{F}_i(u), \quad (1 \le i \le N), \tag{20}$$

$$u^i \in F_i(u^{-i}), \quad \langle(\theta_i)_{u^i}(u), v^i - u^i\rangle_{U_i^*,U_i} \ge 0 \quad \forall\, v^i \in F_i(u^{-i}), \quad (1 \le i \le N). \tag{21}$$

Further, if we replace (C5) by the stronger assumption (C4), cf. Remark 1, then $\widetilde{V}_\alpha$ can be used as a merit function for (21).

Next, we study the relation between the merit functions $\widetilde{V}_\alpha$ and $\widetilde{V}_\alpha^{\mathrm{loc}}$ and a (local) NE.

Theorem 4.2. *Let $u \in U$ satisfy $u \in F(u)$, let $F_i(u^{-i})$ be convex, $i \in [N]$, and let (B1) hold. Then*

1. *If u is a local NE on $\widetilde{F}(u)$ in the sense that it fulfills $\theta_i(u) \leq \theta_i(v^i, u^{-i})$ for all $v^i \in \widetilde{F}_i(u)$ and all i, then there holds $\widetilde{V}_\alpha^{\mathrm{loc}}(u) = 0$.*
2. *If $\theta_i(\cdot, u^{-i})$ is pseudoconvex at u^i on $\widetilde{F}_i(u)$ for all i and if $\widetilde{V}_\alpha^{\mathrm{loc}}(u) = 0$ holds, then u is a local NE on $\widetilde{F}(u)$.*
3. *If u is a NE, then there holds $\widetilde{V}_\alpha(u) = 0 = \widetilde{V}_\alpha^{\mathrm{loc}}(u)$.*
4. *If $\theta_i(\cdot, u^{-i})$ is pseudoconvex at u^i on $F_i(u^{-i})$ for all i, then the following equivalences hold true:*

$$\widetilde{V}_\alpha^{\mathrm{loc}}(u) = 0 \iff u \text{ is a NE} \iff \widetilde{V}_\alpha(u) = 0.$$

Proof. Similar to Remark 1, we first note that since $\frac{\alpha}{2}\|\iota_{H_i}(\cdot) - \iota_{H_i}(u^i)\|_{H_i}^2 + q_i(\cdot - u^i)$ is nonnegative and, at u^i, has vanishing function value and derivative, the pseudoconvexity of $\theta_i(\cdot, u^{-i})$ at u^i on $\widetilde{F}_i(u)$ (or $F_i(u)$) implies (C5) (or (C4)).

1. Let u be a local NE. Theorem 4.1, part 1, yields $\widetilde{V}_\alpha^{\mathrm{loc}}(u) \geq 0$. Further, we get

$$\begin{aligned}
\widetilde{V}_\alpha^{\mathrm{loc}}(u) &= \sum_i \sup_{v^i \in \widetilde{F}_i(u)} \left[\theta_i(u) - \theta_i(v^i, u^{-i}) - \tfrac{\alpha}{2}\|\iota_{H_i}(v^i) - \iota_{H_i}(u^i)\|_{H_i}^2 - q_i(v^i - u^i)\right] \\
&\leq \sum_i \sup_{v^i \in \widetilde{F}_i(u)} \left[\theta_i(u) - \theta_i(v^i, u^{-i})\right] \leq 0.
\end{aligned}$$

2. By 2. of Theorem 4.1 it follows $\langle(\theta_i)_{u^i}(u), v^i - u^i\rangle_{U_i^*, U_i} \geq 0$ for all $v^i \in \widetilde{F}_i(u)$ and all i. Therefore, we obtain $\theta_i(u) \leq \theta_i(v^i, u^{-i})$ for all $v^i \in \widetilde{F}_i(u)$ and all i by the pseudoconvexity of $\theta_i(\cdot, u^{-i})$ at u^i on $\widetilde{F}_i(u)$.

3. $\widetilde{V}_\alpha(u) = 0$ is proved like in part 1, replacing $\widetilde{F}_i(u)$ by $F_i(u^{-i})$. The rest follows from $0 \leq \widetilde{V}_\alpha^{\mathrm{loc}}(u) \leq \widetilde{V}_\alpha(u)$.

4. For the first implication, we use 2. and 3. of Theorem 4.1 to show $\langle(\theta_i)_{u^i}(u), v^i - u^i\rangle_{U_i^*, U_i} \geq 0$ for all $v^i \in F_i(u^{-i})$. The pseudoconvexity of $\theta_i(\cdot, u^{-i})$ at u^i on $F_i(u^{-i})$ then yields that u is a NE. The second implication follows from part 3. Finally, using $0 \leq \widetilde{V}_\alpha^{\mathrm{loc}}(u) \leq \widetilde{V}_\alpha(u)$ completes the proof. □

Remark 2. Since for all $\beta \geq \alpha \geq 0$, all i, and all $v^i \in F_i(u^{-i})$, it holds

$$-\tfrac{\beta}{2}\|\iota_{H_i}(v^i) - \iota_{H_i}(u^i)\|_{H_i}^2 \leq -\tfrac{\alpha}{2}\|\iota_{H_i}(v^i) - \iota_{H_i}(u^i)\|_{H_i}^2,$$

we obtain $\widetilde{V}_\beta(u) \leq \widetilde{V}_\alpha(u) \leq \widetilde{V}_0(u)$ and $\widetilde{V}_\beta^{\mathrm{loc}}(u) \leq \widetilde{V}_\alpha^{\mathrm{loc}}(u) \leq \widetilde{V}_0^{\mathrm{loc}}(u)$ for $u \in F(u)$. Thus, for any $u \in F(u)$ we get, for all $\alpha \geq 0$, the following implications

$$\begin{aligned}
&u \text{ local NE} &&\Longrightarrow \quad \widetilde{V}_0^{\mathrm{loc}}(u) = 0 &&\Longrightarrow \quad \widetilde{V}_\alpha^{\mathrm{loc}}(u) = 0,\\
&u \text{ NE} &&\Longrightarrow \quad \widetilde{V}_0(u) = 0 &&\Longrightarrow \quad \widetilde{V}_\alpha(u) = 0.
\end{aligned}$$

Reverse directions of these implications can be proved under pseudoconvexity assumptions, cf. Theorem 4.2.

Altogether, if $u \in U$ with $u \in F(u)$ and if $\theta_i(\cdot, u^{-i})$ is pseudoconvex at u^i on $\widetilde{F}_i(u)$ for all

i, then we have shown that, for any $\alpha \geq 0$, there holds

$$\begin{aligned}\widetilde{V}_0^{\mathrm{loc}}(u) = 0 \iff \widetilde{V}_\alpha^{\mathrm{loc}}(u) = 0 &\iff \langle (\theta_i)_{u^i}(u), v^i - u^i \rangle_{U_i^*, U_i} \geq 0 \quad \forall\, v^i \in \widetilde{F}_i(u),\ i \in [N],\\ &\iff \langle (\theta_i)_{u^i}(u), v^i - u^i \rangle_{U_i^*, U_i} \geq 0 \quad \forall\, v^i \in F_i(u^{-i}),\ i \in [N],\\ &\iff u \text{ local NE.}\end{aligned}$$

If $\theta_i(\cdot, u^{-i})$ is even pseudoconvex at u^i on $F_i(u^{-i})$ for all i, we obtain, for any $\alpha \geq 0$,

$$\widetilde{V}_0^{\mathrm{loc}}(u) = 0 \iff \widetilde{V}_\alpha^{\mathrm{loc}}(u) = 0 \iff \widetilde{V}_\alpha(u) = 0 \iff \widetilde{V}_0(u) = 0 \iff u \text{ NE.}$$

4.2. Regularized and localized Nikaido–Isoda merit functions

In the following, we consider the particular feasible set $F_i(u^{-i}) := \{v^i \in U_i : (v^i, u^{-i}) \in \mathcal{X}\}$ and the functionals $V_\alpha : U \to \mathbb{R}$ and $V_\alpha^{\mathrm{loc}} : U \to \mathbb{R}$. We prove that these functionals are Nikaido–Isoda merit functions corresponding to a (quasi-)VI. Further, we end up with a relation between the Nikaido–Isoda merit functions and (local) NoE.

Theorem 4.3. *Let $\alpha \geq 0$ and, for assertions 2–4, let (B1) be satisfied and let $\mathcal{X}$ be convex. Also, for "$\Rightarrow$" in 2., let (C3) hold and for "$\Rightarrow$" in part 4, let (C2) hold. Then, for all $u \in \mathcal{X}$, there holds*

1. $V_\alpha(u) \geq V_\alpha^{\mathrm{loc}}(u) \geq 0.$
2. $\sum_i \langle (\theta_i)_{u^i}(u), v^i - u^i \rangle_{U_i^*, U_i} \geq 0 \; \forall\, v \in \widetilde{\mathcal{X}}(u) \iff V_\alpha^{\mathrm{loc}}(u) = 0.$
3. $\sum_i \langle (\theta_i)_{u^i}(u), v^i - u^i \rangle_{U_i^*, U_i} \geq 0 \; \forall\, v \in \mathcal{X} \iff \sum_i \langle (\theta_i)_{u^i}(u), v^i - u^i \rangle_{U_i^*, U_i} \geq 0 \; \forall\, v \in \widetilde{\mathcal{X}}(u).$
4. $V_\alpha^{\mathrm{loc}}(u) = 0 \iff V_\alpha(u) = 0.$

Proof. Consider any $u \in \mathcal{X}$.

1. Choosing $v = u \in \widetilde{\mathcal{X}}(u)$, we obtain

$$V_\alpha(u) \geq V_\alpha^{\mathrm{loc}}(u) \geq \sum_i [\theta_i(u) - \theta_i(u^i, u^{-i}) - \tfrac{\alpha}{2} \|\iota_{H_i}(u^i) - \iota_{H_i}(u^i)\|_{H_i}^2 - q_i(u^i - u^i)] = 0.$$

2. $\Rightarrow$: It holds

$$\big\langle (\widetilde{\Psi}_\alpha(u, v) + \sum_i q_i(v^i - u^i))_v|_{v=u}, v - u \big\rangle_{U^*, U} = \sum_i \langle (\theta_i)_{u^i}(u), v^i - u^i \rangle_{U_i^*, U_i}.$$

By the pseudoconvexity of $\widetilde{\Psi}_\alpha(u, \cdot) + \sum_i q_i(\cdot - u^i)$ at u on $\widetilde{\mathcal{X}}(u)$ and the QVI on the left of 2, this yields

$$\sum_i \theta_i(u) = \widetilde{\Psi}_\alpha(u, u) + \sum_i q_i(u^i - u^i) \leq \widetilde{\Psi}_\alpha(u, v) + \sum_i q_i(v^i - u^i) \quad \forall\, v \in \widetilde{\mathcal{X}}(u).$$

Hence, we obtain $V_\alpha^{\mathrm{loc}}(u) = 0$.

$\Leftarrow$: By $V_\alpha^{\mathrm{loc}}(u) = 0$ we have $\sum_i \theta_i(u) \leq \widetilde{\Psi}_\alpha(u, w) + \sum_i q_i(w^i - u^i)$ for all $w \in \widetilde{\mathcal{X}}(u)$. By the convexity of $\widetilde{\Psi}_\alpha(u, \cdot) + \sum_i q_i(\cdot - u^i)$ at u on $\widetilde{\mathcal{X}}(u)$ we can consider $w := u + t(v - u) \in \widetilde{\mathcal{X}}(u)$

for $u, v \in \widetilde{\mathcal{X}}(u)$ and $t \in (0,1]$ to obtain

$$0 \leq \widetilde{\Psi}_\alpha(u, u+t(v-u))+\sum_i q_i(t(v^i-u^i))-\sum_i \theta_i(u) = \sum_i \langle (\theta_i)_{u^i}(u), t(v^i-u^i)\rangle_{U_i^*,U_i}+\mathrm{o}(t).$$

Multiplying by $\frac{1}{t}$ and taking the limit $t \to 0$ yields the inequality on the right hand side of 2.

3. $\Rightarrow$: By $\widetilde{\mathcal{X}}(u) \subset \mathcal{X}$ this direction is obvious.

$\Leftarrow$: Let the inequality on the right hold and consider any $v \in \mathcal{X}$. In the case $v \in \widetilde{\mathcal{X}}(u)$ the left inequality clearly holds. Now consider $v \in \mathcal{X} \backslash \widetilde{\mathcal{X}}(u)$ and choose $t \in (0,1]$ with $t\|v-u\|_U \leq R$. Then, by convexity of $\mathcal{X}$, there holds $\tilde{v} := u + t(v-u) \in \widetilde{\mathcal{X}}(u)$ and thus

$$\sum_i \langle (\theta_i)_{u^i}(u), v^i - u^i \rangle_{U_i^*,U_i} = \frac{1}{t}\sum_i \langle (\theta_i)_{u^i}(u), \tilde{v}^i - u^i \rangle_{U_i^*,U_i} \geq 0.$$

4. $\Rightarrow$: Let $V_\alpha^{\mathrm{loc}}(u) = 0$. By 2. and 3. it holds $\langle (\widetilde{\Psi}_\alpha)_v(u,v)|_{v=u}, v-u\rangle_{U^*,U} = \sum_i \langle (\theta_i)_{u^i}(u), v^i - u^i \rangle_{U_i^*,U_i} \geq 0$ for all $v \in \mathcal{X}$, and the pseudoconvexity of $\widetilde{\Psi}_\alpha(u,\cdot)$ at u on $\mathcal{X}$ thus yields $\sum_i \theta_i(u) = \widetilde{\Psi}_\alpha(u,u) \leq \widetilde{\Psi}_\alpha(u,v)$ for all $v \in \mathcal{X}$, which implies $V_\alpha(u) = 0$.

$\Leftarrow$: This follows from $0 \leq V_\alpha^{\mathrm{loc}}(u) \leq V_\alpha(u)$. □

Under the assumptions (B1), (C3), and that $\mathcal{X}$ is convex, Theorem 4.3, parts 2 and 3, yields that V_α^{loc} can be used as a merit function for the (Q)VI $\sum_i \langle (\theta_i)_{u^i}(u), v^i - u^i \rangle_{U_i^*,U_i} \geq 0$ posed either on the set $\mathcal{X}$ or on the set $\widetilde{\mathcal{X}}(u)$. Under the additional assumption (C2), V_α can be used as a merit function for this VI on the set $\mathcal{X}$.

In the following, we consider the connection between these merit functions and a (local) NoE.

Theorem 4.4. *Let $\alpha \geq 0$, $\mathcal{X}$ be convex, let (B1) hold, and consider $u \in \mathcal{X}$.*

1. *If u is a local NoE on $\widetilde{\mathcal{X}}(u)$, i.e., if there holds $\sum_i \theta_i(u) \leq \sum_i \theta_i(v^i, u^{-i})$ for all $v \in \widetilde{\mathcal{X}}(u)$, then $V_\alpha^{\mathrm{loc}}(u) = 0$ holds.*
2. *If $\sum_i \theta_i(\cdot, u^{-i})$ is pseudoconvex at u on $\widetilde{\mathcal{X}}(u)$ and if $V_\alpha^{\mathrm{loc}}(u) = 0$ holds, then u is a local NoE on $\widetilde{\mathcal{X}}(u)$.*
3. *If u is a NoE, then $V_\alpha^{\mathrm{loc}}(u) = 0 = V_\alpha(u)$ holds.*
4. *If $\sum_i \theta_i(\cdot, u^{-i})$ is pseudoconvex at u on $\mathcal{X}$, then the following equivalences hold true*

$$V_\alpha^{\mathrm{loc}}(u) = 0 \iff u \text{ NoE} \iff V_\alpha(u) = 0.$$

Proof. We first note that if $\sum_i \theta_i(\cdot, u^{-i})$ is pseudoconvex at u on $\widetilde{\mathcal{X}}(u)$ (or $\mathcal{X}$), then this implies (C3) (or (C2)).

1. Let u be a local NoE on $\widetilde{\mathcal{X}}(u)$. Then

$$0 \leq V_\alpha^{\mathrm{loc}}(u) \leq \sup_{v \in \widetilde{\mathcal{X}}(u)} \sum_i \left[\theta_i(u) - \theta_i(v^i, u^{-i})\right] \leq 0.$$

2. By Theorem 4.3, part 2, we have $\sum_i \langle (\theta_i)_{u^i}(u), v^i - u^i \rangle_{U_i^*,U_i} \geq 0$ for all $v \in \widetilde{\mathcal{X}}(u)$. Since $\sum_i \theta_i(\cdot, u^{-i})$ is pseudoconvex at u on $\widetilde{\mathcal{X}}(u)$, we get $\sum_i \theta_i(u) \leq \sum_i \theta_i(v^i, u^{-i})$ for all $v \in \widetilde{\mathcal{X}}(u)$. Consequently, $u \in \mathcal{X}$ is a local NoE on $\widetilde{\mathcal{X}}(u)$.

3. $V_\alpha(u) = 0$ is proved like in part 1, replacing $\widetilde{\mathcal{X}}(u)$ by $\mathcal{X}$. The rest follows from $0 \leq V_\alpha^{\text{loc}}(u) \leq V_\alpha(u)$.

4. Let $V_\alpha^{\text{loc}}(u) = 0$. Then we obtain $\sum_i \langle (\theta_i)_{u^i}(u), v^i - u^i \rangle_{U_i^*,U_i} \geq 0$ for all $v \in \mathcal{X}$ by Theorem 4.3, parts 2 and 3. The pseudoconvexity of $\sum_i \theta_i(\cdot, u^{-i})$ at u on $\mathcal{X}$ yields $\sum_i \theta_i(u) \leq \sum_i \theta_i(v^i, u^{-i})$ for all $v \in \mathcal{X}$. Thus, u is a NoE. The second implication follows from part 3 and $0 \leq V_\alpha^{\text{loc}}(u) \leq V_\alpha(u) = 0$ concludes the proof. $\square$

Remark 3. As in Remark 2 we get that for $\beta \geq \alpha \geq 0$ and $u \in \mathcal{X}$ there hold $V_\beta(u) \leq V_\alpha(u) \leq V_0(u)$ and $V_\beta^{\text{loc}}(u) \leq V_\alpha^{\text{loc}}(u) \leq V_0^{\text{loc}}(u)$. Further, for all $u \in \mathcal{X}$ and $\alpha \geq 0$, there holds

$$\begin{array}{lllll} u \text{ local NoE} & \Longrightarrow & V_0^{\text{loc}}(u) = 0 & \Longrightarrow & V_\alpha^{\text{loc}}(u) = 0, \\ u \text{ NoE} & \Longrightarrow & V_0(u) = 0 & \Longrightarrow & V_\alpha(u) = 0. \end{array}$$

In the first line, the first implication can be converted to an equivalence if $q_i \equiv 0$. All implications can be converted to equivalences under suitable pseudoconvexity assumptions, cf. Theorem 4.4.

Summarizing, if $\sum_i \theta_i(\cdot, u^{-i})$ is pseudoconvex at $u \in \mathcal{X}$ on $\widetilde{\mathcal{X}}(u)$, we have shown the following (pseudoconvexity is needed only for part of the implications):

$$\begin{aligned} V_0^{\text{loc}}(u) = 0 &\Longleftrightarrow V_\alpha^{\text{loc}}(u) = 0, \\ &\Longleftrightarrow \sum_i \langle (\theta_i)_{u^i}(u), v^i - u^i \rangle_{U_i^*,U_i} \geq 0 \quad \forall\, v \in \widetilde{\mathcal{X}}(u), \\ &\Longleftrightarrow \sum_i \langle (\theta_i)_{u^i}(u), v^i - u^i \rangle_{U_i^*,U_i} \geq 0 \quad \forall\, v \in \mathcal{X}, \\ &\Longleftrightarrow u \in \mathcal{X} \text{ local NoE.} \end{aligned}$$

If $\sum_i \theta_i(\cdot, u^{-i})$ is even pseudoconvex at $u \in \mathcal{X}$ on $\mathcal{X}$, we can continue this as follows

$$\begin{aligned} &\Longleftrightarrow u \text{ NoE} \\ &\Longleftrightarrow V_\alpha(u) = 0. \end{aligned}$$

Moreover, if $\sum_i \theta_i(\cdot, u^{-i})$ is convex on $\mathcal{X}$, we have shown the existence of a fixed point $u \in \mathcal{X}$ of the corresponding solution maps in Section 3. Therefore, we obtain $V_\alpha(u) = 0$. Thus, $u \in \mathcal{X}$ is a NoE by Theorem 4.4.2.

4.3. Difference of two regularized Nikaido–Isoda functions

From finite-dimensional GNEP theory it is known (cf., e.g., [15, 30]) that taking a difference $V_{\alpha\beta}(u) = V_\alpha(u) - V_\beta(u)$ of regularized Nikaido–Isoda merit functions, where $\beta > \alpha \geq 0$, results in a merit function that, written in our notation, can be used on the whole space U rather than only on $\mathcal{X}$. Similiar results can also be developed in our setting. Due to space, we do not elaborate the details here.

5. Differentiability of regularized Nikaido–Isoda function

In this section, we study the differentiability of the regularized merit functions $\widetilde{V}_\alpha$, V_α, and of their localized modifications $\widetilde{V}_\alpha^{\text{loc}}$ and V_α^{loc}.

Standing Assumption. Let the Assumptions 2.1 and 2.2, (B2c) hold. Further, let $\mathcal{X} \subset U$ and $\mathcal{X}_i \subset U_i$ be nonempty, convex, closed, and bounded and, hence, compact with respect to the weak sequential topology.

We consider regularized Nikaido–Isoda merit functions with sets $F_i(u^{-i}) = \mathcal{X}_i$ and $\mathcal{X}$, respectively. Therefore, one can apply a suitable version of Danskin's theorem, see Appendix A. Later on, we discuss the differentiability of (localized) merit functions that involve penalty or barrier terms.

5.1. *Differentiability without localization term*

We work on the fixed sets $F_i(u^{-i}) = \mathcal{X}_i$ and $\mathcal{X}$, respectively. This will enable us to apply a suitable version of Danskin's theorem, see Appendix A.

We start by considering the regularized Nikaido–Isoda merit function

$$\widetilde{V}_\alpha(u) = \sum_i \max_{v^i \in \mathcal{X}_i} \left[\theta_i(u) - \theta_i(v^i, u^{-i}) - \tfrac{\alpha}{2}\|\iota_{H_i}(v^i) - \iota_{H_i}(u^i)\|_{H_i}^2\right] \tag{22}$$

and prove its differentiability by Danskin's theorem, see Proposition A.10. To this end, we show that each term of the sum is differentiable. Since, with

$$f^i(u, v^i) := -\theta_i(v^i, u^{-i}) - \tfrac{\alpha}{2}\|\iota_{H_i}(v^i) - \iota_{H_i}(u^i)\|_{H_i}^2, \tag{23}$$

it holds

$$\max_{v^i \in \mathcal{X}_i} \left[\theta_i(u) - \theta_i(v^i, u^{-i}) - \tfrac{\alpha}{2}\|\iota_{H_i}(v^i) - \iota_{H_i}(u^i)\|_{H_i}^2\right] = \theta_i(u) + \max_{v^i \in \mathcal{X}_i} f^i(u, v^i), \tag{24}$$

we consider the terms $\theta_i(u)$ and $\max_{v^i \in \mathcal{X}_i} f^i(u, v^i)$ separately. The mapping $u \mapsto \theta_i(u)$ is continuously differentiable with

$$\langle(\theta_i)_u(u), h\rangle_{U^*,U} = \langle(\tilde{\theta}_i)_{\tilde{u}}(\iota_{\widetilde{U}}(u)), \iota_{\widetilde{U}}(h)\rangle_{\widetilde{U}^*,\widetilde{U}} + \gamma(u^i, h^i)_{U_i}$$

for all $h \in U$. For the second term in (24), we apply Danskin's theorem, see Proposition A.10 (with f^i, U, u, U_i, and v^i playing the roles of f, X, x, Z, and z, respectively; τ_X and τ_Z are the weak topologies of U and U_i, respectively). For this, we have to satisfy Assumption A.1. We define $f_1^i : U \times U_i \to \mathbb{R}$ and $f_2^i : U_i \to \mathbb{R}$ by

$$f_1^i(u, v^i) = -\tilde{\theta}_i(\iota_{\widetilde{U}}(v^i, u^{-i})) - \frac{\alpha}{2}\|\iota_{H_i}(v^i) - \iota_{H_i}(u^i)\|_{H_i}^2, \quad f_2^i(v^i) = -\frac{\gamma}{2}\|v^i\|_{U_i}^2.$$

Now, f_1^i is weakly sequentially continuous in both arguments and f_2^i is weakly sequentially upper semicontinuous. Thus, $f^i = f_1^i + f_2^i$ is continuous in u and upper semicontinuous in both variables with respect to the weak sequential topology. There holds

$$\langle(f^i)_u(u, v^i), h\rangle_{U^*,U} = -\sum_{j \neq i}\left[\langle(\tilde{\theta}_i)_{\tilde{u}^j}(\iota_{\widetilde{U}}(v^i, u^{-i})), \iota_{\widetilde{U}_j}(h^j)\rangle_{\widetilde{U}_j^*,\widetilde{U}_j}\right]$$

$$+ \alpha(\iota_{H_i}(v^i) - \iota_{H_i}(u^i), \iota_{H_i}(h^i))_{H_i}.$$

We see that $(f^i)_u$ is continuous in both arguments from the weak sequential topology to the norm topology of U^*. Thus, Assumption A.1 is fulfilled and we can apply Danskin's theorem (Proposition A.10) to f^i. If one chooses $\alpha \geq 0$ sufficiently large such that, on an open set A, the solution maps $u \in A \mapsto v_{f^i}(u)$ to $\max_{v^i \in \mathcal{X}_i} f^i(u, v^i)$ are single-valued for all $i \in [N]$, then the value functions $u \mapsto \max_{v^i \in \mathcal{X}_i} f^i(u, v^i)$, $i \in [N]$, are continuously differentiable on A by Danskin's theorem (Proposition A.10) with the derivative

$$\begin{aligned}\langle (\max_{v^i \in \mathcal{X}_i} f^i(u, v^i))_u, h \rangle_{U^*U} = -\sum_{j \neq i} &\langle (\tilde{\theta}_i)_{\tilde{u}^j}(\iota_{\widetilde{U}}(v^i_{f^i}(u), u^{-i})), \iota_{\widetilde{U}_j}(h^j) \rangle_{\widetilde{U}_j^*, \widetilde{U}_j} \\ &+ \alpha(\iota_{H_i}(v^i_{f^i}(u)) - \iota_{H_i}(u^i), \iota_{H_i}(h^i))_{H_i}\end{aligned}$$

and the derivative is sequentially continuous from the weak topology of U to the norm topology of U^*.

Thus, we have shown that $\widetilde{V}_\alpha(u)$ as defined in (22) is continuously differentiable on A:

Theorem 5.1. *Let the Standing Assumption of this section hold. Further, let $\alpha \geq 0$ and the open set A be such that the solution maps $u \in A \mapsto v_{f^i}(u)$ to $\max_{v^i \in \mathcal{X}_i} f^i(u, v^i)$, where f^i is defined in (23), are single-valued for all $i \in [N]$. Then $\widetilde{V}_\alpha$ is continuously differentiable on A and, with $v_f(u) \in U$, $v_f(u)^i = v_{f^i}(u)$, there holds*

$$\begin{aligned}\langle \widetilde{V}_\alpha'(u), h \rangle_{U^*, U} = \sum_{i \in [N]} \Big[&\langle (\tilde{\theta}_i)_{\tilde{u}}(\iota_{\widetilde{U}}(u)), \iota_{\widetilde{U}}(h) \rangle_{\widetilde{U}^*, \widetilde{U}} \\ &- \sum_{j \neq i} \langle (\tilde{\theta}_i)_{\tilde{u}^j}(\iota_{\widetilde{U}}(v_{f^i}(u), u^{-i})), \iota_{\widetilde{U}_j}(h^j) \rangle_{\widetilde{U}_j^*, \widetilde{U}_j} \Big] \\ &+ \gamma(u, h)_U + \alpha(\iota_H(v_f(u)) - \iota_H(u), \iota_H(h))_H.\end{aligned}$$

Remark 4. Under weaker assumptions, some results of Appendix A are still applicable. For instance, if $\tilde{\theta}_i$ is only continuous, then, without requiring the single-valuedness of the solution map $u \mapsto v_{f^i}(u)$ for (23), Proposition A.5 still yields that $\max_{v^i \in \mathcal{X}_i} f^i(u, v^i)$ is weakly sequentially continuous and, thus, $\widetilde{V}_\alpha(u)$ is continuous with respect to the norm topology and weakly sequentially lower semicontinuous.

Next, we consider the regularized Nikaido–Isoda merit function for NoE,

$$V_\alpha(u) := \sup_{v \in \mathcal{X}} \sum_i \left[\theta_i(u) - \theta_i(v^i, u^{-i}) - \tfrac{\alpha}{2} \|\iota_{H_i}(v^i) - \iota_{H_i}(u^i)\|_{H_i}^2 \right].$$

Theorem 5.2. *Let the Standing Assumption of this section hold. Let $\alpha \geq 0$ and the open set A be such that the solution map $u \in A \mapsto v_f(u)$ to $\max_{v \in \mathcal{X}} f(u, v)$ is single-valued, where $f : U \times U \to \mathbb{R}$, $f(u, v) = f_1(u, v) + f_2(v)$ and $f_1 : U \times U \to \mathbb{R}$, $f_2 : U \to \mathbb{R}$ are defined by*

$$f_1(u, v) = -\sum_i \tilde{\theta}_i(\iota_{\widetilde{U}}(v^i, u^{-i})) - \frac{\alpha}{2} \|\iota_H(v) - \iota_H(u)\|_H^2, \quad f_2(v) = -\frac{\gamma}{2} \|v\|_U^2.$$

Then V_α is continuously differentiable on A and there holds

$$\begin{aligned}\langle V_\alpha'(u), h\rangle_{U^*,U} = &\sum_{i\in[N]} \Big[\langle(\tilde{\theta}_i)_{\tilde{u}}(\iota_{\widetilde{U}}(u)), \iota_{\widetilde{U}}(h)\rangle_{\widetilde{U}^*,\widetilde{U}} \\ &\qquad - \sum_{j\neq i}\langle(\tilde{\theta}_i)_{\tilde{u}^j}(\iota_{\widetilde{U}}(v_f^i(u), u^{-i})), \iota_{\widetilde{U}_j}(h^j)\rangle_{\widetilde{U}_j^*,\widetilde{U}_j}\Big] \\ &+ \gamma(u,h)_U + \alpha(\iota_H(v_f(u)) - \iota_H(u), \iota_H(h))_H.\end{aligned} \tag{25}$$

Proof. We proceed as above and split the supremum into the terms $\sum_i \theta_i(u)$ and $\sup_{v\in\mathcal{X}} f(u,v)$. The mapping $u \mapsto \sum_i \theta_i(u)$ is continuously differentiable with

$$\big\langle\big(\sum_i \theta_i(u)\big)_u, h\big\rangle_{U^*,U} = \sum_i \big[\langle(\tilde{\theta}_i)_{\tilde{u}}(\iota_{\widetilde{U}}(u)), \iota_{\widetilde{U}}(h)\rangle_{\widetilde{U}^*,\widetilde{U}}\big] + \gamma(u,h)_U.$$

We apply Danskin's theorem to show the continuous differentiability of the second term. For this, we use the structure $f(u,v) = f_1(u,v) + f_2(v)$ in essentially the same way as in the proof of Theorem 5.1. We obtain that f is continuous in u and upper semicontinuous in both arguments with respect to the weak sequential topology. Further, f is differentiable in u with continuous derivative $f_u = (f_1)_u$ in the weak sequential topology. Since, by assumption, the solution map $u \mapsto v_f(u)$ of $\max_{v\in\mathcal{X}} f(u,v)$ is single-valued on the open set $A \subset U$, Proposition A.10 yields the continuous differentiability of $u \mapsto \max_{v\in\mathcal{X}} f(u,v)$ with $\langle(\max_{v\in\mathcal{X}} f(u,v))_u, h\rangle_{U,U^*} = \langle f_u(u, v_f(u)), h\rangle_{U,U^*}$ and the derivative is continuous from the weak sequential topology of U to the norm topology of U^*. Thus, V_α is continuously differentiable on A and (25) holds true. □

5.2. *Differentiability with localization term*

We now develop similar results for the regularized and localized Nikaido–Isoda merit functions

$$\widetilde{V}_\alpha^{\mathrm{loc}}(u) = \sum_i \sup_{v^i\in\widetilde{F}_i(u)} \big[\theta_i(u) - \theta_i(v^i, u^{-i}) - \tfrac{\alpha}{2}\|\iota_{H_i}(v^i) - \iota_{H_i}(u^i)\|_{H_i}^2 - q_i(v^i - u^i)\big],$$
$$V_\alpha^{\mathrm{loc}}(u) = \sup_{v\in\widetilde{\mathcal{X}}(u)} \sum_i [\theta_i(u) - \theta_i(v^i, u^{-i}) - \tfrac{\alpha}{2}\|\iota_{H_i}(v^i) - \iota_{H_i}(u^i)\|_{H_i}^2 - q_i(v^i - u^i)].$$

Here, the Standing Assumption is still in effect and $\widetilde{F}_i(u)$, $\widetilde{\mathcal{X}}(u)$ are as at the beginning of Section 4. In general, the u-dependent constraint $v^i \in \widetilde{F}_i(u)$ would cause nonsmoothness of $\widetilde{V}_\alpha^{\mathrm{loc}}$ and similarly for V_α^{loc}. This is the reason why we introduce a C^1 penalty or barrier term, expressed by q_i, to enforce locality in a way that does not destroy differentiability.

We can write $\widetilde{V}_\alpha^{\mathrm{loc}}(u) = \sum_{i\in[N]}[\theta_i(u) + \phi_i(u)]$, where

$$\phi_i(u) = \max_{v^i\in\widetilde{F}_i(u)} f^i(u, v^i), \qquad f^i(u,v^i) = f_1^i(u,v^i) + f_2^i(v^i), \tag{26}$$
$$f_1^i(u,v^i) = -\tilde{\theta}_i(\iota_{\widetilde{U}}(v^i, u^{-i})) - \frac{\alpha}{2}\|\iota_{H_i}(v^i) - \iota_{H_i}(u^i)\|_{H_i}^2 - q_i(v^i - u^i), \quad f_2^i(v^i) = -\frac{\gamma}{2}\|v^i\|^2.$$

The functions $q_i : U_i \to [0,\infty]$ and the sets $\widetilde{F}_i(u) \subset F_i(u^{-i})$ and $\widetilde{\mathcal{X}}(u) \subset \widetilde{\mathcal{X}}$ are as in Section 2, but we need some additional structure.

Assumption 5.3. There exist a nonempty open set $A \subset U$, closed convex sets $C_i \subset U_i$ with $0 \in C_i$, and $\rho, R > 0$ such that, with the choices for $\alpha \geq 0$, q_i, and for the convex sets $\widetilde{F}_i(u) \subset U_i$, $i \in [N]$, there hold:

(L1) The function $q_i : U_i \to [0, \infty]$ is convex with $q_i(0) = 0$ and continuously differentiable on the interior of its domain $\mathrm{dom}(q_i)$, $i \in [N]$.

(L2) The function q_i and its Fréchet derivative are weakly to strongly sequentially continuous on $\mathrm{int}(\mathrm{dom}(q_i))$, $i \in [N]$.

(L3) For all $u \in A$, there hold $\overline{B}_R(0) \cup \mathrm{cl}(C_i + B_\rho(0)) \subset \mathrm{dom}(q_i)$ and $F_i(u^{-i}) \cap \overline{B}_R(u^i) \subset \widetilde{F}_i(u) \subset F_i(u^{-i})$.

(L4) For any $u \in A$ and all $i \in [N]$, the problem $\max_{v^i \in \widetilde{F}_i(u)} f^i(u, v^i)$ has a unique solution $v_{f^i}(u)$, where f^i is defined in (26). Furthermore, there holds $v_{f^i}(u) \in u^i + C_i$.

Theorem 5.4. *Let the Standing Assumption and Assumption 5.3 hold. Then the function $\widetilde{V}_\alpha^{\mathrm{loc}}$ is continuously differentiable on the set A with derivative*

$$\begin{aligned}\langle (\widetilde{V}_\alpha^{\mathrm{loc}})'(u), h\rangle_{U^*,U} = \sum_{i \in [N]} \Big[&\langle (\tilde{\theta}_i)_{\tilde{u}}(\iota_{\widetilde{U}}(u)), \iota_{\widetilde{U}}(h)\rangle_{\widetilde{U}^*,\widetilde{U}} \\ &- \sum_{j \neq i} \langle (\tilde{\theta}_i)_{\tilde{u}^j}(\iota_{\widetilde{U}}(v_{f^i}(u), u^{-i})), \iota_{\widetilde{U}_j}(h^j)\rangle_{\widetilde{U}_j^*,\widetilde{U}_j} + \langle q_i'(v_{f^i}(u) - u^i), h^i\rangle_{U_i^*,U_i} \Big] \\ &+ \gamma(u,h)_U + \alpha(\iota_H(v_f(u)) - \iota_H(u), \iota_H(h))_H,\end{aligned} \tag{27}$$

where, for $i \in [N]$, $v_{f^i}(u)$ is the, by (L4), unique solution of the problem $\max_{v^i \in \widetilde{F}_i(u)} f^i(u, v^i)$, and $v_f(u) \in U$ is defined by $v_f(u)_i = v_{f^i}(u)$.

Proof. Let $\hat{u} \in A$ be arbitrarily fixed and let $i \in [N]$. We set $r = \rho/2$ and define the closed, convex, bounded set $Y_i = \mathcal{X}_i \cap \mathrm{cl}(C_i + B_r(\hat{u}^i))$. For any $\hat{u} \in A$ and all $u \in B_r(\hat{u})$, the assumptions yield $v_{f^i}(u) \in u^i + C_i \subset B_r(\hat{u}^i) + C_i$. This shows $v_{f^i}(u) \in Y_i$. We thus obtain

$$\phi_i(u) = \max_{v^i \in \widetilde{F}_i(u)} f^i(u, v^i) = \max_{v^i \in Y_i} f^i(u, v^i)$$

with a unique, single-valued solution operator $u \in B_\rho(\hat{u}) \mapsto v_{f^i}(u) \in Y_i$.

We now prove that ϕ_i is continuously differentiable on $B_r(\hat{u})$. We apply Proposition A.10 with $X = U$, $W = B_r(\hat{u})$, $\tau_X =$ weak topology of U, $Z = \hat{u}^i + \mathrm{dom}(q_i)$, $\tau_Z =$ topology induced by weak topology of U_i, $Y = Y_i$, $f = f^i$. Since $Y_i \subset Z$ and Y_i is weakly sequentially compact in U_i, it is also weakly sequentially compact in Z. For any $(u, v^i) \in W \times Y_i$, there holds

$$v^i - u^i \in v^i - \hat{u}^i + B_r(0) \subset \mathrm{cl}(C_i + B_r(\hat{u}^i)) - \hat{u}^i + B_r(0) \subset C_i + B_\rho(0) \subset \mathrm{int}(\mathrm{dom}(q_i)).$$

On $W \times Y_i$, the function f_1^i is weakly sequentially continuous in both arguments and f_2^i is weakly sequentially upper semicontinuous in v^i on $U_i \supset Y_i$. Thus, f^i is weakly sequentially upper semicontinuous in both variables on $W \times Y_i$ and $f^i(\cdot, v^i)$ is weakly sequentially continuous on W for all $v^i \in Y_i$. The u-derivative of f^i is weakly to strongly sequentially continuous on $W \times Y_i$ and Y_i is weakly sequentially compact. Thus, Proposition A.10 yields that ϕ_i is continuously differentiable on W with the derivative $f_u^i(u, v_{f^i}(u))$ (we also get that the derivative is weakly to strongly continuous on W, but we do not pursue this further here). Since $\hat{u} \in A$ was arbitrary, ϕ_i is thus continuously differentiable on A. This then also holds for $\widetilde{V}_\alpha^{\mathrm{loc}} = \sum_i [\theta_i + \phi_i]$. The formula (27) is obtained by writing out the terms of $\langle (\theta_i)_u, h\rangle_{U^*,U}$ and of (cf. Danskin's theorem, Proposition A.10)

$$\langle (\phi_i)_u, h\rangle_{U^*,U} = \langle (f_i)_u(v_{f^i}(u), u^{-i}), h\rangle_{U^*,U}.$$

□

In a very similar way, we can prove an analogous result for V_α^{loc}. In this case, there holds $V_\alpha^{\mathrm{loc}}(u) = \phi(u) + \sum_i \theta_i(u)$ with $\phi(u) = \max_{v\in\widetilde{X}(u)} f(u,v)$,

$$\phi(u) = \max_{v\in\widetilde{X}(u)} f(u,v), \quad f(u,v) = f_1(u,v) + f_2(v), \tag{28}$$
$$f_1(u,v) = -\sum_i \left[\tilde{\theta}_i(\iota_{\widetilde{U}}(v^i, u^{-i})) + q_i(v^i - u^i)\right] - \tfrac{\alpha}{2}\|\iota_H(v) - \iota_H(u)\|_H^2, \quad f_2(v) = -\tfrac{\gamma}{2}\|v\|_U^2.$$

Theorem 5.5. *Let the Standing Assumption and Assumption 5.3 hold, but with (L3) and (L4) replaced by*

(L3′) *For all $u \in A$, there hold $\overline{B}_R(0) \cup \mathrm{cl}(C_i + B_\rho(0)) \subset \mathrm{dom}(q_i)$ and $\mathcal{X} \cap \overline{B}_R(u) \subset \widetilde{\mathcal{X}}(u)$.*

(L4′) *For any $u \in A$ and all $i \in [N]$, the problem $\max_{v\in\widetilde{\mathcal{X}}(u)} f(u,v)$, where f is defined in (28), has a unique solution $v_f(u)$. Furthermore, there holds $v_f^i(u) \in u^i + C_i$ for all $i \in [N]$.*

Then the function V_α^{loc} is continuously differentiable on the set A with derivative

$$\begin{aligned}\langle (V_\alpha^{\mathrm{loc}})'(u), h\rangle_{U^*,U} = \sum_{i\in[N]} \Big[&\langle (\tilde{\theta}_i)_{\tilde{u}}(\iota_{\widetilde{U}}(u)), \iota_{\widetilde{U}}(h)\rangle_{\widetilde{U}^*,\widetilde{U}} \\ &- \sum_{j\neq i} \langle (\tilde{\theta}_i)_{\tilde{u}^j}(\iota_{\widetilde{U}}(v_f^i(u), u^{-i})), \iota_{\widetilde{U}_j}(h^j)\rangle_{\widetilde{U}_j^*,\widetilde{U}_j} + \langle q_i'(v_f^i(u) - u^i), h^i\rangle_{U_i^*,U_i}\Big] \\ &+ \gamma(u,h)_U + \alpha(\iota_H(v_f(u)) - \iota_H(u), \iota_H(h))_H,\end{aligned}$$

where $v_f(u)$ is the, by (L4′), unique solution of the problem $\max_{v\in\widetilde{\mathcal{X}}(u)} f(u,v)$.

Appendix A. Danskin's Theorem

In [8], J. M. Danskin derived a result about the differentiability of optimal value functions. This result was extended to more general setting in, e.g., [3, 22]. In the following, we develop a generalized version of Danskin's theorem that fits the needs of this paper and might be of independent interest.

We work under the following assumptions, where the first set of results will require (D1)–(D4) while from Proposition A.6 on, with the exception of Proposition A.9, we will require (D1)–(D5).

Assumption A.1.

(D1) *$(X, \|\cdot\|_X)$ is a normed space, $W \subset X$ is open, and (X, τ_X) is a Hausdorff topological vector space with a topology τ_X such that the identity on X is $\|\cdot\|_X$–τ_X–continuous,*

(D2) *(Z, τ_Z) is a Hausdorff topological space,*

(D3) *$Y \subset Z$ is a sequentially compact set with respect to τ_Z,* (D4) $f : W \times Y \to \mathbb{R}$ *is sequentially upper semicontinuous with respect to* $\tau_X \times \tau_Z$ *and* $f(\cdot, z) : W \to \mathbb{R}$ *is sequentially continuous with respect to* τ_X *for all* $z \in Y$,

(D5) $f : W \times Y \to \mathbb{R}$ *is Fréchet differentiable in the first variable and* $f_x : W \times Y \to X^*$ *is sequentially continuous with respect to* $\tau_X \times \tau_Z$.

Unless otherwise stated, all properties, e.g., compactness, closedness or convergence, are considered with respect to the norm topology. We define $g : W \to \mathbb{R}$ and the set-valued function $M : W \rightrightarrows Z$ by

$$g(x) = \max_{z \in Y} f(x, z), \qquad M(x) := \{z \in Y : g(x) = f(x, z)\}.$$

Remark 5. A canonical choice for τ_X could be $\|\cdot\|_X$-topology. The introduction of τ_X allows to establish continuity results for g, M, and possibly the derivative of g with respect to a topology τ_X on X that can be weaker than the norm topology.

Proposition A.2. *Under Assumption A.1, (D1)–(D4), the function* g *is well-defined and* $M(x)$ *is nonempty for all* $x \in W$.

Proof. Let $x \in W$ be arbitrarily fixed. We prove that the supremum $\sup_{z \in Y} f(x, z)$ is attained. Let $\{z_k\}_{k\in\mathbb{N}} \subset Y$ be a maximizing sequence, i.e., $f(x, z_k) \to \sup_{z\in Y} f(x, z) \in \overline{\mathbb{R}}$. Since Y is sequentially compact with respect to τ_Z, there exist a subsequence $\{z_{k_l}\}_{l\in\mathbb{N}} \subset \{z_k\}_{k\in\mathbb{N}} \subset Y$ and $z \in Y$ such that $z_{k_l} \to z \in Y$ with respect to τ_Z. Further, we obtain the estimate

$$\sup_{y \in Y} f(x, y) = \lim_k f(x, z_k) = \lim_l f(x, z_{k_l}) = \limsup_l f(x, z_{k_l}) \le f(x, z),$$

by the sequential upper semicontinuity of $f(x, \cdot)$ with respect to τ_Z. Thus, the supremum is attained at $z \in Y$, g is well-defined, and $z \in M(x) \neq \emptyset$. $\square$

Proposition A.3. *Under Assumption A.1, (D1)–(D4), the set* $M(x) \subset Y \subset Z$, $x \in W$, *is sequentially compact with respect to* τ_Z.

Proof. Let $\{z_k\}_{k\in\mathbb{N}} \subset M(x)$. By sequential compactness of $Y \supset M(x)$ with respect to τ_Z, there exists a subsequence $\{z_{k_l}\}_{l\in\mathbb{N}} \subset \{z_k\}_{k\in\mathbb{N}} \subset Y$ such that $z_{k_l} \to z \in Y$ with respect to τ_Z. We use the sequential upper semicontinuity of $f(x, \cdot)$ with respect to τ_Z and $z_{k_l} \in M(x)$, implying $f(x, z_{k_l}) = g(x)$ for all l, to obtain

$$g(x) \ge f(x, z) \ge \limsup_l f(x, z_{k_l}) = g(x).$$

This proves $f(x, z) = g(x)$ and $z \in M(x)$. Thus, $M(x) \subset Z$ is sequentially compact with respect to τ_Z. $\square$

Proposition A.4. *Let Assumption A.1, (D1)–(D4) hold. Then the set-valued mapping* $M : W \rightrightarrows Z$ *is sequentially closed with respect to* $\tau_X \times \tau_Z$ *in the following sense:*

$$\{x_k\} \subset W, \ \ x_k \xrightarrow{\tau_X} x \in W, \ \ z_k \in M(x_k), \ \ z_k \xrightarrow{\tau_Z} z \quad \Longrightarrow \quad z \in M(x).$$

Proof. Let $\{x_k\}_{k\in\mathbb{N}} \subset W$ be a sequence such that $x_k \to x \in W$ with respect to τ_X. Further, we consider $z_k \in M(x_k)$, $k \in \mathbb{N}$, such that $z_k \to z$ with respect to τ_Z. We have to verify

$z \in M(x)$. It holds for all $y \in Y$

$$f(x,z) \geq \limsup_k f(x_k, z_k) \geq \limsup_k f(x_k, y) = f(x,y),$$

where we have used the sequential upper semicontinuity with respect to $\tau_X \times \tau_Z$ in the first step and the sequential continuity of $f(\cdot, z)$ with respect to τ_X in the last equality. Thus, we arrive at $f(x,z) \geq f(x,y)$ for all $y \in Y$ and hence $z \in M(x)$. $\square$

Proposition A.5. *Under Assumption A.1, (D1)–(D4), the function $g : W \to \mathbb{R}$ is sequentially continuous with respect to τ_X and thus, continuous in the norm topology.*

Proof. We argue by contradiction. Assume that g is not sequentially continuous on W with respect to τ_X. Then there exists a sequence $\{x_k\} \subset W$ and $x \in W$ such that $x_k \to x$ with respect to τ_X and some $\varepsilon > 0$ such that $|g(x_k) - g(x)| \geq \varepsilon$ for all $k \in \mathbb{N}$. Further, let $z_k \in M(x_k)$, $k \in \mathbb{N}$. Since $Y \subset Z$ is sequentially compact with respect to τ_Z, there exists a subsequence $\{z_{k_l}\}_{l \in \mathbb{N}} \subset \{z_k\}_{k \in \mathbb{N}}$ such that $z_{k_l} \to z$ with respect to τ_Z. By sequential closedness of $M : W \rightrightarrows Z$ we obtain $z \in M(x)$.

If it holds $g(x_{k_l}) - g(x) \geq \varepsilon$, we obtain by sequential upper semicontinuity of f with respect to $\tau_X \times \tau_Z$ the following contradiction:

$$g(x) + \varepsilon \leq \limsup_l g(x_{k_l}) = \limsup_l f(x_{k_l}, z_{k_l}) \leq f(x,z) = g(x).$$

If it holds $g(x_{k_l}) - g(x) \leq -\varepsilon$, we obtain

$$\begin{aligned} g(x) - \varepsilon \geq \limsup_l g(x_{k_l}) &= \limsup_l f(x_{k_l}, z_{k_l}) \geq \limsup_l f(x_{k_l}, z) \\ &= \lim_l f(x_{k_l}, z) = f(x,z) = g(x) \end{aligned}$$

by the sequential continuity of $f(\cdot, z)$ with respect to τ_X, which is a contradiction. Hence, g is sequentially continuous with respect to τ_X. $\square$

From here on, except for Proposition A.9, we additionally assume that also (D5) of Assumption A.1 is satisfied.

Proposition A.6. *Under Assumption A.1, the function $g : W \to \mathbb{R}$ is locally Lipschitz continuous.*

Proof. We consider the function $\|f_x(\cdot,\cdot)\|_{X^*} : W \times Y \to \mathbb{R}$, which fulfills (D4) of Assumption A.1. Indeed, since $f_x : W \times Y \to X^*$ is sequentially continuous with respect to $\tau_X \times \tau_Z$, we see that $\|f_x(\cdot,\cdot)\|_{X^*} : W \times Y \to \mathbb{R}$ is sequentially upper semicontinuous with respect to $\tau_X \times \tau_Z$ and $\|f_x(\cdot, z)\|_{X^*} : W \to \mathbb{R}$ is continuous with repsect to τ_X for all $z \in Y$. Thus, we get the continuity of $x \in (W, \tau_X) \mapsto \max_{z \in Y} \|f_x(x,z)\|_{X^*}$ by Proposition A.5. By Assumption A.1, (D1), this also implies continuity of this function from $(W, \|\cdot\|_X)$ to $\mathbb{R}$. Hence, fixing $x_0 \in W$, there exist $\delta > 0$ and $L > 0$ with $B_\delta(x_0) \subset W$ and $\max_{z \in Y} \|f_x(y,z)\|_{X^*} \leq L$ for all $y \in B_\delta(x_0)$.

Consider $x_1, x_2 \in B_\delta(x_0)$ and let $z_1 \in M(x_1)$, $z_2 \in M(x_2)$. Without loss of generality, let $g(x_1) \geq g(x_2)$. It holds

$$\begin{aligned} |g(x_1) - g(x_2)| &= g(x_1) - g(x_2) = f(x_1, z_1) - f(x_2, z_2) \\ &\leq f(x_1, z_1) - f(x_2, z_1) = \langle f_x((1-t)x_1 + tx_2, z_1), x_1 - x_2 \rangle_{X^*,X}, \end{aligned}$$

with a suitable $t \in [0,1]$ by the mean value theorem. This and the convexity of $B_\delta(x_0)$ yield

$$|g(x_1) - g(x_2)| \le \|f_x(x_1 + t(x_2 - x_1), z_1)\|_{X^*} \|x_1 - x_2\|_X \le L\|x_1 - x_2\|_X. \qquad \square$$

Proposition A.7. *Under Assumption A.1, the function $g : W \to \mathbb{R}$ is directionally differentiable with the directional derivative*

$$g'(x,h) = \max_{z \in M(x)} \langle f_x(x,z), h\rangle_{X^*,X} \text{ for all } h \in X.$$

Proof. We prove that the limit $\lim_{t\downarrow 0} \frac{g(x+th)-g(x)}{t}$ exists and equals

$$\max_{z\in M(x)} \langle f_x(x,z), h\rangle_{X^*,X}.$$

Let $\tau > 0$ be such that $x + [0,\tau]h \subset W$. Now, let $\{t_k\}_{k\in\mathbb{N}} \subset (0,\tau]$ be any sequence with $t_k \downarrow 0$. We estimate the terms $\limsup_k \frac{g(x+t_kh)-g(x)}{t_k}$ and $\liminf_k \frac{g(x+t_kh)-g(x)}{t_k}$. Let $x_k := x + t_kh$ and $z_k \in M(x_k)$, $k \in \mathbb{N}$. Further, let $z \in M(x)$ be arbitrary. It holds

$$g(x_k) - g(x) = f(x_k, z_k) - f(x,z) \ge f(x_k, z) - f(x,z).$$

By the mean value theorem there exists some $\lambda \in [0,1]$ such that

$$f(x_k, z) - f(x,z) = \langle f_x(x + (1-\lambda)t_kh, z), t_kh\rangle_{X^*,X}.$$

Thus, we obtain the estimate

$$\liminf_k \frac{g(x_k)-g(x)}{t_k} \ge \liminf_k \langle f_x(x + (1-\lambda)t_kh, z), h\rangle_{X^*,X} = \langle f_x(x,z), h\rangle_{X^*,X},$$

by the sequential τ_X-, and thus, by (D1), also $\|\cdot\|_X$-continuity of $f_x(\cdot, z)$. Since this inequality holds for all $z \in M(x)$, it yields

$$\liminf_k \frac{g(x_k)-g(x)}{t_k} \ge \max_{z\in M(x)} \langle f_x(x,z), h\rangle_{X^*,X}.$$

For the other direction, we bound $\limsup_k \frac{g(x_k)-g(x)}{t_k}$ from above. To this end, we assume that there exists some $\varepsilon > 0$ such that it holds

$$\limsup_k \frac{g(x_k)-g(x)}{t_k} \ge \max_{z\in M(x)} \langle f_x(x,z), h\rangle_{X^*,X} + \varepsilon.$$

By the sequential compactness of Y with respect to τ_Z, there exist a subsequence $\{z_{k_l}\}_{l\in\mathbb{N}} \subset \{z_k\}_{k\in\mathbb{N}} \subset Y$ and $z_0 \in Z$ such that $z_{k_l} \to z_0$ with respect to τ_Z. From $z_{k_l} \in M(x_{k_l})$, $l \in \mathbb{N}$, we obtain $z_0 \in M(x)$ by the sequential closedness of $M : W \rightrightarrows Z$.

Let $z \in M(x)$ be arbitrary. We have

$$g(x_{k_l}) - g(x) = f(x_{k_l}, z_{k_l}) - f(x,z) \le f(x_{k_l}, z_{k_l}) - f(x, z_{k_l}).$$

Further, by the mean value theorem, there exists some $\lambda \in [0, 1]$ such that

$$f(x_{k_l}, z_{k_l}) - f(x, z_{k_l}) = \langle f_x(x + (1-\lambda)t_{k_l}h, z_{k_l}), t_{k_l}h\rangle_{X^*,X},$$

and using the sequential continuity of f_x with respect to $\|\cdot\|_X \times \tau_Z$ (cf. (D1) and (D5)), we get

$$\limsup_l \frac{g(x_{k_l}) - g(x)}{t_{k_l}} \leq \limsup_l \langle f_x(x + (1-\lambda)t_{k_l}h, z_{k_l}), h\rangle_{X^*,X} = \langle f_x(x, z_0), h\rangle_{X^*,X}.$$

We arrive at the contradiction $z_0 \in M(x)$ and

$$\max_{z\in M(x)} \langle f_x(x, z), h\rangle_{X^*,X} + \varepsilon \leq \limsup_l \frac{g(x_{k_l}) - g(x)}{t_{k_l}} \leq \langle f_x(x, z_0), h\rangle_{X^*,X}.$$

Taking both estimates together, we have proved

$$\limsup_k \frac{g(x_k) - g(x)}{t_k} \leq \max_{z\in M(x)} \langle f_x(x, z), h\rangle_{X^*,X} \leq \liminf_k \frac{g(x_k) - g(x)}{t_k}.$$

Since $t_k \downarrow 0$ was arbitrary, this shows

$$g'(x, h) = \lim_{t\downarrow 0} \frac{g(x + th) - g(x)}{t} = \max_{z\in M(x)} \langle f_x(x, z), h\rangle_{X^*,X}. \qquad \square$$

Proposition A.8. *Under Assumption A.1, if, at $x \in W$, the set $M(x) = \{m(x)\}$ is a singleton, then $g : W \to \mathbb{R}$ is Gâteaux differentiable at x and there holds*

$$g'(x, h) = \langle f_x(x, m(x)), h\rangle_{X^*,X}. \tag{A1}$$

Proof. By Proposition A.7, g is directionally differentiable at x with $g'(x, h)$ as in (A1). We see that $g'(x, \cdot)$ is linear and bounded, thus g is Gâteaux differentiable at x. $\square$

Proposition A.9. *Let Assumption A.1, (D1)–(D4) hold. If the correspondence $M : W \rightrightarrows Z$ is single-valued at $x \in W$, i.e., $M(x) = \{m(x)\}$ with $m(x) \in Y$, then M is sequentially τ_X to τ_Z continuous at x in the sense that, for any $\{x_k\}_{k\in\mathbb{N}} \subset W$ and $z_k \in M(x_k)$ such that $x_k \to x$ with respect to τ_X, there holds $z_k \to m(x)$ with respect to τ_Z.*

Proof. Assume that the assertion is wrong. Then there exist a τ_Z-neighborhood U_Z of $m(x)$ and sequences $\{x_k\}_{k\in\mathbb{N}} \subset W$, $z_k \in M(x_k)$ such that $x_k \to x$ with respect to τ_X and $z_k \notin U_Z$ for all $k \in \mathbb{N}$.

Since $Y \subset Z$ is sequentially compact with respect to τ_Z, we can select a subsequence $\{z_{k_l}\}_{l\in\mathbb{N}} \subset \{z_k\}_{k\in\mathbb{N}}$ such that $z_{k_l} \to z$ with respect to τ_Z. Then, by sequential closedness of M, there holds $z \in M(x)$, hence $z = m(x)$. Since U_Z is an open τ_Z-neighborhood of $z = m(x)$ and $z_{k_l} \to_{\tau_Z} z$, we get a contradiction to $z_{k_l} \notin U_Z$ for all l. Hence, M is sequentially τ_X to τ_Z continuous at x. $\square$

Proposition A.10 (Danskin)**.** *Under Assumption A.1, if the correspondence $M : W \rightrightarrows Z$ is single-valued on an $\|\cdot\|$-open set $U_X \subset W$, then $g : U_X \to \mathbb{R}$ is Fréchet differentiable and the derivative $g' : U_X \to X^*$ is continuous from (U_X, τ_X) to X^*, hence also continuous from $(U_X, \|\cdot\|_X)$ to X^*.*

Proof. Let $M(\cdot) = \{m(\cdot)\}$ on U_X. By Proposition A.7 and Proposition A.8 we conclude that $g : W \to \mathbb{R}$ is Gâteaux differentiable on U_X with derivative $g'(x) = f_x(x, m(x))$ for all $x \in U_X$. Further, for any $x \in U_X$, $M : W \rightrightarrows Z$ is sequentially τ_X to τ_Z continuous at x by Proposition A.9 and hence, $m : U_X \to Z$ is sequentially τ_X to τ_Z continuous. Therefore, $g'(x) = f_x(\cdot, m(\cdot)) : U_X \to X^*$ is continuous from (U_X, τ_X) to X^* and thus also from $(U_X, \|\cdot\|_X)$ to X^*. Hence, g is continuously Gâteaux differentiable on $(U_X, \|\cdot\|_X)$, which is equivalent to continuous Fréchet differentiability on $(U_X, \|\cdot\|_X)$. Since $g'(x) = f_x(\cdot, m(\cdot))$ is also continuous from (U_X, τ_X) to X^*, the proof is complete. □

Acknowledgements

This work was supported by the German Research Foundation (DFG) under grant number UL 348/8-1 within the priority program "Non-smooth and Complementarity-based Distributed Parameter Systems: Simulation and Hierarchical Optimization" (SPP 1962).

References

[1] K.J. Arrow and G. Debreu, *Existence of an equilibrium for a competitive economy*, Econometrica (1954), pp. 265–290.

[2] C. Berge, *Topological Spaces: Including a Treatment of Multi-Valued Functions, Vector Spaces, and Convexity*, Courier Corporation, 1963.

[3] J.F. Bonnans and A. Shapiro, *Perturbation Analysis of Optimization Problems*, Springer, New York, 2013.

[4] A. Bressan and K. Han, *Nash equilibria for a model of traffic flow with several groups of drivers*, ESAIM Control Optim. Calc. Var. 18 (2012), pp. 969–986.

[5] A. Bressan and K. Han, *Existence of optima and equilibria for traffic flow on networks*, Netw. Heterog. Media 8 (2013), pp. 627–648.

[6] A. Bressan and K.T. Nguyen, *Optima and equilibria for traffic flow on networks with backward propagating queues.*, Netw. Heterog. Media 10 (2015), pp. 717–748.

[7] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, New York, 2010.

[8] J.M. Danskin, *The theory of max-min, with applications*, SIAM J. Appl. Math. 14 (1966), pp. 641–664.

[9] P. Di Barba, *Multiobjective Shape Design in Electricity and Magnetism*, Springer, New York, 2009.

[10] A. Dreves, C. Kanzow, and O. Stein, *Nonsmooth optimization reformulations of player convex generalized Nash equilibrium problems*, J. Global Optim. 53 (2012), pp. 587–614.

[11] A. Dreves, A. von Heusinger, C. Kanzow, and M. Fukushima, *A globalized Newton method for the computation of normalized Nash equilibria*, J. Global Optim. 56 (2013), pp. 327–340.

[12] F. Facchinei, A. Fischer, and V. Piccialli, *On generalized Nash games and variational inequalities*, Oper. Res. Lett. 35 (2007), pp. 159–164.

[13] F. Facchinei and C. Kanzow, *Generalized Nash equilibrium problems*, Ann. Oper. Res. 175 (2010), pp. 177–211.

[14] F. Facchinei and J.S. Pang, *Nash equilibria: The variational approach*, in *Convex Optimization in Signal Processing and Communications*, D.P. Palomar and Y.C. Eldar, eds., Cambridge Univ. Press, Cambridge, 2010, pp. 443–493.

[15] A. Fischer, M. Herrich, and K. Schönefeld, *Generalized Nash equilibrium problems – recent advances and challenges*, Pesquisa Operacional 34 (2014), pp. 521–558.

[16] I.L. Glicksberg, *A further generalization of the Kakutani fixed point theorem, with application to Nash equilibrium points*, Proc. Amer. Math. Soc. 3 (1952), pp. 170–174.

[17] P.T. Harker, *Generalized Nash games and quasi-variational inequalities*, European J. Oper. Res. 54 (1991), pp. 81–94.

[18] M. Hintermüller, T. Surowiec, and A. Kämmmler, *Generalized Nash equilibrium problems in Banach spaces: Theory, Nikaido–Isoda-based path-following methods, and applications*, SIAM J. Optim. 25 (2015), pp. 1826–1856.

[19] M. Hintermüller and T. Surowiec, *A PDE-constrained generalized Nash equilibrium problem with pointwise control and state constraints*, Pacific J. Math. 9 (2013), pp. 251–273.

[20] C. Kanzow, V. Karl, D. Steck, and D. Wachsmuth, *The multiplier-penalty method for generalized Nash equilibrium problems in Banach spaces*, SIAM J. Optim. 29 (2019), pp. 767–793.

[21] C. Kanzow and D. Steck, *Quasi-variational inequalities in Banach spaces: Theory and augmented Lagrangian methods*, SIAM J. Optim. 29 (2019), pp. 3174–3200.

[22] B. Luderer, L. Minchenko, and T. Satsura, *Multivalued Analysis and Nonlinear Programming Problems with Perturbations*, Vol. 66, Springer, New York, 2013.

[23] J.F. Nash, *Equilibrium points in n-person games*, Proc. Nat. Acad. Sci. U.S.A. 36 (1950), pp. 48–49.

[24] R. Nessah and G. Tian, *On the existence of Nash equilibrium in discontinuous games*, Econom. Theory 61 (2016), pp. 515–540.

[25] H. Nikaidô, K. Isoda, *et al.*, *Note on non-cooperative convex games*, Pacific J. Math. 5 (1955), pp. 807–815.

[26] J.S. Pang and M. Fukushima, *Quasi-variational inequalities, generalized Nash equilibria, and multi-leader-follower games*, Comput. Manag. Sci. 2 (2005), pp. 21–56.

[27] J.B. Rosen, *Existence and uniqueness of equilibrium points for concave n-person games*, Econometrica (1965), pp. 520–534.

[28] Z. Tang, J.A. Désidéri, and J. Périaux, *Multicriterion aerodynamic shape design optimization and inverse problems using control theory and Nash games*, J. Optim. Theory Appl. 135 (2007), pp. 599–622.

[29] A. Von Heusinger and C. Kanzow, *SC^1 optimization reformulations of the generalized Nash equilibrium problem*, Optim. Methods Softw. 23 (2008), pp. 953–973.

[30] A. von Heusinger and C. Kanzow, *Optimization reformulations of the generalized Nash equilibrium problem using Nikaido–Isoda-type functions*, Comput. Optim. Appl. 43 (2009), pp. 353–377.

[31] A. Von Heusinger and C. Kanzow, *Relaxation methods for generalized Nash equilibrium problems with inexact line search*, J. Optim. Theory App. 143 (2009), pp. 159–183.

[32] R. Williams, *Sufficient conditions for Nash equilibria in n-person games over reflexive Banach spaces*, J. Optim. Theory Appl. 30 (1980), pp. 383–394.

[33] J. Yu, *On Nash equilibria in n-person games over reflexive Banach spaces*, J. Optim. Theory Appl. 73 (1992), pp. 211–214.

[34] J. Zowe and S. Kurcyusz, *Regularity and stability for the mathematical programming problem in Banach spaces*, Appl. Math. Optim. 5 (1979), pp. 49–62.