Deutsche Forschungsgemeinschaft
Priority Programme 1962

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SPP 1962
Non-smooth and Complementarity-based Distributed Parameter Systems: Simulation and Hierarchical Optimization

Preprint Number SPP1962-175

received on September 6, 2021

Edited by
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New stationarity conditions between strong and M-stationarity for mathematical programs with complementarity constraints

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September 3, 2021
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Abstract

We introduce new first-order necessary conditions for mathematical programs with complementarity constraints (MPCCs), which lie between strong and M-stationarity and have a relatively simple description. We show that they hold for local minimizers under the rather weak constraint qualification MPCC-GCQ. As a generalization, we also get a class of stationarity conditions that lie between strong and C-stationarity and show that they also hold for local minimizers under MPCC-GCQ. We also present similar results for mathematical programs with vanishing constraints (MPVCs), and a very simple and elementary proof of M-stationarity for local minimizers of MPVCs.

Keywords: Mathematical program with complementarity constraints, Mathematical program with vanishing constraints, Necessary optimality conditions, M-stationarity, Guignard constraint qualification

MSC (2020): 90C33, 90C30

1 Introduction

We consider mathematical programs with complementarity constraints, or MPCCs for short, which are nonlinear optimization problems of the form

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \quad & f(x) \\ \text{s.t.} \quad & g(x) \le 0, \quad h(x) = 0, \\ & G(x) \ge 0, \quad H(x) \ge 0, \quad G(x)^\top H(x) = 0. \end{aligned} \tag{MPCC}$$

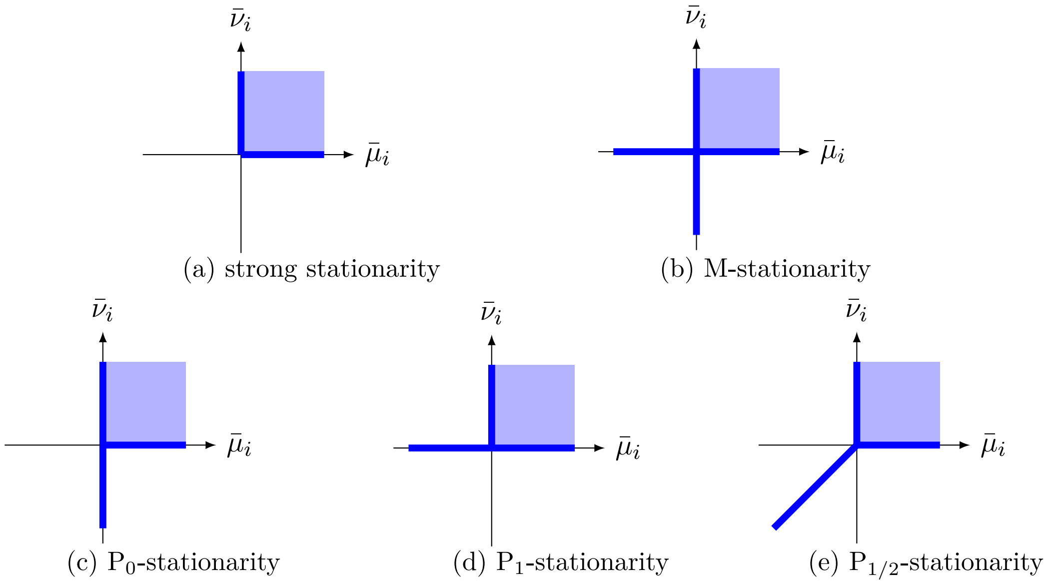


Figure 1: geometric illustration of M-/S-stationarity and the new stationarity conditions for MPCCs, with $i \in \{1, \ldots, p\}$ such that $G_i(\bar x) = H_i(\bar x) = 0$, where $\bar\mu_i$, $\bar\nu_i$ are the Lagrange multipliers that correspond to $G_i(\bar x) \geq 0$, $H_i(\bar x) \geq 0$

Here, $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^l$, $h : \mathbb{R}^n \to \mathbb{R}^m$, $G, H : \mathbb{R}^n \to \mathbb{R}^p$ are differentiable functions. For this class of problems, problem-tailored first-order necessary conditions have been studied extensively in the literature. One such stationarity condition is called strong stationarity (see Definition 2.3 (e), illustrated in Figure 1(a)), which is satisfied for local minimizers under the relatively strong constraint qualification MPCC-LICQ. However, in [Scheel, Scholtes, 2000, Example 3] an example is given where the data is linear but strong stationarity does not hold for the local minimizer.

Another well-known stationarity condition is M-stationarity (see Definition 2.3 (d), illustrated in Figure 1(b)), which is satisfied for local minimizers under the relatively weak constraint qualification MPCC-GCQ (see Definition 2.2 (a)).

The main contribution of this paper is a stationarity condition, which we call *P-stationarity with respect to* α or P_α*-stationarity*, where $\alpha \in \{0, 1\}^p$ is an (a-priory given) parameter vector. This stationarity condition is defined in Definition 2.5 (b) and the special cases $\alpha = 0$ and $\alpha = 1$ are illustrated in Figures 1(c) and 1(d). To the best of our knowledge, this stationarity condition is new. In Theorem 3.4 we are able to show that P_α-stationarity holds for local minimizers of (MPCC) under MPCC-GCQ. The letter "P" stands for the Poincaré–Miranda theorem, which is an important ingredient in the proof. As can be seen in Figure 1, this stationarity condition lies strictly between strong and M-stationarity. Due to the parameter $\alpha \in \{0, 1\}^p$, we actually get 2^p new stationarity conditions.

There are other stationarity conditions that are between strong and M-stationarity. One of them is linearized B-stationarity, defined in Definition 2.3 (f). Others are extended

M-stationarity, strong M-stationarity, $\mathcal{Q}_M$-stationarity, and linearized M-stationarity, see [Gfrerer, 2014; Benko, Gfrerer, 2017; Gfrerer, 2018]. However, we are not aware of a stationarity condition in the literature which has a nice geometrical illustration as those in Figure 1 and which lies between strong and M-stationarity.

One important idea for the proof of P_α-stationarity was taken from [Harder, 2020]: Under MPCC-GCQ, local minimizers satisfy a system of so-called A_α-stationarity, see Proposition 3.1. Then, one can consider convex combinations of the corresponding multipliers to obtain better stationarity conditions. While in [Harder, 2020] this led to a new elementary proof of M-stationarity, in the present paper we utilize the Poincaré–Miranda theorem to establish the stronger P_α-stationarity for all $\alpha \in \{0,1\}^p$. The Poincaré–Miranda theorem is mentioned in Theorem 3.2 and is a generalization of the intermediate value theorem to higher dimensions. It is equivalent to the Brouwer fixed-point theorem.

The idea to combine various A_α-stationarity systems was also used for the concept of $\mathcal{Q}$-stationarity, see [Benko, Gfrerer, 2016; Benko, Gfrerer, 2017]. However, this concept does not lead directly to P_α-stationarity.

Our method based on the Poincaré–Miranda theorem allows for some generalization. We define P_d-stationarity also for the case where $d \in [0,1]^p$ is a vector in Definition 2.5 (c). This stationarity condition is illustrated in Figure 1(e) for the constant vector $d = 1/2$ and lies between strong stationarity and the so-called C-stationarity (see Definition 2.3 (c)). We are able to generalize Theorem 3.4 and show that (for all $d \in [0,1]^p$) P_d-stationarity holds for local minimizers under MPCC-GCQ in Theorem 4.2 (b). In Theorem 4.2 (a) we state that an even more general class of stationarity condition holds under MPCC-GCQ. This allows for some unusual stationarity conditions, such as the one depicted in Figure 5.

We can also transfer the results from MPCCs to mathematical programs with vanishing constraints, or MPVCs for short. These are nonlinear optimization problems of the form

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \quad & f(x) \\ \text{s.t.} \quad & g(x) \le 0, \qquad h(x) = 0, \\ & H_i(x) \ge 0, \quad G_i(x)\, H_i(x) \le 0 \qquad \forall i \in \{1, \ldots, p\}, \end{aligned} \tag{MPVC}$$

where f, g, h, G, H are the same type of functions as for MPCCs. This problem has also been studied frequently in the literature and was introduced in [Achtziger, Kanzow, 2007]. Due to some similarities in the stationarity systems, it is relatively easy to apply the results from MPCCs also to MPVCs. This is done in Section 6, where we show that the MPVC-version of P_d-stationarity holds for all $d \in [0,1]^p$ for local minimizers under the weak constraint qualification MPVC-GCQ (see Definition 2.2 (b)). This also covers the case of P_α-stationarity for all $\alpha \in \{0,1\}^p$. Again, these stationarity conditions for MPVCs are new to the best of our knowledge.

In Section 5, we also provide a very short and elementary proof of M-stationarity for local minimizers of MPVCs under MPVC-GCQ. Although this is an already well-known stationarity condition, we decided to include this proof as it is very short and the required tools are available anyways. The proof is even simpler than the recent elementary proof of M-stationarity for MPCCs in [Harder, 2020] and relies on the observation that A_0-stationarity (defined in Definition 2.6 (a)) trivially implies M-stationarity in the case of MPVCs. The proof does not rely on the Poincaré–Miranda theorem or other complicated theory and, as far as we know, is significantly shorter than existing proofs. We even use a constraint qualification which is more general than MPVC-GCQ in (5.1), as demonstrated in Example 5.2.

As simple corollaries of our main results, we also provide some results for the relations among the newly introduced stationarity conditions, see Corollaries 2.7, 4.3 and 6.3.

2 Definitions

Let us make some definitions that relate to stationarity conditions and constraint qualifications for (MPCC) and (MPVC). For $x \in \mathbb{R}^n$ and $\alpha \in \{0,1\}^p$ we define the index sets

$$\begin{aligned}
I^l &:= \{1, \ldots, l\}, \quad I^m := \{1, \ldots, m\}, \quad I^p := \{1, \ldots, p\},\\
I^g(x) &:= \{i \in I^l \mid g_i(x) = 0\},\\
I^{+0}(x) &:= \{i \in I^p \mid G_i(x) > 0 \wedge H_i(x) = 0\},\\
I^{0+}(x) &:= \{i \in I^p \mid G_i(x) = 0 \wedge H_i(x) > 0\},\\
I^{-0}(x) &:= \{i \in I^p \mid G_i(x) < 0 \wedge H_i(x) = 0\},\\
I^{-+}(x) &:= \{i \in I^p \mid G_i(x) < 0 \wedge H_i(x) > 0\},\\
I^{00}(x) &:= \{i \in I^p \mid G_i(x) = 0 \wedge H_i(x) = 0\},\\
I^{00}_{\alpha=0}(x) &:= \{i \in I^p \mid G_i(x) = 0 \wedge H_i(x) = 0 \wedge \alpha_i = 0\},\\
I^{00}_{\alpha=1}(x) &:= \{i \in I^p \mid G_i(x) = 0 \wedge H_i(x) = 0 \wedge \alpha_i = 1\}.
\end{aligned}$$

Note that if x is a feasible point of (MPCC), then $I^{+0}(x)$, $I^{0+}(x)$, $I^{00}(x)$ form a partition of I^p. Likewise, if x is a feasible point of (MPVC), then $I^{+0}(x)$, $I^{-0}(x)$, $I^{-+}(x)$, $I^{0+}(x)$, $I^{00}(x)$ form a partition of I^p. In any case, $I^{00}_{\alpha=0}(x)$ and $I^{00}_{\alpha=1}(x)$ form a partition of $I^{00}(x)$. Let us mention that in some papers on MPVCs the notation is reversed, i.e. they use $I^{+-}(x)$ instead of $I^{-+}(x)$, but for the sake of consistency we chose to use the same notation for MPCCs and MPVCs.

2.1 Constraint Qualifications

In preparation for the definition of MPCC-GCQ and MPVC-GCQ we introduce some cones.

Definition 2.1. (a) We define the *tangent cone* at a point $\bar{x} \in F$ to a closed set $F \subset \mathbb{R}^n$ via

$$\mathcal{T}_F(\bar{x}) := \left\{ d \in \mathbb{R}^n \;\middle|\; \begin{array}{r} \exists \{x_k\}_{k\in\mathbb{N}} \subset F,\ \exists \{t_k\}_{k\in\mathbb{N}} \subset (0,\infty): \\ x_k \to \bar{x},\ t_k \downarrow 0,\ t_k^{-1}(x_k - \bar{x}) \to d \end{array} \right\}.$$

If F is the feasible set of (MPCC), then we denote the tangent cone by $\mathcal{T}_{\mathrm{MPCC}}(\bar{x})$. Likewise, if F is the feasible set of (MPVC), then we denote the tangent cone by $\mathcal{T}_{\mathrm{MPVC}}(\bar{x})$.

(b) We define the *MPCC-linearized tangent cone* $\mathcal{T}^{\mathrm{lin}}_{\mathrm{MPCC}}(\bar{x}) \subset \mathbb{R}^n$ at $\bar{x} \in \mathbb{R}^n$ via

$$\mathcal{T}^{\mathrm{lin}}_{\mathrm{MPCC}}(\bar{x}) := \left\{ d \in \mathbb{R}^n \;\middle|\; \begin{array}{rl} \nabla g_i(\bar{x})^\top d \le 0 & \forall i \in I^g(\bar{x}), \\ \nabla h_i(\bar{x})^\top d = 0 & \forall i \in I^m, \\ \nabla G_i(\bar{x})^\top d = 0 & \forall i \in I^{0+}(\bar{x}), \\ \nabla H_i(\bar{x})^\top d = 0 & \forall i \in I^{+0}(\bar{x}), \\ \nabla G_i(\bar{x})^\top d \ge 0 & \forall i \in I^{00}(\bar{x}), \\ \nabla H_i(\bar{x})^\top d \ge 0 & \forall i \in I^{00}(\bar{x}), \\ (\nabla G_i(\bar{x})^\top d)(\nabla H_i(\bar{x})^\top d) = 0 & \forall i \in I^{00}(\bar{x}) \end{array} \right\}.$$

(c) We define the *MPVC-linearized tangent cone* $\mathcal{T}^{\mathrm{lin}}_{\mathrm{MPVC}}(\bar{x}) \subset \mathbb{R}^n$ at $\bar{x} \in \mathbb{R}^n$ via

$$\mathcal{T}^{\mathrm{lin}}_{\mathrm{MPVC}}(\bar{x}) := \left\{ d \in \mathbb{R}^n \;\middle|\; \begin{array}{rl} \nabla g_i(\bar{x})^\top d \le 0 & \forall i \in I^g(\bar{x}), \\ \nabla h_i(\bar{x})^\top d = 0 & \forall i \in I^m, \\ \nabla G_i(\bar{x})^\top d \le 0 & \forall i \in I^{0+}(\bar{x}), \\ \nabla H_i(\bar{x})^\top d = 0 & \forall i \in I^{+0}(\bar{x}), \\ \nabla H_i(\bar{x})^\top d \ge 0 & \forall i \in I^{00}(\bar{x}) \cup I^{-0}(\bar{x}), \\ (\nabla G_i(\bar{x})^\top d)(\nabla H_i(\bar{x})^\top d) \le 0 & \forall i \in I^{00}(\bar{x}) \end{array} \right\}.$$

Note that in many instances of (MPCC) or (MPVC), these cones are nonconvex sets. Recall that the polar cone C° of a set $C \subset \mathbb{R}^n$ is defined via

$$C^\circ := \{ d \in \mathbb{R}^n \mid d^\top y \le 0 \quad \forall y \in C \}.$$

Now we are ready to give the definition of MPCC-GCQ and MPVC-GCQ.

Definition 2.2. (a) Let $\bar{x} \in \mathbb{R}^n$ be a feasible point of (MPCC). We say that $\bar{x}$ satisfies the *MPCC-tailored Guignard constraint qualification*, or *MPCC-GCQ*, if

$$\mathcal{T}_{\mathrm{MPCC}}(\bar{x})^\circ = \mathcal{T}^{\mathrm{lin}}_{\mathrm{MPCC}}(\bar{x})^\circ$$

holds. If $\mathcal{T}_{\mathrm{MPCC}}(\bar{x}) = \mathcal{T}^{\mathrm{lin}}_{\mathrm{MPCC}}(\bar{x})$ holds then we say that $\bar{x}$ satisfies *MPCC-ACQ*.

(b) Let $\bar{x} \in \mathbb{R}^n$ be a feasible point of (MPVC). We say that $\bar{x}$ satisfies the *MPVC-tailored Guignard constraint qualification*, or *MPVC-GCQ*, if

$$\mathcal{T}_{\text{MPVC}}(\bar{x})^\circ = \mathcal{T}^{\text{lin}}_{\text{MPVC}}(\bar{x})^\circ$$

holds. If $\mathcal{T}_{\text{MPVC}}(\bar{x}) = \mathcal{T}^{\text{lin}}_{\text{MPVC}}(\bar{x})$ holds then we say that $\bar{x}$ satisfies *MPVC-ACQ*.

The definition for MPCC-GCQ can also be found in [Flegel, Kanzow, Outrata, 2007, (41)], where it is called MPEC-GCQ. The definition of MPVC-GCQ and MPVC-ACQ can also be found in [Hoheisel, Kanzow, 2008, Definition 2.8].

Clearly, MPCC-ACQ implies MPCC-GCQ. We mention that there are also other stronger constraint qualifications (such as MPCC-MFCQ if g, h, G, H are continuously differentiable) which imply MPCC-ACQ or MPCC-GCQ and are easier to verify, see e.g. [Ye, 2005, Theorem 3.2]. In particular, we emphasize that MPCC-GCQ and MPCC-ACQ are satisfied at every feasible point of (MPCC) if the functions g, h, G, H are affine. The same statements are true for MPVCs.

2.2 Stationarity conditions

We continue with the definition of tailored stationarity systems.

Definition 2.3. Let $\bar{x} \in \mathbb{R}^n$ be a feasible point of (MPCC).

(a) We call $\bar{x}$ a *weakly stationary* or *W-stationary* point of (MPCC) if there exist multipliers $\bar{\lambda} \in \mathbb{R}^l$, $\bar{\eta} \in \mathbb{R}^m$, $\bar{\mu}, \bar{\nu} \in \mathbb{R}^p$ with

$$\begin{aligned} \nabla f(\bar{x}) + \sum_{i \in I^l} \bar{\lambda}_i \nabla g_i(\bar{x}) + \sum_{i \in I^m} \bar{\eta}_i \nabla h_i(\bar{x}) - \sum_{i \in I^p} \bar{\mu}_i \nabla G_i(\bar{x}) - \sum_{i \in I^p} \bar{\nu}_i \nabla H_i(\bar{x}) = 0,& \\ \forall i \in I^g(\bar{x}): \quad \bar{\lambda}_i \geq 0,& \\ \forall i \in I^l \setminus I^g(\bar{x}): \quad \bar{\lambda}_i = 0,& \\ \forall i \in I^{+0}(\bar{x}): \quad \bar{\mu}_i = 0,& \\ \forall i \in I^{0+}(\bar{x}): \quad \bar{\nu}_i = 0.& \end{aligned}$$

(b) We call $\bar{x}$ an *A-stationary* point if it is weakly stationary and the multipliers $\bar{\mu}, \bar{\nu}$ satisfy the additional condition $\bar{\mu}_i \geq 0 \vee \bar{\nu}_i \geq 0$ for all $i \in I^{00}(\bar{x})$.

(c) We call $\bar{x}$ a *C-stationary* point if it is weakly stationary and the multipliers $\bar{\mu}, \bar{\nu}$ satisfy the additional condition $\bar{\mu}_i \bar{\nu}_i \geq 0$ for all $i \in I^{00}(\bar{x})$.

(d) We call $\bar{x}$ an *M-stationary* point if it is weakly stationary and the multipliers $\bar{\mu}, \bar{\nu}$ satisfy the additional condition $(\bar{\mu}_i \geq 0 \wedge \bar{\nu}_i \geq 0) \vee \bar{\mu}\bar{\nu} = 0$ for all $i \in I^{00}(\bar{x})$.

(e) We call $\bar{x}$ a *strongly stationary* or *S-stationary* point if it is weakly stationary and the multipliers $\bar{\mu}, \bar{\nu}$ satisfy the additional condition $\bar{\mu}_i \geq 0 \wedge \bar{\nu}_i \geq 0$ for all $i \in I^{00}(\bar{x})$.

(f) We call $\bar{x}$ a *linearized B-stationary* point if $-\nabla f(\bar{x}) \in \mathcal{T}^{\text{lin}}_{\text{MPCC}}(\bar{x})^\circ$ holds.

Parts (a) to (e) of this definition can also be found in [Ye, 2005, Definitions 2.3–2.7]. The letters "A", "C" and "M" stand for "alternative", "Clarke" and "Mordukhovich", respectively. The definition of linearized B-stationarity appears in [Flegel, Kanzow, 2005, (25)]. It is often only referred to as B-stationarity, see, e.g., [Scheel, Scholtes, 2000, Section 2.1]. However, since the condition $-\nabla f(\bar{x}) \in \mathcal{T}_{\text{MPCC}}(\bar{x})^\circ$ (which appears in Lemma 2.8 (a)) is sometimes also called B-stationarity, we prefer the name "linearized B-stationarity" for $-\nabla f(\bar{x}) \in \mathcal{T}^{\text{lin}}_{\text{MPCC}}(\bar{x})^\circ$, to avoid confusion.

Let us define analogous stationarity conditions for MPVCs.

Definition 2.4. Let $\bar{x} \in \mathbb{R}^n$ be a feasible point of (MPVC).

(a) We call $\bar{x}$ a *weakly stationary* or *W-stationary* point of (MPVC) if there exist multipliers $\bar{\lambda} \in \mathbb{R}^l$, $\bar{\eta} \in \mathbb{R}^m$, $\bar{\mu}, \bar{\nu} \in \mathbb{R}^p$ with

$$\begin{aligned}
\nabla f(\bar{x}) + \sum_{i \in I^l} \bar{\lambda}_i \nabla g_i(\bar{x}) + \sum_{i \in I^m} \bar{\eta}_i \nabla h_i(\bar{x}) + \sum_{i \in I^p} \bar{\mu}_i \nabla G_i(\bar{x}) - \sum_{i \in I^p} \bar{\nu}_i \nabla H_i(\bar{x}) &= 0, \\
\forall i \in I^g(\bar{x}): \quad \bar{\lambda}_i &\geq 0, \\
\forall i \in I^l \setminus I^g(\bar{x}): \quad \bar{\lambda}_i &= 0, \\
\forall i \in I^{+0}(\bar{x}) \cup I^{-0}(\bar{x}) \cup I^{-+}(\bar{x}): \quad \bar{\mu}_i &= 0, \\
\forall i \in I^{0+}(\bar{x}) \cup I^{00}(\bar{x}): \quad \bar{\mu}_i &\geq 0, \\
\forall i \in I^{0+}(\bar{x}) \cup I^{-+}(\bar{x}): \quad \bar{\nu}_i &= 0, \\
\forall i \in I^{-0}(\bar{x}): \quad \bar{\nu}_i &\geq 0.
\end{aligned}$$

(b) We call $\bar{x}$ an *A-stationary* point if it is weakly stationary and the multipliers $\bar{\mu}, \bar{\nu}$ satisfy the additional condition $\bar{\mu}_i = 0 \vee \bar{\nu}_i \geq 0$ for all $i \in I^{00}(\bar{x})$.

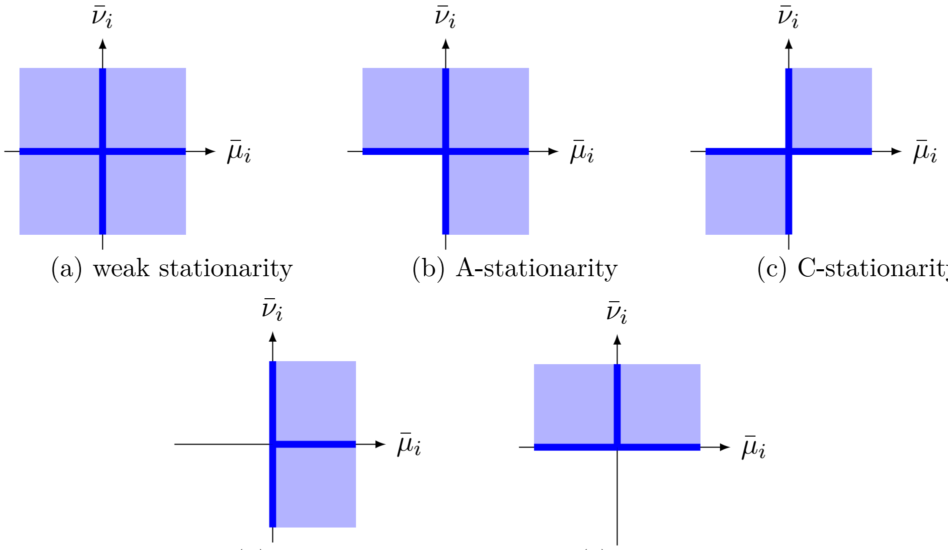
(c) We call $\bar{x}$ a *T-stationary* or *C-stationary* point if it is weakly stationary and the multipliers $\bar{\mu}, \bar{\nu}$ satisfy the additional condition $\bar{\mu}_i \bar{\nu}_i \leq 0$ for all $i \in I^{00}(\bar{x})$.

(d) We call $\bar{x}$ an *M-stationary* point if it is weakly stationary and the multipliers $\bar{\mu}, \bar{\nu}$ satisfy the additional condition $\bar{\mu}_i \bar{\nu}_i = 0$ for all $i \in I^{00}(\bar{x})$.

(e) We call $\bar{x}$ a *strongly stationary* or *S-stationary* point if it is weakly stationary and the multipliers $\bar{\mu}, \bar{\nu}$ satisfy the additional condition $\bar{\mu}_i = 0 \wedge \bar{\nu}_i \geq 0$ for all $i \in I^{00}(\bar{x})$.

(f) We call $\bar{x}$ a *linearized B-stationary* point if $-\nabla f(\bar{x}) \in \mathcal{T}^{\text{lin}}_{\text{MPVC}}(\bar{x})^\circ$ holds.

The definition of W,T,M,S-stationarity can be found in [Hoheisel, Kanzow, Schwartz, 2012, Definition 2.3]. As T-stationarity is an analogue to C-stationarity for MPCCs, we assign C-stationarity as a synonymous name for it. We are not aware of a previous definition of



(a) weak stationarity (b) A-stationarity (c) C-stationarity (d) A_0-stationarity (e) A_1-stationarity

Figure 2: geometric illustration of W-, A-, C-, A_α-stationarity for MPCCs, $i \in I^{00}(\bar{x})$

A-stationarity in the literature, but we included it as an analogy to A-stationarity for MPCCs.

Let us also introduce some new stationarity conditions for (MPCC).

Definition 2.5. Let $\bar{x} \in \mathbb{R}^n$ be a feasible point of (MPCC) and $\alpha \in \{0,1\}^p$, $d \in [0,1]^p$ be given.

(a) We call $\bar{x}$ *A_α-stationary* if it is weakly stationary and the multipliers $\bar{\mu}$, $\bar{\nu}$ satisfy the additional condition

$$\begin{aligned} \bar{\mu}_i &\geq 0 \qquad \forall i \in I^{00}_{\alpha=0}(\bar{x}), \\ \bar{\nu}_i &\geq 0 \qquad \forall i \in I^{00}_{\alpha=1}(\bar{x}). \end{aligned}$$

(b) We call $\bar{x}$ *P-stationary with respect to α* or *P_α-stationary* if it is weakly stationary and the multipliers $\bar{\mu}$, $\bar{\nu}$ satisfy the additional condition

$$\begin{aligned} (\bar{\mu}_i \geq 0 \wedge \bar{\nu}_i \geq 0) \vee \bar{\mu}_i = 0 \qquad \forall i \in I^{00}_{\alpha=0}(\bar{x}), \\ (\bar{\mu}_i \geq 0 \wedge \bar{\nu}_i \geq 0) \vee \bar{\nu}_i = 0 \qquad \forall i \in I^{00}_{\alpha=1}(\bar{x}). \end{aligned}$$

(c) We call $\bar{x}$ *P-stationary with respect to d* or *P_d-stationary* if it is weakly stationary and the multipliers $\bar{\mu}$, $\bar{\nu}$ satisfy the additional condition

$$(\bar{\mu}_i \geq 0 \wedge \bar{\nu}_i \geq 0) \vee (1 - d_i)\bar{\mu}_i = d_i \bar{\nu}_i \qquad \forall i \in I^{00}(\bar{x}).$$

It is easy to see that the definitions of part (b) and part (c) are consistent if $\alpha = d$. Nonetheless, we decided to write down part (b), as it is an important special case (because it is stronger than M-stationarity). Throughout the article, we use the variable α for vectors in $\{0,1\}^p$, and d for vectors in $[0,1]^p$.

The A$_\alpha$-stationarity condition was already observed in [Flegel, Kanzow, 2005, p. 610] when A-stationarity was introduced, but we use the name A$_\alpha$-stationarity for disambiguation.

Let us make the analogous definitions for (MPVC).

Definition 2.6. Let $\bar{x} \in \mathbb{R}^n$ be a feasible point of (MPVC) and $\alpha \in \{0,1\}^p$, $d \in [0,1]^p$ be given.

(a) We call $\bar{x}$ *A$_\alpha$-stationary* if it is weakly stationary and the multipliers $\bar{\mu}$, $\bar{\nu}$ satisfy

$$\begin{aligned} \bar{\mu}_i &= 0 \qquad \forall i \in I^{00}_{\alpha=0}(\bar{x}), \\ \bar{\nu}_i &\geq 0 \qquad \forall i \in I^{00}_{\alpha=1}(\bar{x}). \end{aligned}$$

(b) We call $\bar{x}$ *P$_\alpha$-stationary* if it is weakly stationary and the multipliers $\bar{\mu}$, $\bar{\nu}$ satisfy

$$\begin{aligned} \bar{\mu}_i &= 0 \qquad \forall i \in I^{00}_{\alpha=0}(\bar{x}), \\ (\bar{\mu}_i = 0 \wedge \bar{\nu}_i \geq 0) \vee \bar{\nu}_i &= 0 \qquad \forall i \in I^{00}_{\alpha=1}(\bar{x}). \end{aligned}$$

(c) We call $\bar{x}$ P$_d$-stationary if it is weakly stationary and the multipliers $\bar{\mu}$, $\bar{\nu}$ satisfy

$$(\bar{\mu}_i = 0 \wedge \bar{\nu}_i \geq 0) \vee (1 - d_i)\bar{\mu}_i = -d_i\bar{\nu}_i \qquad \forall i \in I^{00}(\bar{x}).$$

Recall that $\bar{\mu}_i \geq 0$ holds for $i \in I^{00}(\bar{x})$ for weakly stationary points of (MPVC). Again, one can see that parts (b) and parts (c) are consistent if $d = \alpha$.

With the exception of linearized B-stationarity, the stationarity conditions of Definitions 2.3 to 2.6 are illustrated in Figures 1 to 4.

The following relations for the new stationarity conditions follow directly from the definitions.

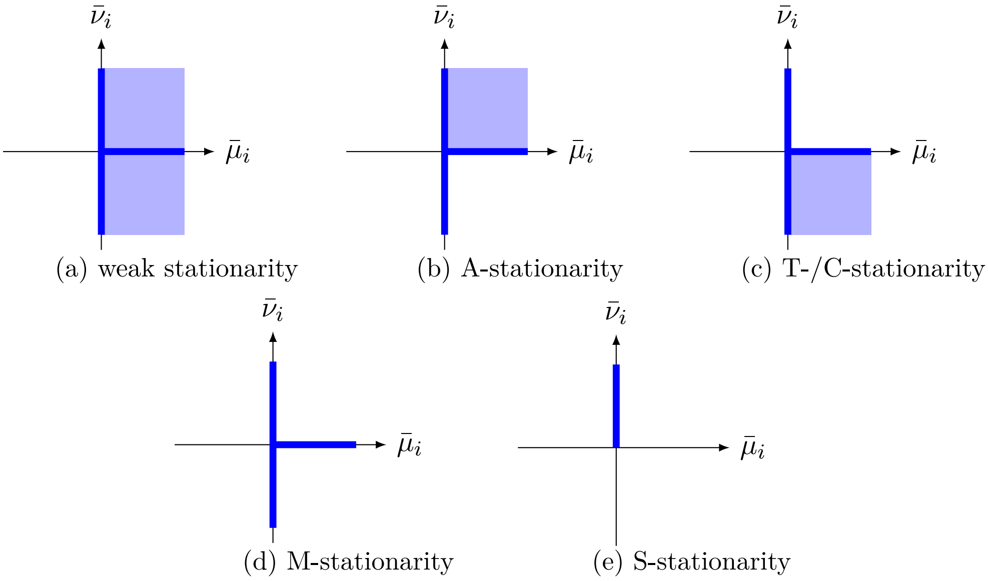
Corollary 2.7. Let $\bar{x}$ be a feasible point of (MPCC) or (MPVC).

(a) If $\bar{x}$ is P$_\alpha$-stationary for some $\alpha \in \{0,1\}^p$, then it is A$_\alpha$-stationary.

(b) The point $\bar{x}$ is A-stationary if and only if there exists some $\alpha \in \{0,1\}^p$ such that $\bar{x}$ is A$_\alpha$-stationary.

(c) The point $\bar{x}$ is C-stationary if and only if there exists some $d \in [0,1]^p$ such that $\bar{x}$ is P$_d$-stationary.

(d) The point $\bar{x}$ is M-stationary if and only if there exists some $\alpha \in \{0,1\}^p$ such that $\bar{x}$ is P$_\alpha$-stationary.



(a) weak stationarity (b) A-stationarity (c) T-/C-stationarity (d) M-stationarity (e) S-stationarity

Figure 3: geometric illustration of stationarity conditions for MPVCs with $i \in I^{00}(\bar x)$

(e) In the case of MPVCs, A_0-stationarity is the same as P_0-stationarity.

2.3 Auxiliary optimization problems

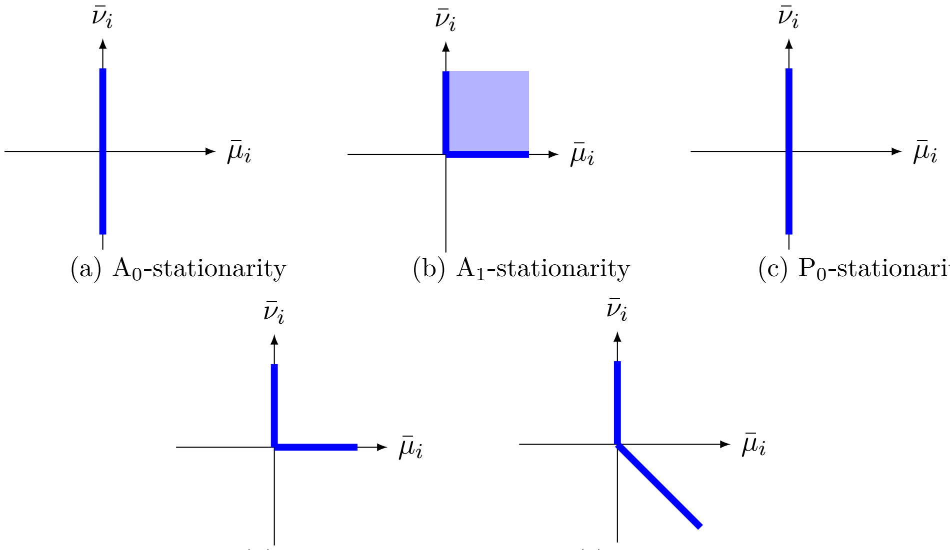
If $\bar x$ is a feasible point of (MPCC) and $\alpha \in \{0,1\}^p$, then we introduce the auxiliary nonlinear optimization problem

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \quad & f(x) \\ \text{s.t.} \quad & g(x) \le 0, \quad h(x) = 0, \\ & G_i(x) \ge 0, \quad H_i(x) = 0 \qquad \forall i \in I^{+0}(\bar x) \cup I^{00}_{\alpha=0}(\bar x), \\ & G_i(x) = 0, \quad H_i(x) \ge 0 \qquad \forall i \in I^{0+}(\bar x) \cup I^{00}_{\alpha=1}(\bar x). \end{aligned} \tag{NLP(α)}$$

Note that the feasible set of this auxiliary problem is a subset of the feasible set of (MPCC). This optimization problem can also be found in [Pang, Fukushima, 1999, Section 2].

Similarly, if $\bar x$ is a feasible point of (MPVC), then we introduce the auxiliary nonlinear optimization problem

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \quad & f(x) \\ \text{s.t.} \quad & g(x) \le 0, \quad h(x) = 0, \\ & \qquad\qquad H_i(x) = 0 \qquad \forall i \in I^{+0}(\bar x) \cup I^{00}_{\alpha=0}(\bar x), \\ & G_i(x) \le 0, \quad H_i(x) \ge 0 \qquad \forall i \in I^{-0}(\bar x) \cup I^{-+}(\bar x) \cup I^{0+}(\bar x) \cup I^{00}_{\alpha=1}(\bar x). \end{aligned} \tag{NLP(α)}$$



(a) A_0-stationarity (b) A_1-stationarity (c) P_0-stationarity (d) P_1-stationarity (e) $P_{1/2}$-stationarity

Figure 4: geometric illustration of stationarity conditions for MPVCs with $i \in I^{00}(\bar{x})$

This optimization problem can also be found in [Hoheisel, Kanzow, 2008, (7)].

Note that these auxiliary problems depend on $\bar{x}$ (in both the MPCC and the MPVC case). It will be clear from context, whether the MPCC-version of (NLP(α)) or the MPVC-version of (NLP(α)) is meant.

In both cases we denote the tangent cone at $\bar{x}$ to the feasible set of (NLP(α)) by $\mathcal{T}_{\text{NLP}(\alpha)}(\bar{x})$, and the standard linearization cone by $\mathcal{T}^{\text{lin}}_{\text{NLP}(\alpha)}(\bar{x})$. Note that $\mathcal{T}^{\text{lin}}_{\text{NLP}(\alpha)}(\bar{x})$ is a convex cone, whereas $\mathcal{T}^{\text{lin}}_{\text{MPCC}}(\bar{x})$ and $\mathcal{T}^{\text{lin}}_{\text{MPVC}}(\bar{x})$ are not always convex. Recall that the *Guignard constraint qualification* or *GCQ* is satisfied at $\bar{x}$ for (NLP(α)) if $\mathcal{T}_{\text{NLP}(\alpha)}(\bar{x})^\circ = \mathcal{T}^{\text{lin}}_{\text{NLP}(\alpha)}(\bar{x})^\circ$.

Finally, we state a simple lemma with well-known facts from the basic theory of nonlinear programming.

Lemma 2.8. (a) If $\bar{x}$ is a local minimizer of (MPCC), then $-\nabla f(\bar{x}) \in \mathcal{T}_{\text{MPCC}}(\bar{x})^\circ$ holds. If $\bar{x}$ is a local minimizer of (MPVC), then $-\nabla f(\bar{x}) \in \mathcal{T}_{\text{MPVC}}(\bar{x})^\circ$ holds.

(b) For some $\alpha \in \{0,1\}^p$ and feasible $\bar{x}$, $-\nabla f(\bar{x}) \in \mathcal{T}^{\text{lin}}_{\text{NLP}(\alpha)}(\bar{x})^\circ$ holds if and only if there exist multipliers $(\eta^\alpha, \lambda^\alpha, \mu^\alpha, \nu^\alpha)$ that satisfy the KKT conditions of (NLP(α)), which is the same as the system for A_α-stationarity (this holds for both of the MPCC and MPVC version of (NLP(α))).

Proof. Part (a) can be shown using the definition of the tangent cone and polar cone, and part (b) can be shown by calculating $\mathcal{T}^{\text{lin}}_{\text{NLP}(\alpha)}(\bar{x})^\circ$, e.g. using Farkas' Lemma. That the KKT system of (NLP(α)) is the same as the system for A_α-stationarity can be seen by writing down both systems. For more detailed proofs, we refer to the standard literature.

3 New stationarity systems between M- and S-stationarity

Let us start by stating that linearized B-stationarity and A_α-stationarity are indeed a stationarity condition under MPCC-GCQ. The result can (partially) be obtained from the proof of [Flegel, Kanzow, 2005, Theorem 3.4], and A_α-stationarity was also shown in [Harder, 2020, Proposition 3.1]. The equivalence was also mentioned in [Scheel, Scholtes, 2000, Section 2.1]. We include a proof for convenience.

Proposition 3.1. Let $\bar{x} \in \mathbb{R}^n$ be a feasible point of (MPCC).

(a) If $\bar{x}$ is a local minimizer of (MPCC) that satisfies MPCC-GCQ, then $\bar{x}$ is linearized B-stationary.

(b) The point $\bar{x}$ is linearized B-stationary if and only if it is A_α-stationary for all $\alpha \in \{0,1\}^p$.

In particular, if $\bar{x}$ is a local minimizer of (MPCC) that satisfies MPCC-GCQ, then it is A_α-stationary for all $\alpha \in \{0,1\}^p$.

Proof. For part (a), we obtain $-\nabla f(\bar{x}) \in \mathcal{T}_{\text{MPCC}}(\bar{x})^\circ = \mathcal{T}^{\text{lin}}_{\text{MPCC}}(\bar{x})^\circ$ from Lemma 2.8 (a) and MPCC-GCQ. Thus, $\bar{x}$ is linearized B-stationary.
For part (b), we first observe the equality

$$\mathcal{T}^{\text{lin}}_{\text{MPCC}}(\bar{x}) = \bigcup_{\alpha \in \{0,1\}^p} \mathcal{T}^{\text{lin}}_{\text{NLP}(\alpha)}(\bar{x}),$$

which can be shown by direct calculations or obtained from [Flegel, Kanzow, 2005, Lemma 3.1]. Therefore, linearized B-stationarity can be written as

$$-\nabla f(\bar{x}) \in \mathcal{T}^{\text{lin}}_{\text{MPCC}}(\bar{x})^\circ = \Bigg(\bigcup_{\alpha \in \{0,1\}^p} \mathcal{T}^{\text{lin}}_{\text{NLP}(\alpha)}(\bar{x}) \Bigg)^\circ = \bigcap_{\alpha \in \{0,1\}^p} \mathcal{T}^{\text{lin}}_{\text{NLP}(\alpha)}(\bar{x})^\circ.$$

Thus, $\bar{x}$ is linearized B-stationary if and only if $-\nabla f(\bar{x}) \in \mathcal{T}^{\text{lin}}_{\text{NLP}(\alpha)}(\bar{x})^\circ$ for all $\alpha \in \{0,1\}^p$. However, the latter condition is equivalent to A_α-stationarity of $\bar{x}$ by Lemma 2.8 (b).

In [Harder, 2020], the idea was to consider convex combinations of the multipliers corresponding to the A_α-stationarity system. Here, we will use the same idea, but aim for stronger results. An important ingredient is the Poincaré–Miranda theorem which is a generalization of the intermediate value theorem.

Theorem 3.2 (Poincaré–Miranda Theorem)**.** Let $h : [0,1]^p \to \mathbb{R}^p$ be a continuous function such that

$$\begin{aligned} h_i(y) &\leq 0 \qquad \text{if } y_i = 0, \\ h_i(y) &\geq 0 \qquad \text{if } y_i = 1 \end{aligned}$$

holds for all $i \in I^p$. Then there exists a point $\bar{y} \in [0,1]^p$ with $h(\bar{y}) = 0$.

We refer to [Kulpa, 1997] for a proof of this theorem. We mention that this theorem is equivalent to the well-known Brouwer fixed-point theorem, see [Miranda, 1940].

Lemma 3.3. Let $\bar{x} \in \mathbb{R}^n$ be a point and let $\psi : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^p$ be a continuous function with the property

$$\begin{aligned} \psi_i(a,b) &\leq 0 \quad \text{if } a_i \geq 0 \\ \psi_i(a,b) &\geq 0 \quad \text{if } b_i \geq 0 \end{aligned} \qquad \forall\, a, b \in \mathbb{R}^p,\ i \in I^{00}(\bar{x}). \tag{3.1}$$

Furthermore, for all $\alpha \in \{0,1\}^p$, let points $(\mu^\alpha, \nu^\alpha) \in A^\alpha$ be given, where the set A^α is described via

$$A^\alpha := \{(\mu, \nu) \in \mathbb{R}^{2p} \mid \mu_i \geq 0\ \forall i \in I^{00}_{\alpha=0}(\bar{x}),\ \nu_i \geq 0\ \forall i \in I^{00}_{\alpha=1}(\bar{x})\}. \tag{3.2}$$

Then there exists a point $(\bar{\mu}, \bar{\nu})$ in the convex hull of $\{(\mu^\alpha, \nu^\alpha) \mid \alpha \in \{0,1\}^p\}$ with

$$\psi_i(\bar{\mu}, \bar{\nu}) = 0 \qquad \forall i \in I^{00}(\bar{x}).$$

Proof. Let us define the function

$$\hat{g} : [0,1]^p \times \{0,1\}^p \to [0, \infty), \quad (y, \alpha) \mapsto \Bigg(\prod_{i \in I^p,\, \alpha_i = 1} y_i \Bigg) \cdot \Bigg(\prod_{i \in I^p,\, \alpha_i = 0} (1 - y_i) \Bigg).$$

Note that for each $y \in [0,1]^p$ there exists some $\beta \in \{0,1\}^p$ with $\hat{g}(y, \beta) > 0$. Therefore, the normalized function

$$g : [0,1]^p \times \{0,1\}^p \to [0,1], \quad (y, \alpha) \mapsto \frac{\hat{g}(y, \alpha)}{\sum_{\beta \in \{0,1\}^p} \hat{g}(y, \beta)}$$

is well-defined and has the property $\sum_{\alpha \in \{0,1\}^p} g(y, \alpha) = 1$. Note that $\hat{g}(\cdot, \alpha)$ and $g(\cdot, \alpha)$ are continuous for each $\alpha \in \{0,1\}^p$. Our next goal is to apply Theorem 3.2 to the function

$$h : [0,1]^p \to \mathbb{R}^p, \quad y \mapsto \begin{cases} \psi_i\Big(\sum_{\alpha \in \{0,1\}^p} g(y, \alpha)(\mu^\alpha, \nu^\alpha) \Big) & \text{if } i \in I^{00}(\bar{x}) \\ 0 & \text{if } i \in I^p \setminus I^{00}(\bar{x}). \end{cases}$$

Clearly, h is continuous. Let us verify that the required sign conditions for h hold. For $i \in I^p \setminus I^{00}(\bar{x})$, we have $h_i(y) = 0$ and the assumption is satisfied. Let $i \in I^{00}(\bar{x})$ and

$y \in [0,1]^p$ be given. We first consider the case that $y_i = 0$ holds. For $\alpha \in \{0,1\}^p$ with $\alpha_i = 1$ we have $\hat{g}(y,\alpha) = 0$ and $g(y,\alpha) = 0$. We obtain

$$\sum_{\alpha\in\{0,1\}^p} g(y,\alpha)\mu_i^\alpha = \sum_{\alpha\in\{0,1\}^p, i\in I^{00}_{\alpha=0}(\bar{x})} g(y,\alpha)\mu_i^\alpha \geq 0$$

from $(\mu^\alpha, \nu^\alpha) \in A^\alpha$. Together with (3.1) this implies

$$h_i(y) = \psi_i\Big(\sum_{\alpha\in\{0,1\}^p} g(y,\alpha)(\mu^\alpha,\nu^\alpha)\Big) \leq 0.$$

The other case with $y_i = 1$ works similarly: There, for $\alpha \in \{0,1\}^p$ with $\alpha_i = 0$ we have $\hat{g}(y,\alpha) = 0$ and $g(y,\alpha) = 0$. We obtain

$$\sum_{\alpha\in\{0,1\}^p} g(y,\alpha)\nu_i^\alpha = \sum_{\alpha\in\{0,1\}^p, i\in I^{00}_{\alpha=1}(\bar{x})} g(y,\alpha)\nu_i^\alpha \geq 0$$

from $(\mu^\alpha, \nu^\alpha) \in A^\alpha$. Together with (3.1) this implies

$$h_i(y) = \psi_i\Big(\sum_{\alpha\in\{0,1\}^p} g(y,\alpha)(\mu^\alpha,\nu^\alpha)\Big) \geq 0.$$

Therefore, Theorem 3.2 can be applied, which yields a point $\bar{y} \in [0,1]^p$ with $h(\bar{y}) = 0$. We define

$$(\bar{\mu},\bar{\nu}) := \sum_{\alpha\in\{0,1\}^p} g(\bar{y},\alpha)(\mu^\alpha,\nu^\alpha).$$

Due to the properties of g, the point $(\bar{\mu},\bar{\nu})$ is indeed a point in the convex hull of $\{(\mu^\alpha,\nu^\alpha) \mid \alpha \in \{0,1\}^p\}$. For $i \in I^{00}(\bar{x})$, we also obtain the remaining condition by

$$\psi_i(\bar{\mu},\bar{\nu}) = \psi_i\Big(\sum_{\alpha\in\{0,1\}^p} g(\bar{y},\alpha)(\mu^\alpha,\nu^\alpha)\Big) = h_i(\bar{y}) = 0.$$

We mention that any feasible function ψ satisfies $\psi_i(a,b) = 0$ if $a_i \geq 0 \wedge b_i \geq 0$, which is the area that corresponds to the system of strong stationarity. By choosing a suitable function ψ, we can use the previous lemma to show that P_α-stationarity is a first-order necessary optimality condition under MPCC-GCQ.

Theorem 3.4. Suppose $\bar{x}$ is an A_α-stationary point of (MPCC) for all $\alpha \in \{0,1\}^p$. Then $\bar{x}$ is a P_α-stationary point for all $\alpha \in \{0,1\}^p$.
In particular, if $\bar{x}$ is a local minimizer of (MPCC) that satisfies MPCC-GCQ, then it is P_α-stationary for all $\alpha \in \{0,1\}^p$.

Proof. Let $\beta \in \{0,1\}^p$ be given. We will show that $\bar{x}$ is P_β-stationarity. For each $\alpha \in \{0,1\}^p$, let $(\lambda^\alpha, \eta^\alpha, \mu^\alpha, \nu^\alpha)$ be multipliers which satisfy the system of A_α-stationarity. We want to apply Lemma 3.3. By definition, we have $(\mu^\alpha,\nu^\alpha) \in A^\alpha$, where A^α is defined as in (3.2). We use the function $\psi : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^p$ which is given by

$$\psi_i(a,b) := \begin{cases} \max(-a_i, \min(0, b_i)) & \text{if } \beta_i = 0 \\ \min(b_i, \max(0, -a_i)) & \text{if } \beta_i = 1 \end{cases}$$

for all $i \in I^p$, $a, b \in \mathbb{R}^p$. It can be checked that these functions are continuous and satisfy (3.1). Thus, we can apply Lemma 3.3 and there exists a convex combination $(\bar{\lambda}, \bar{\eta}, \bar{\mu}, \bar{\nu})$ of the multipliers $(\lambda^\alpha, \eta^\alpha, \mu^\alpha, \nu^\alpha)$ such that $\psi_i(\bar{\mu}, \bar{\nu}) = 0$ holds for all $i \in I^{00}(\bar{x})$. Let us check that $(\bar{\mu}, \bar{\nu})$ satisfy the additional conditions for P_β-stationarity. For $i \in I^{00}_{\beta=0}(\bar{x})$ we have

$$\begin{aligned} 0 = \psi_i(\bar{\mu}, \bar{\nu}) &= \max(-\bar{\mu}_i, \min(0, \bar{\nu}_i)) \\ &\iff (-\bar{\mu}_i = 0 \wedge \min(0, \bar{\nu}_i) \le 0) \vee (\min(0, \bar{\nu}_i) = 0 \wedge -\bar{\mu}_i \le 0) \\ &\iff (\bar{\mu}_i \ge 0 \wedge \bar{\nu}_i \ge 0) \vee \bar{\mu}_i = 0. \end{aligned}$$

Similarly, for $i \in I^{00}_{\beta=1}(\bar{x})$ we have

$$\begin{aligned} 0 = \psi_i(\bar{\mu}, \bar{\nu}) &= \min(\bar{\nu}_i, \max(0, -\bar{\mu}_i)) \\ &\iff (\bar{\nu}_i = 0 \wedge \max(0, -\bar{\mu}_i) \ge 0) \vee (\max(0, -\bar{\mu}_i) = 0 \wedge \bar{\nu}_i \ge 0) \\ &\iff (\bar{\mu}_i \ge 0 \wedge \bar{\nu}_i \ge 0) \vee \bar{\nu}_i = 0. \end{aligned}$$

It remains to show that $(\bar{\lambda}, \bar{\eta}, \bar{\mu}, \bar{\nu})$ satisfies the system of weak stationarity. This, however, is true due to the convex nature of the system of weak stationarity and because $(\lambda^\alpha, \eta^\alpha, \mu^\alpha, \nu^\alpha)$ satisfies the system of weak stationarity for all $\alpha \in \{0,1\}^p$. Due to Proposition 3.1, the P_β-stationarity condition is also satisfied if $\bar{x}$ is a local minimizer of (MPCC) that satisfies MPCC-GCQ.

This result will be generalized in Section 4 by considering other choices for ψ. While the P_α-stationarity also follows directly from Theorem 4.2 (b), we believe it is useful to also present this simpler proof for the interesting case of stationarity conditions between strong and M-stationarity.

Some equivalences involving P_α-stationarity are shown in Corollary 4.3.

4 Other new stationarity conditions for MPCCs

We can generalize the approach in Section 3 to obtain more stationarity conditions under MPCC-GCQ. However, these stationarity conditions do not necessarily lie between strong and M-stationarity, but only between strong and C-stationarity. As a special case, we obtain P_d-stationarity of local minimizers under MPCC-GCQ for all $d \in [0,1]^p$.

Lemma 4.1. For each $i \in I^p$, let $C_i \subset (-\infty, 0]^2$ be a closed, connected and unbounded set with $0 \in C_i$. Furthermore, let $\bar{x} \in \mathbb{R}^n$ be a point and for all $\alpha \in \{0,1\}^p$, let points $(\mu^\alpha, \nu^\alpha) \in A^\alpha$ be given, where the set A^α is defined in (3.2). Then there exists a point $(\bar{\mu}, \bar{\nu})$ in the convex hull of $\{(\mu^\alpha, \nu^\alpha) \mid \alpha \in \{0,1\}^p\}$ with

$$(\bar{\mu}_i \geq 0 \wedge \bar{\nu}_i \geq 0) \vee (\bar{\mu}_i, \bar{\nu}_i) \in C_i \qquad \forall i \in I^{00}(\bar{x}).$$

Proof. Let $i \in I^{00}(\bar{x})$ be fixed. We define $\hat{C} := C_i \cup [0, \infty)^2 \subset \mathbb{R}^2$, which is again a closed and connected set.
Let $O_1, O_2 \subset \mathbb{R}^2$ be the connected components of $\mathbb{R}^2 \setminus \hat{C}$ with the properties $(-1, 1) \in O_1$ and $(1, -1) \in O_2$. The sets O_1, O_2 are open because $\mathbb{R}^2$ is a locally connected space and connected components in a locally connected space are open. In order to apply Lemma 3.3, we want to construct functions $\psi_i : \mathbb{R}^2 \to \mathbb{R}$ that are 0 only on $\hat{C}$ and satisfy some sign conditions. This requires that O_1 and O_2 are different connected components. We will show that the open sets O_1 and O_2 are different connected components. Suppose by contradiction that $O_1 = O_2$. Since O_1 is an open and connected set, it is also path-connected. Thus, there exists a path from $(-1, 1)$ to $(1, -1)$ in O_1. Let K_1 denote the image of the path. Since $\hat{C}$ is closed and K_1 is compact, there exists a minimum distance $d > 0$ of K_1 from $\hat{C}$. We define the open set $O_{\hat{C}} := \{y \in \mathbb{R}^2 \mid \operatorname{dist}(y, \hat{C}) < d/2\}$. This set is also connected: a nontrivial connected component G of $O_{\hat{C}}$ would lead to a nontrivial connected component $G \cap \hat{C}$ of $\hat{C}$.
Since C_i is unbounded, there exists a point $\hat{x} \in C_i$ with $\hat{x}_1 \leq y_1 \ \forall y \in K_1$ or $\hat{x}_2 \leq y_2 \ \forall y \in K_1$. Without loss of generality we assume that $\hat{x}_1 \leq y_1 \ \forall y \in K_1$ holds (otherwise one could just exchange coordinates for the rest of the proof that $O_1 \neq O_2$). Likewise, there exists a point $(\hat{z}_1, 0) \in [0, \infty)^2 \subset \hat{C}$ with $\hat{z}_1 \geq y_1 \ \forall y \in K_1$. Since $O_{\hat{C}}$ is open and connected, we can find a path $\gamma^2 : [0, 1] \to O_{\hat{C}}$ with $\gamma^2(0) = \hat{x}$ and $\gamma^2(1) = (\hat{z}_1, 0)$. We also define $\gamma^1(0) := (-1, 2 + \sup\{\gamma_2^2(a) \mid a \in [0, 1]\}) \in O_1$ and $\gamma^1(1) := (1, \inf\{\gamma_2^2(a) \mid a \in [0, 1]\} - 2) \in O_2 = O_1$, and choose a path $\gamma^1 : [0, 1] \to O_1$ that goes from $\gamma^1(0)$ to $(-1, 1)$ in a straight line, then continuous through K_1 to the point $(1, -1)$, and goes to $\gamma^1(1)$ in a straight line from there. Note that the minimum distance of $\gamma^1([0, 1])$ to $\hat{C}$ is still equal to $d > 0$.
We define the continuous function $h : [0, 1]^2 \to \mathbb{R}^2$ via $h(y) = \gamma^2(y_1) - \gamma^1(y_2)$. Then the conditions of Theorem 3.2 are true due to $\gamma_1^2(0) = \hat{x}_1 \leq \gamma_1^1(a) \leq \hat{z}_1 = \gamma_1^2(1)$ and $\gamma_2^1(1) \leq \gamma_2^2(a) \leq \gamma_2^1(0)$ for all $a \in [0, 1]$. Thus, there exists a point $\bar{y} \in [0, 1]^2$ with $h(\bar{y}) = 0$, i.e. $\gamma^2(\bar{y}_1) = \gamma^1(\bar{y}_2)$. Thus, the paths intersect, which leads to to $d = \operatorname{dist}(\gamma^1([0, 1]), \hat{C}) \leq \operatorname{dist}(\gamma^1([0, 1]), \gamma^2(\bar{y}_1)) + \operatorname{dist}(\gamma^2(\bar{y}_1), \hat{C}) < 0 + d/2$, which is a contradiction. Hence, our assumption $O_1 = O_2$ was wrong and they are different connected components. Finally, we define

$$\hat{\psi}_i : \mathbb{R}^2 \to \mathbb{R}, \quad y \mapsto \begin{cases} \operatorname{dist}(y, \hat{C}) & \text{if } y \in O_1 \\ -\operatorname{dist}(y, \hat{C}) & \text{if } y \notin O_1 \end{cases}.$$

Because O_1 is a connected component of $\mathbb{R}^2 \setminus \hat{C}$, the function $\hat{\psi}_i$ is continuous. Since O_1 and O_2 are different connected components and O_2 is open, we have $\hat{\psi}_i(y) < 0$ on O_2.

Because of $[0,\infty)\times\mathbb{R}\subset\hat{C}\cup O_2$ and $\mathbb{R}\times[0,\infty)\subset\hat{C}\cup O_1$ it follows that $\hat{\psi}_i(y_1,y_2)\le 0$ if $y_1\ge 0$ and $\hat{\psi}_i(y_1,y_2)\ge 0$ if $y_2\ge 0$ holds. Therefore, the function

$$\psi:\mathbb{R}^p\times\mathbb{R}^p\to\mathbb{R}^p,\quad (a,b)\mapsto\left(\hat{\psi}_i(a_i,b_i)\right)_{i\in I^p}$$

satisfies the assumptions of Lemma 3.3 and we can apply the lemma. Thus, there exists a point $(\bar{\mu},\bar{\nu})$ in the convex hull of $\{(\mu^\alpha,\nu^\alpha)\mid\alpha\in\{0,1\}^p\}$ with $\psi_i(\bar{\mu},\bar{\nu})=0$ for all $i\in I^{00}(\bar{x})$. The result then follows from the definition of ψ_i and $\hat{C}$, in particular the equivalence

$$\psi_i(a,b)=0 \iff \hat{\psi}_i(a_i,b_i)=0 \iff (a_i,b_i)\in\hat{C} \iff (a_i\ge 0\wedge b_i\ge 0)\vee(a_i,b_i)\in C_i$$

for all $a,b\in\mathbb{R}^p$, $i\in I^{00}(\bar{x})$.

Theorem 4.2. Suppose $\bar{x}$ is an A_α-stationary point of (MPCC) for all $\alpha\in\{0,1\}^p$. Then we have the following conditions.

(a) For each $i\in I^p$, let $C_i\subset(-\infty,0]^2$ be a closed, connected and unbounded set with $0\in C_i$. Then there exists multipliers $(\bar{\eta},\bar{\lambda},\bar{\mu},\bar{\nu})$ that satisfy the system of weak stationarity and

$$(\bar{\mu}_i\ge 0\wedge\bar{\nu}_i\ge 0)\vee(\bar{\mu}_i,\bar{\nu}_i)\in C_i\qquad\forall i\in I^{00}(\bar{x}).$$

(b) The point $\bar{x}$ is P_d-stationary for all $d\in[0,1]^p$.

In particular, these stationarity conditions are satisfied for local minimizers $\bar{x}$ if MPCC-GCQ holds at $\bar{x}$.

Proof. We start with part (a). For each $\alpha\in\{0,1\}^p$, let $(\lambda^\alpha,\eta^\alpha,\mu^\alpha,\nu^\alpha)$ be multipliers which satisfy the system of A_α-stationarity. We want to apply Lemma 4.1. By definition, we have $(\mu^\alpha,\nu^\alpha)\in A^\alpha$, where A^α is defined as in (3.2). Thus, an application of Lemma 4.1 yields a convex combination $(\bar{\lambda},\bar{\eta},\bar{\mu},\bar{\nu})$ of the multipliers $(\lambda^\alpha,\eta^\alpha,\mu^\alpha,\nu^\alpha)$ which satisfies $(\bar{\mu}_i\ge 0\wedge\bar{\nu}_i\ge 0)\vee(\bar{\mu}_i,\bar{\nu}_i)\in C_i$ for all $i\in I^{00}(\bar{x})$. Furthermore, $(\bar{\lambda},\bar{\eta},\bar{\mu},\bar{\nu})$ satisfies the system of weak stationarity due to the convex nature of the weak stationarity system and because $(\lambda^\alpha,\eta^\alpha,\mu^\alpha,\nu^\alpha)$ satisfy the system of weak stationarity for all $\alpha\in\{0,1\}^p$.
Part (b) follows from part (a) with the choice $C_i:=\{(a,b)\in(-\infty,0]^2\mid(1-d_i)a=d_ib\}$, which is indeed a closed, connected and unbounded set with $0\in C_i$.
Due to Proposition 3.1, these stationarity conditions are also satisfied if $\bar{x}$ is a local minimizer of (MPCC) that satisfies MPCC-GCQ.

Clearly, part (b) was only a special case of part (a) in Theorem 4.2, but we included it because it is a more natural condition.

As a simple corollary, let us state some relations among the new stationarity conditions.

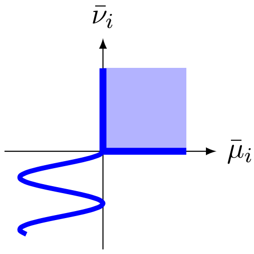


Figure 5: geometric illustration of the stationarity condition of Theorem 4.2 (a) with $i \in I^{00}(\bar{x})$, $C_i = \{(-\sin^2(5a), a) \mid a \leq 0\}$

Corollary 4.3. Let $\bar{x}$ be a feasible point of (MPCC). The following are equivalent.

(a) $\bar{x}$ is linearized-B-stationary,

(b) $\bar{x}$ is A_α-stationary for all $\alpha \in \{0,1\}^p$,

(c) $\bar{x}$ is P_d-stationary for all $d \in [0,1]^p$,

(d) $\bar{x}$ is P_α-stationary for all $\alpha \in \{0,1\}^p$.

Proof. The equivalence (a) $\iff$ (b) was already stated in Proposition 3.1 (b), and (b) $\implies$ (c) was shown in Theorem 4.2 (b). Finally, the implication (c) $\implies$ (d) is trivial and (d) $\implies$ (b) follows from Corollary 2.7 (a).

5 A simple proof of M-stationarity for MPVCs

We turn our attention to MPVCs. As one can see in Figures 3(d) and 4(a), the feasible set for the multipliers $(\bar{\mu}_i, \bar{\nu}_i)$ for A_0-stationarity is a subset of the feasible set for M-stationarity. In particular, A_0-stationarity implies M-stationarity. Note that such a relation does not hold for MPCCs. Since A_α-stationarity is usually easy to show under MPVC-GCQ (see also Proposition 6.1), this will lead to a very simple and short proof of M-stationarity under MPVC-GCQ. This is a known result, see, e.g. [Hoheisel, Kanzow, 2008, Theorem 3.4], but our proof is much simpler. Furthermore, GCQ for (NLP(0)) also works as a constraint qualification for M-stationarity.

Theorem 5.1. Let $\bar{x}$ be a minimizer of (MPVC). Suppose that

$$\text{MPVC-GCQ} \quad \text{or} \quad \text{GCQ for (NLP(0))} \tag{5.1}$$

holds at $\bar{x}$. Then $\bar{x}$ is A_0-stationary. In particular, $\bar{x}$ is an M-stationary point.

Proof. We have $-\nabla f(\bar{x}) \in \mathcal{T}_{\text{MPVC}}(\bar{x})^\circ$ due to Lemma 2.8 (a). In the case that MPVC-GCQ holds, we can use $\mathcal{T}^{\text{lin}}_{\text{NLP}(0)}(\bar{x}) \subset \mathcal{T}^{\text{lin}}_{\text{MPVC}}(\bar{x})$ to obtain $\mathcal{T}_{\text{MPVC}}(\bar{x})^\circ = \mathcal{T}^{\text{lin}}_{\text{MPVC}}(\bar{x})^\circ \subset \mathcal{T}^{\text{lin}}_{\text{NLP}(0)}(\bar{x})^\circ$. In the case that GCQ holds for (NLP(0)), we can use $\mathcal{T}_{\text{NLP}(0)}(\bar{x}) \subset \mathcal{T}_{\text{MPVC}}(\bar{x})$ to obtain $\mathcal{T}_{\text{MPVC}}(\bar{x})^\circ \subset \mathcal{T}_{\text{NLP}(0)}(\bar{x})^\circ = \mathcal{T}^{\text{lin}}_{\text{NLP}(0)}(\bar{x})^\circ$. In both cases we have $-\nabla f(\bar{x}) \in \mathcal{T}^{\text{lin}}_{\text{NLP}(0)}(\bar{x})^\circ$. By Lemma 2.8 (b), $\bar{x}$ is A_0-stationary.
If $(\bar{\lambda}, \bar{\eta}, \bar{\mu}, \bar{\nu})$ satisfies the system of A_0-stationarity, then we have $\bar{\mu}_i = 0$ for all $i \in I^{00}(\bar{x})$ and therefore the multipliers also satisfy the system of M-stationarity.

Note that this proof does not rely on the more complicated methods from Sections 3 and 4 or on advanced techniques from variational analysis such as the limiting normal cone. And because (5.1) is a weaker condition than MPVC-GCQ (see Example 5.2 below), we have even generalized the result of M-stationarity under MPVC-GCQ slightly. We mention that a similarly elementary method was used to show M-stationarity for *mathematical programs with switching constraints* (MPSCs) in [Mehlitz, 2019, Theorem 5.1].

One might wonder what the relationship is between MPVC-GCQ and GCQ for (NLP(0)). The following two counterexamples show that neither implies the other.

Example 5.2. We consider the setting with $n = 2$, $p = l = m = 1$, $G(x) = x_1$, $H(x) = x_2$, and $\bar{x} = (0, 0)$.

(a) If $h(x) = x_2^2 - x_1$ and $g(x) = 0$, then MPVC-GCQ does not hold at $\bar{x}$, but GCQ holds for (NLP(0)) at $\bar{x}$.

(b) If $h(x) = x_1^2 - x_2$, and $g(x) = x_1$. Then MPVC-GCQ holds at $\bar{x}$, but GCQ does not hold for (NLP(0)) at $\bar{x}$.

Proof. We have $I^p = I^{00}(\bar{x}) = \{1\}$. For part (a), the point $\bar{x} = (0, 0)$ is the only feasible point of (MPVC) and (NLP(0)). Thus, we have $\mathcal{T}_{\text{MPVC}}(\bar{x}) = \mathcal{T}_{\text{NLP}(0)} = \{(0, 0)\}$. One can also calculate $\mathcal{T}^{\text{lin}}_{\text{MPVC}}(\bar{x}) = \{0\} \times [0, \infty)$ and $\mathcal{T}^{\text{lin}}_{\text{NLP}(0)}(\bar{x}) = \{(0, 0)\}$. Taking polar cones yields the claim.
For part (b), the feasible set of (NLP(0)) is again just $\{\bar{x}\}$, but the feasible set of (MPVC) is $\{x \in \mathbb{R}^2 \mid x_1^2 = x_2, x_1 \leq 0\}$. Thus, we have $\mathcal{T}_{\text{MPVC}}(\bar{x}) = (-\infty, 0] \times \{0\}$ and $\mathcal{T}_{\text{NLP}(0)} = \{(0, 0)\}$. For the linearization cones, one can calculate $\mathcal{T}^{\text{lin}}_{\text{MPVC}}(\bar{x}) = \mathcal{T}_{\text{NLP}(0)}(\bar{x}) = (-\infty, 0] \times \{0\}$. Taking polar cones yields the claim.

6 New stationarity conditions for MPVCs

In this section we will use the results of Sections 3 and 4 and apply them in the setting of MPVCs. This will lead to new stationarity conditions for MPVCs. An important result will be that P_α-stationarity is a first-order necessary optimality condition under MPVC-GCQ for all $\alpha \in \{0, 1\}^p$.

We start with an analogue to Proposition 3.1 for MPVCs.

Proposition 6.1. Let $\bar{x} \in \mathbb{R}^n$ be a feasible point of (MPVC).

(a) If $\bar{x}$ is a local minimizer of (MPVC) that satisfies MPVC-GCQ, then $\bar{x}$ is linearized B-stationary.

(b) The point $\bar{x}$ is linearized B-stationary if and only if it is A_α-stationary for all $\alpha \in \{0,1\}^p$.

In particular, if $\bar{x}$ is a local minimizer of (MPVC) that satisfies MPVC-GCQ, then it is A_α-stationary for all $\alpha \in \{0,1\}^p$.

Proof. For part (a), we obtain $-\nabla f(\bar{x}) \in \mathcal{T}_{\mathrm{MPVC}}(\bar{x})^\circ = \mathcal{T}^{\mathrm{lin}}_{\mathrm{MPVC}}(\bar{x})^\circ$ from Lemma 2.8 (a) and MPVC-GCQ. Thus, $\bar{x}$ is linearized B-stationary.
For part (b), we first observe the equality

$$\mathcal{T}^{\mathrm{lin}}_{\mathrm{MPVC}}(\bar{x}) = \bigcup_{\alpha \in \{0,1\}^p} \mathcal{T}^{\mathrm{lin}}_{\mathrm{NLP}(\alpha)}(\bar{x}),$$

which can be shown by direct calculations or obtained from [Hoheisel, Kanzow, 2008, Lemma 2.4]. Therefore, linearized B-stationarity can be written as

$$-\nabla f(\bar{x}) \in \mathcal{T}^{\mathrm{lin}}_{\mathrm{MPVC}}(\bar{x})^\circ = \Bigg(\bigcup_{\alpha \in \{0,1\}^p} \mathcal{T}^{\mathrm{lin}}_{\mathrm{NLP}(\alpha)}(\bar{x}) \Bigg)^\circ = \bigcap_{\alpha \in \{0,1\}^p} \mathcal{T}^{\mathrm{lin}}_{\mathrm{NLP}(\alpha)}(\bar{x})^\circ.$$

Thus, $\bar{x}$ is linearized B-stationary if and only if $-\nabla f(\bar{x}) \in \mathcal{T}^{\mathrm{lin}}_{\mathrm{NLP}(\alpha)}(\bar{x})^\circ$ for all $\alpha \in \{0,1\}^p$. However, the latter condition is equivalent to A_α-stationarity of $\bar{x}$ by Lemma 2.8 (b).

Now we come to the main result for MPVCs, which is an analogue of Theorems 3.4 and 4.2. The proof is not very difficult, since it is possible to apply Lemma 4.1.

Theorem 6.2. Suppose $\bar{x}$ is an A_α-stationary point of (MPVC) for all $\alpha \in \{0,1\}^p$. Then we have the following conditions.

(a) For each $i \in I^p$, let $C_i \subset (-\infty, 0]^2$ be a closed, connected and unbounded set with $0 \in C_i$. Then there exists multipliers $(\bar{\eta}, \bar{\lambda}, \bar{\mu}, \bar{\nu})$ that satisfy the system of weak stationarity and

$$(\bar{\mu}_i = 0 \wedge \bar{\nu}_i \geq 0) \vee (-\bar{\mu}_i, \bar{\nu}_i) \in C_i \qquad \forall i \in I^{00}(\bar{x}).$$

(b) The point $\bar{x}$ is P_d-stationary for all $d \in [0,1]^p$.

(c) The point $\bar{x}$ is P_α-stationary for all $\alpha \in \{0,1\}^p$.

In particular, these stationarity conditions are satisfied for local minimizers $\bar{x}$ if MPVC-GCQ holds at $\bar{x}$.

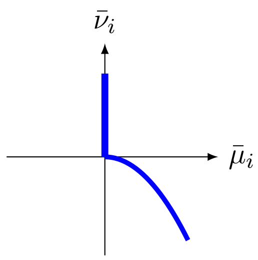


Figure 6: geometric illustration of the stationarity condition of Theorem 6.2 (a) with $i \in I^{00}(\bar{x})$, $C_i = \{(a, -a^2) \mid a \le 0\}$

Proof. We start with part (a). For each $\alpha \in \{0,1\}^p$, let $(\lambda^\alpha, \eta^\alpha, \mu^\alpha, \nu^\alpha)$ be multipliers which satisfy the system of A_α-stationarity.
We want to apply Lemma 4.1 with $-\mu$ instead of μ. Indeed, we have $(-\mu^\alpha, \nu^\alpha) \in A^\alpha$ due to the A_α-stationarity. Thus, Lemma 4.1 can be applied and there exists a convex combination $(\bar{\lambda}, \bar{\eta}, \bar{\mu}, \bar{\nu})$ of the multipliers $(\lambda^\alpha, \eta^\alpha, \mu^\alpha, \nu^\alpha)$ such that $(-\bar{\mu}_i \ge 0 \wedge \bar{\nu}_i \ge 0) \vee (-\bar{\mu}_i, \bar{\nu}_i) \in C_i$ for all $i \in I^{00}(\bar{x})$ holds. Because $\mu_i^\alpha \ge 0$ holds for all $i \in I^{00}(\bar{x})$ and $\alpha \in \{0,1\}^p$, the same holds for $\bar{\mu}_i$. Thus, the condition $-\bar{\mu}_i \ge 0$ can be equivalently replaced by $\bar{\mu}_i = 0$ if $i \in I^{00}(\bar{x})$. It remains to show that $(\bar{\lambda}, \bar{\eta}, \bar{\mu}, \bar{\nu})$ satisfies the system of weak stationarity. This, however, is true due to the convex nature of the system of weak stationarity.
Part (b) follows from part (a) with the choice $C_i := \{(a,b) \in (-\infty, 0]^2 \mid (1-d_i)a = d_i b\}$, which is indeed a closed, connected and unbounded set with $0 \in C_i$.
Part (c) then follows trivially from part (b).
Due to Proposition 6.1, these stationarity conditions are also satisfied if $\bar{x}$ is a local minimizer of (MPVC) that satisfies MPVC-GCQ.

As a corollary, we obtain an analogue to Corollary 4.3.

Corollary 6.3. Let $\bar{x}$ be a feasible point of (MPCC). The following are equivalent.

(a) $\bar{x}$ is linearized-B-stationary,

(b) $\bar{x}$ is A_α-stationary for all $\alpha \in \{0,1\}^p$,

(c) $\bar{x}$ is P_d-stationary for all $d \in [0,1]^p$,

(d) $\bar{x}$ is P_α-stationary for all $\alpha \in \{0,1\}^p$.

Proof. The equivalence (a) $\iff$ (b) was already stated in Proposition 6.1 (b), and (b) $\implies$ (c) was shown in Theorem 6.2 (b). Finally, the implication (c) $\implies$ (d) is obvious and (d) $\implies$ (b) follows from Corollary 2.7 (a).

7 Conclusion and outlook

We introduced new first-order necessary stationarity conditions for MPCCs and MPVCs. In particular, we were able to introduce a new stationarity condition which lies strictly between strong and M-stationarity. We also provided a simple, elementary, and short proof of M-stationarity for MPVCs in Section 5.

In the future, it might be interesting to investigate to what extend the methods from this article can be generalized to mathematical programs with disjunctive constraints (MPDCs), which is a problem class more general than MPCCs or MPVCs.

Our proof of P_α-stationarity (and the other new stationarity conditions) was based on the Poincaré–Miranda theorem, which is not constructive. Thus, it would be interesting to know whether a more constructive proof or other alternative proofs can be found.

It is unclear whether the ideas from this article can be used in Lebesgue or Sobolev spaces. Here, a problem might be that infinite-dimensional generalizations of the Poincaré–Miranda theorem or the Brouwer fixed-point theorem usually require compactness of the set or the function.

Let us describe another open question. For $i \in I^p$, consider sets $D_i \subset \mathbb{R}^2$. Can we characterize the sets D_i such that weak stationarity and $(\bar\mu_i, \bar\nu_i) \in D_i \; \forall i \in I^{00}(\bar x)$ is a stationarity conditions (under MPCC-GCQ or MPVC-GCQ)? Using Theorem 4.2 (a) and Theorem 6.2 (a), one can describe already a large variety of such sets D_i, but it is not clear whether these are all possibilities for stationarity conditions described by a set D_i.

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