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Gerd Wachsmuth

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SPP1962 at Weierstrass Institute for Applied Analysis and Stochastics (WIAS)
Leibniz Institute in the Forschungsverbund Berlin e.V.
Mohrenstraße 39, 10117 Berlin, Germany
E-Mail: spp1962@wias-berlin.de

World Wide Web: http://spp1962.wias-berlin.de/

Maximal monotone operators with non-maximal graphical limit

Gerd Wachsmuth*

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We present a counterexample showing that the graphical limit of maximally monotone operators might not be maximally monotone. We also characterize the directional differentiability of the resolvent of an operator B in terms of existence and maximal monotonicity of the proto-derivative of B.

Keywords: Maximal monotone operator, graphical limit, proto-derivative, directional differentiability

MSC (2020): 49J53, 49J52, 49K40, 47H04

1 Introduction

By means of a counterexample we show that the (non-empty) graphical limit of maximally monotone operators may fail to be maximally monotone. This was raised as an open question in Adly, Rockafellar, 2020, Remark 6. We also shed some light on the relation of proto-differentiability of an operator and directional differentiability of its resolvent. Throughout this work, we use standard notation, see, e.g., Adly, Rockafellar, 2020.

2 Graphical limits of maximally monotone operators

For all $n \in \mathbb{N}$, we define the auxiliary function $f_n\colon [0, \infty) \to \mathbb{R}$ via

$$f_n(t) := \begin{cases} 0 & \text{if } t \le 2^{-n-1} \text{ or } t \ge 2^{-n+2}, \\ 2\bigl(t - 2^{-n-1}\bigr) & \text{if } 2^{-n-1} \le t \le 2^{-n}, \\ 2^{-n} & \text{if } 2^{-n} \le t \le 2^{-n+1}, \\ \frac{1}{2}\bigl(2^{-n+2} - t\bigr) & \text{if } 2^{-n+1} \le t \le 2^{-n+2}. \end{cases}$$

*Brandenburgische Technische Universität Cottbus–Senftenberg, Institute of Mathematics, 03046 Cottbus, Germany, wachsmuth@b-tu.de, https://www.b-tu.de/fg-optimale-steuerung/team/prof-gerd-wachsmuth, ORCID: 0000-0002-3098-1503

Each f_n is globally Lipschitz continuous with Lipschitz constant 2, see also Figure 1.

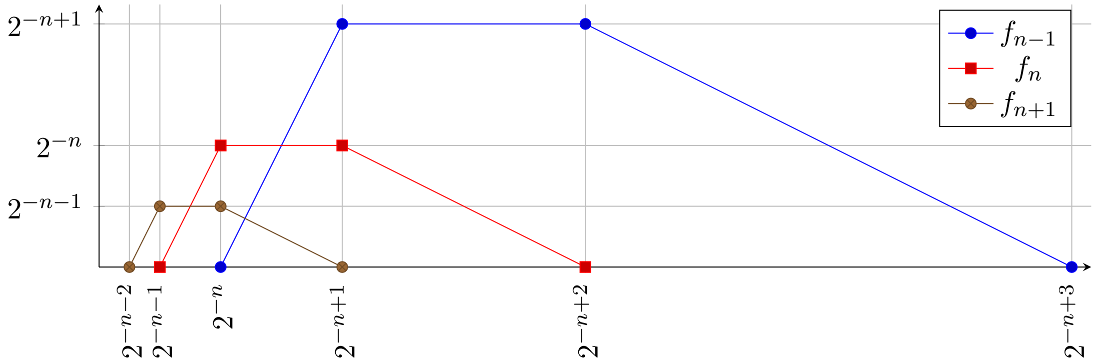

Figure 1: Plot of the functions f_n.

Let ℓ^2 be the Hilbert space of square-summable sequences (an analogue construction is possible in every infinite-dimensional Hilbert space). We define $T\colon \ell^2 \to \ell^2$ via

$$T(x) := \sum_{n=1}^{\infty} f_n(\|x\|) e_n,$$

where $(e_n)_n$ is the canonical orthonormal basis of ℓ^2. This operator is well defined, since for each $x \in \ell^2$, the sum contains at most three non-vanishing terms.

Lemma 1. The operator T is globally Lipschitz continuous on ℓ^2 with Lipschitz constant at most $\sqrt{17}/2$. Moreover, for each $x \in \ell^2$ with $\|x\| \le 1$ we have $\|T(x)\| \ge \|x\|/2$.

Proof. First of all, it can be checked easily that T is continuous on ℓ^2. Now, let $x, y \in \ell^2$ be given and we denote $(x, y) := \{\lambda x + (1 - \lambda) y \mid \lambda \in (0, 1)\}$.

We are going to utilize the mean value inequality from Penot, 2013, Theorem 2.7. Since the functions f_n are directionally differentiable, T is directionally differentiable on $(x, y) \setminus \{0\}$ and

$$T'(z; y - x) = \sum_{n=1}^{\infty} f_n'(\|z\|; \langle z, y - x\rangle / \|z\|) e_n \qquad \forall z \in (x, y) \setminus \{0\}.$$

This yields the estimate

$$\|T'(z; y - x)\|^2 \le \|y - x\|^2 \sum_{n=1}^{\infty} |f_n'(\|z\|; \operatorname{sign}\langle z, y - x\rangle)|^2 \qquad \forall z \in (x, y) \setminus \{0\}.$$

By construction of f_n, the sum contains at most two distinct addends from $\{\frac{1}{2}, 2\}$. Thus,

$$\|T'(z; y - x)\| \le \sqrt{2^2 + 1/2^2} \|y - x\| = \sqrt{17}/2 \, \|y - x\| \qquad \forall z \in (x, y) \setminus \{0\}.$$

Now, Penot, 2013, Theorem 2.7 (together with the remark afterwards) yields the estimate $\|T(y) - T(x)\| \le \frac{\sqrt{17}}{2}\|y - x\|$ for all $x, y \in \ell^2$.

For every $t \in [0, 1]$, there exists $n \in \mathbb{N}$ such that $f_n(t) \ge t/2$, cf. Figure 1. This implies the second claim.

Combining Lemma 1 with Bauschke, Combettes, 2011, Example 20.26, we find that $\operatorname{Id} + \alpha T$ is maximally monotone for all $\alpha \in \mathbb{R}$ with $|\alpha| \le 2/\sqrt{17}$. For an arbitrary α in this range, we set $B := \operatorname{Id} + \alpha T$. For all $m \in \mathbb{N}$, we define the operator $B_m \colon \ell^2 \to \ell^2$ via

$$B_m(x) := m B(x/m) = x + \alpha m T(x/m).$$

It is easy to check that all the operators B_m are again maximally monotone. However, their graphical limit fails to be maximally monotone in an extreme way.

Theorem 2. Let the maximally monotone operators $B_m \colon \ell^2 \to \ell^2$ be given as above. Then, the graphical limit of B_m as $m \to \infty$ is the operator $Z \colon \ell^2 \rightrightarrows \ell^2$, defined via $\operatorname{graph}(Z) = \{(0, 0)\}$.

Proof. We start by the computation of the outer limit of $\operatorname{graph}(B_m)$. For $(x, y) \in \limsup_{m\to\infty} \operatorname{graph}(B_m)$, we find a sequence $((x_{m_k}, y_{m_k}))_k$ with $(x_{m_k}, y_{m_k}) \in \operatorname{graph}(B_{m_k})$ and $x_{m_k} \to x$, $y_{m_k} \to y$. In particular, we have

$$y_{m_k} = x_{m_k} + \alpha m_k T(x_{m_k}/m_k).$$

Since $(x_{m_k})_k$ is bounded, we have $x_{m_k}/m_k \to 0$. Now, the structure of T implies that

$$[T(x_{m_k}/m_k)]_n = 0 \qquad \text{for } k \text{ large enough}$$

for each fixed n. Since $y_{m_k} - x_{m_k} = \alpha m_k T(x_{m_k}/m_k)$ converges, the limit can only attain the value 0 and, thus, we have $x = y$. Since $x_{m_k}/m_k \to 0$, we know $\|m_k T(x_{m_k}/m_k)\| \ge \|x_{m_k}\|/2$. Together with $y_{m_k} - x_{m_k} = \alpha m_k T(x_{m_k}/m_k) \to 0$, this gives $x_{m_k} \to 0$. Thus, $(x, y) = (0, 0)$ is the only point in $\limsup_{m\to\infty} \operatorname{graph}(B_m)$. Moreover, $(0, 0) \in \operatorname{graph}(B_m)$ shows that the limit of $\operatorname{graph}(B_m)$ is $\{(0, 0)\}$.

Clearly, the same argument can be used for the operators $B_\tau \colon \ell^2 \to \ell^2$, $\tau \in (0, 1)$, defined via $B_\tau(x) = \tau^{-1} B(\tau x)$ and for the limiting process $\tau \searrow 0$. Note that B_τ is just the finite difference appearing in the definition of the proto-derivative of B at 0 relative to 0.

Corollary 3. The maximal monotone mapping B is proto-differentiable at 0 and the proto-derivative at 0 relative to $0 = B(0)$ is given by the non-maximally monotone operator Z from Theorem 2.

3 Directional differentiability of resolvents

Let H be a (real) Hilbert space. For a maximally monotone $B\colon H \rightrightarrows H$, we denote by $J_B\colon H \to H$ its single-valued resolvent, i.e., $J_B := (\operatorname{Id} + B)^{-1}$. The next result characterizes the directional differentiability of J_B.

Theorem 4. Let $B\colon H \rightrightarrows H$ be maximally monotone. For $y \in H$ set $x := J_B(y)$. Then, the following are equivalent.

(i) B is proto-differentiable at x relative to $y - x \in B(x)$ and the proto-derivative $D_p B(x \mid y - x)\colon H \rightrightarrows H$ is maximally monotone,

(ii) J_B is directionally differentiable at y, i.e., the limit $J_B'(y; h) = \lim_{\tau \searrow 0} \frac{J_B(y+\tau h) - J_B(y)}{\tau}$ exists for all $h \in H$.

Proof. "⇒" follows from Adly, Rockafellar, 2020, Theorem 1 by setting $A(t, x) := x$, $B(t, x) := B(x)$, $\xi(t) := y + t\,h$.

"⇐": From Adly, Rockafellar, 2020, Remark 5, we get that J_B is proto-differentiable at y for $x = J_B(y)$. Consequently, Adly, Rockafellar, 2020, Lemma 2 implies that B is proto-differentiable at x relative to $y - x$. Moreover, we get the formula

$$D_p J_B(y \mid x) = \{J'(y; \cdot)\} = (\operatorname{Id} + D_p B(x \mid y - x))^{-1}$$

linking the derivatives of B and J_B. This shows that the resolvent of the monotone operator $D_p B(x \mid y - x)$ is single-valued. By Minty's theorem Bauschke, Combettes, 2011, Theorem 21.1, we find that $D_p B(x \mid y - x)$ is maximally monotone.

Theorems 2 and 4 show that the requirement of the proto-derivative of B being maximally monotone in Adly, Rockafellar, 2020, Theorem 1 cannot be dropped. The same result can be proved in the t-dependent case considered in Adly, Rockafellar, 2020.

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References

Adly, Samir; Rockafellar, Ralph Tyrrell (2020). "Sensitivity analysis of maximally monotone inclusions via the proto-differentiability of the resolvent operator". *Mathematical Programming.* DOI: 10.1007/s10107-020-01515-z.

Bauschke, Heinz H.; Combettes, Patrick L. (2011). *Convex analysis and monotone operator theory in Hilbert spaces.* Berlin: Springer. DOI: 10.1007/978-1-4419-9467-7.

Penot, Jean-Paul (2013). *Calculus Without Derivatives.* Vol. 266. Graduate Texts in Mathematics. Springer, New York. DOI: 10.1007/978-1-4614-4538-8.