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# LIPSCHITZ STABILITY AND HADAMARD DIRECTIONAL DIFFERENTIABILITY FOR ELLIPTIC AND PARABOLIC OBSTACLE-TYPE QUASI-VARIATIONAL INEQUALITIES\*

CONSTANTIN CHRISTOF<sup>†</sup> AND GERD WACHSMUTH<sup>‡</sup>

**Abstract.** This paper is concerned with the sensitivity analysis of a class of parameterized fixed-point problems that arise in the context of obstacle-type quasi-variational inequalities. We prove that, if the operators in the considered fixed-point equation satisfy a positive superhomogeneity condition, then the maximal and minimal element of the solution set of the problem depend locally Lipschitz continuously on the involved parameters. We further show that, if certain concavity conditions hold, then the maximal solution mapping is Hadamard directionally differentiable and its directional derivatives are precisely the minimal solutions of suitably defined linearized fixed-point equations. In contrast to prior results, our analysis requires neither a Dirichlet space structure, nor restrictive assumptions on the mapping behavior and regularity of the involved operators, nor sign conditions on the directions that are considered in the directional derivatives. Our approach further covers the elliptic and parabolic setting simultaneously and also yields Hadamard directional differentiability results in situations in which the solution set of the fixed-point equation is a continuum and a characterization of directional derivatives via linearized auxiliary problems is provably impossible. To illustrate that our results can be used to study interesting problems arising in practice, we apply them to establish the Hadamard directional differentiability of the solution operator of a nonlinear elliptic quasi-variational inequality, which emerges in impulse control and in which the obstacle mapping is obtained by taking essential infima over certain parts of the underlying domain, and of the solution mapping of a parabolic quasi-variational inequality, which involves boundary controls and in which the state-to-obstacle relationship is described by a partial differential equation.

**Key words.** sensitivity analysis, fixed-point equation, quasi-variational inequality, Lipschitz stability, Hadamard directional differentiability, optimal control, order approach, impulse control

**AMS subject classifications.** 35J87, 35K86, 47J20, 49J40, 49K40, 90C31

**1. Introduction and summary of results.** The aim of this paper is to study parameterized fixed-point problems of the form

$$(F) \quad y \in L^2(X), \quad y = S(\Phi(y), u).$$

Here,  $L^2(X)$  denotes the standard  $L^2$ -space on a complete measure space  $(X, \Sigma, \mu)$  endowed with the partial order induced by the  $\mu$ -a.e.-sense;  $S: \bar{P} \times U \rightarrow L^2_+(X)$  is a function that maps elements of a partially ordered set  $\bar{P}$ , which possesses a largest element  $\bar{p}$ , and a set  $U$  into  $L^2_+(X)$  and is nondecreasing in its first argument; and  $\Phi: L^2(X) \rightarrow \bar{P}$  is a nondecreasing map with values in  $P := \bar{P} \setminus \{\bar{p}\}$ . For the precise assumptions on the quantities in (F), we refer the reader to [subsection 2.1](#).

Our prime interest is in the derivation of conditions that ensure the local Lipschitz continuity and/or directional differentiability of certain selections from the (in general set-valued) solution mapping  $\mathbb{S}: U \rightrightarrows L^2(X)$ ,  $u \mapsto \{y \mid y = S(\Phi(y), u)\}$ , associated with (F). The main application that we have in mind is that the function  $S$  is

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the solution operator of an elliptic or parabolic obstacle-type variational inequality, i.e., the function that maps an obstacle  $p \in \bar{P}$  and a right-hand side  $u \in U$  to the solution of a variational inequality of the first kind that involves a unilateral constraint set of the form  $\{v \leq p\}$ , cf. the examples in [section 6](#). In this situation, the fixed-point problem (F) is equivalent to a so-called obstacle-type quasi-variational inequality (QVI) in which the bound defining the admissible set depends implicitly on the problem solution. Variational inequalities with such a structure arise, for instance, in the areas of mechanics, superconductivity, and thermoforming, see [\[2, 3, 8, 31, 32\]](#) and the references therein. As a prototypical example, we mention the following elliptic quasi-variational inequality that emerges in impulse control and that was one of the first QVIs to be formulated when this problem class was introduced by Lions and Bensoussan in the nineteen-seventies, cf. [\[10, section VIII-2\]](#): Given an open bounded nonempty set  $\Omega \subset \mathbb{R}^d$ ,  $d \in \mathbb{N}$ , a constant  $\kappa \geq 0$ , a function  $c_0 \in L^0_+(\mathbb{R}^d)$ , a nondecreasing, globally Lipschitz continuous, convex function  $f: \mathbb{R} \rightarrow \mathbb{R}$  with  $f(0) = 0$  (acting as a superposition operator), and a  $u \in H^{-1}_+(\Omega) := \{z \in H^{-1}(\Omega) \mid \langle z, v \rangle \geq 0 \text{ for all } 0 \leq v \in H^1_0(\Omega)\}$ , find a function  $0 \leq y \in H^1_0(\Omega)$  satisfying

$$(1.1) \quad y \leq \Theta(y) \quad \text{and} \quad \langle -\Delta y + f(y) - u, v - y \rangle \geq 0 \quad \forall v \in H^1_0(\Omega), v \leq \Theta(y),$$

where the obstacle  $\Theta(y)$  is defined by

$$(1.2) \quad \Theta(y)(x) := \kappa + \operatorname{ess\,inf}_{0 \leq \xi \in \mathbb{R}^d, x+\xi \in \Omega} c_0(\xi) + y(x + \xi) \quad \text{for a.a. } x \in \Omega.$$

Here,  $\Delta: H^1_0(\Omega) \rightarrow H^{-1}(\Omega)$  denotes the distributional Laplacian,  $H^1_0(\Omega)$  and  $H^{-1}(\Omega)$  are defined as usual, and  $\langle \cdot, \cdot \rangle$  denotes the dual pairing. For more details on the above problem, its background, and its reformulation as a fixed-point equation of the form (F), we refer the reader to [\[11, 12, 26, 29, 30\]](#) and [subsection 6.2](#).

Due to the various processes in physics and economics that can be described by QVIs, there has been an increasing interest in the optimal control of this class of variational inequalities, cf. [\[1, 6, 19, 37\]](#) and the references therein. Studying optimization problems with QVI-constraints, however, turns out to be a challenging task. Because of the set-valuedness of the solution mapping of (F), the formulation of reasonable optimal control problems for such a fixed-point equation typically requires working with certain distinguished selections from the solution set (e.g., minimal and maximal elements, cf. [Theorem 2.4](#)), and because of the implicit and often highly nontrivial and nonsmooth dependence of the constraint set on the problem solution (cf. the function  $\Theta$  in (1.2)), the derivation of necessary optimality conditions is far from straightforward for QVIs even in those situations where the solution set  $\mathbb{S}(u)$  can be proved to be a singleton. Despite these difficulties, there have been several contributions in the recent years that have tried to establish stability and directional differentiability results for obstacle-type quasi-variational inequalities and, by doing so, to lay the foundation for the study of optimal control problems governed by QVIs. We mention exemplarily [\[2, 4, 5\]](#), which establish the directional differentiability of the solution maps of elliptic and parabolic obstacle-type quasi-variational inequalities in signed (i.e., nonnegative or nonpositive) directions by means of an approximation argument and classical results on ordinary variational inequalities; [\[6\]](#), which proves the continuity of the minimal and maximal solution mappings of elliptic obstacle-type QVIs in the  $L^2$ -spaces; and [\[37\]](#), which establishes the directional differentiability of the solution operators of elliptic QVIs in all directions under a smallness assumption on the obstacle mapping and by means of the results of [\[17\]](#). Unfortunately, all of

the above papers have in common that they require very restrictive and, at times, even unrealistic assumptions on the involved operators and quantities. See, e.g., the conditions on the sign of the directions appearing in the derivatives in [2, Theorem 1], [4, Assumption 28], and [5, Theorems 3.6, 4.4]; the assumptions on the image, the complete continuity, and the size of  $\Phi$  and its derivatives in [2, Assumptions (A2), (A3), (A5)], [4, Assumptions 28, 32, 34], [5, Theorems 3.6, 4.4], [6, Assumption 1], and [37, Assumption 3.1]; and the comments in [1, Remark 2], which emphasize that compactness assumptions on the obstacle map are a main bottleneck in the study of obstacle-type QVIs. We remark that all of these conditions in particular prevent the differentiability results of [2, 5, 37] from being applicable to the problem (1.1).

The aim of this paper is to demonstrate that, if the operators  $S$  and  $\Phi$  possess certain pointwise curvature properties, which are present in many situations, then it is possible to establish very strong Lipschitz stability and directional differentiability results for the solution map of a fixed-point equation of the type (F) that do not suffer from the above problems. Our main results are as follows. (See the references in brackets for the precise assumptions and statements.)

- **(Local Lipschitz continuity)** We show that, if  $P$  is a subset of a real vector space,  $U$  is a subset of  $L_+^\infty(Y)$  for some complete measure space  $(Y, \Xi, \eta)$ ,  $S$  is nondecreasing in both of its arguments, and  $S$  and  $\Phi$  satisfy a superhomogeneity condition, then the minimal and maximal solution map of (F) are locally Lipschitz continuous on the set  $\{u \in U \mid u \geq c \text{ for some constant } c > 0\}$  as functions from  $L^\infty(Y)$  into all  $L^q(X)$ -spaces that  $S$  maps into. (See [Theorem 3.2](#) and [\(3.5\)](#).)
- **(Concavity of the maximal solution operator)** We show that, if  $P$  and  $U$  are convex subsets of real vector spaces and  $S: P \times U \rightarrow L_+^2(X)$  and  $\Phi: L_+^2(X) \rightarrow P$  are concave functions, then the maximal solution map of (F) is concave and pointwise  $\mu$ -a.e. directionally differentiable. (See [Proposition 4.2](#), [Theorem 4.3](#).)
- **(Hadamard directional differentiability of the maximal solution map)** For problems (F) that are covered by both our Lipschitz continuity and our concavity result, we prove that the maximal solution map is Hadamard directionally differentiable on the set  $\{u \in U \mid u \geq c \text{ for some constant } c > 0\}$  as a function from  $L^\infty(Y)$  into all  $L^q(X)$ -spaces that  $S$  maps into. (See [Corollary 4.4](#).)
- **(Unique characterization of directional derivatives)** For problems (F) that satisfy a strengthened concavity assumption (which also ensures that  $\mathbb{S}(u)$  is a singleton), we establish that the derivatives of the solution map  $\mathbb{S}: U \rightarrow L_+^2(X)$  are uniquely characterized by the condition that they are the smallest elements of the solution sets of certain linearized fixed-point equations. (See [Theorem 5.4](#).)

Note that the differentiability results in [Theorem 4.3](#) and [Corollary 4.4](#) indeed do not require any conditions on the signs of the directions in the derivatives, on the smallness of  $\Phi$  (or its differentiability), or on the existence of an underlying Dirichlet space structure, cf. [2, 5, 37]. As we will see in [section 6](#), because of this, our theorems are in particular able to cover the elliptic and the parabolic setting simultaneously and to even yield Hadamard directional differentiability results in situations in which the solution set  $\mathbb{S}(u)$  of (F) is a continuum and a characterization of derivatives via linearized auxiliary problems is provably impossible. We remark that this is in stark contrast to, e.g., [2, 37], in which the used smallness assumptions imply that solutions of (F) are locally unique, and that [Theorems 3.2](#) and [5.4](#) and [Corollary 4.4](#) improve, for instance, [6, Theorem 4] and [2, Theorem 1] under [2, Condition (A2b)]. Our theorems also seem to be the first results that allow to establish the Hadamard

directional differentiability of the solution operators of parabolic obstacle-type QVIs in all directions and to uniquely characterize the associated directional derivatives. Additional comments on this topic can be found in the subsequent sections after the respective theorems. Lastly, we would like to mention that our analysis allows to prove the local Lipschitz continuity and Hadamard directional differentiability of the maximal solution map  $M$  of the nonlinear elliptic impulse control problem (1.1) (which is equal to  $\mathbb{S}$  in the case  $\kappa > 0$ ) on the set  $\{u \in L^\infty(\Omega) \mid u \geq c \text{ for a constant } c > 0\}$  as a function from  $L^\infty(\Omega)$  into all  $L^q(\Omega)$ -spaces, see [Theorems 6.5](#) and [6.6](#). Again, this paper seems to be the first contribution to accomplish this.

**Content of the remaining sections.** We conclude this introduction with a brief overview of the structure of the remainder of the paper.

[Section 2](#) is concerned with preliminaries. Here, we clarify the notation, state our standing assumptions, and discuss the solvability of (F) as well as the existence and properties of smallest and largest elements of its solution set.

[Section 3](#) addresses the Lipschitz stability of the minimal and maximal solution map of (F). See [Theorem 3.2](#) for the main result of this section.

In [section 4](#), we prove that the maximal solution operator of (F) is indeed concave when the functions  $S$  and  $\Phi$  possess this property, see [Proposition 4.2](#). In [Theorem 4.3](#) and [Corollary 4.4](#), we then study the consequences that this observation has for the directional differentiability properties of the maximal solution map.

[Section 5](#) establishes the already mentioned characterization result for directional derivatives by means of a linearized auxiliary problem, see [Theorem 5.4](#).

[Section 6](#) contains three examples that illustrate that our Lipschitz continuity and directional differentiability results can be used to study interesting problems and are also relevant for applications. In [subsection 6.1](#), we begin the discussion with a simple one-dimensional model quasi-variational inequality that demonstrates that the Hadamard directional differentiability result in [Corollary 4.4](#) covers cases in which the considered QVI possesses a continuum of solutions. This example in particular shows that the strengthened concavity assumption in [Theorem 5.4](#) is necessary and that, without it, the characterization of derivatives via linearized auxiliary problems may be impossible. The subsequent [subsection 6.2](#) is concerned with the application of our theory to the QVI (1.1). Here, we show that [Theorem 3.2](#) and [Corollary 4.4](#) indeed allow to prove the Hadamard directional differentiability of the maximal solution map of this problem, see [Theorems 6.5](#) and [6.6](#). In [subsection 6.3](#), we finally consider an evolution QVI with boundary controls in which the obstacle mapping arises from a parabolic partial differential equation. This example shows that all of the results in [sections 2](#) to [5](#) are also applicable to time-dependent problems.

**2. Preliminaries.** This section is concerned with preliminaries. [Subsection 2.1](#) clarifies the notation, rigorously formulates the considered problem, and collects our standing assumptions. [Subsection 2.2](#) contains results on the solvability of (F) and the properties and existence of minimal and maximal solutions.

**2.1. Notation, standing assumptions, and problem setting.** Throughout this paper,  $(X, \Sigma, \mu)$  denotes a complete measure space. The  $L^q$ -spaces on  $(X, \Sigma, \mu)$  are denoted by  $L^q(X)$ ,  $1 \leq q \leq \infty$ , and the vector space of equivalence classes of real-valued measurable functions on  $X$  by  $L^0(X)$  (so that  $L^q(X) \subset L^0(X)$  for all  $q \in \{0\} \cup [1, \infty]$ ). We equip the spaces  $L^q(X)$ ,  $1 \leq q \leq \infty$ , with the usual norms  $\|\cdot\|_{L^q(X)}$  and endow  $L^q(X)$  for all  $q \in \{0\} \cup [1, \infty]$  with the partial order induced by the  $\mu$ -a.e.-sense, i.e.,  $v_1 \geq v_2 : \iff v_1 \geq v_2 \text{ } \mu\text{-a.e. in } X$ . For later use, we also introduce

the abbreviations  $L^{[r,s]}(X) := L^r(X) \cap L^s(X)$  for all  $1 \leq r \leq s \leq \infty$  (equipped with the norm  $\|\cdot\|_{L^{[r,s]}(X)} := \|\cdot\|_{L^r(X)} + \|\cdot\|_{L^s(X)}$ ),  $L_+^q(X) := \{v \in L^q(X) \mid v \geq 0\}$  for all  $q \in \{0\} \cup [1, \infty]$ ,  $L_\oplus^\infty(X) := \{v \in L^\infty(X) \mid v \geq c \text{ for a constant } c > 0\}$ , and  $L_+^{[r,s]}(X) := L_+^r(X) \cap L_+^s(X)$ . Note that we have  $L^{[r,s]}(X) = \bigcap_{q \in [r,s]} L^q(X)$  by Hölder's inequality. With  $L^0(X, (-\infty, \infty])$ , we denote the set of equivalence classes of extended real-valued measurable functions on  $X$  with values in  $(-\infty, \infty]$ . Sometimes, we require a second complete measure space, which we denote by  $(Y, \Xi, \eta)$ . We use the same notations and conventions for the spaces  $L^q(Y)$  as for the spaces  $L^q(X)$ .

In all what follows, the symbol  $\bar{P}$  denotes a set that is equipped with a partial order  $\leq$  and possesses a largest element  $\bar{p} \in \bar{P}$ . We always assume that  $\bar{P}$  contains at least two elements so that the set  $P := \bar{P} \setminus \{\bar{p}\}$  is nonempty. Recall that a partial order is a binary relation that is reflexive ( $p \leq p$  for all  $p \in \bar{P}$ ), antisymmetric (if  $p_1 \leq p_2$  and  $p_2 \leq p_1$ , then  $p_1 = p_2$  for all  $p_1, p_2 \in \bar{P}$ ), and transitive (if  $p_1 \leq p_2$  and  $p_2 \leq p_3$ , then  $p_1 \leq p_3$  for all  $p_1, p_2, p_3 \in \bar{P}$ ) and that a largest element  $\bar{p}$  of a partially ordered set  $\bar{P}$  is an element satisfying  $\bar{p} \in \bar{P}$  and  $p \leq \bar{p}$  for all  $p \in \bar{P}$ . Due to the antisymmetry of  $\leq$ , such an element is always unique. In our applications, the element  $\bar{p}$  can be understood as  $\infty$  and  $P = \bar{P} \setminus \{\bar{p}\}$  is the nonnegative cone in a real vector space. Finally, with  $U$  we denote a nonempty set (the set of parameters).

As already mentioned in [section 1](#), the main objective of this paper is to study parameterized fixed-point equations of the form

$$(F) \quad y \in L^2(X), \quad y = S(\Phi(y), u).$$

Our standing assumptions on the operators  $S$  and  $\Phi$  in [\(F\)](#) are as follows:

- $S: \bar{P} \times U \rightarrow L_+^2(X)$  and it holds  $u \in U, p_1, p_2 \in \bar{P}, p_1 \leq p_2 \Rightarrow S(p_1, u) \leq S(p_2, u)$ .
- $\Phi: L^2(X) \rightarrow P$  and it holds  $v_1, v_2 \in L^2(X), v_1 \leq v_2 \Rightarrow \Phi(v_1) \leq \Phi(v_2)$ .

Tangible examples of operators  $S$  and  $\Phi$  that arise in the context of obstacle-type QVIs and satisfy the above conditions can be found in [section 6](#). For the sake of brevity, we will sometimes also work with the shorthand notation  $T_u(v) := S(\Phi(v), u)$  in this paper. Note that, using the map  $T_u: L^2(X) \rightarrow L_+^2(X)$ , the problem [\(F\)](#) can be recast as  $y = T_u(y)$  for all  $u \in U$ . We remark that the above standing assumptions on  $(X, \Sigma, \mu)$ ,  $\bar{P}$ ,  $P$ ,  $U$ ,  $S$ , and  $\Phi$  will be complemented with additional conditions in the subsequent sections, see [Assumptions 3.1](#), [4.1](#) and [5.1](#). In the statements of our theorems, we will always make precise if such assumptions are needed.

**2.2. Existence of solutions via an order approach.** To establish that the problem [\(F\)](#) possesses a nonempty set of solutions under the standing assumptions of [subsection 2.1](#), one can use a classical order approach based on the theorem of Birkhoff-Tartar, see [\[8, section 15.2.2\]](#). As we need several results on minimal and maximal solutions that are obtained from this method of proof for our sensitivity analysis, we present the arguments in detail in this subsection.

**LEMMA 2.1.** *The map  $T_u: L^2(X) \rightarrow L_+^2(X)$  is nondecreasing, i.e., for all  $u \in U$  and all  $v_1, v_2 \in L^2(X)$  with  $v_1 \leq v_2$ , we have  $T_u(v_1) \leq T_u(v_2)$ .*

*Proof.* Due to our assumptions on  $\Phi$  and  $S$ ,  $v_1 \leq v_2$  implies  $\Phi(v_1) \leq \Phi(v_2)$  and  $T_u(v_1) = S(\Phi(v_1), u) \leq S(\Phi(v_2), u) = T_u(v_2)$ . This proves the claim.  $\square$

Next, we recall two classical concepts.

**DEFINITION 2.2** (sub- and supersolutions). *Let  $u \in U$  be fixed. A  $v \in L^2(X)$  is called a subsolution of [\(F\)](#) with parameter  $u$  if  $v \leq T_u(v)$  and a supersolution of [\(F\)](#) with parameter  $u$  if  $v \geq T_u(v)$ .*



Using the monotonicity properties of  $S$  and  $\Phi$ , the nonnegativity of  $S$ , and the largest element  $\bar{p}$  of  $\bar{P}$ , we obtain bounds on the solutions of  $(\mathbf{F})$ .

**LEMMA 2.3.** *Let  $u \in U$  be fixed and consider the problem  $(\mathbf{F})$  with parameter  $u$ . Then  $S(\bar{p}, u) \in L^2_+(X)$  is a supersolution of  $(\mathbf{F})$  and the zero function is a subsolution of  $(\mathbf{F})$ . Further, for every subsolution  $v$  of  $(\mathbf{F})$ , it holds  $v \leq S(\bar{p}, u)$  and, for every supersolution  $v$  of  $(\mathbf{F})$ , it holds  $v \geq 0$ . In particular, all solutions  $y$  of  $(\mathbf{F})$  (should they exist) satisfy  $0 \leq y \leq S(\bar{p}, u)$ .*

*Proof.* Due to the mapping properties of  $S$  and  $\Phi$ , it holds  $0 \leq S(\Phi(0), u) = T_u(0)$  and  $S(\bar{p}, u) \geq S(\Phi(S(\bar{p}, u)), u) = T_u(S(\bar{p}, u))$ . Thus, the zero function is indeed a subsolution of  $(\mathbf{F})$  and  $S(\bar{p}, u)$  is indeed a supersolution of  $(\mathbf{F})$ . For all subsolutions  $v$  of  $(\mathbf{F})$ , we further have  $v \leq T_u(v) = S(\Phi(v), u) \leq S(\bar{p}, u)$ , and, for all supersolutions  $v$  of  $(\mathbf{F})$ , it holds  $v \geq T_u(v) = S(\Phi(v), u) \geq 0$ . This completes the proof.  $\square$

Via the theorem of Birkhoff-Tartar, we can now prove the existence of minimal and maximal solutions.

**THEOREM 2.4** (solvability of  $(\mathbf{F})$ ). *Suppose that a  $u \in U$  is given and consider the problem  $(\mathbf{F})$  with parameter  $u$ . Then the set of solutions  $\mathbb{S}(u) \subset L^2(X)$  of  $(\mathbf{F})$  is nonempty. Further, there exist unique solutions  $m(u), M(u) \in \mathbb{S}(u)$  of  $(\mathbf{F})$  such that, for every subsolution  $v$  of  $(\mathbf{F})$ , it holds  $v \leq M(u)$  and such that, for every supersolution  $v$  of  $(\mathbf{F})$ , it holds  $m(u) \leq v$ . In particular,*

$$(2.1) \quad m(u) \leq y \leq M(u) \quad \forall y \in \mathbb{S}(u).$$

*Proof.* Since  $T_u: L^2(X) \rightarrow L^2(X)$  is nondecreasing by [Lemma 2.1](#) and since the functions  $0$  and  $S(\bar{p}, u)$  provide a sub- and a supersolution for  $(\mathbf{F})$  with  $0 \leq S(\bar{p}, u)$  by [Lemma 2.3](#), the existence of solutions of  $(\mathbf{F})$  follows immediately from the theorem of Birkhoff-Tartar, see [[8](#), section 15.2.2, Proposition 2]. This theorem also implies the existence of elements  $m(u), M(u) \in \mathbb{S}(u)$  such that, for every solution  $y$  of  $(\mathbf{F})$  with  $0 \leq y \leq S(\bar{p}, u)$ , we have  $m(u) \leq y \leq M(u)$ . Since [Lemma 2.3](#) yields that all  $y \in \mathbb{S}(u)$  have to satisfy  $0 \leq y \leq S(\bar{p}, u)$ , [\(2.1\)](#) and the uniqueness of  $m(u)$  and  $M(u)$  now follow immediately. Consider now an arbitrary but fixed subsolution  $v \in L^2(X)$  of  $(\mathbf{F})$ . Then [Lemma 2.3](#) implies that  $v \leq S(\bar{p}, u)$  holds, and we may again invoke the theorem of Birkhoff-Tartar to deduce that  $(\mathbf{F})$  admits at least one solution  $y$  with  $v \leq y \leq S(\bar{p}, u)$ . According to [\(2.1\)](#), this solution  $y$  has to satisfy  $y \leq M(u)$  which implies  $v \leq M(u)$ . All subsolutions are thus smaller than  $M(u)$  as claimed. To prove that  $m(u)$  is smaller than every supersolution, we can use the same arguments.  $\square$

In the above situation, the solutions  $m(u)$  and  $M(u)$  are called the minimal and the maximal solution of  $(\mathbf{F})$ , respectively. Note that this notation makes sense since [\(2.1\)](#) implies that  $\mathbb{S}(u) \subset L^2(X)$  possesses a smallest and a largest element and since, as a consequence, the minimal and the maximal element of  $\mathbb{S}(u)$  are unique. We would like to point out that [Theorem 2.4](#) shows that the set of solutions of  $(\mathbf{F})$  can be studied as a whole and that it is not necessary to restrict the analysis to those solutions  $y$  of  $(\mathbf{F})$  that satisfy  $\underline{y} \leq y \leq \bar{y}$  for the sub- and supersolutions  $\underline{y}$  and  $\bar{y}$  appearing in the theorem of Birkhoff-Tartar, cf. [[6](#), Theorem 2]. Having established the solvability of  $(\mathbf{F})$ , we now turn our attention to questions of Lipschitz stability.

**3. Lipschitz continuity of the minimal and maximal solution map.** To prove Lipschitz stability estimates for the minimal and the maximal solution mapping  $m: U \rightarrow L^2(X)$  and  $M: U \rightarrow L^2(X)$  associated with  $(\mathbf{F})$ , we require additional assumptions on the involved sets, spaces, and functions.



ASSUMPTION 3.1 (additional assumptions for proving local Lipschitz continuity).

In addition to the standing assumptions in [subsection 2.1](#), we require the following:

- (i)  $P$  is a subset of a real vector space satisfying  $\lambda p \in P$  for all  $p \in P$ ,  $\lambda \in (0, 1]$ .
- (ii)  $U$  is a subset of  $L^{\infty}_+(Y)$  for some complete measure space  $(Y, \Xi, \eta)$  and it holds  $\lambda u \in U$  for all  $u \in U$ ,  $\lambda \in (0, 1]$ .
- (iii)  $S$  satisfies  $\lambda S(p, u) \leq S(\lambda p, \lambda u)$  for all  $p \in P$ ,  $u \in U$ , and  $\lambda \in (0, 1]$ , and it holds  $S(p, u_1) \leq S(p, u_2)$  for all  $u_1, u_2 \in U$  with  $u_1 \leq u_2$  and all  $p \in P$ .
- (iv)  $\Phi$  satisfies  $\lambda \Phi(v) \leq \Phi(\lambda v)$  for all  $v \in L^2_+(X)$ ,  $\lambda \in (0, 1]$ .

For examples of problems satisfying [Assumption 3.1](#), we refer to [section 6](#). We remark that a superhomogeneity condition analogous to that in [Assumption 3.1\(iv\)](#) has already been used in [[6](#), [Theorem 4](#)]. Compare also with the earlier work [[25](#)] in this context. In the situation of [Assumption 3.1](#), we can employ [Theorem 2.4](#) to establish the following result.

THEOREM 3.2 (Lipschitz continuity of the minimal and maximal solution map).

Suppose that [Assumption 3.1](#) holds and let  $q \in [1, \infty]$  be an exponent satisfying  $S(p, u) \in L^q(X)$  for all  $p \in P$  and all  $u \in U$ . Then, for every  $u \in U$  satisfying  $u \geq c$  in  $L^{\infty}(Y)$  for some constant  $c > 0$  and every  $v \in U$  satisfying  $\|u - v\|_{L^{\infty}(Y)} \leq c - \rho$  for some  $0 < \rho < c$ , the minimal and maximal solutions  $m(u)$ ,  $m(v)$ ,  $M(u)$ , and  $M(v)$  of [\(F\)](#) associated with  $u$  and  $v$  satisfy the stability estimates

$$(3.1) \quad \|m(u) - m(v)\|_{L^q(X)} \leq \frac{1}{\rho} \|m(u)\|_{L^q(X)} \|u - v\|_{L^{\infty}(Y)}$$

and

$$(3.2) \quad \|M(u) - M(v)\|_{L^q(X)} \leq \frac{1}{\rho} \|M(u)\|_{L^q(X)} \|u - v\|_{L^{\infty}(Y)}.$$

*Proof.* Suppose that  $u, v \in U$  with constants  $0 < \rho < c$  as in the theorem are given. Set  $\varepsilon := \|u - v\|_{L^{\infty}(Y)} < c$  and  $\lambda := 1 - \varepsilon/c \in (0, 1]$ . Then it holds

$$(\lambda - 1)u \leq (\lambda - 1)c = -\varepsilon \leq v - u \leq \varepsilon = (1 - \lambda)c \leq \frac{1 - \lambda}{\lambda}c \leq \left(\frac{1}{\lambda} - 1\right)u$$

and, as a consequence,  $\lambda u \leq v \leq \lambda^{-1}u$  in  $L^{\infty}(Y)$ . Define  $y := M(u) \in L^2_+(X)$  and  $z := M(v) \in L^2_+(X)$ . From

$$\lambda y = \lambda S(\Phi(y), u) \leq S(\lambda \Phi(y), \lambda u) \leq S(\lambda \Phi(y), v) \leq S(\Phi(\lambda y), v) = T_v(\lambda y),$$

we get that  $\lambda y$  is a subsolution of [\(F\)](#) with parameter  $v$ . Since  $z = M(v)$ , this implies  $\lambda y \leq z$ , see [Theorem 2.4](#). Analogously, we obtain

$$\lambda z = \lambda S(\Phi(z), v) \leq S(\lambda \Phi(z), \lambda v) \leq S(\lambda \Phi(z), u) \leq S(\Phi(\lambda z), u) = T_u(\lambda z),$$

so that  $\lambda z$  is a subsolution of [\(F\)](#) with parameter  $u$ . Again by [Theorem 2.4](#), this yields  $\lambda z \leq M(u) = y$ . In summary, we have now proved that  $(\lambda - 1)y \leq z - y \leq (\lambda^{-1} - 1)y$ . Since  $|\lambda - 1| = 1 - \lambda \leq \lambda^{-1} - 1$ , it follows

$$(3.3) \quad |M(v) - M(u)| \leq (\lambda^{-1} - 1)|M(u)| = \frac{\varepsilon}{c - \varepsilon} |M(u)| \leq \frac{1}{\rho} \|u - v\|_{L^{\infty}(Y)} |M(u)|$$

$\mu$ -a.e. in  $X$ . For the minimal solutions, we can argue along the same lines: Set  $y := m(u) \in L^2_+(X)$  and  $z := m(v) \in L^2_+(X)$ . Then the properties of  $S$  and  $\Phi$  yield

$$\lambda S(\Phi(\lambda^{-1}y), v) \leq S(\lambda \Phi(\lambda^{-1}y), \lambda v) \leq S(\lambda \Phi(\lambda^{-1}y), u) \leq S(\Phi(y), u) = y$$

and

$$\lambda S(\Phi(\lambda^{-1}z), u) \leq S(\lambda\Phi(\lambda^{-1}z), \lambda u) \leq S(\lambda\Phi(\lambda^{-1}z), v) \leq S(\Phi(z), v) = z.$$

The functions  $\lambda^{-1}y$  and  $\lambda^{-1}z$  are thus supersolutions of (F) with parameters  $v$  and  $u$ , respectively, and we again obtain from [Theorem 2.4](#) that  $\lambda^{-1}y \geq m(v) = z$  and  $\lambda^{-1}z \geq m(u) = y$  holds and, as a consequence, that  $(\lambda - 1)y \leq z - y \leq (\lambda^{-1} - 1)y$ . This estimate and the same calculation as in [\(3.3\)](#) yield

$$(3.4) \quad |m(v) - m(u)| \leq \frac{1}{\rho} \|u - v\|_{L^\infty(Y)} |m(u)|$$

$\mu$ -a.e. in  $X$ . To finish the proof, it now suffices to integrate (or take the essential supremum) in [\(3.3\)](#) and [\(3.4\)](#) and to use that the assumptions on  $S$  and the equation (F) imply that  $m(u)$ ,  $m(v)$ ,  $M(u)$ , and  $M(v)$  are elements of  $L^q(X)$ .  $\square$

*Remark 3.3.*

- (i) The choice  $q = 2$  is always possible in [Theorem 3.2](#) by our standing assumptions on the mapping behavior of  $S$ , see [subsection 2.1](#).
- (ii) It is easy to check that the estimates [\(3.1\)](#) and [\(3.2\)](#) imply that, for every  $u \in U \cap L^\infty_\oplus(Y)$ , there exist constants  $C, r > 0$  satisfying

$$(3.5) \quad \|m(v_1) - m(v_2)\|_{L^q(X)} + \|M(v_1) - M(v_2)\|_{L^q(X)} \leq C \|v_1 - v_2\|_{L^\infty(Y)}$$

for all  $v_1, v_2 \in U$  with  $\|v_i - u\|_{L^\infty(Y)} \leq r$ ,  $i = 1, 2$ . The maps  $m$  and  $M$  are thus indeed locally Lipschitz continuous on the set  $U \cap L^\infty_\oplus(Y)$ .

- (iii) [Theorem 3.2](#) generalizes [\[6, Theorem 4\]](#) in the sense that it shows that the minimal and maximal solution map  $m$  and  $M$  of (F) are not only continuous as functions from  $U \cap L^\infty_\oplus(Y)$  into the space  $L^2(X)$  but even locally Lipschitz continuous as functions from  $U \cap L^\infty_\oplus(Y)$  into every space  $L^q(X)$ ,  $1 \leq q \leq \infty$ , that the operator  $S$  maps into. [Theorem 3.2](#) further illustrates that this local Lipschitz stability relies solely on the order properties in [Assumption 3.1](#) and does not require, e.g., the assumption that there is an underlying Gelfand triple structure, that the map  $\Phi$  possesses complete continuity properties, or that the map  $S$  is positively homogeneous, cf. [\[6, section 2.1, Assumption 1\]](#). As we will see in [section 6](#), the lack of these restrictions in particular allows us to apply [Theorem 3.2](#) to parabolic quasi-variational inequalities. Note that our proof of [Theorem 3.2](#) is also much simpler than the one in [\[6\]](#).
- (iv) If, in the situation of [Theorem 3.2](#), it is known that there exists a reflexive Banach space  $V \subset L^2(X)$  such that  $V$  is continuously embedded into  $L^2(X)$  and such that, for every  $u \in U \cap L^\infty_\oplus(Y)$ , there exist constants  $C, r > 0$  with  $\|S(p, v)\|_V \leq C$  for all  $v \in U$  with  $\|u - v\|_{L^\infty(Y)} \leq r$  and all  $p \in P$ , then it follows immediately that every sequence  $\{u_n\} \subset U \cap L^\infty_\oplus(Y)$  which converges in  $L^\infty(Y)$  to a point  $u \in U \cap L^\infty_\oplus(Y)$  not only satisfies  $m(u_n) \rightarrow m(u)$  and  $M(u_n) \rightarrow M(u)$  in  $L^2(X)$  but also  $m(u_n) \rightharpoonup m(u)$  and  $M(u_n) \rightharpoonup M(u)$  in  $V$ . Indeed, in this case, the sequences  $m(u_n) = S(\Phi(m(u_n)), u_n)$  and  $M(u_n) = S(\Phi(M(u_n)), u_n)$  are clearly bounded in  $V$  and the convergences  $m(u_n) \rightharpoonup m(u)$  and  $M(u_n) \rightharpoonup M(u)$  in  $V$  are straightforwardly obtained by applying the theorem of Banach-Alaoglu and trivial contradiction arguments. We will get back to this topic in [section 6](#), where we will show that such additional convergence properties are in particular available for elliptic and parabolic QVIs of obstacle type.

**4. Directional differentiability of  $M$  via pointwise concavity.** Next, we consider questions of differentiability. We would like to point out that we will not approach this topic by using concepts like polyhedricity etc. known from the analysis of elliptic variational inequalities, see [23, 27, 36], but rather by considering pointwise curvature properties similar to those already exploited in [Theorem 3.2](#). This approach to the sensitivity analysis of nonsmooth systems has already been used in [13, 14] to establish directional differentiability results for solution operators of obstacle-type evolution variational inequalities and is very natural for problems of the form (F) whose solvability can also be discussed with pointwise arguments, see [subsection 2.2](#). As in the last section, we require some additional assumptions for our analysis.

ASSUMPTION 4.1 (additional assumptions for proving pointwise differentiability). *In addition to the standing assumptions in [subsection 2.1](#), we require the following:*

- (i)  $P$  and  $U$  are convex subsets of real vector spaces.
- (ii)  $S$  satisfies  $\lambda S(p_1, u_1) + (1 - \lambda)S(p_2, u_2) \leq S(\lambda p_1 + (1 - \lambda)p_2, \lambda u_1 + (1 - \lambda)u_2)$  for all  $p_1, p_2 \in P$ ,  $u_1, u_2 \in U$ , and  $\lambda \in [0, 1]$ .
- (iii)  $\Phi$  satisfies  $\lambda \Phi(v_1) + (1 - \lambda)\Phi(v_2) \leq \Phi(\lambda v_1 + (1 - \lambda)v_2)$  for all  $v_1, v_2 \in L_+^2(X)$  and  $\lambda \in [0, 1]$ .

For examples of problems that satisfy the above conditions, we again refer the reader to [section 6](#).

PROPOSITION 4.2 (pointwise concavity of the maximal solution map  $M$ ). *Suppose that [Assumption 4.1](#) holds. Then, for all  $u_1, u_2 \in U$  and all  $\lambda \in [0, 1]$ , it is true that*

$$(4.1) \quad \lambda M(u_1) + (1 - \lambda)M(u_2) \leq M(\lambda u_1 + (1 - \lambda)u_2).$$

*Proof.* Let  $\lambda \in [0, 1]$  and  $u_1, u_2 \in U$  be given. From the properties of  $S$  and  $\Phi$ , we obtain that

$$(4.2) \quad \begin{aligned} \lambda M(u_1) + (1 - \lambda)M(u_2) &= \lambda S(\Phi(M(u_1)), u_1) + (1 - \lambda)S(\Phi(M(u_2)), u_2) \\ &\leq S(\lambda \Phi(M(u_1)) + (1 - \lambda)\Phi(M(u_2)), \lambda u_1 + (1 - \lambda)u_2) \\ &\leq S(\Phi(\lambda M(u_1) + (1 - \lambda)M(u_2)), \lambda u_1 + (1 - \lambda)u_2) \\ &= T_{\lambda u_1 + (1 - \lambda)u_2}(\lambda M(u_1) + (1 - \lambda)M(u_2)). \end{aligned}$$

The function  $\lambda M(u_1) + (1 - \lambda)M(u_2)$  is thus a subsolution of (F) with parameter  $\lambda u_1 + (1 - \lambda)u_2$  and we may invoke [Theorem 2.4](#) to arrive at (4.1).  $\square$

As a straightforward consequence of [Proposition 4.2](#), we obtain the following directional differentiability result.

THEOREM 4.3 (pointwise directional differentiability of the map  $M$ ). *Suppose that [Assumption 4.1](#) holds. Assume further that a  $u \in U$  and an  $h \in \mathbb{R}^+(U - u)$  satisfying  $u + \tau h \in U$  for all  $\tau \in (0, \tau_0)$ ,  $\tau_0 > 0$ , are given. Define*

$$\delta_\tau := \frac{M(u + \tau h) - M(u)}{\tau}, \quad \tau \in (0, \tau_0).$$

*Then there exists a unique  $\delta \in L^0(X, (-\infty, \infty])$  such that, for every  $\{\tau_n\} \subset (0, \tau_0)$  satisfying  $\tau_n \rightarrow 0$ , it holds  $\delta_{\tau_n} \rightarrow \delta$  pointwise  $\mu$ -a.e. in  $X$ .*

*Proof.* Suppose that  $0 < \tau_2 < \tau_1 < \tau_0$  are given. Then it holds  $u + \tau_2 h = (1 - \tau_2/\tau_1)u + (\tau_2/\tau_1)(u + \tau_1 h)$  and it follows from (4.1) that

$$M(u + \tau_2 h) \geq \left(1 - \frac{\tau_2}{\tau_1}\right) M(u) + \frac{\tau_2}{\tau_1} M(u + \tau_1 h).$$

The last inequality can also be written as

$$(4.3) \quad \delta_{\tau_2} = \frac{M(u + \tau_2 h) - M(u)}{\tau_2} \geq \frac{M(u + \tau_1 h) - M(u)}{\tau_1} = \delta_{\tau_1}.$$

This shows that the family  $\{\delta_\tau\}$  is  $\mu$ -a.e. nonincreasing in  $X$  w.r.t.  $\tau > 0$  in the sense that  $\delta_{\tau_2} \geq \delta_{\tau_1}$  holds  $\mu$ -a.e. in  $X$  for all  $0 < \tau_2 < \tau_1 < \tau_0$ . Consider now an  $N \in \mathbb{N}$  with  $1/N < \tau_0$  and define  $\delta \in L^0(X, (-\infty, \infty])$  via

$$\delta(x) := \sup_{n \geq N} \delta_{1/n}(x) \quad \text{for } \mu\text{-a.a. } x \in X.$$

Then (4.3) yields that  $\delta_{1/n} \rightarrow \delta$  holds  $\mu$ -a.e. in  $X$  for  $N \leq n \rightarrow \infty$ . Now, let a sequence  $\{\tau_k\} \subset (0, \tau_0)$  with  $\tau_k \rightarrow 0$  be given. For every  $k \in \mathbb{N}$ , there exists  $N \leq n \in \mathbb{N}$  with  $\tau_k \geq 1/n$ . This yields  $\delta_{\tau_k} \leq \delta_{1/n}$  and, therefore,  $\delta_{\tau_k} \leq \delta$   $\mu$ -a.e. in  $X$ . Moreover, for every  $N \leq n \in \mathbb{N}$  there exists  $K \in \mathbb{N}$  such that  $k \geq K$  implies  $\tau_k \leq 1/n$  and, thus,  $\delta_{\tau_k} \geq \delta_{1/n}$   $\mu$ -a.e. in  $X$ . Together with the first inequality, this yields  $\delta \geq \delta_{\tau_k} \geq \delta_{1/n}$  for all  $k \geq K$   $\mu$ -a.e. in  $X$  and, since  $n \geq N$  was arbitrary,  $\delta_{\tau_k} \rightarrow \delta$   $\mu$ -a.e. in  $X$  for all  $\{\tau_k\} \subset (0, \tau_0)$  with  $\tau_k \rightarrow 0$  as claimed. The uniqueness of  $\delta$  is trivial.  $\square$

In situations in which both the Lipschitz stability result in [Theorem 3.2](#) and the directional differentiability result in [Theorem 4.3](#) are applicable, we immediately get Hadamard directional differentiability for the maximal solution map  $M$  of [\(F\)](#).

**COROLLARY 4.4** (Hadamard directional differentiability of the map  $M$ ). *Suppose that [Assumptions 3.1](#) and [4.1](#) hold and let  $1 \leq r \leq 2 \leq s \leq \infty$  be numbers such that  $S(p, u) \in L^{[r, s]}(X)$  holds for all  $p \in P$ ,  $u \in U$ . Then the maximal solution map  $M$  of [\(F\)](#) is Hadamard directionally differentiable on the set  $U \cap L_{\oplus}^\infty(Y)$  in the following sense: For every  $u \in U \cap L_{\oplus}^\infty(Y)$  and every  $h \in \mathbb{R}^+(U - u)$ , there exists a unique  $M'(u; h) \in L^{[r, s]}(X)$  such that, for all  $\{\tau_n\} \subset (0, \infty)$  and  $\{h_n\} \subset \mathbb{R}^+(U - u)$  satisfying  $\tau_n \rightarrow 0$ ,  $\|h - h_n\|_{L^\infty(Y)} \rightarrow 0$ , and  $u + \tau_n h_n \in U$  for all  $n$ , we have*

$$(4.4) \quad \frac{M(u + \tau_n h_n) - M(u)}{\tau_n} \rightarrow M'(u; h) \text{ in } L^q(X) \text{ for all } q \in [r, s] \setminus \{\infty\}$$

and

$$(4.5) \quad \frac{M(u + \tau_n h_n) - M(u)}{\tau_n} \xrightarrow{*} M'(u; h) \text{ in } L^1(X)^* \text{ in the case } s = \infty.$$

Here, (4.5) is to be understood in the sense of equation (4.7) below.

*Proof.* Suppose that  $u$ ,  $h$ ,  $\{h_n\}$ , and  $\{\tau_n\}$  are as in the statement of the corollary. We assume w.l.o.g. that  $u + \tau_n h \in U$  holds for all  $n$ , set  $\delta_n := (M(u + \tau_n h) - M(u))/\tau_n$ , and denote with  $\delta \in L^0(X, (-\infty, \infty])$  the unique limit function from [Theorem 4.3](#) associated with  $u$  and  $h$ . Due to [Theorem 4.3](#), we know that  $\delta_n \rightarrow \delta$  holds  $\mu$ -a.e. in  $X$  and, due to [Theorem 3.2](#), that  $\|\delta_n\|_{L^{[r, s]}(X)} \leq C$  holds for a constant  $C \geq 0$ . In the case  $s = \infty$ , this yields  $\|\delta\|_{L^\infty(X)} \leq C$ , and, for all  $s$ , by the lemma of Fatou,

$$\|\delta\|_{L^q(X)}^q = \int_X \lim_{n \rightarrow \infty} |\delta_n|^q d\mu \leq \liminf_{n \rightarrow \infty} \int_X |\delta_n|^q d\mu \leq C^q \quad \forall q \in [r, s] \setminus \{\infty\}.$$

In summary, the above shows that, under the assumptions of the corollary,  $\delta$  is real-valued  $\mu$ -a.e. in  $X$  and can be identified with an element of  $L^{[r, s]}(X)$ . Next, we establish (4.4) for all  $q \in [r, s] \setminus \{\infty\}$ . Using that  $0 \leq |\delta - \delta_n| = \delta - \delta_n \leq \delta - \delta_1$

holds  $\mu$ -a.e. in  $X$  for all sufficiently large  $n$  by (4.3), the convergence  $\delta_n \rightarrow \delta$  in  $L^q(X)$  for all  $q \in [r, s] \setminus \{\infty\}$  follows immediately from the dominated convergence theorem. Using [Theorem 3.2](#) (or, more precisely, its consequence (3.5)), we further get

$$(4.6) \quad \begin{aligned} & \left\| \frac{M(u + \tau_n h_n) - M(u)}{\tau_n} - \delta \right\|_{L^q(X)} \\ & \leq \left\| \frac{M(u + \tau_n h) - M(u)}{\tau_n} - \delta \right\|_{L^q(X)} + \left\| \frac{M(u + \tau_n h_n) - M(u + \tau_n h)}{\tau_n} \right\|_{L^q(X)} \\ & \leq \|\delta_n - \delta\|_{L^q(X)} + C \|h_n - h\|_{L^\infty(Y)} \end{aligned}$$

for all large enough  $n$  with some constant  $C > 0$ . The right-hand side of this estimate converges to zero for  $n \rightarrow \infty$ . This proves (4.4) with  $M'(u; h) := \delta \in L^{[r, s]}(X)$ .

It remains to prove (4.5) in the case  $s = \infty$ . For this exponent, we obtain from the same arguments as in (4.6) that the sequence  $(M(u + \tau_n h_n) - M(u))/\tau_n$  is bounded in  $L^\infty(X)$  and pointwise  $\mu$ -a.e. convergent to  $M'(u; h) := \delta \in L^{[r, \infty]}(X)$ . In combination with the dominated convergence theorem, this yields

$$(4.7) \quad \lim_{n \rightarrow \infty} \int_X z \left( \frac{M(u + \tau_n h_n) - M(u)}{\tau_n} - M'(u; h) \right) d\mu = 0 \quad \forall z \in L^1(X).$$

This establishes the desired convergence (4.5) and completes the proof.  $\square$

Recall that the canonical mapping of  $L^\infty(X)$  into the topological dual of  $L^1(X)$  is an isometric isomorphism if (and only if) the measure  $\mu$  is localizable, see [22, 243G]. This is in particular the case if  $\mu$  is  $\sigma$ -finite, see [22, 211L]. If one of these conditions holds, then the space  $L^1(X)^*$  in (4.5) can be replaced by  $L^\infty(X)$ .

Note that the choice  $r = s = 2$  is always allowed in [Corollary 4.4](#) by our standing assumptions on the map  $S$ , see [subsection 2.1](#). It is further easy to check that, if a Lipschitz estimate for  $M$  in a reflexive Banach space  $V$  is available that is continuously embedded into  $L^2(X)$ , then [Corollary 4.4](#) implies that the difference quotients on the left-hand side of (4.4) also converge weakly in  $V$  to  $M'(u; h)$ , cf. [Remark 3.3\(iv\)](#). Lastly, we would like to point out that the arguments that we have used in this section only work for the maximal solution map  $M$ . Indeed, by proceeding along the lines of (4.2), one only obtains that  $\lambda m(u_1) + (1 - \lambda)m(u_2)$  is a subsolution of (F) with parameter  $\lambda u_1 + (1 - \lambda)u_2$  for all  $u_1, u_2 \in U$  and  $\lambda \in [0, 1]$  and this information does not allow to conclude that  $m$  is convex or concave, cf. [Theorem 2.4](#). To obtain differentiability results for the minimal solution map  $m$ , one can use, e.g., the results of [37]. This, however, requires far more restrictive assumptions.

**5. Unique characterization of directional derivatives.** Having established the directional differentiability of the maximal solution operator  $M$ , we next aim at deriving an auxiliary problem that uniquely characterizes the derivatives  $M'(u; h)$ . In view of classical results for differential equations, one would expect that such an auxiliary problem can be obtained by “differentiating” the left- and the right-hand side of (F). Given sufficiently smooth  $S$  and  $\Phi$ , this means that the natural candidate for an auxiliary problem for  $\delta := M'(u; h)$  is (at least formally) the fixed-point equation  $\delta = \Psi'((M(u), u); (\delta, h))$ , where  $\Psi$  denotes the composition  $\Psi(v, u) := S(\Phi(v), u)$ . Unfortunately, it turns out that this linearized problem may fail to contain any form of useful information even in cases in which [Corollary 4.4](#) is applicable and the directional derivatives of  $M$  exist. In fact, we have an instance of (F) for which the equation  $\delta = \Psi'((M(u), u); (\delta, h))$  reduces to  $\delta = \delta$ , see [subsection 6.1](#). To arrive at an auxiliary problem that is meaningful, we have to strengthen [Assumptions 3.1](#) and [4.1](#).

ASSUMPTION 5.1 (additional assumptions for the characterization of derivatives).

In addition to the standing assumptions in [subsection 2.1](#), we require the following:

- (i)  $P$  is a convex subset of a real vector space, it holds  $0 \in P$ , and the partial order on  $P$  satisfies  $\lambda p_1 + (1 - \lambda)p_2 \geq \lambda p_1$  for all  $p_1, p_2 \in P$  and  $\lambda \in [0, 1]$ .
- (ii)  $U$  is a convex subset of  $L_+^\infty(Y)$  for some complete measure space  $(Y, \Xi, \eta)$  and it holds  $0 \in U$ .
- (iii)  $S$  satisfies  $\lambda S(p_1, u_1) + (1 - \lambda)S(p_2, u_2) \leq S(\lambda p_1 + (1 - \lambda)p_2, \lambda u_1 + (1 - \lambda)u_2)$  for all  $p_1, p_2 \in P$ ,  $u_1, u_2 \in U$ , and  $\lambda \in [0, 1]$ ; it holds  $S(p, u_1) \leq S(p, u_2)$  for all  $u_1, u_2 \in U$  with  $u_1 \leq u_2$  and all  $p \in P$ ; and we have  $S(p, u) \in L_+^\infty(X)$  for all  $p \in P$  and all  $u \in U$ .
- (iv) There exists an  $\varepsilon > 0$  such that  $\lambda \Phi(v_1) + (1 - \lambda)\Phi(v_2) \leq \Phi(\lambda v_1 + (1 - \lambda)v_2)$  holds for all  $\lambda \in [0, 1]$  and all  $v_1, v_2 \in L^2(X)$  with  $v_1 \geq -\varepsilon$  and  $v_2 \geq -\varepsilon$ .

Note that the above conditions imply in particular that [Assumptions 3.1](#) and [4.1](#) are satisfied (as one may easily check by combining the properties in [Assumption 5.1](#) with the standing assumptions from [subsection 2.1](#)). [Assumption 5.1](#) also entails that the problem [\(F\)](#) is uniquely solvable for all  $u \in U$  as the following result shows.

PROPOSITION 5.2 (unique solvability of [\(F\)](#)). *Suppose that [Assumption 5.1](#) holds. Then [\(F\)](#) is uniquely solvable for all  $u \in U$ . In particular, it holds  $m \equiv M$  on  $U$ .*

*Proof.* For all  $v \in L_+^{[2, \infty]}(X)$  and all  $\alpha \in [0, 1]$ , we can find a  $\beta \in (\alpha, 1)$  such that  $-\varepsilon \leq v(\alpha - \beta)/(1 - \beta) \leq 0$  holds for the  $\varepsilon > 0$  in [Assumption 5.1\(iv\)](#). Due to the mapping properties of  $\Phi$  and our assumptions on  $P$  and its partial order, this implies

$$\Phi(\alpha v) = \Phi\left(\beta v + (1 - \beta)\left(\frac{\alpha - \beta}{1 - \beta}\right)v\right) \geq \beta\Phi(v) + (1 - \beta)\Phi\left(\left(\frac{\alpha - \beta}{1 - \beta}\right)v\right) \geq \beta\Phi(v).$$

We may thus conclude that, for all  $v \in L_+^{[2, \infty]}(X)$  and all  $\alpha \in [0, 1]$ , there exists a  $\beta \in (\alpha, 1)$  satisfying  $\Phi(\alpha v) \geq \beta\Phi(v)$ . To prove the uniqueness of solutions of [\(F\)](#), we can now use an argument of [\[25\]](#). Suppose that  $u \in U$  is fixed and that [\(F\)](#) possesses two solutions, i.e., that there exist  $y_1, y_2$  with  $S(\Phi(y_1), u) = y_1 \neq y_2 = S(\Phi(y_2), u)$ . Then it holds  $y_1, y_2 \in L_+^{[2, \infty]}(X)$  by the properties of  $S$  and we may assume w.l.o.g. that  $y_1 \not\leq y_2$ . In this situation, the number  $\alpha := \sup\{\gamma \in \mathbb{R} \mid \gamma y_1 \leq y_2\}$  has to satisfy  $\alpha \in [0, 1)$  and we may choose a  $\beta \in (\alpha, 1)$  with  $\Phi(\alpha y_1) \geq \beta\Phi(y_1)$  to obtain

$$\begin{aligned} y_2 = S(\Phi(y_2), u) &\geq S(\Phi(\alpha y_1), u) \geq S(\beta\Phi(y_1), u) \geq \beta S(\Phi(y_1), u) + (1 - \beta)S(0, u) \\ &\geq \beta S(\Phi(y_1), u) = \beta y_1. \end{aligned}$$

Here, we have used the monotonicity properties of  $S$  and  $\Phi$  and the concavity and nonnegativity of  $S$ . The above implies  $\beta y_1 \leq y_2$  and, by the properties of  $\alpha$  and  $\beta$ ,  $\beta \leq \alpha < \beta$  which is impossible. Solutions of [\(F\)](#) are thus indeed unique. As the solvability of [\(F\)](#) has been established in [Theorem 2.4](#), this completes the proof.  $\square$

We remark that [\(F\)](#) can possess multiple solutions if [Assumption 5.1\(iv\)](#) only holds with  $\varepsilon = 0$ , see the example in [subsection 6.1](#). Next, we prove an auxiliary result on the properties of the composition  $\Psi(v, u) := S(\Phi(v), u)$ .

LEMMA 5.3 (pointwise concavity and directional differentiability of  $\Psi$ ). *Suppose that [Assumption 5.1](#) holds. Define  $\Psi: L^2(X) \times U \rightarrow L_+^2(X)$ ,  $\Psi(v, u) := S(\Phi(v), u)$ . Then, for all  $\lambda \in [0, 1]$ , all  $u_1, u_2 \in U$ , and all  $v_1, v_2 \in L^2(X)$  which satisfy  $v_1 \geq -\varepsilon$  and  $v_2 \geq -\varepsilon$ , where the constant  $\varepsilon > 0$  is given by [Assumption 5.1\(iv\)](#), it holds*

$$\lambda \Psi(v_1, u_1) + (1 - \lambda)\Psi(v_2, u_2) \leq \Psi(\lambda v_1 + (1 - \lambda)v_2, \lambda u_1 + (1 - \lambda)u_2).$$

Further, for all tuples  $(v, u) \in L_+^2(X) \times U$  and  $(h_1, h_2) \in L^{[2, \infty]}(X) \times \mathbb{R}^+(U - u)$  satisfying  $u + \tau_0 h_2 \in U$  for a  $\tau_0 > 0$ , there exists a unique element  $\Psi'((v, u); (h_1, h_2))$  of the set  $L^0(X, (-\infty, \infty])$  such that the difference quotients

$$(5.1) \quad \frac{\Psi(v + \tau h_1, u + \tau h_2) - \Psi(v, u)}{\tau} \in L^2(X), \quad \tau \in (0, \tau_0),$$

converge  $\mu$ -a.e. to  $\Psi'((v, u); (h_1, h_2))$  along every sequence  $\{\tau_n\} \subset (0, \tau_0)$  with  $\tau_n \rightarrow 0$ .

*Proof.* If  $\lambda \in [0, 1]$ ,  $u_1, u_2 \in U$ , and  $v_1, v_2 \in L^2(X)$  as in the first part of the lemma are given, then it follows from our assumptions on  $S$  and  $\Phi$  that

$$\begin{aligned} \Psi(\lambda v_1 + (1 - \lambda)v_2, \lambda u_1 + (1 - \lambda)u_2) &= S(\Phi(\lambda v_1 + (1 - \lambda)v_2), \lambda u_1 + (1 - \lambda)u_2) \\ &\geq S(\lambda \Phi(v_1) + (1 - \lambda)\Phi(v_2), \lambda u_1 + (1 - \lambda)u_2) \\ &\geq \lambda S(\Phi(v_1), u_1) + (1 - \lambda)S(\Phi(v_2), u_2) \\ &= \lambda \Psi(v_1, u_1) + (1 - \lambda)\Psi(v_2, u_2). \end{aligned}$$

This proves the concavity of  $\Psi$  on  $\{v \in L^2(X) \mid v \geq -\varepsilon\} \times U$ . To prove the pointwise  $\mu$ -a.e. directional differentiability of  $\Psi$ , we can proceed analogously to [Theorem 4.3](#). Indeed, if tuples  $(v, u) \in L_+^2(X) \times U$  and  $(h_1, h_2) \in L^{[2, \infty]}(X) \times \mathbb{R}^+(U - u)$  as in the second part of the lemma are given, then the nonnegativity of  $v$  and the  $L^\infty(X)$ -regularity of  $h_1$  imply that  $v + \tau h_1 \geq -\varepsilon$  holds for all  $0 < \tau < \varepsilon/\|h_1\|_{L^\infty(X)}$ . Due to the concavity of  $\Psi$  on  $\{v \in L^2(X) \mid v \geq -\varepsilon\} \times U$ , this yields that the difference quotients in (5.1) are nonincreasing w.r.t.  $\tau \in (0, \min(\tau_0, \varepsilon/\|h_1\|_{L^\infty(X)}))$ , cf. (4.3). Using exactly the same arguments as in the proof of [Theorem 4.3](#), the existence of a  $\Psi'((v, u); (h_1, h_2))$  with the desired properties now follows immediately.  $\square$

We are now in the position to prove the main result of this section.

**THEOREM 5.4** (unique characterization of directional derivatives). *Suppose that [Assumption 5.1](#) holds and that  $r \in [1, 2]$  is an exponent satisfying  $S(p, u) \in L^r(X)$  for all  $p \in P$ ,  $u \in U$ . Denote the unique solution of (F) with parameter  $u \in U$  by  $\mathbb{S}(u)$  and let  $\Psi: L^2(X) \times U \rightarrow L_+^2(X)$  be defined as in [Lemma 5.3](#). Then the function  $\mathbb{S}: U \rightarrow L_+^{[r, \infty]}(X)$  is Hadamard directionally differentiable on the set  $U \cap L_{\oplus}^\infty(Y)$  in the sense that the assertion of [Corollary 4.4](#) holds for  $\mathbb{S}$  with the numbers  $r$  and  $s = \infty$ . Further, the directional derivative  $\delta := \mathbb{S}'(u; h) \in L^{[r, \infty]}(X)$  at a point  $u \in U \cap L_{\oplus}^\infty(Y)$  in a direction  $h \in \mathbb{R}^+(U - u)$  satisfying  $u + \tau_0 h \in U$  for a  $\tau_0 > 0$  is characterized by the condition that it is the (necessarily unique) smallest element of the set*

$$\mathcal{A} := \left\{ \zeta \in L^{[r, \infty]}(X) \left| \begin{array}{l} \exists \{\zeta_n\} \subset L^{[r, \infty]}(X), \{\tau_n\} \subset (0, \tau_0): \\ \zeta_n \leq \zeta_{n+1} \text{ and } \tau_{n+1} \leq \tau_n \text{ for all } n, \\ \tau_n \rightarrow 0 \text{ for } n \rightarrow \infty, \zeta_n \rightarrow \zeta \text{ in } L^q(X) \text{ for all } q \in [r, \infty), \\ \mathbb{S}(u) + \tau_n \zeta_n \text{ is supersol. of (F) with par. } u + \tau_n h \text{ for all } n, \\ \Psi'((\mathbb{S}(u), u); (\zeta_n, h)) - \zeta_n \rightarrow 0 \text{ } \mu\text{-a.e. in } X \text{ for } n \rightarrow \infty \end{array} \right. \right\}$$

w.r.t. the pointwise  $\mu$ -a.e. partial order on  $X$ . This set  $\mathcal{A}$  is a superset of the set

$$\mathcal{B} := \{ \zeta \in L^{[r, \infty]}(X) \mid \zeta = \Psi'((\mathbb{S}(u), u); (\zeta, h)) \text{ } \mu\text{-a.e. in } X \},$$

and, if  $\Psi$  is semicontinuous in the sense that, for every  $w \in U$ , we have

$$(5.2) \quad \begin{aligned} \{z_n\} \text{ bounded in } L^{[r, \infty]}(X), \quad z_n \leq z_{n+1} \quad \forall n, \quad z_n \rightarrow z \text{ in } L^q(X) \quad \forall q \in [r, \infty) \\ \Rightarrow \limsup_{n \rightarrow \infty} \Psi(z_n, w) \geq \Psi(z, w) \text{ } \mu\text{-a.e. in } X, \end{aligned}$$

then it holds  $\mathcal{A} = \mathcal{B}$  and  $\delta = \mathbb{S}'(u; h)$  is the smallest element of the set  $\mathcal{B}$ .



*Proof.* The unique solvability of (F) has been proved in Proposition 5.2, the  $L_+^{[r,\infty]}(X)$ -regularity of  $\mathbb{S}(u)$  for all  $u \in U$  follows from (F) and our assumptions on  $S$ , and the asserted Hadamard directional differentiability of  $\mathbb{S}$  on  $U \cap L_{\oplus}^{\infty}(Y)$  and the  $L^{[r,\infty]}(X)$ -regularity of the directional derivatives follow immediately from Corollary 4.4 and the identity  $\mathbb{S} \equiv M$ . It remains to prove the characterization result. To this end, let us assume that a  $u \in U \cap L_{\oplus}^{\infty}(Y)$ , an  $h \in \mathbb{R}^+(U - u)$  satisfying  $u + \tau_0 h \in U$  for some  $\tau_0 > 0$ , and an arbitrary but fixed sequence  $\{\tau_n\} \subset (0, \tau_0)$  satisfying  $\tau_n \rightarrow 0$  and  $\tau_{n+1} \leq \tau_n$  for all  $n$  are given and define

$$(5.3) \quad \delta := \mathbb{S}'(u; h) \in L^{[r,\infty]}(X), \quad \delta_n := \frac{\mathbb{S}(u + \tau_n h) - \mathbb{S}(u)}{\tau_n} \in L^{[r,\infty]}(X) \quad \forall n \in \mathbb{N}.$$

We first show that  $\delta \in \mathcal{A}$ . Since  $\{\tau_n\}$  is nonincreasing, since  $\mathbb{S} \equiv M$  holds, and since Corollary 4.4 can be applied with  $r$  and  $s = \infty$ , we obtain from (4.3) that  $\delta_n \leq \delta_{n+1}$  holds for all  $n$  and from (4.4) that  $\delta_n \rightarrow \delta$  holds in  $L^q(X)$  for all  $q \in [r, \infty)$ . Due to the definition of  $\delta_n$ , we further have  $\mathbb{S}(u) + \tau_n \delta_n = \mathbb{S}(u + \tau_n h)$  so that  $\mathbb{S}(u) + \tau_n \delta_n$  is a supersolution of (F) with parameter  $u + \tau_n h$ . This proves that  $\delta$ ,  $\{\delta_n\}$ , and  $\{\tau_n\}$  satisfy all of the conditions in  $\mathcal{A}$  except for the last one. To get this last condition, we note that the concavity of  $\Psi$  on  $\{v \in L^2(X) \mid v \geq -\varepsilon\} \times U$ , the nonnegativity of  $\mathbb{S}(u)$ , the fact that  $\{\delta_n\}$  is bounded in  $L^{[2,\infty]}(X)$  by Theorem 3.2, and (5.3) yield that

$$(5.4) \quad \begin{aligned} \delta_n &= \frac{\mathbb{S}(u + \tau_n h) - \mathbb{S}(u)}{\tau_n} = \frac{\Psi(\mathbb{S}(u + \tau_n h), u + \tau_n h) - \Psi(\mathbb{S}(u), u)}{\tau_n} \\ &= \frac{\Psi(\mathbb{S}(u) + \tau_n \delta_n, u + \tau_n h) - \Psi(\mathbb{S}(u), u)}{\tau_n} \leq \Psi'((\mathbb{S}(u), u); (\delta_n, h)) \end{aligned}$$

holds  $\mu$ -a.e. in  $X$  for all large enough  $n$ . Here, we have again used that pointwise  $\mu$ -a.e. concavity implies that difference quotients are majorized by the directional derivatives that they approximate, cf. (4.3). Since  $\delta_n \leq \delta_k \leq \delta$  holds for all  $n \leq k$  and since  $\Psi$  is nondecreasing in its first argument by the properties of  $S$  and  $\Phi$ , we further obtain

$$(5.5) \quad \begin{aligned} \delta \geq \delta_k &= \frac{\Psi(\mathbb{S}(u) + \tau_k \delta_k, u + \tau_k h) - \Psi(\mathbb{S}(u), u)}{\tau_k} \\ &\geq \frac{\Psi(\mathbb{S}(u) + \tau_k \delta_n, u + \tau_k h) - \Psi(\mathbb{S}(u), u)}{\tau_k} \rightarrow \Psi'((\mathbb{S}(u), u); (\delta_n, h)) \end{aligned}$$

for all  $k \geq n$ , where the limit is  $\mu$ -a.e. and for  $k \rightarrow \infty$ , see Lemma 5.3. From (5.4), (5.5), and Theorem 4.3, we obtain that  $0 \leq \Psi'((\mathbb{S}(u), u); (\delta_n, h)) - \delta_n \leq \delta - \delta_n \rightarrow 0$  holds  $\mu$ -a.e. in  $X$  for  $n \rightarrow \infty$ . This shows that  $\delta$  is indeed an element of  $\mathcal{A}$ .

Suppose now that  $\zeta$  is an arbitrary element of  $\mathcal{A}$  with associated sequences  $\{\zeta_n\}$  and  $\{\tau_n\}$  and let  $\delta_n$  and  $\delta$  be defined as in (5.3). Then  $\mathbb{S}(u) + \tau_n \zeta_n$  is a supersolution of (F) with parameter  $u + \tau_n h$  and it holds  $\mathbb{S}(u) + \tau_n \zeta_n \geq m(u + \tau_n h) = \mathbb{S}(u + \tau_n h) = \mathbb{S}(u) + \tau_n \delta_n$  for all  $n$  by Theorem 2.4 and Proposition 5.2. Thus,  $\delta_n \leq \zeta_n$  and, after passing to the limit,  $\delta \leq \zeta$ . This shows that  $\delta$  is the smallest element of  $\mathcal{A}$ .

It remains to study the inclusions between  $\mathcal{A}$  and  $\mathcal{B}$ . To this end, let us suppose that an element  $\zeta$  of  $\mathcal{B}$  is given. Then the concavity of  $\Psi$  on  $\{v \in L^2(X) \mid v \geq -\varepsilon\} \times U$  and the nonnegativity of  $\mathbb{S}(u)$  yield that, for all sufficiently small  $\tau > 0$ , we have

$$(5.6) \quad \zeta = \Psi'((\mathbb{S}(u), u); (\zeta, h)) \geq \frac{\Psi(\mathbb{S}(u) + \tau \zeta, u + \tau h) - \Psi(\mathbb{S}(u), u)}{\tau}.$$

Since (5.6) can be recast as  $\mathbb{S}(u) + \tau \zeta \geq S(\Phi(\mathbb{S}(u) + \tau \zeta), u + \tau h)$  due to the identity  $\mathbb{S}(u) = \Psi(\mathbb{S}(u), u)$ , it follows that  $\mathbb{S}(u) + \tau \zeta$  is a supersolution of (F) with parameter

$u + \tau h \in U$  for all sufficiently small  $\tau > 0$ . By choosing a nonincreasing sequence  $\{\tau_n\} \subset (0, \tau_0)$  that converges to zero and has a sufficiently small first element and by defining  $\zeta_n := \zeta$ , it now follows immediately that  $\zeta \in \mathcal{A}$  and, thus,  $\mathcal{B} \subset \mathcal{A}$ .

Let us now finally assume that (5.2) holds and that a  $\zeta \in \mathcal{A}$  with associated sequences  $\{\zeta_n\}$  and  $\{\tau_n\}$  is given. Then the monotonicity of  $\{\zeta_n\}$ , the convergence  $\zeta_n \rightarrow \zeta$  in  $L^q(X)$  for all  $q \in [r, \infty)$ , and the regularity  $\zeta \in L^{[r, \infty]}(X)$  yield that  $\zeta_n \rightarrow \zeta$  holds  $\mu$ -a.e. in  $X$ , that  $\zeta_1 \leq \zeta_n \leq \zeta$  holds for all  $n$ , and that  $\{\zeta_n\}$  is bounded in  $L^{[r, \infty]}(X)$ . In combination with Lemma 5.3 and the fact that  $\Psi$  is nondecreasing in its first argument by the properties of  $S$  and  $\Phi$ , this allows us to deduce that

$$\begin{aligned} \Psi'((S(u), u); (\zeta, h)) &= \lim_{k \rightarrow \infty} \frac{\Psi(S(u) + \tau_k \zeta, u + \tau_k h) - \Psi(S(u), u)}{\tau_k} \\ (5.7) \quad &\geq \lim_{k \rightarrow \infty} \frac{\Psi(S(u) + \tau_k \zeta_n, u + \tau_k h) - \Psi(S(u), u)}{\tau_k} \\ &= \Psi'((S(u), u); (\zeta_n, h)) \end{aligned}$$

holds  $\mu$ -a.e. in  $X$  for all  $n \in \mathbb{N}$ . Since the convergence  $\Psi'((S(u), u); (\zeta_n, h)) - \zeta_n \rightarrow 0$   $\mu$ -a.e. in  $X$  in the last condition of  $\mathcal{A}$  and the convergence  $\zeta_n \rightarrow \zeta$   $\mu$ -a.e. in  $X$  imply that  $\Psi'((S(u), u); (\zeta_n, h)) \rightarrow \zeta$  holds  $\mu$ -a.e. in  $X$ , (5.7) yields  $\Psi'((S(u), u); (\zeta, h)) \geq \zeta$   $\mu$ -a.e. in  $X$ . To show that we also have the reverse inequality, we note that the convergence  $\Psi'((S(u), u); (\zeta_n, h)) \rightarrow \zeta$   $\mu$ -a.e. in  $X$ , the same majorization argument as in (5.4) and (5.6), the properties of  $\{\zeta_n\}$ , and (5.2) imply that

$$\begin{aligned} \zeta &= \lim_{n \rightarrow \infty} \Psi'((S(u), u); (\zeta_n, h)) \geq \limsup_{n \rightarrow \infty} \frac{\Psi(S(u) + \tau_l \zeta_n, u + \tau_l h) - \Psi(S(u), u)}{\tau_l} \\ (5.8) \quad &\geq \frac{\Psi(S(u) + \tau_l \zeta, u + \tau_l h) - \Psi(S(u), u)}{\tau_l} \end{aligned}$$

holds for all sufficiently large  $l \in \mathbb{N}$ . Here, all limits etc. have to be understood in the  $\mu$ -a.e.-sense. By letting  $l$  go to infinity in (5.8), we obtain that  $\zeta \geq \Psi'((S(u), u); (\zeta, h))$  and, thus, that  $\zeta \in L^{[r, \infty]}(X)$ ,  $\zeta = \Psi'((S(u), u); (\zeta, h))$   $\mu$ -a.e. in  $X$ , and  $\zeta \in \mathcal{B}$ . This shows that  $\mathcal{A} = \mathcal{B}$  holds under condition (5.2) and completes the proof.  $\square$

By our standing assumptions, we can always choose  $r = 2$  in Theorem 5.4, see subsection 2.1. If the measure space  $(X, \Sigma, \mu)$  is finite, then the  $L^\infty$ -regularity of  $S$  in Assumption 5.1(iii) and Hölder's inequality imply that Theorem 5.4 can be invoked with  $r = 1$ . (In this case, we have  $L^{[r, \infty]}(X) = L^\infty(X)$  for all  $r \in [1, \infty]$ .) Note that, due to the last condition in the definition of  $\mathcal{A}$  and the inclusion  $\mathcal{B} \subset \mathcal{A}$ , the set  $\mathcal{A}$  can be interpreted as a set of generalized solutions of the QVI  $\zeta = \Psi'((S(u), u); (\zeta, h))$ . Under the pointwise semicontinuity condition (5.2), this set of “limiting” solutions coincides with the ordinary solution set  $\mathcal{B}$ . Finally, we would like to point out that the monotonicity properties of  $\Psi$  and  $\{z_n\}$  yield that the right-hand side of (5.2) can be replaced by “ $\Psi(z_n, w) \rightarrow \Psi(z, w)$   $\mu$ -a.e. in  $X$ ” without changing the strength of this requirement. The semicontinuity condition (5.2) is thus, in fact, an assumption on the pointwise  $\mu$ -a.e. continuity of  $\Psi$  along certain sequences.

**6. Applications and examples.** In this section, we illustrate by means of three examples that our Hadamard directional differentiability and Lipschitz continuity results can be applied to many interesting problems. We begin with a counterexample which demonstrates that Theorems 3.2 and 4.3, Proposition 4.2, and Corollary 4.4 also cover situations in which (F) possesses a continuum of solutions and that the linearized fixed-point equations  $\delta = \Psi'((M(u), u); (\delta, h))$  may indeed fail to contain any form of useful information.

**6.1. One-dimensional example.** Define  $X := Y := \{0\}$ ,  $\Sigma := \Xi := \mathcal{P}(X)$ , and  $\mu := \eta := \delta_0$ , where  $\mathcal{P}(X)$  denotes the power set of  $X$  and  $\delta_0$  the Dirac measure at zero. Then the spaces  $(L^q(X), \|\cdot\|_{L^q(X)})$  and  $(L^q(Y), \|\cdot\|_{L^q(Y)})$ ,  $1 \leq q \leq \infty$ , can be identified with  $(\mathbb{R}, |\cdot|)$  and the partial order on  $L^q(X)$  and  $L^q(Y)$  is just the usual one on  $\mathbb{R}$ . We further set  $\bar{P} := [0, \infty]$  endowed with its canonical order,  $\bar{p} := \infty$ ,  $P := [0, \infty)$ , and  $U := [0, \infty)$ . It is easy to check that these choices satisfy all of the assumptions on  $(X, \Sigma, \mu)$ ,  $(Y, \Xi, \eta)$ ,  $\bar{P}$ ,  $P$ , and  $U$  in [subsection 2.1](#) and [Assumptions 3.1, 4.1](#) and [5.1](#). As the map  $S$ , we consider the solution operator  $S: [0, \infty] \times [0, \infty) \rightarrow [0, \infty)$ ,  $(p, u) \mapsto y$ , of the one-dimensional obstacle-type variational inequality

$$y \in \mathbb{R}, \quad y \leq p, \quad y(v - y) \geq u(v - y) \quad \forall v \in \mathbb{R}, \quad v \leq p,$$

i.e.,  $S(p, u) = \min(p, u)$  for all  $(p, u) \in [0, \infty] \times [0, \infty)$ . The map  $\Phi: \mathbb{R} \rightarrow [0, \infty)$  is chosen as an arbitrary but fixed real-valued function satisfying

$$(6.1) \quad \Phi(s) = 0 \text{ for } s < 0, \quad \Phi(s) = s \text{ for } s \in [0, 1], \quad \Phi(s) < s \text{ for } s > 1,$$

which is nondecreasing on  $\mathbb{R}$  and concave and continuously differentiable on  $(0, \infty)$ . (Such a  $\Phi$  can be constructed easily.) It is straightforward to verify that these  $S$  and  $\Phi$  satisfy all of the conditions in [subsection 2.1](#) and [Assumptions 3.1, 4.1](#) and [5.1](#) except for the concavity of  $\Phi$  on the set  $[-\varepsilon, \infty)$  for some  $\varepsilon > 0$  in [Assumption 5.1\(iv\)](#). Note that it is not possible to recover this strengthened concavity condition by redefining  $\Phi$  on the set  $(-\infty, 0)$  since the required monotonicity and nonnegativity of  $\Phi$  and the identity  $\Phi(0) = 0$  already imply  $\Phi(s) = 0$  for all  $s \leq 0$ . Consider now the QVI

$$(6.2) \quad y \in \mathbb{R}, \quad y \leq \Phi(y), \quad y(v - y) \geq u(v - y) \quad \forall v \in \mathbb{R}, \quad v \leq \Phi(y).$$

Then the solution set  $\mathbb{S}(u)$  of (6.2) for a given  $u \in U$  is identical to the set of solutions of the fixed-point equation  $y = S(\Phi(y), u)$ , which is precisely of the form (F), and it is easy to check that  $\mathbb{S}(u) = [0, \min(u, 1)]$  holds for all  $u \in U$ . In particular,  $m(u) = 0$  and  $M(u) = \min(u, 1)$  for all  $u \in U$ . Note that these explicit formulas show that the functions  $m: U \rightarrow [0, \infty)$  and  $M: U \rightarrow [0, \infty)$  are locally Lipschitz continuous on  $(0, \infty)$  and that  $M$  is concave and Hadamard directionally differentiable on  $(0, \infty)$  as predicted by [Theorem 3.2](#), [Proposition 4.2](#), and [Corollary 4.4](#). As (6.2) satisfies all of the assumptions in [sections 2 to 4](#), this demonstrates that our analysis indeed covers situations in which the considered QVI (or, more generally, fixed-point problem) possesses a continuum of solutions and the maximal and minimal element of the solution set are not isolated. This is in contrast to the results of [\[2, 37\]](#) which only apply to problems with locally unique solutions, see [\[37, Theorem 4.2\]](#). The example (6.2) also shows that the assumption of concavity of  $\Phi$  on an interval of the form  $[-\varepsilon, \infty)$  for some  $\varepsilon > 0$  in [Assumption 5.1\(iv\)](#) cannot be dropped in the uniqueness result of [Proposition 5.2](#). Next, we demonstrate that the same is true for the characterization of the directional derivatives of  $M$  by means of the linearized auxiliary problems in [Theorem 5.4](#). If we naively differentiate the fixed-point equation  $y = \min(\Phi(y), u)$  associated with (6.2) at some  $y, u > 0$ , then we arrive at the problem

$$(6.3) \quad \delta = \Psi'((y, u); (\delta, h)) = \begin{cases} \Phi'(y) \delta & \text{for } \Phi(y) < u \\ \min(\Phi'(y) \delta, h) & \text{for } \Phi(y) = u \\ h & \text{for } \Phi(y) > u. \end{cases}$$

Here,  $\Psi: \mathbb{R} \times [0, \infty) \rightarrow [0, \infty)$  again denotes the composition  $\Psi(v, u) := S(\Phi(v), u)$ . For all  $u \in (1, \infty)$ , the identity (6.3) simplifies for  $y = M(u) = \min(u, 1) = 1$  to the

degenerate equation  $\delta = \delta$  which does not contain any form of useful information. This shows that it can, in general, not be expected that the directional derivatives of the maximal solution map  $M$  of a fixed-point problem of the form (F) can be characterized by means of the equations  $\delta = \Psi'(M(u), u; (\delta, h))$  and that such a failure of the linearized auxiliary problems may occur even in situations in which the differentiability results in [Theorem 4.3](#) and [Corollary 4.4](#) are applicable and the directional derivatives of  $M$  exist. In particular, the assumption that  $\Phi$  is concave on a set slightly larger than  $L_+^2(X)$  in [Assumption 5.1\(iv\)](#) cannot be dropped for [Theorem 5.4](#) to be true. Note that the function  $\Phi$  in (6.1) trivially satisfies (5.2) so that the generalization of the solution set of the linearized auxiliary problem in [Theorem 5.4](#) does not have any impact here. Finally, we would like to point out that, by considering the continuous and nondecreasing function  $\Phi: \mathbb{R} \rightarrow [0, \infty)$  given by

$$\Phi(s) = 1 \text{ for } s < 1, \quad \Phi(s) = s \text{ for } s \in [1, 2], \quad \Phi(s) = 2 \text{ for } s > 2,$$

in (6.2) instead of a function with the properties in (6.1), one can proceed along exactly the same lines as above to obtain an example of a problem (F) that is covered by [Theorem 3.2](#) and satisfies  $\mathbb{S}(u) = [\min(u, 1), \min(u, 2)]$ ,  $m(u) = \min(u, 1)$ , and  $M(u) = \min(u, 2)$  for all  $u \in [0, \infty)$ . This shows that our Lipschitz stability result also covers situations in which the minimal solution mapping is not constant zero (as in the case of (6.1)) and in which the estimates (3.1) and (3.2) are both nontrivial.

**6.2. Application in impulse control.** Next, we consider the impulse control problem (1.1). Suppose that  $\Omega \subset \mathbb{R}^d$ ,  $d \in \mathbb{N}$ , is a bounded open nonempty set, let  $H_0^1(\Omega)$  and  $H^{-1}(\Omega)$  be defined as usual (see [7]), let  $\langle \cdot, \cdot \rangle$  be the dual pairing between elements of  $H^{-1}(\Omega)$  and  $H_0^1(\Omega)$ , let  $\Delta: H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$  denote the distributional Laplacian, let  $\kappa \geq 0$  and  $c_0 \in L_+^0(\mathbb{R}^d)$  be arbitrary but fixed, and let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a nondecreasing, globally Lipschitz continuous, convex function satisfying  $f(0) = 0$ . Given a  $u \in H_+^{-1}(\Omega) := \{z \in H^{-1}(\Omega) \mid \langle z, v \rangle \geq 0 \text{ for all } 0 \leq v \in H_0^1(\Omega)\}$ , we are interested in the nonlinear elliptic quasi-variational inequality

$$(6.4) \quad y \in H_0^1(\Omega), \quad 0 \leq y \leq \Theta(y), \quad \langle -\Delta y + f(y) - u, v - y \rangle \geq 0 \quad \forall v \in H_0^1(\Omega), v \leq \Theta(y),$$

where  $\Theta(y)$  is the function defined by

$$(6.5) \quad \Theta(y)(x) := \kappa + \operatorname{ess\,inf}_{0 \leq \xi \in \mathbb{R}^d, x + \xi \in \Omega} c_0(\xi) + y(x + \xi) \quad \text{for a.a. } x \in \Omega.$$

Here, the inequality  $\xi \geq 0$  has to be understood componentwise, the inequalities  $0 \leq v$ ,  $0 \leq y \leq \Theta(y)$ , and  $v \leq \Theta(y)$  are meant to hold in the a.e.-sense, and  $f$  acts by superposition. Note that the above problem formulation is analogous to that considered in [10, section VIII-2] (with more general  $\kappa$ ,  $c_0$ , and  $u$  and an additional nonlinearity  $f$ ). Variations of the above setting (involving, e.g., Neumann boundary conditions) can also be found in the literature, see [11, 12, 26, 29, 30], and may be discussed with the same techniques that we use in the following. Before we start with the analysis of the solution mapping of (6.4), we briefly check that the formula (6.5) is sensible. To this end, we prove the measurability of a partial essential infimum.

**LEMMA 6.1.** *Let  $(X_1, \Sigma_1)$  be a measurable space and let  $(X_2, \Sigma_2, \mu_2)$  be a  $\sigma$ -finite measure space. Assume that the function  $F: X_1 \times X_2 \rightarrow [-\infty, \infty]$  is measurable w.r.t. the product  $\sigma$ -algebra  $\Sigma_1 \otimes \Sigma_2$ . Then the function  $G: X_1 \rightarrow [-\infty, \infty]$  defined via*

$$G(x_1) := \operatorname{ess\,inf}_{x_2 \in X_2} F(x_1, x_2) \quad \forall x_1 \in X_1$$

is measurable as well. The same assertion holds true if  $(X_1, \Sigma_1, \mu_1)$  and  $(X_2, \Sigma_2, \mu_2)$  are complete,  $\sigma$ -finite measure spaces and  $F$  is measurable w.r.t. the completion of the product  $\sigma$ -algebra  $\Sigma_1 \otimes \Sigma_2$  w.r.t. the product measure  $\mu_1 \otimes \mu_2$ .

*Proof.* For every  $c \in \mathbb{R}$ , we have  $M_c := F^{-1}([-\infty, c]) \in \Sigma_1 \otimes \Sigma_2$ . Denote the indicator function of  $M_c$  with  $\chi_{M_c}$ . Then the function  $h_c: X_1 \rightarrow [0, \infty]$  defined via

$$h_c(x_1) := \int_{X_2} \chi_{M_c}(x_1, x_2) d\mu_2(x_2) = \mu_2(\{x_2 \in X_2 \mid (x_1, x_2) \in M_c\}) \quad \forall x_1 \in X_1$$

is measurable by [18, Proposition 5.1.3] or [21, Satz V.1.3]. By definition, we have  $G(x_1) < c$  if and only if  $h_c(x_1) > 0$ . Hence, the set  $G^{-1}([-\infty, c]) = h_c^{-1}((0, \infty])$  belongs to  $\Sigma_1$  for all  $c \in \mathbb{R}$  and, thus,  $G$  is measurable. In the complete case, we can argue similarly by using [18, Exercise 5.2.6] or [21, Satz V.2.4].  $\square$

**COROLLARY 6.2** (well-definedness of  $\Theta$ ). *The formula (6.5) yields a well-defined operator  $\Theta: \{v \in L^2(\Omega) \mid v \geq -\kappa\} \rightarrow L_+^0(\Omega)$ .*

*Proof.* Let  $v \in L^2(\Omega)$  with  $v \geq -\kappa$  be given. First of all, it is clear that  $\Theta(v)$  does not depend on the representative of  $v$ . The measurability of  $\Theta(v)$  follows from Lemma 6.1 applied to  $X_1 := \Omega$ ,  $X_2 := \{\xi \in \mathbb{R}^d \mid \xi \geq 0\}$  (both equipped with the Lebesgue  $\sigma$ -algebra and the Lebesgue measure), and

$$F(x, \xi) := \begin{cases} c_0(\xi) + v(x + \xi) & \text{if } x + \xi \in \Omega \\ +\infty & \text{if } x + \xi \notin \Omega \end{cases} \quad \forall x \in \Omega, \xi \geq 0.$$

From  $c_0 \geq 0$  and  $v \geq -\kappa$ , we get  $\Theta(v) \geq 0$ . Finally,  $\Theta(v) < \infty$  follows since  $c_0$  and  $v$  are almost everywhere finite and  $\Omega$  is open.  $\square$

We remark that the map  $\Theta$  may fail to preserve Sobolev regularity even if  $c_0$  is smooth, cf. [28, Remark 3-2.1]. We omit discussing additional regularity properties of the functions  $\Theta(v)$  because they are not needed for our analysis. Next, we collect some classical results on the solution operator of the obstacle problem, cf. [33].

**LEMMA 6.3** (properties of the obstacle problem). *Let  $H_0^1(\Omega)$  be endowed with the partial order  $v \geq w : \iff v \geq w$  a.e. in  $\Omega$  (analogously to  $L^q(\Omega)$ ,  $q \in \{0\} \cup [1, \infty]$ ) and let  $H^{-1}(\Omega)$  be endowed with the partial order  $v \geq w : \iff \langle v - w, z \rangle \geq 0$  for all  $0 \leq z \in H_0^1(\Omega)$ . Denote the function that is equal to infinity a.e. in  $\Omega$  by  $\infty$  and add it to  $L_+^0(\Omega)$  as the largest element. Define  $U := H_+^{-1}(\Omega)$ ,  $\bar{P} := L_+^0(\Omega) \cup \{\infty\}$ , and  $P := L_+^0(\Omega)$ , and consider for  $u \in U$  and  $p \in \bar{P}$  the problem*

$$(6.6) \quad y \in H_0^1(\Omega), \quad y \leq p, \quad \langle -\Delta y + f(y) - u, v - y \rangle \geq 0 \quad \forall v \in H_0^1(\Omega), v \leq p.$$

*Then this nonlinear variational inequality has a unique solution  $y = S(p, u) \in H_0^1(\Omega)$  for all  $p \in \bar{P}$  and all  $u \in U$  and the following is true:*

- (i) *It holds  $S(p, u) \in L_+^2(\Omega)$  for all  $p \in \bar{P}$ ,  $u \in U$ .*
- (ii) *It holds  $S(p_1, u_1) \leq S(p_2, u_2)$  for all  $p_1, p_2 \in \bar{P}$ ,  $u_1, u_2 \in U$ ,  $p_1 \leq p_2$ ,  $u_1 \leq u_2$ .*
- (iii) *It holds  $\lambda S(p_1, u_1) + (1 - \lambda)S(p_2, u_2) \leq S(\lambda p_1 + (1 - \lambda)p_2, \lambda u_1 + (1 - \lambda)u_2)$  for all  $p_1, p_2 \in P$ ,  $u_1, u_2 \in U$ ,  $\lambda \in [0, 1]$ .*
- (iv) *It holds  $S(p, u) \in L_+^q(\Omega)$  for all  $p \in \bar{P}$ ,  $u \in L_+^q(\Omega)$ ,  $q \in [1, \infty] \cap (d/2, \infty]$ .*

*Proof.* The unique solvability of (6.6) for all  $p \in \bar{P}$  and  $u \in U$  follows from [33, Theorem 4-3.1]. To prove (ii), let  $p_1, p_2 \in \bar{P}$ ,  $u_1, u_2 \in U$  with  $p_1 \leq p_2$ ,  $u_1 \leq u_2$  be given and set  $y_1 := S(p_1, u_1)$  and  $y_2 := S(p_2, u_2)$ . From Stampacchia's lemma, see [7, Theorem 5.8.2], and  $p_1 \leq p_2$ , it follows that  $H_0^1(\Omega) \ni y_1 - \max(0, y_1 - y_2) \leq y_1 \leq p_1$

and  $H_0^1(\Omega) \ni y_2 + \max(0, y_1 - y_2) = \max(y_1, y_2) \leq p_2$  holds. This allows us to use (6.6), the monotonicity of  $f$ , and the inequality  $u_1 \leq u_2$  to obtain

$$\begin{aligned} 0 &\leq \langle -\Delta y_1 + f(y_1) - u_1, -\max(0, y_1 - y_2) \rangle + \langle -\Delta y_2 + f(y_2) - u_2, \max(0, y_1 - y_2) \rangle \\ &\leq \langle -\Delta(y_2 - y_1), \max(0, y_1 - y_2) \rangle. \end{aligned}$$

Using [7, Theorem 5.8.2] and the inequality of Poincaré-Friedrichs on the right-hand side of the last estimate yields  $\max(0, y_1 - y_2) = 0$ . Thus,  $y_1 \leq y_2$  and (ii) is proved. Since we trivially have  $S(0, 0) = 0$ , this also shows (i). To prove (iii), we can argue along the same lines: If  $p_1, p_2 \in P = L_+^0(\Omega)$ ,  $u_1, u_2 \in U$ , and  $\lambda \in [0, 1]$  are given and if  $y_1 := S(p_1, u_1)$ ,  $y_2 := S(p_2, u_2)$ , then it also holds  $\lambda p_1 + (1 - \lambda)p_2 \in L_+^0(\Omega) \subset \bar{P}$  and  $\lambda u_1 + (1 - \lambda)u_2 \in U$ . We define  $y_{12} := S(\lambda p_1 + (1 - \lambda)p_2, \lambda u_1 + (1 - \lambda)u_2)$  and  $w := \lambda y_1 + (1 - \lambda)y_2 - y_{12}$ . To establish (iii), we have to show that  $w \leq 0$ . From [7, Theorem 5.8.2], we obtain  $\max(0, w) \in H_0^1(\Omega)$  and it is easy to check that  $w, y_1$ , and  $y_2$  satisfy  $y_1 - \max(0, w) \leq p_1$ ,  $y_2 - \max(0, w) \leq p_2$ , and  $y_{12} + \max(0, w) \leq \lambda p_1 + (1 - \lambda)p_2$ . This allows us to use these functions as test functions in the variational inequalities satisfied by  $y_1, y_2$ , and  $y_{12}$ , respectively. By multiplying the resulting inequalities for  $y_1$  and  $y_2$  by  $\lambda$  and  $(1 - \lambda)$ , respectively, we arrive at the estimates

$$\begin{aligned} \langle -\lambda \Delta y_1 + \lambda f(y_1) - \lambda u_1, -\max(0, w) \rangle &\geq 0, \\ \langle -(1 - \lambda) \Delta y_2 + (1 - \lambda) f(y_2) - (1 - \lambda) u_2, -\max(0, w) \rangle &\geq 0, \\ \langle -\Delta y_{12} + f(y_{12}) - \lambda u_1 - (1 - \lambda) u_2, \max(0, w) \rangle &\geq 0. \end{aligned}$$

Adding these inequalities and exploiting the convexity and monotonicity of  $f$  yields

$$\begin{aligned} 0 &\leq \langle -\Delta w + \lambda f(y_1) + (1 - \lambda) f(y_2) - f(y_{12}), -\max(0, w) \rangle \\ &\leq \langle -\Delta w + f(\lambda y_1 + (1 - \lambda) y_2) - f(y_{12}), -\max(0, w) \rangle \\ &\leq \langle -\Delta w, -\max(0, w) \rangle. \end{aligned}$$

Due to [7, Theorem 5.8.2], this implies  $w \leq 0$  as desired. It remains to prove (iv). To this end, suppose that  $p \in \bar{P}$  and  $u \in L_+^q(\Omega)$ ,  $q \in [1, \infty] \cap (d/2, \infty]$ , are arbitrary but fixed, set  $y := S(p, u) \geq 0$ , and define  $y_k := y - \min(k, y) = y - \min(k, \max(y, -k))$  for all  $k \geq 0$ . Then it holds  $H_0^1(\Omega) \ni \min(k, y) \leq p$  for all  $k \geq 0$  and it follows straightforwardly from (6.6), our assumptions on  $f$ , the inequality of Poincaré-Friedrichs, and [7, Theorem 5.8.2] that there exists a constant  $c > 0$  such that  $y_k$  satisfies

$$c \|y_k\|_{H^1(\Omega)}^2 \leq \langle -\Delta y_k, y_k \rangle = \langle -\Delta y, y_k \rangle \leq \int_{\Omega} |u y_k| \, dx \quad \forall k \geq 0.$$

From this estimate and a standard calculation involving the Sobolev embedding  $H_0^1(\Omega) \hookrightarrow L^r(\Omega)$  with  $r = 2d/(d - 2)$  for  $d > 2$ ,  $1 \leq r < \infty$  for  $d = 2$ , and  $r = \infty$  for  $d = 1$  (which can be found, e.g., in [15, Lemma 2.3] and [34, Lemma 2.8]), it follows that there exists a constant  $C > 0$  satisfying  $\|y\|_{L^\infty(\Omega)} \leq C \|u\|_{L^q(\Omega)}$ . This shows (iv) and completes the proof. Note that we do not need any additional assumptions on  $\Omega$  here for the Sobolev embedding to hold due to the zero boundary conditions.  $\square$

Since  $\Theta$  is not defined (and nonnegative) on all of  $L^2(\Omega)$ , we have to truncate its argument to be able to reformulate (6.4) as a problem of the type (F).

LEMMA 6.4 (reformulation of the impulse control problem (6.4)). *Let  $U, P, \bar{P}, S: \bar{P} \times U \rightarrow L_+^2(\Omega)$ , and  $\Theta: \{v \in L^2(\Omega) \mid v \geq -\kappa\} \rightarrow P$  be defined as in Corollary 6.2 and Lemma 6.3, and denote with  $\Phi$  the map  $\Phi: L^2(\Omega) \rightarrow P, v \mapsto \Theta(\max(-\kappa, v))$ .*



Suppose that a  $u \in U$  is given. Then  $y \in H_0^1(\Omega)$  is a solution of (6.4) if and only if it is a solution of the equation  $y = S(\Phi(y), u)$ .

*Proof.* If  $y$  solves (6.4), then it holds  $y \geq 0$  and, by the definitions of  $S$  and  $\Phi$ , we have  $y = S(\Theta(y), u) = S(\Phi(y), u)$ . If, conversely, we start with a solution  $y$  of  $y = S(\Phi(y), u)$ , then it follows from  $\Phi(y) \in \bar{P}$  and Lemma 6.3(i) that  $y \geq 0$  holds and, as a consequence, that  $y = S(\Phi(y), u) = S(\Theta(y), u)$ . This shows that  $y$  also solves (6.4) and completes the proof.  $\square$

As the equation  $y = S(\Phi(y), u)$  has precisely the form (F), we may now use our abstract analysis to arrive at the following two main results of this subsection.

**THEOREM 6.5** (concavity and pointwise directional differentiability on  $H_+^{-1}(\Omega)$ ). *The impulse control problem (6.4) possesses a nonempty set of solutions  $\mathbb{S}(u) \subset H_0^1(\Omega)$  for all  $u \in H_+^{-1}(\Omega)$  and this solution set possesses unique smallest and largest elements  $m(u)$  and  $M(u)$ . The map  $M: H_+^{-1}(\Omega) \rightarrow H_0^1(\Omega)$ ,  $u \mapsto M(u)$ , is concave in the sense that  $\lambda M(u_1) + (1 - \lambda)M(u_2) \leq M(\lambda u_1 + (1 - \lambda)u_2)$  holds a.e. in  $\Omega$  for all  $u_1, u_2 \in H_+^{-1}(\Omega)$  and  $\lambda \in [0, 1]$ . Further,  $M$  is directionally differentiable in the sense that, for all  $u \in H_+^{-1}(\Omega)$  and  $h \in H^{-1}(\Omega)$  satisfying  $u + \tau_0 h \in H_+^{-1}(\Omega)$  for a  $\tau_0 > 0$ , there exists a unique  $M'(u; h) \in L^0(\Omega, (-\infty, \infty])$  such that the difference quotients*

$$\frac{M(u + \tau h) - M(u)}{\tau}, \quad \tau \in (0, \tau_0),$$

converge pointwise a.e. to  $M'(u; h)$  along every sequence  $\{\tau_n\} \subset (0, \tau_0)$  with  $\tau_n \rightarrow 0$ .

*Proof.* Let  $U, P, \bar{P}, S: \bar{P} \times U \rightarrow L_+^2(\Omega)$ ,  $\Theta: \{v \in L^2(\Omega) \mid v \geq -\kappa\} \rightarrow P$ , and  $\Phi: L^2(\Omega) \rightarrow P$  be defined as in Corollary 6.2 and Lemmas 6.3 and 6.4. Then it follows from Lemma 6.3 that  $U, P, \bar{P}$ , and  $S$  satisfy all of the conditions in subsection 2.1 and Assumption 4.1 (with  $X = \Omega$ ,  $\Sigma$  as the Lebesgue  $\sigma$ -algebra on  $\Omega$ , and  $\mu$  as the Lebesgue measure). Using (6.5), it is easy to check that this is also true for  $\Phi$ . Indeed, for all  $v_1, v_2 \in L^2(\Omega)$ ,  $v_1 \leq v_2$ , we clearly have  $\max(-\kappa, v_1) \leq \max(-\kappa, v_2)$  and  $\Phi(v_1) = \Theta(\max(-\kappa, v_1)) \leq \Theta(\max(-\kappa, v_2)) = \Phi(v_2)$  by the definitions of  $\Theta$  and  $\Phi$ . Further, (6.5) implies that  $\Theta$  is pointwise a.e. concave. Thus, for all  $\lambda \in [0, 1]$  and all  $v_1, v_2 \in \{v \in L^2(\Omega) \mid v \geq -\kappa\}$ , we obtain from the definition of  $\Phi$  that  $\Phi(\lambda v_1 + (1 - \lambda)v_2) = \Theta(\lambda v_1 + (1 - \lambda)v_2) \geq \lambda \Theta(v_1) + (1 - \lambda)\Theta(v_2) = \lambda \Phi(v_1) + (1 - \lambda)\Phi(v_2)$ . Theorems 2.4 and 4.3, Proposition 4.2, and Lemma 6.4 now yield the claims.  $\square$

**THEOREM 6.6** (Lipschitz continuity and directional differentiability on  $L_+^\infty(\Omega)$ ). *On the set  $L_+^\infty(\Omega) \subset H_+^{-1}(\Omega)$ , the minimal and maximal solution map of the impulse control problem (6.4) are well-defined as operators  $m, M: L_+^\infty(\Omega) \rightarrow H_0^1(\Omega) \cap L_+^\infty(\Omega)$ . Further, the following is true:*

- (i) *The functions  $m$  and  $M$  are locally Lipschitz continuous on  $L_\oplus^\infty(\Omega)$  in the sense that, for every  $u \in L_\oplus^\infty(\Omega)$ , there exist constants  $C, r > 0$  such that*

$$(6.7) \quad \|m(v_1) - m(v_2)\|_{L^\infty(\Omega)} + \|M(v_1) - M(v_2)\|_{L^\infty(\Omega)} \leq C \|v_1 - v_2\|_{L^\infty(\Omega)}$$

*holds for all  $v_1, v_2 \in L_+^\infty(\Omega)$  satisfying  $\|v_i - u\|_{L^\infty(\Omega)} \leq r$ ,  $i = 1, 2$ .*

- (ii) *The functions  $m$  and  $M$  are weakly continuous on  $L_\oplus^\infty(\Omega)$  in the sense that, for all  $u \in L_\oplus^\infty(\Omega)$  and all  $\{u_n\} \subset L_\oplus^\infty(\Omega)$  with  $u_n \rightarrow u$  in  $L^\infty(\Omega)$ , we have  $m(u_n) \rightarrow m(u)$  and  $M(u_n) \rightarrow M(u)$  in  $H_0^1(\Omega)$ .*

- (iii) *The function  $M$  is Hadamard directionally differentiable on the set  $L_\oplus^\infty(\Omega)$  in the sense that, for all  $u \in L_\oplus^\infty(\Omega)$  and all  $h \in L^\infty(\Omega)$ , there exists a unique  $M'(u; h) \in L^\infty(\Omega)$  such that, for all  $\{\tau_n\} \subset (0, \infty)$ ,  $\{h_n\} \subset L^\infty(\Omega)$  satisfying*



$\tau_n \rightarrow 0$ ,  $\|h - h_n\|_{L^\infty(\Omega)} \rightarrow 0$ , and  $u + \tau_n h_n \in L_+^\infty(\Omega)$  for all  $n$ , we have

$$\frac{M(u + \tau_n h_n) - M(u)}{\tau_n} \rightarrow M'(u; h) \text{ in } L^q(\Omega) \text{ for all } 1 \leq q < \infty$$

and

$$\frac{M(u + \tau_n h_n) - M(u)}{\tau_n} \xrightarrow{*} M'(u; h) \text{ in } L^\infty(\Omega).$$

(iv) If  $\kappa > 0$  holds, then (6.4) is uniquely solvable for all  $u \in L_+^\infty(\Omega)$  and it holds  $m \equiv M \equiv \mathbb{S}$  on  $L_+^\infty(\Omega)$ . In this situation, the directional derivative  $\mathbb{S}'(u; h) = M'(u; h)$  of the solution operator  $\mathbb{S}$  of (6.4) at a point  $u \in L_\oplus^\infty(\Omega)$  in a direction  $h \in L^\infty(\Omega)$  satisfying  $u + \tau_0 h \in L_+^\infty(\Omega)$  for a  $\tau_0 > 0$  is uniquely characterized by the condition that it is the smallest element of the set

$$\left\{ \zeta \in L^\infty(\Omega) \left| \begin{array}{l} \exists \{\zeta_n\} \subset L^\infty(\Omega), \{\tau_n\} \subset (0, \tau_0): \\ \zeta_n \leq \zeta_{n+1} \text{ and } \tau_{n+1} \leq \tau_n \text{ for all } n, \\ \tau_n \rightarrow 0 \text{ for } n \rightarrow \infty, \zeta_n \rightarrow \zeta \text{ in } L^q(\Omega) \text{ for all } q \in [1, \infty), \\ \mathbb{S}(u) + \tau_n \zeta_n \text{ is supersol. of } y = S(\Phi(y), u + \tau_n h) \text{ for all } n, \\ \Psi'((\mathbb{S}(u), u); (\zeta_n, h)) - \zeta_n \rightarrow 0 \text{ a.e. in } \Omega \text{ for } n \rightarrow \infty \end{array} \right. \right\}.$$

Here,  $S$ ,  $\Phi$ , and  $\Psi$  are defined as in Lemmas 5.3 and 6.4.

*Proof.* We consider the same  $P$ ,  $\bar{P}$ ,  $S$ ,  $\Theta$ , and  $\Phi$  as before, but this time we restrict the function  $S$  in the second argument to the set  $\tilde{U} := L_+^\infty(\Omega) \subset H_+^{-1}(\Omega)$ . For this setting, we obtain from Lemma 6.3 and the same arguments as in the proof of Theorem 6.5 that all of the conditions in subsection 2.1 and Assumptions 3.1 and 4.1 are satisfied (with  $X = Y = \Omega$ ,  $\Sigma = \Xi$  as the Lebesgue  $\sigma$ -algebra on  $\Omega$ , and  $\mu = \eta$  as the Lebesgue measure) and, in the case  $\kappa > 0$ , that Assumption 5.1 holds as well (with  $\varepsilon = \kappa$ ). By invoking Proposition 5.2, Theorems 3.2 and 5.4, Corollary 4.4, and Lemmas 6.3 and 6.4 and by exploiting that  $L^{[r, \infty]}(\Omega) = L^\infty(\Omega)$  holds for all  $r \in [1, \infty]$  due to the boundedness of  $\Omega$ , it now follows immediately that  $m$  and  $M$  possess the asserted mapping properties on  $L_+^\infty(\Omega)$ . Note that (6.7) is a consequence of (3.5) here and that the weak continuity of  $m$  and  $M$  as functions from  $L_\oplus^\infty(\Omega)$  to  $H_0^1(\Omega)$  in (ii) is obtained from the arguments outlined in Remark 3.3(iv) and the estimate  $\|S(p, u)\|_{H^1(\Omega)} \leq C(\Omega)\|u\|_{H^{-1}(\Omega)}$  for all  $p \in \bar{P}$  and  $u \in U$  that follows straightforwardly from (6.6) with  $v = 0$ . This completes the proof.  $\square$

We remark that, even in the case  $\kappa > 0$ , it is not possible to invoke the uniqueness result of [25] in the situation of Theorem 6.5 since  $S(p, u) \in L^\infty(\Omega)$  may not hold for all  $p \in L_+^0(\Omega)$  and all  $u \in H_+^{-1}(\Omega)$ . Similarly, in Theorem 6.6(iv), we cannot apply the second part of Theorem 5.4 to conclude that the directional derivatives of  $\mathbb{S}$  are the smallest elements of the ‘‘ordinary’’ solution sets of the QVIs  $\zeta = \Psi'((\mathbb{S}(u), u); (\zeta, h))$  since the map  $\Theta$  lacks pointwise a.e. continuity properties w.r.t. convergence in the  $L^q(\Omega)$ -spaces for  $1 \leq q < \infty$ . Both of these effects make the sensitivity analysis of (6.4) a delicate issue. Before we proceed, we would like to point out that Theorem 6.6 generalizes the result on problem (6.4) obtained in [6, section 7.1.2], where the continuity of the maps  $m$  and  $M$  as functions from  $L^\infty(\Omega)$  to  $L^2(\Omega)$  is proved on the set  $L_\oplus^\infty(\Omega)$  in the case  $d = 1$  and  $f = 0$ . We obtain not only continuity but even local Lipschitz continuity on  $L_\oplus^\infty(\Omega)$  for all dimensions  $d \in \mathbb{N}$ , for the nonlinear QVI, and for  $m$  and  $M$  as functions into the space  $L^\infty(\Omega)$ . Our results further establish the Hadamard directional differentiability of  $M$  on  $L_\oplus^\infty(\Omega)$  into all  $L^q$ -spaces in the sense

of (4.4) and (4.5) for all  $d \in \mathbb{N}$  (without any sign conditions on  $h$ ) and, in the case  $\kappa > 0$ , uniquely characterize the directional derivatives of the solution map  $\mathbb{S}$  of (6.4). At least to the best of the authors' knowledge, [Theorem 6.6](#) is the first result in the literature to accomplish this for the problem (6.4). We are able to prove all of these properties because we do not require any restrictive assumptions on the regularity or complete continuity of the map  $\Phi$ , cf. [[6](#), Assumption 1] and also [[2](#), [3](#), [4](#), [5](#), [37](#)].

**6.3. Parabolic QVIs.** As a third example, we consider a parabolic QVI with boundary controls. Let  $\Omega \subset \mathbb{R}^d$ ,  $d \in \mathbb{N}$ , be a bounded Lipschitz domain with boundary  $\partial\Omega$ , let  $T > 0$  and  $\psi \in L_+^\infty(\Omega)$  be given, and let  $g: \mathbb{R} \rightarrow \mathbb{R}$  be a globally Lipschitz continuous, bounded, nonnegative, nondecreasing function that is concave on  $[0, \infty)$ . Given a  $u \in L_+^\infty((0, T) \times \partial\Omega) \subset L^2((0, T) \times \partial\Omega) \cong L^2(0, T; L^2(\partial\Omega))$ , we are interested in the problem of finding a (potentially weak) solution  $y$  of the variational inequality

$$(6.8) \quad \begin{aligned} y(0) &= 0, & H^1(\Omega) \ni y(t) &\leq \psi + \Phi(y) \text{ a.e. in } \Omega \text{ for a.a. } t \in (0, T), \\ \int_{\Omega} \partial_t y(t)(v - y(t)) + \nabla y(t) \cdot \nabla(v - y(t)) dx - \int_{\partial\Omega} u(t)(v - y(t)) ds &\geq 0 \\ &\forall v \in H^1(\Omega), v \leq \psi + \Phi(y) \text{ a.e. in } \Omega \text{ for a.a. } t \in (0, T) \end{aligned}$$

with  $\Phi$  defined by  $\Phi(y) := w(T)$  and  $w$  as the solution of the heat equation

$$(6.9) \quad \partial_t w - \Delta w = g(y) \text{ in } (0, T) \times \Omega, \quad w = 0 \text{ on } (0, T) \times \partial\Omega, \quad w(0) = 0 \text{ in } \Omega.$$

Here and in what follows, the appearing Lebesgue, Sobolev, and Bochner spaces  $L^q(\Omega)$ ,  $H^1(\Omega)$ ,  $L^2(0, T; L^2(\partial\Omega))$ , etc. are defined as usual, see [[7](#), [9](#), [24](#)], and endowed with the canonical pointwise a.e. partial orders induced by the underlying measure spaces,  $\partial_t$  denotes the time derivative in the Sobolev-Bochner sense,  $\nabla$  is the weak spatial gradient,  $\Delta$  is the distributional spatial Laplacian, and  $g$  acts by superposition. We would like to emphasize that (6.8) is again a model problem. Other boundary conditions, functions  $\Phi$ , etc. can be studied with the same techniques that we use in the following. To transform (6.8) into a problem of the type (F), we define  $X := (0, T) \times \Omega$  (endowed with the Lebesgue measure),  $\bar{P} := L_+^\infty(\Omega) \cup \{\infty\}$  (endowed with the partial order induced by the a.e.-sense on  $\Omega$  and with the function that is  $\infty$  a.e. in  $\Omega$  as the largest element),  $P := L_+^\infty(\Omega)$ , and  $U := L_+^\infty((0, T) \times \partial\Omega)$ . Note that these  $X$ ,  $\bar{P}$ ,  $P$ , and  $U$  satisfy all of the conditions in [subsection 2.1](#) and [Assumptions 3.1](#), [4.1](#) and [5.1](#) (with  $Y := (0, T) \times \partial\Omega$  and  $\eta$  as the completion of the product measure of the Lebesgue measure on  $(0, T)$  and the surface measure on  $\partial\Omega$ ). The next two lemmas establish that the same is true for the maps  $S$  and  $\Phi$  associated with (6.8) and (6.9).

**LEMMA 6.7** (properties of the parabolic obstacle problem). *Let  $P$ ,  $\bar{P}$ ,  $U$ , and  $X$  be defined as above and set  $\tilde{U} := \{v \in C^{0,1}([0, T] \times \partial\Omega) \mid v \geq 0 \text{ and } v|_{\{0\} \times \partial\Omega} = 0\} \subset U$ . Then, for all  $p \in \bar{P}$  and all  $u \in \tilde{U}$ , there exists a unique strong solution  $y = S(p, u) \in H^1(0, T; H^1(\Omega)) \cap W^{1,\infty}(0, T; L^2(\Omega))$  of the parabolic variational inequality*

$$(6.10) \quad \begin{aligned} y(0) &= 0, & H^1(\Omega) \ni y(t) &\leq \psi + p \text{ a.e. in } \Omega \text{ for a.a. } t \in (0, T), \\ \int_{\Omega} \partial_t y(t)(v - y(t)) + \nabla y(t) \cdot \nabla(v - y(t)) dx - \int_{\partial\Omega} u(t)(v - y(t)) ds &\geq 0 \\ &\forall v \in H^1(\Omega), v \leq \psi + p \text{ a.e. in } \Omega \text{ for a.a. } t \in (0, T). \end{aligned}$$

Further, there exists a constant  $C > 0$  satisfying

$$(6.11) \quad \begin{aligned} \|S(p, u_1) - S(p, u_2)\|_{C([0, T]; L^2(\Omega))} + \|S(p, u_1) - S(p, u_2)\|_{L^2(0, T; H^1(\Omega))} \\ \leq C \|u_1 - u_2\|_{L^2((0, T) \times \partial\Omega)} \quad \forall u_1, u_2 \in \tilde{U} \quad \forall p \in \bar{P}. \end{aligned}$$

Due to (6.11), the solution map  $S$  of (6.10) can be extended uniquely by continuity to a function  $S: \bar{P} \times U \rightarrow C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)) \subset L^2(0, T; L^2(\Omega)) \cong L^2(X)$ .

This weak solution operator  $S$  of (6.10) possesses the following properties:

- (i) It holds  $S(p, u) \in L^2_+(X)$  for all  $p \in \bar{P}$ ,  $u \in U$ .
- (ii) It holds  $S(p_1, u_1) \leq S(p_2, u_2)$  for all  $p_1, p_2 \in \bar{P}$ ,  $u_1, u_2 \in U$ ,  $p_1 \leq p_2$ ,  $u_1 \leq u_2$ .
- (iii) It holds  $S(p, u) \in L^2_+(X)$  for all  $p \in P$ ,  $u \in U$ .
- (iv) It holds  $\lambda S(p_1, u_1) + (1 - \lambda)S(p_2, u_2) \leq S(\lambda p_1 + (1 - \lambda)p_2, \lambda u_1 + (1 - \lambda)u_2)$  for all  $p_1, p_2 \in P$ ,  $u_1, u_2 \in U$ , and  $\lambda \in [0, 1]$ .
- (v) It holds  $\|S(p_1, u) - S(p_2, u)\|_{L^\infty(X)} \leq \|p_1 - p_2\|_{L^\infty(\Omega)}$  for all  $p_1, p_2 \in P$ ,  $u \in U$ .

*Proof.* The existence of a unique strong solution  $S(p, u) \in H^1(0, T; H^1(\Omega)) \cap W^{1,\infty}(0, T; L^2(\Omega))$  of (6.10) for all  $p \in \bar{P}$  and all  $u \in \tilde{U}$  and the estimate (6.11) follow from [9, Theorem 4.1] and the embeddings  $C^{0,1}([0, T] \times \partial\Omega) \hookrightarrow H^1(0, T; L^2(\partial\Omega)) \hookrightarrow H^1(0, T; H^1(\Omega)^*)$  and  $L^2((0, T) \times \partial\Omega) \cong L^2(0, T; L^2(\partial\Omega)) \hookrightarrow L^2(0, T; H^1(\Omega)^*)$ . To see that  $S$  admits a unique extension  $S: \bar{P} \times U \rightarrow C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$ , it suffices to note that  $\tilde{U}$  is dense in  $U$  w.r.t. the norm  $\|\cdot\|_{L^2((0, T) \times \partial\Omega)}$  and to use (6.11), see [9, Corollary 4.1]. Suppose now that  $u_1, u_2 \in U$  and  $p_1, p_2 \in \bar{P}$  satisfying  $u_1 \leq u_2$  and  $p_1 \leq p_2$  are given and define  $y_1 := S(p_1, u_1)$  and  $y_2 := S(p_2, u_2)$ . Then it follows from [16, Lemma A.1] and [35] that  $y_1 - \max(0, y_1 - y_2), y_2 + \max(0, y_1 - y_2) \in L^2(0, T; H^1(\Omega)) \cap H^1(0, T; L^2(\Omega))$  holds and we may use (6.10) and the formulas in [16, Lemma A.1] to obtain (analogously to the elliptic case in Lemma 6.3(ii))

$$\begin{aligned}
0 &\leq \int_0^\tau \int_\Omega \partial_t(y_2 - y_1) \max(0, y_1 - y_2) + \nabla(y_2 - y_1) \cdot \nabla \max(0, y_1 - y_2) dx \\
(6.12) \quad &\quad - \int_{\partial\Omega} (u_2 - u_1) \max(0, y_1 - y_2) ds dt \\
&\leq -\frac{1}{2} \|\max(0, y_1(\tau) - y_2(\tau))\|_{L^2(\Omega)}^2 \quad \forall \tau \in [0, T].
\end{aligned}$$

The above shows that  $S(p_1, u_1) \leq S(p_2, u_2)$  holds for all  $u_1, u_2 \in \tilde{U}$  and  $p_1, p_2 \in \bar{P}$  satisfying  $u_1 \leq u_2$  and  $p_1 \leq p_2$ . By approximation in  $L^2((0, T) \times \partial\Omega)$ , this result readily carries over to all  $u_1, u_2 \in U$  and  $p_1, p_2 \in \bar{P}$  with  $u_1 \leq u_2$  and  $p_1 \leq p_2$ , cf. (6.11). (Note that the set  $\tilde{U}$  is stable w.r.t. pointwise truncation and that, as a consequence, given  $u_1, u_2 \in U$  with  $u_1 \leq u_2$ , it is easy to construct approximating sequences in  $\tilde{U}$  whose elements satisfy the same inequality.) This proves (ii). Since we again have  $S(0, 0) = 0$  and since  $S(p, u)(t) \leq \psi + p$  holds a.e. in  $\Omega$  for a.a.  $t \in (0, T)$ , the assertions in (i) and (iii) follow immediately from (ii). To establish (iv), we can proceed completely analogously to the elliptic case in Lemma 6.3(iii) using the results in [16, Lemma A.1], a calculation as in (6.12), and approximation. It remains to establish (v). To this end, let  $u \in \tilde{U}$  and  $p_1, p_2 \in L^2_+(\Omega)$  with associated  $y_1 := S(p_1, u)$ ,  $y_2 := S(p_2, u)$  be given and define  $z := \max(y_2 - y_1 - \|p_1 - p_2\|_{L^\infty(\Omega)}, 0)$ . By choosing the test functions  $y_1 + z$  and  $y_2 - z$  in the variational inequalities for  $y_1$  and  $y_2$  and by adding, integrating, and again exploiting [16, Lemma A.1], we obtain

$$0 \leq \int_0^\tau \int_\Omega \partial_t(y_1 - y_2)z + \nabla(y_1 - y_2) \cdot \nabla z dx dt = - \int_0^\tau \int_\Omega (\partial_t z)z + \nabla z \cdot \nabla z dx dt$$

for all  $\tau \in [0, T]$ . This implies  $z = 0$  and, after plugging in the definition of  $z$  and switching the roles of  $y_1$  and  $y_2$ ,  $|y_1 - y_2| \leq \|p_1 - p_2\|_{L^\infty(\Omega)}$  a.e. in  $X$  as claimed in (v). To finally establish (v) for all  $u \in U$ , we can again use approximation. Note that the Lipschitz continuity estimate (6.11) is sufficient for this purpose as the set  $\{v \in L^2(X) \mid 0 \leq v \leq \|p_1 - p_2\|_{L^\infty(\Omega)}\}$  is closed in  $L^2(X)$ .  $\square$

LEMMA 6.8 (properties of  $\Phi$ ). *Let  $X$  and  $P$  be as before and set  $\Phi(v) := w(T)$  with  $w$  as the solution of the heat equation with right-hand side  $g(v)$  in (6.9). Then  $\Phi$  is well-defined as a function  $\Phi: L^2(X) \rightarrow P$  and the following is true:*

- (i) *For all  $v_1, v_2 \in L^2(X)$ ,  $v_1 \leq v_2$ , it holds  $\Phi(v_1) \leq \Phi(v_2)$ .*
- (ii) *If  $g$  is concave on  $[-\varepsilon, \infty)$ ,  $\varepsilon \geq 0$ , then, for all  $\lambda \in [0, 1]$  and  $v_1, v_2 \in L^2(X)$  with  $v_1 \geq -\varepsilon$ ,  $v_2 \geq -\varepsilon$ , we have  $\lambda\Phi(v_1) + (1-\lambda)\Phi(v_2) \leq \Phi(\lambda v_1 + (1-\lambda)v_2)$ .*
- (iii) *There exist an exponent  $q \in [2, \infty)$  and a constant  $C > 0$  such that, for all  $v_1, v_2 \in L^q(X)$ , we have  $\|\Phi(v_1) - \Phi(v_2)\|_{L^\infty(\Omega)} \leq C\|v_1 - v_2\|_{L^q(X)}$ .*

*Proof.* Due to our assumptions on the function  $g$ , it holds  $g(v) \in L^q_+(\!(0, T) \times \Omega)$  for all  $v \in L^2(X)$  and, since  $\Omega$  is a bounded Lipschitz domain, we obtain from [20, Theorem 3.1] that there exist a  $q \in [2, \infty)$  and a constant  $C > 0$  such that the solution  $w$  of (6.9) with right-hand side  $g(v)$  is in  $C([0, T] \times \bar{\Omega})$  and satisfies

$$\|\Phi(v)\|_{L^\infty(\Omega)} = \|w(T)\|_{C(\bar{\Omega})} \leq \|w\|_{C([0, T] \times \bar{\Omega})} \leq C\|g(v)\|_{L^q(X)} \quad \forall v \in L^2(X).$$

Using standard results (or again [16, Lemma A.1]), it is further easy to check that the solution of a heat equation with homogeneous Dirichlet boundary conditions, the initial condition zero, and a nonnegative right-hand side is nonnegative in  $[0, T] \times \bar{\Omega}$ . This proves that  $\Phi$  is indeed well-defined as a map from  $L^2(X)$  into  $P = L^q_+(\Omega)$ . From the monotonicity and linearity of the solution operator of the heat equation and our assumptions on  $g$ , one also easily obtains (i) and (ii). To finally establish (iii), we note that the global Lipschitz continuity of  $g$  and again [20, Theorem 3.1] yield

$$\|\Phi(v_1) - \Phi(v_2)\|_{L^\infty(\Omega)} \leq C\|g(v_1) - g(v_2)\|_{L^q(X)} \leq C\|g\|_{C^{0,1}(\mathbb{R})}\|v_1 - v_2\|_{L^q(X)}$$

for all  $v_1, v_2 \in L^q(X)$  with the same  $C$  and  $q$  as before. This completes the proof.  $\square$

Using the results of sections 2 to 5, we now get the following for the QVI (6.8).

THEOREM 6.9 (directional differentiability and Lipschitz stability for (6.8)). *The parabolic QVI (6.8) possesses a nonempty set of solutions  $\mathbb{S}(u) \subset C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)) \cap L^q_+(\!(0, T) \times \Omega)$  for all  $u \in L^q_+(\!(0, T) \times \partial\Omega)$ . This solution set possesses unique smallest and largest elements  $m(u)$  and  $M(u)$ . Further, the following is true for the maps  $m, M: L^q_+(\!(0, T) \times \partial\Omega) \rightarrow L^\infty(\!(0, T) \times \Omega)$ :*

- (i) *The function  $M$  is concave, i.e., for all  $u_1, u_2 \in L^q_+(\!(0, T) \times \partial\Omega)$ ,  $\lambda \in [0, 1]$ , we have  $\lambda M(u_1) + (1-\lambda)M(u_2) \leq M(\lambda u_1 + (1-\lambda)u_2)$ .*
- (ii) *The functions  $m$  and  $M$  are locally Lipschitz continuous in the sense that, for all  $u \in L^q_+(\!(0, T) \times \partial\Omega)$ , there exist constants  $C, r > 0$  with*

$$(6.13) \quad \begin{aligned} & \|m(v_1) - m(v_2)\|_{L^\infty(\!(0, T) \times \Omega)} + \|M(v_1) - M(v_2)\|_{L^\infty(\!(0, T) \times \Omega)} \\ & \leq C\|v_1 - v_2\|_{L^\infty(\!(0, T) \times \partial\Omega)} \end{aligned}$$

*for all  $v_1, v_2 \in L^q_+(\!(0, T) \times \partial\Omega)$  satisfying  $\|u - v_i\|_{L^\infty(\!(0, T) \times \partial\Omega)} \leq r$ ,  $i = 1, 2$ .*

- (iii) *The functions  $m$  and  $M$  are weakly continuous in the sense that, for all  $u, u_n \in L^q_+(\!(0, T) \times \partial\Omega)$  satisfying  $u_n \rightharpoonup u$  in  $L^\infty(\!(0, T) \times \partial\Omega)$ , we have  $m(u_n) \rightharpoonup m(u)$  and  $M(u_n) \rightharpoonup M(u)$  in  $L^2(0, T; H^1(\Omega))$ .*
- (iv) *The function  $M$  is Hadamard directionally differentiable on  $L^q_+(\!(0, T) \times \partial\Omega)$  in the sense that, for all  $u \in L^q_+(\!(0, T) \times \partial\Omega)$  and  $h \in L^\infty(\!(0, T) \times \partial\Omega)$ , there exists a unique  $M'(u; h) \in L^\infty(\!(0, T) \times \Omega)$  such that, for all  $\{\tau_n\} \subset (0, \infty)$  and  $\{h_n\} \subset L^\infty(\!(0, T) \times \partial\Omega)$  satisfying  $\tau_n \rightarrow 0$ ,  $\|h - h_n\|_{L^\infty(\!(0, T) \times \partial\Omega)} \rightarrow 0$ , and  $u + \tau_n h_n \in L^q_+(\!(0, T) \times \partial\Omega)$  for all  $n$ , it holds*

$$\frac{M(u + \tau_n h_n) - M(u)}{\tau_n} \rightarrow M'(u; h) \text{ in } L^q(\!(0, T) \times \Omega) \text{ for all } 1 \leq q < \infty$$

and

$$\frac{M(u + \tau_n h_n) - M(u)}{\tau_n} \star M'(u; h) \text{ in } L^\infty((0, T) \times \Omega).$$

- (v) If the function  $g$  is not only concave on  $[0, \infty)$  but even on an interval of the form  $[-\varepsilon, \infty)$ ,  $\varepsilon > 0$ , then the QVI (6.8) possesses a unique solution  $\mathbb{S}(u) = m(u) = M(u)$  for all  $u \in L^{\infty}_+(0, T) \times \partial\Omega$ . In this case, the derivatives  $\mathbb{S}'(u; h) = M'(u; h)$  in (iv) are uniquely characterized by the condition that they are the smallest elements in  $L^\infty((0, T) \times \Omega)$  of the sets

$$\{\zeta \in L^\infty((0, T) \times \Omega) \mid \zeta = \Psi'(\mathbb{S}(u), u); (\zeta, h) \text{ a.e. in } (0, T) \times \Omega\}.$$

Here,  $\Psi$  denotes the composition  $\Psi(v, u) := S(\Phi(v), u)$  of the functions  $S$  and  $\Phi$  in Lemmas 6.7 and 6.8, cf. Lemma 5.3.

*Proof.* Lemmas 6.7 and 6.8 yield that (6.8) (or, more precisely, its reformulation  $y = S(\Phi(y), u)$ ) satisfies all of the conditions in subsection 2.1 and Assumptions 3.1 and 4.1 (with  $X, \bar{P}, P, U, Y, S$ , and  $\Phi$  as before). They further show that there exists a  $q \in [2, \infty)$  such that the function  $L^q(X) \ni v \mapsto \Psi(v, u) := S(\Phi(v), u) \in L^{\infty}_+(X)$  is continuous for every arbitrary but fixed  $u \in U$  and that Assumption 5.1 holds when  $g$  is concave on  $[-\varepsilon, \infty)$  for some  $\varepsilon > 0$ . By combining all of this with Theorems 2.4, 3.2 and 5.4, Propositions 4.2 and 5.2, and Corollary 4.4, the assertions of the theorem follow immediately. Note that, to obtain (6.13) and the weak continuity in (iii), one can use (6.11) and the arguments outlined in points (ii) and (iv) of Remark 3.3.  $\square$

Coupled parabolic systems of the type (6.8) arise, for instance, in the context of thermoforming, see [2, 3, 8, 31, 32]. Note that Theorem 6.9 again does not require any assumptions on the sign of the directions  $h$  or on the operator norms of  $\Phi$  and its derivatives, cf. [4]. At least to the authors' best knowledge, this is the first differentiability result for parabolic QVIs in such a general setting. The same seems to be the case for the characterization of the derivatives  $\mathbb{S}'(u; h)$  in Theorem 6.9(v).

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