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Switching cost aware rounding for relaxations of mixed-integer optimal control problems: the two-dimensional case

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***Abstract*— This article is concerned with a recently proposed switching cost aware rounding (SCARP) strategy in the combinatorial integral decomposition for mixed-integer optimal control problems (MIOCPs). We consider the case of a control variable that is discrete-valued and distributed on a two-dimensional domain.**

While the theoretical results from the one-dimensional case directly apply to the multidimensional setting, the structure of the cost function in the graph-based rounding computation is significantly more involved in the two-dimensional case.

We describe a set up of the computational graph and the traversal algorithm underlying the SCARP strategy that enable a transfer to the two-dimensional setting. We demonstrate the SCARP strategy in this two-dimensional setting using the example of a MIOCP from topology optimization. We compare the graph-based approach to a ground truth computation using an integer linear programming (ILP) solver. The graph-based approach becomes computationally intractable for medium grid sizes. We show that the one-dimensional SCARP algorithm can be employed on a serialization of the grid cells in these cases and still provides an efficient heuristic that yields superior performance compared with that of other rounding heuristics such as sum-up rounding (SUR).

I. INTRODUCTION

Mixed-integer optimal control problems (MIOCPs) have a wide range of applications, including optimization for supply chain or traffic networks [10], [18], gear shifts [9], [17], and chemical engineering [5], [11].

One approach to treating MIOCPs that has been advanced recently is combinatorial integral decomposition [25], [26], [28], where the process of solving the MIOCP is decomposed into two steps:

1) Solve a (discretization of the) continuous relaxation of the MIOCP.
2) Approximate the resulting continuously valued control with a discrete-valued one using a so-called *rounding algorithm*.

By choosing sufficiently fine grids for the involved computations, the approximation error of this procedure can be made arbitrarily small under certain assumptions about the underlying dynamical system [16].

Taking advantage of these approximation principles, researchers have recently used the MIOCP point of view also for topology optimization; see, for example, [12], [19], [22]. Moreover, the focus on the rounding algorithms in the second step of the combinatorial integral decomposition has shifted from fast algorithms such as *sum-up rounding* (SUR) and *next-forced rounding* (NFR), which offer only a certain approximation quality [14], [15], [24], [25], to optimization-based algorithms [2], [3], [26], [27] that can incorporate multiple criteria into the rounding step, an important example being switching costs of the resulting control.

The switching cost aware rounding problem (SCARP) [2] allows computing a discrete-valued control with minimum possible switching costs for a given continuously valued control and a given accuracy by solving an integer linear program (ILP). The approach can be implemented efficiently for problems where the control is distributed in the time domain only, because the problem can be reformulated as a shortest path search in a directed acyclic graph (DAG) [3].

A. Contribution

For control functions that are defined on multidimensional domains, the natural ordering of the grid cells that is used to construct the DAG is not available anymore. We derive a multidimensional SCARP formulation for switching costs on discretizations of multidimensional domains, which are serialized into the DAG structure such that the desired approximation properties still hold.

We evaluate the approach computationally on a topology optimization problem adapted from [13], [19] that is governed by a Helmholtz equation. We compare the graph-based solution of SCARP with the solution obtained by an off-the-shelf IP solver and the SUR algorithm, which tends to produce control functions with high switching costs. We also demonstrate that the graph-based solution can be accelerated significantly and with only a moderate increase in switching costs by using the shortest path approach from one-dimensional problems instead.

B. Structure of the paper

In Section II we present the investigated problem class, our assumptions, and a summary of the approximation framework. In Section III we provide the multidimensional SCARP formulation as an IP and transfer it to the DAG formulation. We describe our computational example and experimental setup in IV. We present our results in Section V and close with concluding remarks in Section VI.

II. THE PROBLEM CLASS

We consider the following class of MIOCPs,

$$\inf_{y,w}\; J(y)+R(w) \text{ s.t.} \begin{cases} y=S(w), \text{and} \\ w \text{ is a binary control,} \end{cases} \tag{P}$$

where a *binary control* is a measurable function $w : \Omega \to \{0,1\}^M$ such that $\sum_{j=1}^M w_j(x) = 1$ for a.a. $x \in \Omega$, cf. [22], where $\Omega \subset \mathbb{R}^d$ is a bounded domain on which the dynamical system is defined. This can be interpreted as a *one-hot encoding* of the different discrete modes and is also known as partial outer convexification [25]. The function S denotes the solution operator of the underlying dynamical system that maps control inputs w to the resulting state vectors y. The function J assigns a cost value to the resulting state vector, and the function R is intended to regularize the control input.

Problem (P) can be relaxed by dropping the binary constraint and admitting $[0,1]^M$-valued control functions. We then have

$$\min_{y,a}\; J(y)+R(a) \text{ s.t.} \begin{cases} y=S(a), \text{and} \\ a \text{ is a relaxed control,} \end{cases} \tag{R}$$

where a *relaxed control* is a measurable function $a : \Omega \to [0,1]^M$ such that $\sum_{j=1}^M a_j(x) = 1$ for a.a. $x \in \Omega$, cf. [22].

A. Underlying Approximation Principle

For sake of brevity, we omit a detailed description of the assumptions and involved function spaces that are required. We summarize the key result of the literature as follows.

Proposition 2.1: Under appropriate assumptions on S, J, and R, the problem (R) admits a minimizer and

$$\inf_{w,y \text{ feas. for (P)}} J(y)+R(w) = \min_{a,y \text{ feas. for (R)}} J(y)+R(a).$$

For a rigorous analysis of possible choices of S, J, and R, we refer the reader to [16], [21].

Proposition 2.1 gives rise to the following approximation method for the infimum of (P), the *combinatorial integral decomposition*.

1) Compute a relaxed control a that approximates the minimum of (R).
2) Compute a binary control w from a using a so-called rounding algorithm.

The rounding algorithms to compute w from a operate on grids – discretizations of Ω – that, when refining them appropriately, allow to obtain $J(y)+R(w) \to J(y)+R(a)$. Suitable assumptions on the grids and admissible refinement strategies are detailed in [22]. We explicitly note, however, that the assumptions can be satisfied by uniform refinements of uniform grids.

B. Integrating Switching Costs into Rounding Algorithms

As mentioned above, the binary controls computed with rounding algorithms usually exhibit high-frequency switching. This is often undesirable because implementation or manufacturing may become increasingly difficult. This phenomenon is unavoidable in the combinatorial integral decomposition approach for high accuracies.

However, a possible remedy has been proposed and evaluated for problems with control functions that vary only in time [2]. Therein, the rounding algorithm is replaced by solving an ILP that minimizes the switching cost while being constrained by a given approximation quality and the current discretization grid.

This article contributes a transfer of this approach to the multidimensional setting. To this end, we need the discretized perspective on binary control functions. For a given set of grid cells $\mathcal{T} = \{T_1, \dots, T_N\}$ that decompose the domain Ω, we consider the binary control functions

$$w(x) \coloneqq \sum_{t=1}^{N} \left(\chi_{T_t}(x) \sum_{j=1}^{M} \omega_{t,j} e_j \right)$$

that are possible outputs of a rounding algorithm operating on the grid $\mathcal{T}$ [22]. Here $\omega \in \{0,1\}^{N \times M}$ is a matrix such that in every row exactly one entry is set to 1, and e_j are the canonical unit basis vectors of $\mathbb{R}^M$. The function $\chi_{T_t}(x)$ denotes the $\{0,1\}$-valued indicator function of the set T_t.

As in [2], we adapt the total variation for our definition of switching costs between the different modes. and denote the set of numbers $\{1,\dots,K\}$ for $K \in \mathbb{N}$ as $[K]$. For a grid cell indexed by $t \in [N]$, let $\mathcal{N}_t \in 2^{\{1,\dots,N\}}$ denote the set of indices of the adjacent grid cells of T_t, specifically those grid cells that connect to T through a set of dimension $d-1$ (an edge in 2D, a surface in 3D). For two indices of adjacent grid cells t and s with different discrete modes i, $j \in [M]$, in other words, $\omega_{t,i} \neq \omega_{s,i}$, we account for a switching cost $c_{ij} \in \mathbb{R}$ that is scaled with the length (area) of the shared interface between the grid cells, which we denote by $\ell_{t,s}$. We choose $c_{ij} = c_{ji}$ for consistency and $c_{ii} = 0$ for all i and j:

$$C_{2D}(w) = C_{2D}(\omega) \coloneqq \frac{1}{2} \sum_{t=1}^{N} \sum_{i=1}^{M} C_{2D}(t,i,\omega), \tag{1}$$

$$C_{2D}(t,i,\omega) \coloneqq \sum_{s \in \mathcal{N}_t} \sum_{j \neq i} \omega_{t,i}\omega_{s,j} c_{ij} \ell_{s,t}. \tag{2}$$

The expression in (2) means that for an adjacent cell s of t the switching cost $c_{ij}\ell_{s,t}$ occurs if the i-th control mode is switched on in cell t and the j-th control mode is switched on in cell s.

III. SWITCHING COST AWARE ROUNDING PROBLEM

Starting from the two-dimensional cost function (1), this section describes both the ILP formulation and the graph-based shortest path approach to compute binary controls w that minimize C_{2D} such that a given approximation quality is observed.

A. Multidimensional ILP for rounding

Following the combinatorial integral decomposition approach, we start from a relaxed control a solving (R). Let $\mathcal{T}$ be a grid on which to define the resulting binary control. Then, a corresponding matrix α can be obtained on the grid $\mathcal{T}$ by averaging,

$$\alpha_{t,i} := \frac{1}{\lambda(T_t)} \int_{T_t} a_i(x)\,\mathrm{d}x$$

for all $t \in [N]$ and $i \in [M]$, where λ denotes the Lebesgue measure. Let $\bar{\lambda} := \max_t \lambda(T_t)$. Minimizing C_{2D}, we can constrain the approximation quality as

$$\left| \sum_{k=1}^{t} \lambda(T_t)(\alpha_{k,i} - \omega_{k,i}) \right| \le \theta \bar{\lambda} \qquad \text{for all } t \in [N] \text{ and all } i \in [M] \quad \text{(Slack)}$$

(see [3], [22]), where the parameter θ balances the approximation quality with the switching costs. As in the one-dimensional case, the term C_{2D} can be modeled with linear integral inequalities [2], yielding the ILP formulation

$$\begin{aligned} \min_{\omega} \quad & C_{2D}(\omega) && \text{(SCARP-ILP-2D)} \\ \text{s.t.} \quad & \sum_{i=1}^{M} \omega_{t,i} = 1 \text{ for all } t \in [N], && \text{(SOS1)} \\ & \text{(Slack) and } \omega_{t,i} \in \{0,1\} \text{ for all } t \in [N] \text{ and } i \in [M]. \end{aligned}$$

B. Switching cost aware rounding in two dimensions

We propose to derive an optimal solution of (SCARP-ILP-2D) by reformulating the ILP as a shortest path problem on a topologically sorted DAG. This is achieved by introducing a labeling function L for binary controls $\omega \in \{0,1\}^{t \times M}$:

$$L(\omega) := \left(\sum_{k=1}^{t} \omega_{k,1},\, \sum_{k=1}^{t} \omega_{k,2},\, \ldots,\, \sum_{k=1}^{t} \omega_{k,M} \right)^T .$$

In the one-dimensional case, the labeling function suffices to construct a DAG such that optimal ILP solution can be obtained by using a shortest path algorithm [3].

The challenge in the multidimensional setting is that for any grid cell t one cannot immediately determine the switching value of all neighboring grid cells $\mathcal{N}_t$. We therefore introduce a prefix vector $p \in [M]^P$, where the size P of this vector depends on the order in which grid cells are visited. A prefix vector contains the control choices made in the neighboring grid cells for the given control ω. We note that an entry in p can be overwritten as soon as all neighbors of the grid cell corresponding to the entry have been visited.

Employing the labels together with corresponding prefixes allows the definition of a DAG for any $\theta > 0$ with vertex set $V = \bigcup_{t=1}^{N} V_t$ consisting of sets

$$V_t(\alpha,\theta) := \left\{ \big(L(\omega), p(\omega)\big) \;\middle|\; \begin{array}{l} \omega \in \{0,1\}^{t\times M} \text{ satisfies} \\ \text{(Slack) for all } i \in [M] \\ \text{at } t \in [N],\, p(\omega) \in [M]^P \end{array} \right\}.$$

The associated set of arcs $A := \bigcup_{t=1}^{N-1} A_t$ comprises

$$A_t := \left\{ \begin{array}{c} \Big(\big(L(\omega^t), p(\omega^t)\big), \\ \big(L(\omega^{t+1}), p(\omega^{t+1})\big) \Big) \\ \in V_t \times V_{t+1} \end{array} \;\middle|\; \begin{array}{l} \|L(\omega^{t+1}) - L(\omega^t)\|_1 = 1, \\ \|p(\omega^{t+1}) - p(\omega^t)\|_1 = 0. \end{array} \right\}$$

In contrast to the one-dimensional case, the worst-case runtime estimate in the multidimensional case now depends on the size of the prefix as well as the number of grid cells and controls.

Proposition 3.1: Let a be a relaxed control, let the grid be uniform, and let $\theta \ge 1$. Then the shortest path approach for (SCARP-ILP-2D) has the worst-case runtime of

$$|V| + |A| \in \mathcal{O}\left(M^P N \sqrt{M} \lfloor 2\theta + 3 \rfloor^{M-1} \right). \tag{3}$$

Proof: For $t \le P$ the number of prefixes per label $L(\omega)$ is bounded by M^t. For any later grid cell $t > P$ at most M^P many different prefixes can exist because previous information does not influence the switching costs at cell t.

Additionally, we have that any vertex in V_t has at most M neighbors among V_{t+1} because one can switch only one control from one grid cell to a neighboring grid cell. The worst-case estimate for the number of vertices in a DAG without the prefix construction, denoted by $|V_t^{1D}(a,\theta)|$, is provided in [3, Thm 33]. This allows us to obtain an estimate on the number of arcs:

$$\begin{aligned} |A| = \sum_{t=1}^{N} |A_t| &< M \sum_{t=1}^{N} M^P |V_t^{1D}(a,\theta)| \\ &\in \mathcal{O}\left(M^P N \sqrt{M} \lfloor 2\theta + 3 \rfloor^{M-1} \right). \end{aligned}$$

The claim now follows inductively from the construction, which states that the inequality $|V| = \sum_{t=1}^{N} V_t < \sum_{t=1}^{N} M^P |V_t^{1D}(a,\theta)|$ holds. ■

Remark 3.2: From Proposition 3.1 we expect exponential runtime for the shortest path approach because the underlying rounding problem is already strongly $\mathcal{NP}$-hard and admits no polynomial time approximation algorithm (already in 1D) [4, Thm. 6.2, Cor. 6.3]. However, one can omit the prefix in the two-dimensional case and use the one-dimensional algorithm. This approach implies that the switching cost is computed with respect to only two instead of four neighboring cells.

IV. COMPUTATIONAL EXAMPLE

We consider the topological optimization problem of designing cloaks for wave functions governed by the Helmholtz equation in 2D. We use a scenario similar to [13], [19], see also Fig. 1.

A. Modeling of the MIOCP

For an incident wave y_0 and a design area $D_s \subset \Omega \subset \mathbb{R}^2$, we seek a function $v : D_s \to \{\nu_1, \nu_2, \nu_3\}$, where $\nu_1 = 0$, $\nu_2 = 0.5$, and $\nu_3 = 1$ are possible material constants with $v(x) = v_1$ indicating that no material is placed at x. The goal is to protect an object in another region $D_o \subset \Omega \subset \mathbb{R}^2$ from the incident wave. Using partial outer convexification, we have the reformulation $v(x) = \sum_{i=1}^{M} w_i(x)\nu_i$ for a.a. $x \in \Omega$, and the considered optimization problem is

$$\inf_{u,w} \quad \frac{1}{2}\|y + y_0\|^2_{L^2(D_o)} + R(w) \tag{P$_H$}$$

$$\begin{aligned}
\text{s.t.} \quad & -\Delta y - k_0^2 y = \left(k_0^2 yq + k_0^2 q y_0\right) \sum_{i=1}^{M} v_i w_i \quad \text{in } \Omega, \\
& (\partial y/\partial n) - i k_0 y = 0 \quad \text{on } \partial\Omega, \\
& w \in L^\infty(\Omega, \mathbb{R}^3), \\
& w(x) \in \{0,1\}^3 \text{ and } \sum_{i=1}^{3} w_i(x) = 1 \text{ for a.a. } x \in D_s, \\
& w(x) = 0 \text{ for a.a. } x \in D \backslash D_s.
\end{aligned}$$

For R we use the relaxed multi-bang regularizer [6], [21].

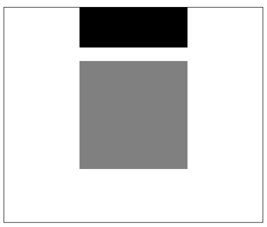

Fig. 1. Domain $\Omega = (0,2)^2$ with the scatterer design area $D_s = (.5, 1.5)^2$ in light gray and the protected area $D_o = (0.5, 1.5) \times (1.625, 2.)$ in black.

After transforming, Proposition 2.1 holds for (P$_H$) as $v(x) = \sum_{i=1}^{M} w_i(x)\nu_i$ with the *bangs* ν_1, ν_2, and ν_3; the weights (costs) $g_1 = 0$, $g_2 = 1$, and $g_3 = 4$; and a general scale of 0.05. By combining Theorem 2.12 in [21] and Proposition 2.7 in [19], the claim of Proposition 2.1 holds for (P$_H$).

B. Discretization and Solution of the Relaxation

The domain is discretized uniformly into $2^8 \times 2^8$ squares, which are decomposed into 4 triangles each. The discretized state equation is then solved with the open-source library FIREDRAKE [23], using PETSC as the backend for numerical linear algebra [1]. The adjoint equation for the first term of the reduced objective is computed by using DOLFIN-ADJOINT [8]. We optimize for a first-order stationary point using a limited-memory quasi-Newton method with BFGS updates, `blmvm`, from the PETSC TAO package [7]. In order to be able to apply gradient-based methods, the regularizer is smoothed slightly by using a Moreau envelope with $\gamma = 0.01$ as described in [21].

C. Computing Binary from Relaxed Controls

We implement the second step of the combinatorial integral decomposition in four ways. We use an adapted version of the shortest path algorithm from [3] for the graph-based formulation as described in Section III. Moreover, we use the shortest path algorithm from [3] directly as a heuristic as described in Remark 3.2. The ILP formulation is solved by using version 9.1 of the GUROBI optimizer; see [20]. We also use a C++ implementation of the SUR algorithm.

For the order of the grid cells in (Slack) provided to the different rounding approaches, we use iterates of Hilbert curves as in [22] to satisfy the requirements of the decomposition approach.

TABLE I

RELATIVE OBJECTIVE ERRORS OF `ILP` / `SCARP-G`, `SCARP-HG`, AND `SUR` TO THE CONTINUOUS RELAXATION OVER SUCCESSIVELY REFINED GRIDS. $N(D_s)$ IS THE NUMBER OF CELLS DECOMPOSING D_s.

$N(D_s)$	`ILP` / `SCARP-G`	`SCARP-HG`	`SUR`
4^1	$4.02 \cdot 10^0$	$\mathbf{3.72 \cdot 10^0}$	$4.62 \cdot 10^0$
4^2	$4.02 \cdot 10^0$	$4.02 \cdot 10^0$	$\mathbf{3.98 \cdot 10^0}$
4^3	$2.14 \cdot 10^0$	$2.61 \cdot 10^0$	$\mathbf{1.76 \cdot 10^0}$
4^4	$\mathbf{5.01 \cdot 10^{-1}}$	$5.69 \cdot 10^{-1}$	$6.01 \cdot 10^{-1}$
4^5	$\mathbf{9.70 \cdot 10^{-2}}$	$1.22 \cdot 10^{-1}$	$1.66 \cdot 10^{-1}$
4^6	$\mathbf{2.78 \cdot 10^{-2}}$	$3.05 \cdot 10^{-2}$	$4.67 \cdot 10^{-2}$

V. COMPUTATIONAL RESULTS

To record and evaluate the behavior of runtime, approximation error, and switching costs for refined grids, we solve the ILP formulation `ILP`, the graph-based algorithm `SCARP-G`, the heuristic graph-based algorithm `SCARP-HG`, and sum-up rounding `SUR` on uniformly refined grids of square cells that decompose D_s for the solution of the continuous relaxation.

All experiments were conducted on a workstation with an AMD Epic 7742 CPU and 96 GB RAM.

The computed relaxed control a exhibits values such that $\sum_{i=1}^{M} a_i v_i$ is close to the bangs 0, 0.5, and 1 in large parts of the domain. We are able to compute the resulting binary controls for `SCARP-HG`, `SUR`, and `ILP` for $N = 4$ to $N = 4096$ squares that decompose D_s. For finer grids, `SCARP-HG` and `SUR` can still be executed but the memory demand of `ILP` exceeds the installed memory of our workstation. The memory demand of `SCARP-G` exceeds the installed memory of our workstation starting from $N = 256$.

Figure 2 shows state and control vectors for the continuous relaxation as well as for `SUR`, `SCARP-HG`, `SCARP-G`, and `ILP` on the finest grid. Differences in the tracking term $y + y_0$ are hardly visible. Blue indicates a value close to zero, indicating that a protection of the region D_o can be established.

For the refined grids on which the roundings are computed, we obtain convergence of the objective values. This can be observed from the resulting objective values in Table I. The switching costs of the design obtained with `SUR` are higher than the ones obtained with `SCARP-HG`, which in turn are higher than the ones obtained with `SCARP-G`, and `ILP`. These results can be observed from the objective values in Table II and the control designs shown in Fig. 3. Jittering in the design is almost eliminated in the designs produced by `SCARP-G` and `ILP`. Notably, the switching costs produced by `SCARP-HG` are only moderately higher than the optimal ones produced by `SCARP-G` and `ILP` (6.1 % for $N = 1024$ and 4.5 % for $N = 4096$).

VI. CONCLUSION

We have given an ILP formulation for switching cost aware rounding in the second step of the combinatorial integral

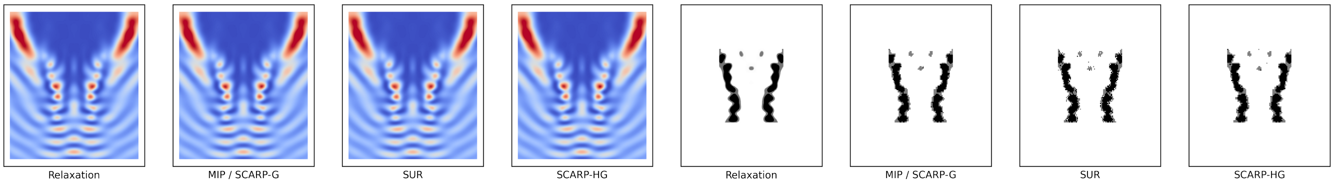

Fig. 2. Resulting amplitude of $y + y_0$ (left) and controls (right) of the Helmholtz cloaking problem.

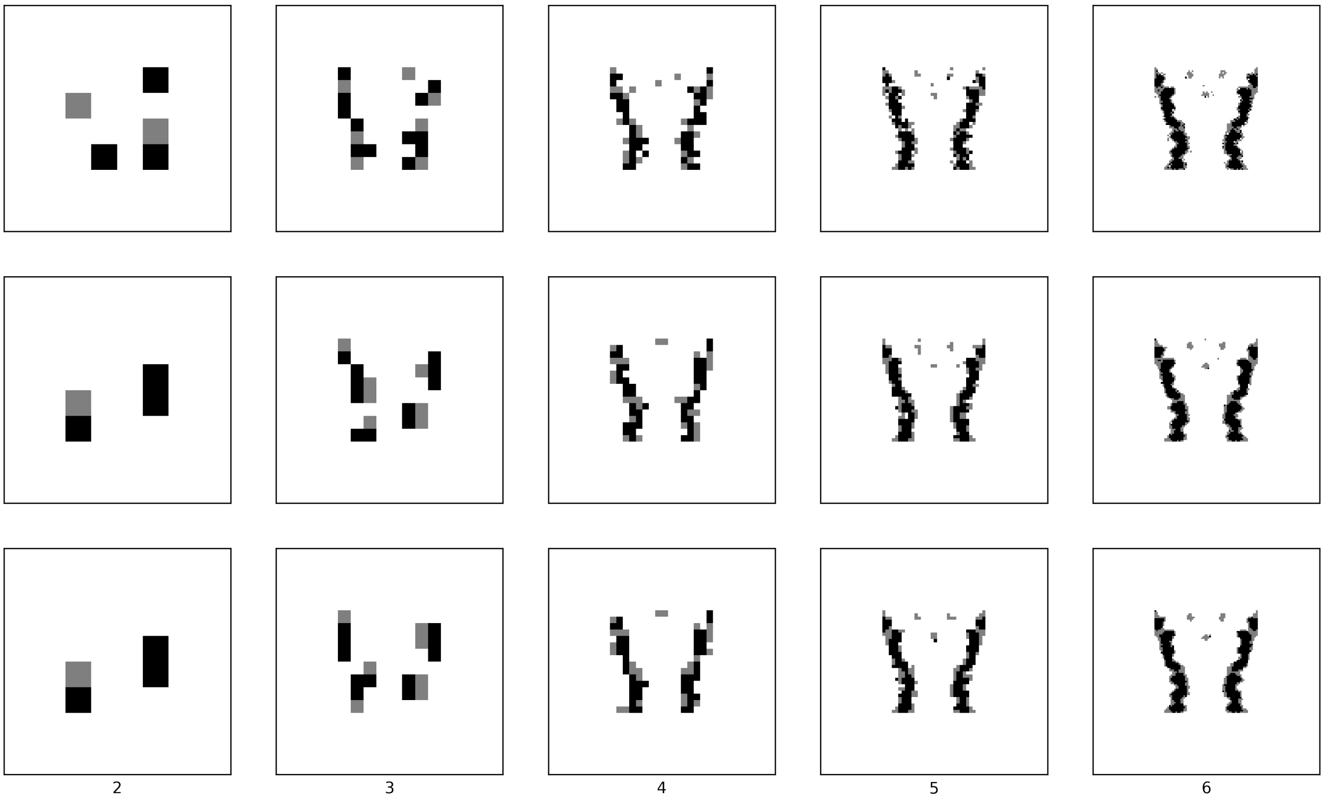

Fig. 3. Control designs for grids 2 to 6 for `SUR` (top row), `SCARP-HG` (center row), and `SCARP-G` / `ILP` (bottom row).

TABLE II

SWITCHING COSTS OF `ILP`, `SCARP-G`, `SCARP-HG`, AND `SUR`. $N(D_s)$ IS THE NUMBER OF CELLS DECOMPOSING D_s.

$N(D_s)$	`ILP` / `SCARP-G`	`SCARP-HG`	`SUR`
4^1	1	1	1
4^2	2	2	3
4^3	4.75	5.13	5.88
4^4	7.25	7.88	9.38
4^5	10.13	10.75	13.34
4^6	12.47	13.03	16.09

TABLE III

COMPUTATIONAL EFFORT FOR `ILP`, `SCARP-G`, `SCARP-HG`, AND `SUR` IN SECONDS. $N(D_s)$ IS THE NUMBER OF CELLS DECOMPOSING D_s.

$N(D_s)$	`ILP`	`SCARP-G`	`SCARP-HG`	`SUR`
4^1	0.09	$1.07 \cdot 10^{-05}$	$2.73 \cdot 10^{-06}$	$4.84 \cdot 10^{-06}$
4^2	0.03	$9.19 \cdot 10^{-05}$	$1.77 \cdot 10^{-05}$	$5.43 \cdot 10^{-06}$
4^3	0.06	344.25	$9.92 \cdot 10^{-05}$	$7.85 \cdot 10^{-04}$
4^4	0.70	–	$8.61 \cdot 10^{-04}$	$2.20 \cdot 10^{-05}$
4^5	6.85	–	$9.95 \cdot 10^{-03}$	$6.29 \cdot 10^{-05}$
4^6	78.87	–	$1.24 \cdot 10^{-01}$	$2.35 \cdot 10^{-04}$

decomposition approach and shown that the DAG construction from the one-dimensional setting can be transferred to the multidimensional one.

The theoretical properties of an improved approximation quality for refined grids and reduced switching over heuristic approaches are validated computationally on a topology optimization problem. Using the relaxed multi-bang regularization in the relaxation yields a relaxed control that is already close to a binary control.

The computational efforts of the shortest path approach and the ILP solve impose a prohibitive computational demand that limits the number of possible grid refinements. In particular, the exact shortest path approach we presented is expensive in the two-dimensional setting. The exponential growth in the number of vertices that is introduced by the prefix vector fills the memory too quickly. A reduction in the size of the graph thus seems to be inevitable for a useful shortest path approach. Further research is necessary to alleviate this problem and find computationally feasible graph-based solution strategies.

However, the results also show that using the one-dimensional shortest path algorithm for the two-dimensional problem—implying that switching costs are considered only with respect to two neighboring grid cells instead of four—may yield a satisfactory result at relatively low computational cost.

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