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Second order optimality conditions for an optimal control problem governed by a regularized phase-field fracture propagation model

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Abstract. We prove second order optimality conditions for an optimal control problem of tracking type for a time-discrete regularized phase-field fracture or damage propagation model. The energy minimization problem describing the fracture process contains a penalization term for violation of the irreversibility condition in the fracture growth process, as well as a viscous regularization corresponding to a time-step restriction in a temporal discretization of the problem. In the control problem, the energy minimization problem is replaced by its Euler-Lagrange equations. While the energy minimization functional is convex due to the viscous approximation, the associated Euler-Lagrange equations are of quasilinear type, making the control problem nonconvex. We prove second order necessary as well as second order sufficient optimality conditions without two-norm discrepancy.

1. Introduction

In this paper, we are interested in second order optimality conditions for an optimal control problem for regularized fracture propagation. More precisely, we consider an optimal control problem of tracking type of the following form: Find a control q in an admissible control set Q_{ad}, with associated state pair $\mathbf{u} := (u, \varphi)$, that satisfies

$$(\mathrm{NLP}^{\gamma,\eta}) \qquad \min_{q\in Q_{\mathrm{ad}},\, \mathbf{u}\in V} \quad J(q,\mathbf{u}) := \frac{1}{2}\|u-u_d\|^2_{L^2(\Omega,\mathbb{R}^2)} + \frac{\alpha}{2}\|q\|^2_{\Gamma}$$

$$(\mathrm{EL}^{\gamma,\eta}) \qquad \text{subject to :} \quad A(\mathbf{u}) + R(\varphi;\gamma) = B(q).$$

The precise functional analytic setting along with a concrete mathematical definition of the operators A, R, and B will be introduced in the next subsection.

Let us give a brief introduction to the model problem, which stems from a bi-level optimization problem with an upper-level tracking type functional and a lower-level variational fracture propagation problem. The latter is an energy minimization problem, which is eventually replaced by its Euler-Lagrange equations. The lower-level fracture propagation problem behind this formulation was considered in [**6**, **7**, **21**]. An Ambrosio-Tortorelli regularization cf. [**3**] is used to avoid the irregular fracture set. This means, that in addition to the displacement u, a phase-field variable φ is introduced. The latter has values $0 \le \varphi \le 1$ and describes the condition of the material at every point in the domain, with $\varphi = 1$ where the material is completely sound, and $\varphi = 0$ where the material is fully broken,

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guaranteeing a smooth transition between those two states. The control q acts as a boundary force. We consider one time-step of a time-discrete but spatially continuous problem formulation. In the energy minimization problem describing the fracture process, a viscous regularization corresponding to a time-step restriction in a temporal discretization of the problem cf. [**32, 40**] is used, that guarantees strict convexity of the lower-level minimization problem. Nevertheless, the Euler-Lagrange equations are of quasilinear type, making the overall control problem nonconvex. Moreover, a violation of the irreversibility condition in the fracture growth process is penalized using the regularization from [**37**]. The corresponding terms appear in the operator R in the differential equations, whereas the differential operator A stems from the actual fracture propagation process. We refer to e.g. the introduction of [**39**] for a more detailed description of the mathematical model.

Our work complements the studies of this model problem that were conducted in [**26, 38–40**]. Compared to previous works on optimal control of fractures cf. [**31, 34**] where paths of fixed length or prescribed fracture paths could be treated, this variational fracture approach is more flexible and also allows the treatment of arbitrary fracture paths and branching cf. [**32**].

The control problem, without viscous regularization but with an additional trivial kernel assumption, has been analyzed with respect to first order necessary optimality conditions in [**39**]. In [**40**], convergence of regularized solutions (with respect to the penalization parameter γ) has been proven, along with estimates on the constraint violations for fixed γ. In [**26**], this convergence result was extended to the dual variables and it was shown, that in the limit, the functions satisfy an optimality system of an MPCC. In the same publication, the sequential quadratic programming (SQP) method was introduced for both the regularized and nonregularized problem, the involved sub-problems were investigated, and it was proven that the limit point of the SQP method, in case of convergence, satisfies the first order optimality system of the regularized original problem. A convergence result of the FEM discretization of a linearized fracture control problem was shown in [**38**].

Related results regarding analysis and numerics of optimization of fractures include [**28**]. Shape optimization was used in [**2**]. We also want to mention [**19, 20**], where control of a viscous-damage model was considered in a continuous setting, and [**44**] and [**4**], where the author investigated necessary conditions of an optimal control problem of a two-field damage model, and strong stationarity of a nonsmooth (viscous damage) coupled system, respectively.

The goal of the present paper is to establish second order necessary and sufficient optimality conditions for the model problem. Second order optimality conditions are an area of active research in the optimal control community. To our knowledge, the first partial differential equation (PDE) related publication for second order sufficient (SSC) optimality conditions has been [**22**]. For an overview about many aspects considered since then we refer to the work [**13**] and the references therein. One of the central difficulties in infinite dimensions is the two-norm discrepancy see [**30**], i.e. if differentiability and coercivity of the second derivative hold in different spaces. This influences many aspects that rely on SSC, such as convergence of finite element discretizations or convergence of solution algorithms, to name two examples. Let us mention some further results, restricting ourselves to mainly the elliptic setting. SSC for the semi-linear case with state-constraints have been established in [**14**]. In [**8**], for the same setting, SSC that are closest to the associated necessary ones have been presented. SSC for a more abstract optimization in Banach spaces have been analyzed in [**12**]. For the quasilinear case, SSC in the control-constrained case, can be found in [**9, 10**]. At this point, we also want to mention [**17**], where SSC of an optimal control problem that is governed by a nonsmooth quasilinear PDE were investigated.

Finally, we also want to give a short overview about related results in the field of MPCCs, since as mentioned above, the model problem converges to an MPCC in the penalization limit. For MPCCs, SSC are a challenging task due to the lack of smoothness in the control-to-state operator. A lot of the recent research thus focused on regularization methods to establish different (M, B, C, strong) stationary concepts cf. [**1, 18**]. In [**25**] a comparison of those concepts for the obstacle problem has been made. The authors of [**42, 43**] tackled the lack of smoothness of the control-to-state operator by investigating generalized derivatives of this operator for the obstacle problem. SSC for the obstacle problem have recently been under investigation in [**16**]. For a control problem governed by variational inequalities, SSC could be established in [**5**] and [**33**]. In [**36**], the authors established SSC for a regularization of an optimal control problem that is governed by an evolution variational inequality. Finally, in a more general context, SSC of a nonsmooth obstacle problems have been analyzed in [**15**].

We finish this section with an outline of the present paper. In Section 2, we provide all details and assumptions for the model problem along with an overview about the notation used. Then, in Section 3, we collect and provide the necessary existence and regularity results for $(\mathrm{EL}^{\gamma,\eta})$. The most important preliminary result is the Lipschitz continuity of the associated control-to-state operator G and its derivatives G' and G'' in Lemma 3.6, Lemma 3.8, and Lemma 3.9. After collecting solvability results and first order necessary conditions for $(\mathrm{NLP}^{\gamma,\eta})$, our main results, the second order necessary conditions of Theorem 4.4 and the second order sufficient conditions of Theorem 4.6 in Section 4, follow from applying optimal results for an abstract setting from [**11**]. In particular, we arrive at a result on sufficient conditions without two-norm discrepancy, following from plain positivity of the second derivative of the reduced functional in a critical point for directions from a cone of critical directions. This means that the gap between necessary and sufficient conditions is minimal, which is a result of the structure of our problem, that ensures that for $(\mathrm{NLP}^{\gamma,\eta})$, positivity and coercivity of the second derivative of the reduced functional are equivalent.

2. Problem Formulation, General Assumptions, and Notation

Let us now introduce the precise assumptions on the model problem along with the general functional analytic setting of this paper. For convenience, we recall the formulation.

Find a control q in a set of admissible controls Q_{ad}, subset of the control space Q, with associated state pair $\mathbf{u} := (u, \varphi)$, that satisfies

$$(\mathrm{NLP}^{\gamma,\eta}) \qquad \min_{q\in Q_{\mathrm{ad}},\, \mathbf{u}\in V} \quad J(q,\mathbf{u}) := \frac{1}{2}\|u-u_d\|^2_{L^2(\Omega,\mathbb{R}^2)} + \frac{\alpha}{2}\|q\|^2_{L^2(\Gamma)}$$

$$(\mathrm{EL}^{\gamma,\eta}) \qquad \text{subject to :} \quad A(\mathbf{u}) + R(\varphi;\gamma) = B(q).$$

Here, the domain Ω is a polygonal subset of $\mathbb{R}^2$ with boundary $\partial\Omega = \Gamma\dot{\cup}\Gamma_D$, i.e. the boundary of Ω is split into a Neumann part Γ where we apply the control q, and a part Γ_D denoting the remaining part of $\partial\Omega$ with homogeneous Dirichlet boundary conditions. As in [**40**, Section 2], we additionally assume throughout that Ω is $W^{2,q}$-regular for the homogeneous Neumann-problem $-\varepsilon\triangle\varphi + \frac{1}{\varepsilon}\varphi = f$, as well as Gröger regularity cf. [**23**] of $\Omega \cup \Gamma$.

The given function $u_d \in L^2(\Omega, \mathbb{R}^2)$ in $(\mathrm{NLP}^{\gamma,\eta})$ denotes a desired displacement, and the Tikhonov parameter α is a fixed positive real number, the cost parameter. The control space is $Q = L^2(\Gamma)$, and the set of admissible controls is defined by simple box constraints as

$$Q_{\mathrm{ad}} := \{q \in Q \quad | \quad q_a \leq q \leq q_b \quad \text{a.e. on } \Gamma\},$$

for q_a, $q_b \in L^\infty(\Gamma)$ and $q_a < q_b$ a.e. on Γ.

For a precise definition of $(\mathrm{EL}^{\gamma,\eta})$, let us first fix some general notation for function spaces. For $p > 2$, $q := p/2 > 1$, we define the spaces

$$\begin{aligned}
V_u &:= H^1_D(\Omega, \mathbb{R}^2) := \{v \in H^1(\Omega; \mathbb{R}^2) \,|\, v = 0 \text{ on } \Gamma_D\}, & V_\varphi &:= H^1(\Omega),\\
W_u &:= W^{1,p}_D(\Omega, \mathbb{R}^2), & W_\varphi &:= W^{2,q}(\Omega),\\
V &:= V_u \times V_\varphi, & W &:= W_u \times W_\varphi,\\
W^\times &:= W^{-1,p}(\Omega, \mathbb{R}^2) \cap \times L^q(\Omega), & Q &:= L^2(\Gamma).
\end{aligned}$$

For the choice of p and q and in $N = 2$ spatial dimensions, note that $W_u \hookrightarrow V_u$ and $W_\varphi \hookrightarrow V_\varphi$ by the Sobolev/Kondrachov embedding theorem. Further $W_\varphi \hookrightarrow L^\infty$, which we will frequently use without further mentioning. Further, choose $s \in (0, 1/2)$ and assume that p and s are chosen such that $H^{1+s} \subset W^{1,p}$.

We want to point out that the definition of the space W_u (thus also of W) and $W^\times$ differs from [**26**], where W_u was defined as $W^{1,p}_D(\Omega, \mathbb{R}^2) \cap H^{1+s}(\Omega, \mathbb{R}^2)$, and $W^\times$ was defined as $(W^{-1,p}(\Omega, \mathbb{R}^2) \cap H^{-1+s}(\Omega, \mathbb{R}^2)) \times L^q(\Omega)$. Here, we will only use these improved regularities when looking at regularity of solutions of $(\mathrm{EL}^{\gamma,\eta})$ and its linearization, and will specifically point out all instances in which we include these spaces. We will denote the respective dual spaces with a superscript *, e.g., V^*. Here and throughout, we understand all spaces to be defined on the domain Ω unless otherwise stated and omit the dependence on Ω for readability.

We will further use the following notation for the scalar product/norm: $(\cdot, \cdot)$ denotes the usual L^2 inner product with corresponding norm $\|\cdot\|$, and $(\cdot, \cdot)_\Gamma$ corresponds to the inner product of the control space $Q = L^2(\Gamma)$. In addition, $\langle \cdot, \cdot \rangle$ stands for a duality pairing where we omit the spaces wherever obvious from the context.

The operators involved in $(\mathrm{EL}^{\gamma,\eta})$ are what we will call the nonlinear phase-field operator $A \colon V \supset W \to V^*$, the penalization operator $R \colon V_\varphi \to V_\varphi^*$, and the control-action operator on the Neumann boundary Γ, $B \colon Q \to V^*$. For a displacement/phase-field pair $\mathbf{u} := (u, \varphi) \in W$, they are defined by

$$\begin{aligned}
\langle A(\mathbf{u}), \mathbf{v}) \rangle : &= \Big(g(\varphi)\mathbb{C}e(u), e(v^u)\Big) + \epsilon(\nabla\varphi, \nabla v^\varphi) - \frac{1}{\epsilon}(1 - \varphi, v^\varphi)\\
&\quad + \eta(\varphi - \varphi^-, v^\varphi) + (1 - \kappa)(\varphi\mathbb{C}e(u) : e(u), v^\varphi),
\end{aligned} \tag{2.1}$$

$$\langle R(\varphi; \gamma), v^\varphi \rangle : = \gamma([(\varphi - \varphi^-)^+]^3, v^\varphi), \tag{2.2}$$

$$\langle B(q), (v^u, v^\varphi) \rangle : = (q, v^u)_\Gamma, \tag{2.3}$$

for any $\mathbf{v} = (v^u, v^\varphi) \in V$, and given phase-field $\varphi^- \in W_\varphi$ with $0 \le \varphi^- \le 1$. In a temporal discrete multi-step model, φ^- would be the phase-field from the previous time-step, and we would make this assumption on an initial phase-field φ^0. Moreover, let the parameters $\kappa, \varepsilon, \gamma > 0$, and $\eta \ge 0$ be given. We will refer to γ as penalization parameter, since the operator R stems from a penalization of the violation of the irreversibility condition for the fracture growth, and to η as (viscosity) regularization parameter, cf. the comments in the introduction. The parameter $\varepsilon > 0$ stems from the phase field modeling of the fracture growth problem and will be considered fixed here. Finally, κ appears in $g(x) := (1-\kappa)x^2 + \kappa$, and $\mathbb{C}$ denotes the (symmetric) rank-4 elasticity tensor cf. [**41**, Section 3]. For further explanation of the forward problem we also refer to the exposition in [**39**].

For further use, we define the operators appearing in the linearized equations. Let $\mathbf{d^u} := (d^u, d^\varphi) \in V$ be a pair of displacement and phase-field functions. For $\mathbf{u} \in W$, we define the operators $A'(\mathbf{u})\colon V \to V^*$ and $R'(\varphi;\gamma)\colon V_\varphi \to V_\varphi^*$ by

$$
\begin{aligned}
\langle A'(\mathbf{u})\mathbf{d^u}, \mathbf{v}\rangle := {}& \Big(g(\varphi)\mathbb{C}e(d^u), e(v^u)\Big) + 2(1-\kappa)(\varphi\mathbb{C}e(u) : e(d^u), v^\varphi) \\
& + (1-\kappa)(d^\varphi\mathbb{C}e(u) : e(u), v^\varphi) + \epsilon(\nabla d^\varphi, \nabla v^\varphi) + \frac{1}{\epsilon}(d^\varphi, v^\varphi) \\
& + \eta(d^\varphi, v^\varphi) + 2(1-\kappa)(\varphi\mathbb{C}e(u)d^\varphi, e(v^u)) \qquad \forall \mathbf{v} \in V,
\end{aligned} \tag{2.4}
$$

$$
\langle R'(\varphi;\gamma)d^\varphi, v^\varphi\rangle := 3\gamma([(\varphi-\varphi^-)^+]^2 d^\varphi, v^\varphi) \qquad \forall \mathbf{v} \in V, \tag{2.5}
$$

as well as $A''(\mathbf{u})\colon W \times W \to W^\times$ by

$$
\begin{aligned}
&\langle A''(\mathbf{u})[\mathbf{d}_1^{\mathbf{u}}, \mathbf{d}_2^{\mathbf{u}}], \mathbf{v}\rangle \\
&\quad = 2(1-\kappa)(d_2^\varphi\mathbb{C}e(u)d_1^\varphi, e(v^u)) + 2(1-\kappa)(d_2^\varphi\mathbb{C}e(d_1^u)\varphi, e(v^u)) \\
&\qquad + 2(1-\kappa)(d_2^\varphi\mathbb{C}e(u) : e(d_1^u), v^\varphi) + 2(1-\kappa)(\varphi\mathbb{C}e(d_2^u)d_1^\varphi, e(v^u)) \\
&\qquad + 2(1-\kappa)(d_1^\varphi\mathbb{C}e(d_2^u) : e(u), v^\varphi) + 2(1-\kappa)(\varphi\mathbb{C}e(d_2^u) : e(d_1^u), v^\varphi) \qquad \forall \mathbf{v} \in V,
\end{aligned} \tag{2.6}
$$

and $R''(\varphi;\gamma)\colon W_\varphi \times W_\varphi \to V_\varphi^*$ by

$$
\langle R''(\varphi;\gamma)[d_1^\varphi, d_2^\varphi], v^\varphi\rangle = 6\gamma([(\varphi-\varphi^-)^+]d_1^\varphi d_2^\varphi, v^\varphi) \qquad \forall \mathbf{v} \in V. \tag{2.7}
$$

In the remainder of this paper, we will tacitly assume that $\eta \ge 0$ is large enough such that all results below hold true. We collect this in the following standing assumption:

Assumption 2.1 (Viscous approximation). Let $\eta \ge 0$ be chosen large enough for all following calculations and results.

3. The Control-to-State Operator and the Objective Functional

We start with the analysis of $(\mathrm{EL}^{\gamma,\eta})$ and the objective functional J. We will collect and extend known results for the PDE, and eventually introduce a well-defined control-to-state mapping $G\colon q \mapsto \mathbf{u}$ due to the regularization effect of a sufficiently large η. We can then introduce the reduced functional $f\colon Q \to \mathbb{R}$ and establish differentiability and Lipschitz properties for G, f, and their first and second order derivatives. The main results are the Lipschitz continuity of this operator G and its derivatives G' and G'' in Section 3.2, which allow to deduce analogous properties for f.

First, let us recall a result on unique solvability of $(\mathrm{EL}^{\gamma,\eta})$ and regularity results for the solution, known from cf. [**24**, **39**, **40**]. While [**39**] had to deal with possible nonuniqueness of solutions for the Euler-Lagrange equations, the presence of a sufficiently large $\eta \geq 0$ as in [**40**] makes the energy minimization problem for the fracture growth strictly convex, and guarantees existence of a unique solution of $(\mathrm{EL}^{\gamma,\eta})$ for any given $q \in Q$.

Lemma 3.1. *Let Assumption 2.1 hold, and let $0 \leq \varphi^- \in W_\varphi$. Then for every $q \in Q$, $(\mathrm{EL}^{\gamma,\eta})$ has a unique weak solution $\mathbf{u} \in W$, such that $\varphi \in L^\infty$ and $0 \leq \varphi \leq 1$. Additionally, $\mathbf{u} \in H^{1+s} \times H^{1+s}$, and the following stability properties are satisfied*

$$\|u\|_{1,p} \leq c\|q\|_\Gamma, \tag{3.1}$$

$$\|\varphi\|_{2,q} \leq c(1 + \|q\|_\Gamma^2 + \|\frac{1}{\varepsilon} + \varepsilon\triangle\varphi^-\|_q) =: \tilde{c}, \tag{3.2}$$

$$\|u\|_{1+s} \leq c\|q\|_\Gamma, \tag{3.3}$$

for a $\tilde{c} = c(\varphi^-, q) < \infty$. In particular, $\tilde{c}$ can be chosen independently of γ and φ.

Proof. We refer to Section 1 and 2 of [**40**] for the existence and regularity results $\mathbf{u} \in W \cap (H^{1+s} \times L^\infty)$. The norm estimates follow from estimate (2.2) therein, as well as [**40**, Corollary 3.9] for arbitrary $\eta \geq 0$. More precisely, unique solvability in W with $0 \leq \varphi \leq 1$ as well as the norm estimate (3.1) is ensured by [**39**, Corollary 4.2] for $\eta = 0$. The proof is not affected by allowing $\eta > 0$. The norm estimate (3.2) was proven in [**40**, Corollary 3.9] for arbitrary $\eta \geq 0$ based on [**24**, Corollary 2, Section 7]. Finally, the H^{1+s}-regularity result along with estimate (3.3) also follows from [**24**, Corollary 2, Section 7]. Again the proof is not affected by allowing $\eta \geq 0$. □

With Lemma 3.1, it is now possible to introduce a control-to-state operator

$$G\colon Q \to W, \quad G(q) = \mathbf{u}, \tag{3.4}$$

associated with the nonlinear PDE $(\mathrm{EL}^{\gamma,\eta})$. Here, $\mathbf{u}$ solves $(\mathrm{EL}^{\gamma,\eta})$ for right-hand-side $q \in Q$. We obtain a usual reduced problem formulation

$$\min_{q \in Q_{\mathrm{ad}}} \quad f(q) := J(q, G(q)), \tag{3.5}$$

where we implicitly use that W is embedded in $L^2(\Omega, \mathbb{R}^2) \times L^2(\Omega)$.

For what follows, we will discuss the linearized equations w.r.t. existence, uniqueness, and sufficient regularity. Unique solvability in V was stated in [**26**, Section 2.2, Proposition 2.1] and proven for similar equations in [**39**] without viscous approximation, but additional trivial kernel assumption $\ker A' = \{0\}$. There, due to the lack of the viscous approximation term $\eta(d^\varphi, v^\varphi)$, only a Fredholm property of A' was proven, that in combination with the trivial kernel assumption ensured coercivity. Replacing this by Assumption 2.1, we extend Lemma 5.1 and 5.2 in [**39**] to obtain existence of unique solutions first in V then in W in the setting $\eta > 0$, and eventually utilize the ideas from the nonlinear setting in [**24**] to guarantee regularity of solutions in the space $(W_u \cap H^{1+s}) \times W_\varphi$. The latter improved regularity result is not needed for our analysis of SSC, but interesting for future research. We state and repeat important technical steps of the proofs for convenience of the reader. To motivate this, note that for instance a direct adaptation of the proof of [**39**, Lemma

5.1] to obtain coercivity of the underlying bilinear form to the case $\eta > 0$ would lead to

$$\langle A'(\mathbf{u})\mathbf{d^u}, \mathbf{d^u}\rangle \geq c\|u\|^2_{V_u} + \|\varphi\|^2_{V_\varphi} - c\|\varphi\|^2_{s,2} + \eta\|\varphi\|^2,$$

where the negative term cannot be absorbed directly into the L^2-regularization term. Therefore, some further technical estimates as in e.g. [**32**] have to be used to prove coercivity in V.

Lemma 3.2. *Let Assumption 2.1 hold and $\mathbf{u} \in W$ be given. Then, for every $\mathbf{f} := (f^u, f^\varphi) \in V^*$, there exists a unique solution $\mathbf{d^u} \in V$ to*

$$\langle A'(\mathbf{u})\mathbf{d^u}, \mathbf{v}\rangle + \langle R'(\varphi;\gamma)d^\varphi, v^\varphi\rangle = \langle \mathbf{f}, \mathbf{v}\rangle \qquad \forall\, \mathbf{v} \in V, \tag{3.6}$$

and it holds

$$\|d^u\|_{1,2} + \|d^\varphi\|_{1,2} \leq c_{coer}\|\mathbf{f}\|_{V^*}.$$

If a forteriori $\mathbf{f} \in W^\times \hookrightarrow V^*$, then additionally $\mathbf{d^u} \in W$, and it holds

$$\|d^u\|_{1,p} + \|d^\varphi\|_{2,q} \leq c(\mathbf{u}, \varphi^-, \eta)\|\mathbf{f}\|_{W^\times}. \tag{3.7}$$

Further, the constant on the right-hand-side of (3.7) *depends on positive integer powers of the norm $\|\mathbf{u}\|_W$. If additionally $\mathbf{u} \in H^{1+s}\times H^{1+s}$ and $f^u \in H^{-1+s}$, then additionally $d^u \in H^{1+s}$.*

Proof. (1) Let us start with unique solvability of (3.6) in V. In comparison to [**39**, Lemma 5.1], the only difference is that due to Assumption 2.1, the bilinear form induced by $\langle A'(\mathbf{u})\cdot, \cdot\rangle$ is coercive. Similar to [**39**], the crucial part is to look at the only possibly nonpositive terms in $\langle A'(\mathbf{u})\mathbf{d^u}, \mathbf{d^u}\rangle$, and show that they can be absorbed into the viscous regularization term. Recalling the definition of the operator in (2.4), we estimate

$$4(1-\kappa)(\varphi\mathbb{C}e(u), d^\varphi e(d^u)) \geq -\,4(1-\kappa)\|\varphi\|_\infty\|u\|_{1,p}\Big(\frac{1}{\delta_1}\,\|d^\varphi\|^2_r + \delta_1\|d^u\|^2_{1,2}\Big), \tag{3.8}$$

which follows from Hölder's and Young's inequality for a $\delta_1 > 0$, since the standard Sobolev embedding guarantees $d^\varphi \in H^1 \hookrightarrow L^r$, for an r such that $\frac{1}{r} + \frac{1}{p} = \frac{1}{2}$, i.e. $r \in (2,\infty)$. Since $r > 2$, the Gagliardo-Nirenberg inequality and Young's inequality ensure

$$\|d^\varphi\|^2_r \leq \tilde{c}^2_\Omega\Big(\|\nabla d^\varphi\|_2^{\frac{r-2}{r}}\|\psi\|_2^{\frac{2}{r}}\Big)^2 \leq c_\Omega\delta_2\|\nabla d^\varphi\|^2_2 + c_\Omega C(\delta_2)\|d^\varphi\|^2_2,$$

with a $\delta_2 > 0$ and exponents $\frac{r}{r-2}$ and $\frac{r}{2}$. Inserting this in (3.8) we find

$$4(1-\kappa)(\varphi\mathbb{C}e(u), d^\varphi e(d^u)) \geq -\,c_{\kappa,\Omega}\|\mathbf{u}\|^2_W\Big(\delta_1\|d^u\|^2_{1,2} + \frac{\delta_2}{\delta_1}\|\nabla d^\varphi\|^2_2 + C(\delta_1,\delta_2)\|d^\varphi\|^2_2\Big),$$

for a constant $C(\delta_1,\delta_2) > 0$. The remaining terms in $\langle A'(\mathbf{u})\mathbf{d^u}, \mathbf{d^u}\rangle$ are handled analogously to [**39**, Lemma 5.1]. With $c_{korn} > 0$ being the constant from Korn's inequality of the second kind for zero boundary functions, we end up at

$$\begin{aligned}\langle A'(\mathbf{u})\mathbf{d^u}, \mathbf{d^u}\rangle \geq{}& \Big(\kappa c_{korn} - \delta_1\, c_\kappa\|\mathbf{u}\|^2_W\Big)\|d^u\|^2_{1,2} + \Big(\varepsilon - c_{\kappa,\Omega}\|\mathbf{u}\|^2_W\frac{\delta_2}{\delta_1}\Big)\|\nabla d^\varphi\|^2_2\\ &+ \Big(C(\varepsilon) + \eta - c_{\kappa,\Omega}\|\mathbf{u}\|^2_W C(\delta_1,\delta_2)\Big)\|d^\varphi\|^2_2.\end{aligned} \tag{3.9}$$

Choosing δ_1 small enough such that $\kappa c_{korn} - \delta_1\, c_\kappa \|\mathbf{u}\|_W^2 > 0$, then choosing δ_2 small enough such that $\varepsilon - c_{\kappa,\Omega}\|\mathbf{u}\|_W^2 \frac{\delta_2}{\delta_1} > 0$, we can then choose η large enough to bound the right-hand-side of (3.9) from below by $c_{coer}(\|d^u\|_{1,2}^2 + \|d^\varphi\|_{1,2}^2)$ for a coercivity constant $c_{coer} > 0$. Thus, by the Lax-Milgram lemma, there exists a unique solution $\mathbf{d^u} := (d^u, d^\varphi) \in V$ of (3.6) with

$$\|d^u\|_{1,2} + \|d^\varphi\|_{1,2} \leq c_{coer}\|\mathbf{f}\|_{V^*}. \tag{3.10}$$

(2) To show the improved regularity result in W, we follow [**24**, **39**] and test (3.6) first with $(0, v^\varphi) \in V$, to obtain

$$\begin{aligned} \varepsilon(\nabla d^\varphi, \nabla v^\varphi) + \frac{1+\varepsilon\eta}{\varepsilon}(d^\varphi, v^\varphi) = & -(1-\kappa)(d^\varphi \mathbb{C}e(u) : e(u), v^\varphi) \\ & - 2(1-\kappa)(\varphi\mathbb{C}e(u) : e(d^u), v^\varphi) \\ & - 3\gamma([(\varphi - \varphi^-)^+]^2 d^\varphi, v^\varphi) + (f^\varphi, v^\varphi) =: (\tilde{g}, v^\varphi), \end{aligned} \tag{3.11}$$

and secondly with $(v^u, 0) \in V$, to obtain

$$(g(\varphi)\mathbb{C}e(d^u), e(v^u)) = -2(1-\kappa)(\varphi\mathbb{C}e(u)d^\varphi, e(v^u)) + (f^u, v^u). \tag{3.12}$$

Note that the term corresponding to the viscous regularization appears on the left-hand-side of (3.11). The right-hand-sides of both equalities can be treated as in [**39**, Lemma 5.2]. We only need to adapt the proof of [**39**, Lemma 5.2] to the presence of data $f \in V^*$ to eventually obtain

$$\begin{aligned} \|\tilde{g}\|_{L^{r'}} &\leq 2\, c_\kappa\, c_{coer}\|\mathbf{f}\|_{V^*}\|\mathbf{u}\|_W^2 + c_\gamma(\|\mathbf{u}\|_W + \|\varphi^-\|_\infty)c_{coer}\|\mathbf{f}\|_{V^*} + c\|\mathbf{f}\|_{V^*} \\ &\leq c_{\kappa,\gamma,\eta} \max\Big(\|\mathbf{u}\|_W, \|\mathbf{u}\|_W^2, \|\varphi^-\|_\infty\Big)\|\mathbf{f}\|_{V^*}, \end{aligned} \tag{3.13}$$

where $r' \in (1,2)$, $1 = \frac{1}{r} + \frac{1}{r'}$. A standard elliptic regularity result, cf. [**45**, Theorem 4.7 and Chapter 7.2.1], applied to (3.11) yields $d^\varphi \in H^1 \cap L^\infty$, and for $c_1(\mathbf{u}, \varphi^-, \eta)$ subsuming the constants of the right-hand-side of (3.13), the following estimate holds

$$\|d^\varphi\|_\infty \leq c_1(\mathbf{u}, \varphi^-, \eta)\|\mathbf{f}\|_{V^*}. \tag{3.14}$$

Using this, we find

$$\begin{aligned} \|\varphi\mathbb{C}e(u)d^\varphi\|_p \leq c\|\varphi\|_\infty\|u\|_{1,p}\|d^\varphi\|_\infty &\leq \|\mathbf{u}\|_W^2\, c_1(\mathbf{u}, \varphi^-, \eta)\,\|\mathbf{f}\|_{V^*} \\ &\leq c_2(\mathbf{u}, \varphi^-, \eta)\,\|\mathbf{f}\|_{W^\times}. \end{aligned} \tag{3.15}$$

Combining this with (3.12), using $f^u \in W^{-1,p}$ and [**27**, Proposition 1.1], we find $d^u \in W^{1,p}$ and

$$\|d^u\|_{1,p} \leq c_2(\mathbf{u}, \varphi^-, \eta)\|\mathbf{f}\|_{W^\times}. \tag{3.16}$$

Using the additional regularity $f^\varphi \in L^q$ and improved regularity results $d^\varphi \in L^\infty$ and $d^u \in W^{1,p}$ from (3.14) and (3.16), setting $r' = q = \frac{p}{2}$ in (3.13), the right-hand-side of (3.11) is in fact in L^q. Instead of (3.13), we obtain

$$\|\tilde{g}\|_q \leq c_1(\mathbf{u}, \varphi^-, \eta)\|\mathbf{f}\|_{W^\times}\|\mathbf{u}\|_W^2 + c_\gamma(\|\mathbf{u}\|_W + \|\varphi^-\|_\infty)c_1(\mathbf{u}, \varphi^-, \eta)\|\mathbf{f}\|_{W^\times} + c\|\mathbf{f}\|_{W^\times}. \tag{3.17}$$

Due to the $W^{2,q}$ regularity of Ω, by the same argument as in [**24**, Corollary 2, Section 7], we find $d^\varphi \in W^{2,q}$, and the following estimate holds

$$\|d^\varphi\|_{2,q} \le c_3(\mathbf{u},\varphi^-,\eta)\|\mathbf{f}\|_{W^\times}, \tag{3.18}$$

where $c_3(\mathbf{u},\varphi^-,\eta)$ subsumes the constant in (3.17). The dependence of the constant in (3.7) on $\mathbf{u}$ is a direct consequence of (3.13), (3.15), and (3.17).

(3) Finally we prove $d^u \in H^{1+s}$, exploiting the additional assumption $(u,\varphi) \in H^{1+s} \times H^{1+s}$ and again the same idea as for $(\mathrm{EL}^{\gamma,\eta})$ from the nonlinear setting of [**24**, Corollary 2, Section 7]. It is important to recognize that in (3.12), we have to ensure that $g(\varphi)$ is a multiplier in the sense of [**24**] on H^s. By [**24**, Lemma 1, Section 5] it suffices that $g(\varphi) \in C^\theta$ for $\theta = 1 + s - \frac{2}{p}$, which itself follows from $\varphi \in C^\theta$. This holds true from the standard Sobolev embedding and our assumption on φ. The right-hand-side of (3.12) is an element of H^{-1+s}, since $f^u \in H^{-1+s}$ by assumption and $\varphi\mathbb{C}(u)d^\varphi \in H^s$, since $\varphi, d^\varphi \in L^\infty$ and $u \in H^{1+s}$. Now from [**24**, Theorem 1, Section 2], we get that in fact $d^u \in H^{1+s}$.

$\square$

3.1. Differentiability results. Differentiability of G has been used in earlier publications, and follows from standard techniques. Since this property is used for the Lipschitz results in Subsection 3.2, we present some steps of the proof.

Proposition 3.3. *Let Assumption 2.1 hold. The control-to-state operator $G\colon Q \to W$ is twice continuously Fréchet differentiable. For the first order derivative, we obtain $G'(q)d^q = \mathbf{d}^\mathbf{u}$, where $\mathbf{d}^\mathbf{u} \in W$ is the weak solution to*

$$\langle A'(\mathbf{u})\mathbf{d}^\mathbf{u}, \mathbf{v}\rangle + \langle R'(\varphi;\gamma)d^\varphi, v^\varphi\rangle = (d^q, v^u)_\Gamma \quad \forall\, \mathbf{v} \in V, \tag{3.19}$$

with $G(q) = \mathbf{u}$. For the second order derivative, we find $G''(q)[d_1^q, d_2^q] = \tilde{\mathbf{d}}^\mathbf{u}$, where $\tilde{\mathbf{d}}^\mathbf{u} \in W$ is the unique weak solution to

$$\begin{aligned}\langle A'(\mathbf{u})\tilde{\mathbf{d}}^\mathbf{u}, \mathbf{v}\rangle + \langle R'(\varphi;\gamma)\tilde{d}^\varphi, v^\varphi\rangle = &- \langle A''(\mathbf{u})[\mathbf{d}_1^\mathbf{u}, \mathbf{d}_2^\mathbf{u}], \mathbf{v}\rangle \\ &- \langle R''(\varphi;\gamma)[d_1^\varphi, d_2^\varphi], v^\varphi\rangle \quad \forall\, \mathbf{v} \in V,\end{aligned} \tag{3.20}$$

where again $G(q) = \mathbf{u}$, as well as $G'(q)d_j^q = \mathbf{d}_j^\mathbf{u}$, $j = 1,2$.

Proof. The claim follows by standard techniques utilizing the implicit function theorem cf. [**46**, Theorem 4.B] applied to $F\colon Q \times W \mapsto W^\times$,

$$\langle F(q,\mathbf{u}), \mathbf{v}\rangle := \langle A(\mathbf{u}), \mathbf{v}\rangle + \langle R(\varphi;\gamma), v^\varphi\rangle - \langle B(q), \mathbf{v}\rangle \qquad \forall\, \mathbf{v} \in V. \tag{3.21}$$

Note that Lemma 3.1 already ensures for every $q \in Q$ the existence of a unique $\mathbf{u} \in W$, such that $F(q,\mathbf{u}) = 0$. The mapping F, and thus G, eventually, is continuously Fréchet differentiable from $Q \times W$ into $W^\times$, with

$$\langle F'(q,\mathbf{u})(d^q,\mathbf{d}^\mathbf{u}), \mathbf{v}\rangle = \langle A'(\mathbf{u})\mathbf{d}^\mathbf{u}, \mathbf{v}\rangle + \langle R'(\varphi,\gamma)d^\varphi, v^\varphi\rangle - \langle B'(q)d^q, \mathbf{v}\rangle \qquad \forall \mathbf{v} \in V. \tag{3.22}$$

The only interesting part of the proof is the differentiability of A from W into $W^\times$. A straight forward calculation verifies

$$\langle A(\mathbf{u}+\mathbf{d}^\mathbf{u}), \mathbf{v}\rangle = \langle A(\mathbf{u}), \mathbf{v}\rangle + \langle A'(\mathbf{u})\mathbf{d}^\mathbf{u}, \mathbf{v}\rangle + \langle \mathrm{rem}_A(\mathbf{u},\mathbf{d}^\mathbf{u}), \mathbf{v}\rangle,$$

with A' as in (2.4) and remainder

$$\begin{aligned}\langle \mathrm{rem}_A(\mathbf{u},\mathbf{d^u}),\mathbf{v}\rangle &= 2(1-\kappa)(\varphi d^\varphi \mathbb{C}e(d^u), e(v^u)) + (1-\kappa)((d^\varphi)^2\mathbb{C}e(u), e(v^u))\\ &\quad + (1-\kappa)((d^\varphi)^2\mathbb{C}e(d^u), e(v^u)) + (1-\kappa)(\varphi\mathbb{C}e(d^u)\colon e(d^u), v^\varphi)\\ &\quad + 2(1-\kappa)(d^\varphi\mathbb{C}e(d^u)\colon e(u), v^\varphi) + (1-\kappa)(d^\varphi\mathbb{C}e(d^u)\colon e(d^u), v^\varphi).\end{aligned}$$

To show that rem_A is of order $o(\|\mathbf{d^u}\|_W)$, we calculate exemplarily

$$\|\varphi d^\varphi \mathbb{C}e(d^u)\|_{0,p} \le c\|\varphi\|_\infty\|d^\varphi\|_\infty\|d^u\|_{1,p} \le c\|\varphi\|_{2,q}\|d^\varphi\|_{2,q}\|d^u u\|_{1,p} \le c\|\mathbf{d^u}\|_W^2\|\mathbf{u}\|_W,$$

as in the proof of Lemma 3.2, cf. (3.15). The remaining terms in $\mathrm{rem}_A(\mathbf{u},\mathbf{d^u})$ can be bounded analogously, overall it holds

$$\|rem_A(\mathbf{u},\mathbf{d^u})\|_{W^\times}/\|\mathbf{d^u}\|_W \to 0 \quad \text{for} \quad \|\mathbf{d^u}\|_W \to 0.$$

A' is also continuous as a mapping $\mathbf{u} \mapsto A'(\mathbf{u})$ from W into $\mathcal{L}(W, W^\times)$, since

$$\|A'(\mathbf{u}) - A'(\bar{\mathbf{u}})\|_{\mathcal{L}(W,W^\times)} \le 8c_\kappa \max(\|\mathbf{u}\|_W, \|\bar{\mathbf{u}}\|_W)\|\mathbf{u}-\bar{\mathbf{u}}\|_W. \tag{3.23}$$

We skip the calcuations, since the steps are very similar to the proof of the Lipschitz-continuity results in Lemma 3.9, which we carry out in detail.

Now, note that

$$F_{\mathbf{u}}(q,\mathbf{u})\mathbf{d^u} = A'(\mathbf{u})\mathbf{d^u} + R'(\varphi;\gamma)d^\varphi,$$

and for $(f^u, f^\varphi) = (B(d^q), 0)$ we see that $F_{\mathbf{u}}(q,\mathbf{u})\mathbf{d^u} = d^q$ is equivalent to (3.6). Thus by Lemma 3.2, $F_{\mathbf{u}}(q,\mathbf{u})$ is invertible in W. Alltogether, this proves that the control-to-state operator G is continuously Fréchet differentiable from Q into W in every $q \in Q$ with derivative $\mathbf{d^u} = G'(q)d^q$ given by (3.19).

Second order continuous Fréchet differentiability of G follows from the continuous differentiability of the operator $\tilde{F}(q,\mathbf{u}) := F'(q,\mathbf{u})(d_1^q, \mathbf{d_1^u})$ from W into $W^\times$. This technical result can be shown analogously to first order differentiability. □

Let us clarify some properties of solutions of (3.20), the PDE corresponding to $G''(q)[d_1^q, d_2^q] = \tilde{\mathbf{d}}^{\mathbf{u}}$.

Corollary 3.4. *Let Assumption 2.1 hold and* $j = 1, 2$. *Then for every* $q, d_j^q \in Q$, *with associated* $G(q) = \mathbf{u} \in W$, $G'(q)d_j^q = \mathbf{d}_j^{\mathbf{u}} \in W$, (3.20) *has a unique weak solution* $\tilde{\mathbf{d}}^{\mathbf{u}} \in W$, *and there exists a constant* $c = c(q) > 0$ *such that*

$$\|\tilde{d}^u\|_{1,p} + \|\tilde{d}^\varphi\|_{2,q} \le c\|d_1^q\|_\Gamma\|d_2^q\|_\Gamma. \tag{3.24}$$

Proof. This follows from Lemma 3.2, by estimating the right-hand-side of (3.20) in $W^\times$ termwise, recalling the definitions of A'' and R'' from (2.6) and (2.7). Using Hölder's inequality with $\frac{1}{q} = \frac{1}{p} + \frac{1}{p}$, recalling that $p > 2$ and $q = \frac{p}{2}$,

$$\begin{aligned}\|d_2^\varphi \mathbb{C}e(u)d_1^\varphi\|_p &\le \|d_2^\varphi\|_\infty\|u\|_{1,p}\|d_1^\varphi\|_\infty \le c(q)\|d_1^q\|_\Gamma\|d_2^q\|_\Gamma,\\ \|d_2^\varphi\mathbb{C}e(u) : e(d_1^u)\|_q &\le \|d_2^\varphi\|_\infty\|u\|_{1,p}\|d_1^u\|_{1,p} \le c(q)\|d_1^q\|_\Gamma\|d_2^q\|_\Gamma,\\ \|(\varphi-\varphi^-)^+ d_1^\varphi d_2^\varphi\|_q &\le c(q)\|d_1^\varphi\|_\infty\|d_2^\varphi\|_\infty \le c(q)\|d_1^q\|_\Gamma\|d_2^q\|_\Gamma,\end{aligned}$$

where in the final inequalities we used the norm-estimates from Lemma 3.1 and Lemma 3.2. The remaining terms are calculated analogously. □

In order to prove first and second order optimality conditions, we collect some rather straightforward results on the reduced objective functional f.

Corollary 3.5. *Let Assumption 2.1 hold, and $q, d_j^q \in Q$, $j = 1, 2$. The reduced functional f is twice continuously Fréchet differentiable from Q into $\mathbb{R}$, with*

$$f'(q)d^q = (u - u_d, d^u) + \alpha(q, d^q)_\Gamma, \tag{3.25}$$

where $\mathbf{u} = G(q)$ and $\mathbf{d^u} = G'(q)d^q$, and

$$f''(q)[d_1^q, d_2^q] = (d_2^u, d_1^u) + (u - u_d, \tilde{d}^u) + \alpha(d_2^q, d_1^q)_\Gamma, \tag{3.26}$$

where $\mathbf{u} = G(q)$, $\mathbf{d}_j^\mathbf{u} = G'(q)d_j^q$, $j = 1, 2$, and $\tilde{\mathbf{d}}^\mathbf{u} = G''(q)[d_1^q, d_2^q]$.

Proof. This immediately follows from Proposition 3.3 and the chain rule. $\square$

3.2. Lipschitz Continuity Results.

Next, we establish Lipschitz continuity of the operators G, G', and G'', and subsequently also of the functionals f, f', and f''. These technical results are the most crucial part on our way to establishing second order sufficient optimality conditions.

Lemma 3.6. *Let $q \in Q$ be given, then for all $\rho > 0$ there exists a constant $c = c(q, \rho) > 0$ such that for $h \in Q$, with $\|h\|_\Gamma \leq \rho$, it holds*

$$\|u_h - u\|_{1,p} + \|\varphi_h - \varphi\|_{2,q} \leq c\|h\|_\Gamma, \tag{3.27}$$

where $\mathbf{u}_h = G(q + h)$ and $\mathbf{u} = G(q)$.

Remark 3.7. Note that the boundedness of Q_{ad} yields a global Lipschitz continuity result for all $q \in Q_{\text{ad}}$.

Proof. Let q, h be as stated. By Proposition 3.3, the control-to-state operator G is continuously Fréchet differentiable from Q into W. Hence in combination with the mean value inequality cf. [**35**, Section 7.3, Proposition 2], for a $t \in [0, 1]$ we can estimate

$$\begin{aligned}\|\mathbf{u}_h - \mathbf{u}\|_W &= \|G(q + h) - G(q)\|_W \leq c\|G'(q + th)\|_{\mathcal{L}(Q,W)}\|h\|_\Gamma \\ &\leq c \sup_{\|d^q\|_\Gamma = 1} \|G'(q + th)d^q\|_W\|h\|_\Gamma.\end{aligned}$$

For any $t \in [0, 1]$, set $W \ni \mathbf{d}_{th}^\mathbf{u} := G'(q + th)d^q$. Utilizing Lemma 3.1 yields

$$\|u_{th}\|_{1,p} + \|\varphi_{th}\|_{2,q} = \|\mathbf{u}_{th}\|_W \leq \tilde{c}(q, \rho), \tag{3.28}$$

for a constant $\tilde{c}(q, \rho) > 0$ independent of t and h since $\|th\|_\Gamma \leq \|h\|_\Gamma \leq \rho$ for all $t \in [0, 1]$. Setting $\mathbf{f} = (B(d^q), 0) \in W^\times$, Lemma 3.2 yields the estimate:

$$\|d_{th}^u\|_{1,p} + \|d_{th}^\varphi\|_{2,q} = \|\mathbf{d}_{th}^\mathbf{u}\|_W \leq c(q, \rho)\|d^q\|_\Gamma \tag{3.29}$$

with a constant $c(q, \rho) > 0$. Collecting all estimates yields the assertion. $\square$

Lemma 3.8. *Let $q, d^q \in Q$ be given, then for all $\rho > 0$ there exists a constant $c = c(q, \rho) > 0$ such that for all $h \in Q$, with $\|h\|_\Gamma \leq \rho$, it holds*

$$\|d_h^u - d^u\|_{1,p} + \|d_h^\varphi - d^\varphi\|_{2,q} \leq c\|h\|_\Gamma\|d^q\|_\Gamma, \tag{3.30}$$

where $\mathbf{d}_h^\mathbf{u} = G'(q + h)d^q$ and $\mathbf{d^u} = G'(q)d^q$.

Proof. Let $q, h, d^q \in Q$ be as stated. Due to the second order Fréchet differentiability of G from Proposition 3.3, estimating as in the proof of Lemma 3.6, yields for a $t \in [0,1]$,

$$\begin{aligned} \|\mathbf{d}_h^{\mathbf{u}} - \mathbf{d}^{\mathbf{u}}\|_W &= \|G'(q+h)d^q - G'(q)d^q\|_W \le c\|G''(q+th)d^q\|_{\mathcal{L}(Q,W)}\|h\|_\Gamma \\ &\le c \sup_{\|d_2^q\|_\Gamma = 1} \|G''(q+th)[d^q, d_2^q]\|_W \|h\|_\Gamma. \end{aligned} \tag{3.31}$$

Again, for any $t \in [0,1]$, set $W \ni \tilde{\mathbf{d}}_{th}^{\mathbf{u}} := G''(q+th)[d^q, d_2^q]$ and utilizing Corollary 3.4 yields

$$\|\tilde{\mathbf{d}}_{th}^{\mathbf{u}}\|_W \le c(q,\rho)\|d_2^q\|_\Gamma \|d^q\|_\Gamma.$$

The assertion is obtained after collecting all estimates. $\square$

Lemma 3.9. *Let $q, d_j^q \in Q$, $j = 1,2$, be given, then for all $\rho > 0$ there exists a constant $c = c(q,\rho) > 0$ such that for all $h \in Q$, with $\|h\|_\Gamma \le \rho$, it holds*

$$\|\tilde{d}_h^u - \tilde{d}^u\|_{1,p} + \|\tilde{d}_h^\varphi - \tilde{d}^\varphi\|_{2,q} \le c\|h\|_\Gamma \|d_1^q\|_\Gamma \|d_2^q\|_\Gamma, \tag{3.32}$$

where $\tilde{\mathbf{d}}_h^{\mathbf{u}} = G''(q+h)[d_1^q, d_2^q]$ and $\tilde{\mathbf{d}}^{\mathbf{u}} = G''(q)[d_1^q, d_2^q]$.

Proof. Let $\tilde{\mathbf{d}}_h^{\mathbf{u}}$ and $\tilde{\mathbf{d}}^{\mathbf{u}}$ be as stated. By linearity of (3.20), it holds

$$\begin{aligned} A'(\mathbf{u})(\tilde{\mathbf{d}}_h^{\mathbf{u}} - \tilde{\mathbf{d}}^{\mathbf{u}}) + R'(\varphi;\gamma)(\tilde{d}_h^\varphi - \tilde{d}^\varphi) = &- \big(A'(\mathbf{u}_h) - A'(\mathbf{u})\big)\tilde{\mathbf{d}}_h^{\mathbf{u}} \\ &- \big(R'(\varphi_h;\gamma) - R'(\varphi;\gamma)\big)\tilde{d}_h^\varphi \\ &- \big(A''(\mathbf{u}_h)[\mathbf{d}_{1,h}^{\mathbf{u}}, \mathbf{d}_{2,h}^{\mathbf{u}}] - A''(\mathbf{u})[\mathbf{d}_1^{\mathbf{u}}, \mathbf{d}_2^{\mathbf{u}}]\big) \\ &- \big(R''(\varphi_h;\gamma)[d_{1,h}^\varphi, d_{2,h}^\varphi] - R''(\varphi;\gamma)[d_1^\varphi, d_2^\varphi]\big) \end{aligned} \tag{3.33}$$

where $G(q+h) = \mathbf{u}_h$, $G(q) = \mathbf{u}$, as well as $G'(q+h)d_j^q = \mathbf{d}_{j,h}^{\mathbf{u}}$ and $G'(q)d_j^q = \mathbf{d}_j^{\mathbf{u}}$.

Again, we want to utilize the norm estimate from Lemma 3.2, thus we estimate the right-hand-side of (3.33) in $W^\times$. Let us start with the difference of the A' terms, in particular, for all test functions $\mathbf{v} \in V$ we obtain

$$\begin{aligned} \langle (A'(\mathbf{u}_h) - A'(\mathbf{u}))\tilde{\mathbf{d}}_h^{\mathbf{u}}, \mathbf{v}\rangle = &\Big((g(\varphi_h) - g(\varphi))\mathbb{C}e(\tilde{d}_h^u), e(v^u)\Big) \\ &+ 2(1-\kappa)((\varphi_h\mathbb{C}e(u_h) - \varphi\mathbb{C}e(u)) : e(\tilde{d}_h^u), v^\varphi) \\ &+ (1-\kappa)((\mathbb{C}e(u_h) : e(u_h) - \mathbb{C}e(u) : e(u))\tilde{d}_h^\varphi, v^\varphi) \\ &+ 2(1-\kappa)((\varphi_h\mathbb{C}e(u_h) - \varphi\mathbb{C}e(u))\tilde{d}_h^\varphi, e(v^u)). \end{aligned} \tag{3.34}$$

For the last term of the right-hand-side, we exploit Lemma 3.6 for $\|\varphi_h - \varphi\|_\infty$ and $\|u_h - u\|_{1,p}$, Lemma 3.1 for $\|u_h\|_{1,p}$ and $\|\varphi\|_\infty$, and Corollary 3.4 for $\|\tilde{d}_h^\varphi\|$, to obtain

$$\begin{aligned} \|(\varphi_h\mathbb{C}e(u_h) - \varphi\mathbb{C}e(u))\tilde{d}_h^\varphi\|_p &\le c\|\varphi_h - \varphi\|_\infty \|u_h\|_{1,p}\|\tilde{d}_h^\varphi\|_\infty + c\|\varphi\|_\infty\|u_h - u\|_{1,p}\|\tilde{d}_h^\varphi\|_\infty \\ &\le c(q,\rho)\|h\|_\Gamma\|d_1^q\|_\Gamma\|d_2^q\|_\Gamma. \end{aligned}$$

The remaining terms on the right-hand-side of (3.34) can be bounded in the same way, overall we obtain

$$\|A'(\mathbf{u}_h) - A'(\mathbf{u}))\tilde{\mathbf{d}}_h^{\mathbf{u}}\|_{W^\times} \le c(q,\rho)\|h\|_\Gamma\|d_1^q\|_\Gamma\|d_2^q\|_\Gamma.$$

For the difference in R', exploiting continuity of $[(\cdot)^+]^2$, and using Lemma 3.6 and Corollary 3.4, we estimate

$$\|(R'(\varphi_h;\gamma) - R'(\varphi;\gamma))\tilde{d}_h^{\varphi}\|_q \leq c\|\varphi_h - \varphi\|_\infty \|\tilde{d}_h^{\varphi}\|_\infty \leq c(q,\rho)\|h\|_\Gamma \|d_1^q\|_\Gamma \|d_2^q\|_\Gamma.$$

Next, we look at the difference in the A'' terms, i.e. at

$$\begin{aligned}
-(A''(\mathbf{u}_h)[\mathbf{d}_{1,h}^{\mathbf{u}}, \mathbf{d}_{2,h}^{\mathbf{u}}] - A''(\mathbf{u})[\mathbf{d}_1^{\mathbf{u}}, \mathbf{d}_2^{\mathbf{u}}]) = &-(A''(\mathbf{u}_h) - A''(\mathbf{u}))[\mathbf{d}_{1,h}^{\mathbf{u}}, \mathbf{d}_{2,h}^{\mathbf{u}}] \\
&- A''(\mathbf{u})[\mathbf{d}_{1,h}^{\mathbf{u}} - \mathbf{d}_1^{\mathbf{u}}, \mathbf{d}_{2,h}^{\mathbf{u}}] \\
&- A''(\mathbf{u})[\mathbf{d}_1^{\mathbf{u}}, \mathbf{d}_{2,h}^{\mathbf{u}} - \mathbf{d}_2^{\mathbf{u}}].
\end{aligned} \tag{3.35}$$

Recalling the definition of the operator A'' from (2.6), we see that in every of the six terms of A'', each of the involved functions $\mathbf{u}, \mathbf{d}_1^{\mathbf{u}}, \mathbf{d}_2^{\mathbf{u}}$ occur linearly. Thus, for all test functions $\mathbf{v} \in V$, for the first term of the right-hand-side of (3.35) it holds

$$\begin{aligned}
-\langle (A''(\mathbf{u}_h) - A''(\mathbf{u}))[\mathbf{d}_{1,h}^{\mathbf{u}}, \mathbf{d}_{2,h}^{\mathbf{u}}], \mathbf{v}\rangle = &-2(1-\kappa)(d_{2,h}^{\varphi}\mathbb{C}e(u_h - u)d_{1,h}^{\varphi}, e(v^u)) \\
&- 2(1-\kappa)(d_{2,h}^{\varphi}\mathbb{C}e(d_{1,h}^u)(\varphi_h - \varphi), e(v^u)) \\
&- 2(1-\kappa)(d_{2,h}^{\varphi}\mathbb{C}e(u_h - u) : e(d_{1,h}^u), v^{\varphi}) \\
&- 2(1-\kappa)((\varphi_h - \varphi)\mathbb{C}e(d_{2,h}^u)d_{1,h}^{\varphi}, e(v^u)) \\
&- 2(1-\kappa)(d_{1,h}^{\varphi}\mathbb{C}e(d_{2,h}^u) : e(u_h - u), v^{\varphi}) \\
&- 2(1-\kappa)((\varphi_h - \varphi)\mathbb{C}e(d_{2,h}^u) : e(d_{1,h}^u), v^{\varphi}).
\end{aligned} \tag{3.36}$$

Let us estimate the last term of the right-hand-side of (3.36), using Hölder's inequality with $\frac{1}{q} = \frac{1}{p} + \frac{1}{p}$. Then, we utilize Lemma 3.6 for $\|\varphi_h - \varphi\|_\infty$, and Lemma 3.2 for $\|d_{1,h}^u\|_{1,p}$ and $\|d_{2,h}^u\|_{1,p}$, to obtain

$$\begin{aligned}
\|(\varphi_h - \varphi)\mathbb{C}e(d_{2,h}^u) : e(d_{1,h}^u)\|_q &\leq \|\varphi_h - \varphi\|_\infty \|d_{2,h}^u\|_{1,p}\|d_{1,h}^u\|_{1,p} \\
&\leq c_(q,\rho)\|h\|_\Gamma \|d_2^q\|_\Gamma \|d_1^q\|_\Gamma.
\end{aligned}$$

Analogously, we bound the remaining terms, and subsequently estimate (3.35) by

$$\| - A''(\mathbf{u}_h)[\mathbf{d}_{1,h}^{\mathbf{u}}, \mathbf{d}_{2,h}^{\mathbf{u}}] + A''(\mathbf{u})[\mathbf{d}_1^{\mathbf{u}}, \mathbf{d}_2^{\mathbf{u}}]\|_{W^\times} \leq c(q,\rho)\|h\|_\Gamma \|d_1^q\|_\Gamma \|d_2^q\|_\Gamma.$$

Due to the continuity of the Nemytskii operator $[(\cdot)^+]$, an easy calculation shows

$$\| - R''(\varphi_h;\gamma)[d_{1,h}^{\varphi}, d_{2,h}^{\varphi}] + \langle R''(\varphi;\gamma)[d_1^{\varphi}, d_2^{\varphi}]\|_q \leq c(q,\rho)\|h\|_\Gamma \|d_1^q\|_\Gamma \|d_2^q\|_\Gamma.$$

Combining all estimates, the right-hand-side of (3.33) can be bounded in the $W^\times$-norm by $c(q,\rho)\|d_1^q\|_\Gamma \|d_2^q\|_\Gamma \|h\|_\Gamma$, the claim now follows from Lemma 3.2.

□

Lipschitz continuity results for the reduced objective functional and its derivatives are now easily obtained.

Lemma 3.10. *Let $q, d_j^q \in Q$, $j = 1,2$, be given, then for every $\rho > 0$, there exists a constant $c_L = c(\rho) > 0$ such that for all $h \in Q$ it holds*

$$|f(q+h) - f(q)| \leq c_L\|h\|_\Gamma, \tag{3.37}$$

$$|f'(q+h)d_1^q - f'(q)d_1^q| \leq c_L\|h\|_\Gamma \|d_1^q\|_\Gamma, \tag{3.38}$$

$$|f''(q+h)[d_1^q, d_2^q] - f''(q)[d_1^q, d_2^q]| \leq c_L\|h\|_\Gamma \|d_1^q\|_\Gamma \|d_2^q\|_\Gamma, \tag{3.39}$$

provided that $\max(\|q\|_\Gamma, \|h\|_\Gamma) \le \rho$.

Proof. For q, h, d_j^q, $j = 1,2$ as stated, we will denote $G(q+h) = \mathbf{u}_h$, $G(q) = \mathbf{u}$, $G'(q+h)d_j^q = \mathbf{d}_{j,h}^{\mathbf{u}}$, $G'(q)d_j^q = \mathbf{d}_j^{\mathbf{u}}$, $G''(q+h)[d_1^q, d_2^q] = \tilde{\mathbf{d}}_h^{\mathbf{u}}$ and finally $G''(q)[d_1^q, d_2^q] = \tilde{\mathbf{d}}^{\mathbf{u}}$. All three results follow from the definition of f from (3.5) and the representation of f' and f'' from Lemma 3.5, combined with the norm- and Lipschitz-estimates for G, G', and G''. For the reduced functional, a quick standard calculation yields

$$\begin{aligned}
|f(q+h)-f(q)| = {} & \frac{1}{2}\Big|(u_h - u, u_h) + (u_h - u, u) - 2(u_h - u, u_d) + 2\alpha(q,h)_\Gamma + \alpha(h,h)_\Gamma\Big| \\
& \le c\Big(\|u_h - u\|_{1,p}\|u_h\|_{1,p} + \|u_h - u\|_{1,p}\|u\|_{1,p} + \|u_h - u\|_{1,p}\|u_d\| \\
& \quad + \|q\|_\Gamma\|h\|_\Gamma + \|h\|_\Gamma\|h\|_\Gamma\Big) \\
& \le 5c(q,\rho)\|h\|_\Gamma,
\end{aligned}$$

where in the last inequality we used Lemma 3.6 to estimate $\|u_h - u\|_{1,p}$, Lemma 3.1 for $\|u_h\|_{1,p}$ and $\|u\|_{1,p}$, and the boundedness of u_d in L^2.

For the first derivative, we estimate similarly

$$\begin{aligned}
|(f'(q+h) - f'(q))d_1^q| = {} & \Big|(u_h - u, d_{1,h}^u) + (u, d_{1,h}^u - d_1^u) - (u_d, d_{1,h}^u - d_1^u) + \alpha(h, d_1^q)_\Gamma\Big| \\
& \le c\Big(\|u_h - u\|_{1,p}\|d_{1,h}^u\|_{1,p} + \|u\|_{1,p}\|d_{1,h}^u - d_1^u\|_{1,p} \\
& \quad + \|u_d\|\|d_{1,h}^u - d_1^u\|_{1,p} + \|h\|_\Gamma\|d_1^q\|_\Gamma\Big) \\
& \le 4c(q,\rho)\|h\|_\Gamma\|d_1^q\|_\Gamma,
\end{aligned}$$

also using Lemma 3.8 and Lemma 3.2.

Finally, for the second derivative, it holds

$$\begin{aligned}
|f''(q+h)[d_1^q, d_2^q] - f''(q)[d_1^q, d_2^q]| = {} & \Big|(d_{1,h}^u - d_1^u, d_{2,h}^u) + (d_1^u, d_{2,h}^u - d_2^u) \\
& \quad + (u_h - u, \tilde{d}_h^u) + (u, \tilde{d}_h^u - \tilde{d}^u) - (u_d, \tilde{d}_h^u - \tilde{d}^u)\Big| \\
& \le c\Big(\|d_{1,h}^u - d_1^u\|_{1,p}\|d_{2,h}^u\|_{1,p} + \|d_1^u\|_{1,p}\|d_{2,h}^u - d_2^u\|_{1,p} \\
& \quad + \|u_h - u\|_{1,p}\|\tilde{d}_h^u\|_{1,p} + \|u\|_{1,p}\|\tilde{d}_h^u - \tilde{d}^u\|_{1,p} \\
& \quad + \|u_d\|\|\tilde{d}_h^u - \tilde{d}^u\|_{1,p}\Big) \\
& \le 5c(q,\rho)\|h\|_\Gamma\|d_1^q\|_\Gamma\|d_2^q\|_\Gamma.
\end{aligned}$$

Here, we additionally used Lemma 3.9. □

3.3. The Adjoint Equation. To conclude our preliminary results, we collect results for the adjoint equation for later use. Recalling that $\mathbb{C}$ is the rank-4 elasticity tensor $\mathbb{C}e(v) : e(z) = \mathbb{C}e(z) : e(v)$ and hence symmetric for all $v, z \in V_u$, c.f. [**41**, Lemma 3.1], it is clear that the operators A' and R' are in fact self-adjoint. Then, Lemma 3.2 guarantees that under Assumption 2.1, for every $\mathbf{u} \in W$ and $\mathbf{f}_{\text{adj}} = (f_{\text{adj}}^u, f_{\text{adj}}^\varphi) \in W^\times$, there exists a unique solution $\mathbf{z} = (z^u, z^\varphi) \in W$ to

$$\langle (A'(\mathbf{u}))^*\mathbf{z}, \mathbf{v}\rangle + \langle (R'(\varphi;\gamma))^* z^\varphi, v^\varphi\rangle = \langle \mathbf{f}_{\text{adj}}, \mathbf{v}\rangle \qquad \forall \mathbf{v} \in V, \tag{3.40}$$

and

$$\|z^u\|_{1,p} + \|z^\varphi\|_{2,q} \le c(\mathbf{u}, \varphi^-, \eta)\|\mathbf{f}_{\text{adj}}\|_{W^\times}. \tag{3.41}$$

Further, the first derivative f', see (3.25), can equivalently be expressed by

$$f'(q)d^q = (B^*\mathbf{z} + \alpha q, d^q)_\Gamma, \tag{3.42}$$

where $\mathbf{z} = (z^u, z^\varphi) \in W$ is the unique weak solution to

$$(A'(\mathbf{u})^*\mathbf{z}) + R'(\varphi;\gamma)^* z^\varphi = u - u_d. \tag{3.43}$$

This equivalence of (3.25) and (3.42) in combination with (3.43) follows from usual calculations for adjoint operators, cf. e.g. to the first order optimality conditions obtained in [**39**] for the case $\eta = 0$. Likewise, by standard calculations, we obtain an expression of the second derivative f'' from (3.26) with the help of the adjoint state, resulting in

$$f''(q)[d_1^q, d_2^q] = (d_2^u, d_1^u) + \alpha(d_2^q, d_1^q)_\Gamma - \langle A''(\mathbf{u})[\mathbf{d_1^u}, \mathbf{d_2^u}], \mathbf{z}\rangle - \langle R''(\varphi;\gamma)[d_1^\varphi, d_2^\varphi], z^\varphi\rangle,$$

where $G(q) = \mathbf{u} \in W$, $G'(q)d_j^q = \mathbf{d}_j^\mathbf{u} \in W$, $j = 1, 2$, and $\mathbf{z} \in W$ is the solution to (3.43).

For completeness we will quickly establish an auxiliary result for the adjoint equations here.

Corollary 3.11. *Let $q \in Q$ and $\mathbf{f_h}, \mathbf{f} \in W^\times$ be given, then for every $\rho > 0$, there exists a constant $c = c(q, \rho) > 0$, such that for all $h \in Q$, with $\|h\|_\Gamma \le \rho$, it holds*

$$\|z_h^u - z^u\|_{1,p} + \|z_h^\varphi - z^\varphi\|_{2,q} \le c\,\|\mathbf{f_h} - \mathbf{f}\|_{W^\times} + c\|h\|_\Gamma\|\mathbf{f_h}\|_{W^\times}, \tag{3.44}$$

where $\mathbf{z}_h \in W$ is the solution to (3.40) for $W \ni \mathbf{u}_h = G(q+h)$ and $\mathbf{f}_{adj} = \mathbf{f_h}$, and $\mathbf{z} \in W$ is the solution to (3.40) for $W \ni \mathbf{u} = G(q)$ and $\mathbf{f}_{adj} = \mathbf{f} \in W^\times$, respectively.

Proof. The difference $\mathbf{z}_h - \mathbf{z}$ fulfills

$$\begin{aligned}(A'(\mathbf{u}))^*(\mathbf{z}_h - \mathbf{z}) + (R'(\varphi;\gamma))^*(z_h^\varphi - z^\varphi) &= \mathbf{f_h} - \mathbf{f} - (A'(\mathbf{u}_h) - A'(\mathbf{u}))^*\mathbf{z}_h \\ &\quad - (R'(\varphi_h;\gamma) - R'(\varphi;\gamma))^* z_h^\varphi.\end{aligned} \tag{3.45}$$

Then, the claim follows from (3.41) combined with Lemma 3.8. □

Note that Corollary 3.11 combined with Lemma 3.6 easily provides a Lipschitz result for the adjoint equations with $\mathbf{f} = G(q)$ and $\mathbf{f_h} = G(q+h)$ in the right-hand-side.

4. Second Order Optimality Conditions for the Optimal Control Problem

After having completed the analysis of the lower-level problem ($\mathrm{EL}^{\gamma,\eta}$) we are now in the position to return to the optimal control problem ($\mathrm{NLP}^{\gamma,\eta}$). First, let us state an existence result for global solutions to ($\mathrm{NLP}^{\gamma,\eta}$).

Proposition 4.1. *There exists at least one global minimizer $\bar{q} \in Q_{ad}$ with associated state $\bar{\mathbf{u}} = (\bar{u}, \bar{\varphi}) \in W$ to ($\mathrm{NLP}^{\gamma,\eta}$).*

Proof. In [**39**, Theorem 4.3], the proof has been carried out for a setting similar to ours, yet without inequality constraints. However, the admissible set Q_{ad} is a simple, closed and convex subset of L^2, hence the standard existence proof from e.g. [**45**, Theorem 2.14] or [**29**, Theorem 1.45] can be applied, utilizing Lemma 3.1 for a boundedness result of the sequence (u_k, φ_k), associated with a minimizing sequence $(q_k) \subset Q_{\text{ad}}$ in $W^{1,p} \times H^1$. □

Due to nonconvexity of $(\mathrm{NLP}^{\gamma,\eta})$ we characterize local minimizers.

Definition 4.2. We say that a control $\bar q \in Q_{\mathrm{ad}}$ is a local minimizer of $(\mathrm{NLP}^{\gamma,\eta})$ if there exists an L^2-neighborhood $\mathcal{B}(\bar q)$ of $\bar q$ such that there holds

$$f(\bar q) \le f(q) \quad \forall\, q \in \mathcal{B}(\bar q) \cap Q_{\mathrm{ad}}. \tag{4.1}$$

Such a solution $\bar q \in Q_{\mathrm{ad}}$ with associated state $\bar u = G(\bar q) \in W$ satisfies first order necessary optimality conditions, which we briefly state below.

Lemma 4.3. *If $\bar q \in Q_{ad}$ is a local minimizer with associated state $\bar{\mathbf{u}} \in W$, then there exists an adjoint state $\bar{\mathbf{z}} = (\bar z^u, \bar z^\varphi) \in W$ such that*

$$A(\bar{\mathbf{u}}) + R(\bar\varphi;\gamma) = B\bar q \in V^*, \tag{$\mathrm{EL}^{\gamma,\eta}$}$$

$$(A'(\bar{\mathbf{u}}))^*\bar{\mathbf{z}} + R'(\bar\varphi;\gamma)\bar z^\varphi = \bar u - u_d \in V^*, \tag{$\mathrm{AE}^{\gamma,\eta}$}$$

$$(B^*\bar{\mathbf{z}} + \alpha\bar q, q - \bar q)_\Gamma \ge 0 \qquad \forall\, q \in Q_{ad}. \tag{$\mathrm{VE}^{\gamma,\eta}$}$$

Proof. This is a straight forward extension of the necessary optimality conditions from [**39**,**40**] for settings without control constraints to the control-constrained setting. Since Q_{ad} is convex a local minimizer $\bar q$ satisfies the variational inequality

$$f'(\bar q)(q - \bar q) \ge 0 \quad \forall\, q \in Q_{\mathrm{ad}}. \tag{4.2}$$

Using (3.42) for $d^q = q - \bar q$ results in the stated optimality system. $\square$

Now we can prove our main results, the second order optimality conditions, by applying an abstract result from [**11**]. We start by defining the cone of critical directions.

$$\mathbf{C}(\bar q)\{d^q \in Q\colon\, d^q(x) \begin{cases} \ge 0 & \text{if } \bar q(x) = q_a(x), \\ \le 0 & \text{if } \bar q(x) = q_b(x), \\ = 0 & \text{if } f'(\bar q)(x)d^q(x)) \ne 0. \end{cases} \qquad \}.$$

The following second order necessary result for $(\mathrm{NLP}^{\gamma,\eta})$ then holds.

Theorem 4.4. *Let $\bar q \in Q_{ad}$ be a locally optimal control to* $(\mathrm{NLP}^{\gamma,\eta})$. *Then, it holds*

$$f''(\bar q)[d^q, d^q] \ge 0 \qquad \forall d^q \in \boldsymbol{C}(\bar q), \tag{4.3}$$

Proof. We check that the assumptions of [**11**, Theorem 2.3] are valid. From Corollary 3.5, Lemma 3.2, and Corollary 3.4, we obtain boundedness of f' and f''. In particular there exists $M_1, M_2 > 0$ such that

$$|f'(q)d_1^q| \le M_1\|d_1^q\|_\Gamma \qquad \text{and} \qquad |f''(q)[d_1^q, d_2^q]| \le M_2\|d_1^q\|_\Gamma\|d_2^q\|_\Gamma,$$

for all $d_1^q, d_2^q \in Q$. Moreover by Lemma 3.10, f' and f'' are locally Lipschitz continuous in Q for all $d_1^q, d_2^q \in Q$ and $q \in Q_{\mathrm{ad}} \cap B_Q(\bar q)$, where $B_Q(\bar q)$ is a $Q = L^2(\Gamma)$ - neighborhood of $\bar q$. Next, we point out that the tracking type objective functional results in a Legendre form $\mathcal{Q}\colon d^q \mapsto f''(\bar q)(d^q)^2$ from $L^2(\Gamma) \to \mathbb{R}$. Choosing $U_2 = U_\infty = Q$ and $\mathcal{K} = Q_{\mathrm{ad}}$, all conditions of Assumption (A2) in [**11**] are satisfied. Furthermore, for the choice of the cone of critical directions $\mathbf{C}(\bar q)$, we refer to Remark 2.4 of the same paper. Hence all prerequisites of [**11**, Theorem 2.3] are satisfied, now applying the theorem yields the assertion. $\square$

The result on second order sufficient optimality conditions for (NLP$^{\gamma,\eta}$), which does not involve a two-norm discrepancy, follows directly from [**11**, Theorem 2.5]. We will assume the following second order sufficient condition.

Assumption 4.5 (SSC). Let $\bar{q} \in Q_{\mathrm{ad}}$, together with the associated state and adjoint state $\bar{\mathbf{u}}, \bar{\mathbf{z}} \in W$, fulfill the first order necessary conditions given in Lemma 4.3. We assume

$$f''(\bar{q})[d^q, d^q] > 0 \qquad \forall\, d^q \in \mathbf{C}(\bar{q}) \setminus \{0\}. \tag{4.4}$$

Theorem 4.6. *Let $\bar{q} \in Q_{ad}$, with associated state $\bar{\mathbf{u}}$ and adjoint state $\bar{\mathbf{z}}$, satisfy Assumption 4.5. Then, there exist constants $\epsilon \geq 0$ and $\sigma \geq 0$ such that the quadratic growth condition*

$$f(q) \geq f(\bar{q}) + \sigma \|q - \bar{q}\|_\Gamma^2$$

holds for every $q \in Q_{ad}$ with $\|q - \bar{q}\|_\Gamma \leq \epsilon$. In particular, this means that $\bar{q}$ is a locally optimal control in the sense of L^2.

Proof. We can directly apply [**11**, Theorem 2.5], since the control $\bar{q}$ satisfies the variational inequality (4.2) and Assumption (A2) of [**11**] is satisfied, as already shown in Theorem 4.4. □

Remark 4.7. Comparing the conditions (4.3) and (4.4), we see that the gap between the necessary and sufficient second order conditions is minimal. This stems from the structure of our main problem (NLP$^{\gamma,\eta}$). In fact, since the reduced objective functional f satisfies Assumption (A2) of [**11**], the positivity assumption (4.4) and the coercivity condition

$$\exists\, c > 0 \quad \text{such that} \quad f''(\bar{q})[d^q, d^q] \geq\; c\, \|d^q\|_Q \quad \forall d^q \in \mathbf{C}(\bar{q}),$$

are equivalent. For a proof of this result, we refer to [**11**, Lemma 2.6].

Remark 4.8. We recall $f(q) := J(q, \mathbf{u})$, where $\mathbf{u} = G(q)$. Let $\bar{q}$, with associated $\bar{\mathbf{u}}$ and $\bar{\mathbf{z}}$ satisfy Assumption 4.5, then the quadratic growth condition in Theorem 4.6 can be written equivalently as

$$J(q, \mathbf{u}) \geq J(\bar{q}, \bar{\mathbf{u}}) + +\sigma \|q - \bar{q}\|_\Gamma^2,$$

for every $q \in Q_{\mathrm{ad}}$ with $\|q - \bar{q}\|_\Gamma \leq \epsilon$, and ϵ and σ as in Theorem 4.6.

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References

[1] A. Ahmad, K. Deckelnick, and M. Hinze, *Global minima for optimal control of the obstacle problem*, ESAIM: Control Optim. Calc. Var. (2020). to appear.

[2] G. Allaire, F. Jouve, and N. van Goethem, *Damage and fracture evolution in brittle materials by shape optimization methods*, J. Comput. Phys. **230** (2011), 5010–5044.

[3] L. Ambrosio and V.M. Tortorelli, *On the approximation of free discontinuity problems*, Boll. Unione Math. Ital., 1 (1992), 105–123.

[4] L. M. Betz, *Strong stationarity for optimal control of a non-smooth coupled system: Application to a viscous evolutionary VI coupled with an elliptic PDE*, SIAM J. Optim. **29** (2019), no. 4, 3069 –3099.

[5] T. Betz and C. Meyer, *Second-order sufficient optimality conditions for optimal control of static elastoplasticity with hardening*, ESAIM Control Optim. Calc. Var. **21** (2014), 271 –300.

[6] B. Bourdin, G. A. Francfort, and J.-J. Marigo, *Numerical experiments in revisited brittle fracture*, J. Mech. Phys. Solids **48** (2000), no. 4, 797–826.
[7] ———, *The variational approach to fracture*, J. Elast. **91** (2008), no. 1–3, 1–148.
[8] E. Casas, J.C. de los Reyes, and F. Tröltzsch, *Sufficient second-order optimality conditions for semiliear control problems with pointwise state constraints*, SIAM J. Optim. **19** (2008), 616–643.
[9] E. Casas and V. Dhamo, *Optimality conditions for a class of optimal boundary control problems with quasilinear elliptic equations*, Control Cybern. **40** (2011), 457–490.
[10] E. Casas and F. Tröltzsch, *First- and second-order optimality conditions for a class of optimal control problems with quasilinear elliptic equations*, SIAM J. Control Optim. **48** (2009), 688 –718.
[11] ———, *A general theorem on error estimates with application to a quasilinear elliptic control problem*, Comput. Optim. Appl. **53** (2012), 173 –206.
[12] ———, *Second order analysis for optimal control problems: improving results expected from abstract theory*, SIAM J. Optim. **22** (2012), 261 –279.
[13] ———, *Second order optimality conditions and their role in PDE control*, Jahresber. Dtsch. Math.-Ver. **117** (2015), 3–44.
[14] E. Casas, F. Tröltzsch, and A. Unger, *Second order sufficient optimality conditions for some state-constrained control problems of semilinear elliptic equations*, SIAM J. Control Optim. **38** (2000), 1369 –1391.
[15] C. Christof and G. Wachsmuth, *No-gap second-order conditions via a directional curvature functional*, SIAM J. Optim. **28** (2018), no. 3, 2097 –2130.
[16] ———, *On second-order optimality conditions for optimal control problems governed by the obstacle problem*, Optimization (2020). To appear.
[17] C. Clason, V.H. Nhu, and A. Rösch, *No-gap second-order optimality conditions for optimal control of a non-smooth quasilinear elliptic equation*, ESAIM: Control Optim. Calc. Var. (2021). Accepted.
[18] ———, *Optimal control of a non-smooth quasilinear elliptic equation*, Math. Control Relat. Fields (2021).
[19] M.H. Farshbaf-Shaker and C. Heinemann, *A phase-field approach for optimal boundary control of damage processes in two-dimensional viscoelastic media*, Math. Models Methods Appl. Sci. **25** (2015), 2749–2793.
[20] ———, *Necessary conditions of first-order for an optimal boundary control problem for viscous damage processes in 2D*, ESIAM Control Optim. Calc. Var. **24** (2018), 479–603.
[21] G. A. Francfort and J.-J. Marigo, *Revisiting brittle fracture as an energy minimization problem*, J. Mech. Phys. Solids **46** (1998), no. 8, 1319–1342.
[22] H Goldberg and F Tröltzscht, *Second order sufficient optimality conditions for a class of control problems governed by non-linear integral equations with application to parabolic boundary control*, Optimization **20** (1989), 687–698.
[23] K. Gröger, *A $W^{1,p}$-estimate for solutions to mixed boundary value problems for second order elliptic differential equations*, Math. Ann. **283** (1989), no. 4, 679–687.
[24] R. Haller-Dintelmann, H. Meinlschmidt, and W. Wollner, *Higher regularity for solutions to elliptic systems in divergence form subject to mixed boundary conditions*, Ann. Math. Pura Appl. **198** (2018), 1227 –1241.
[25] F. Harder and G. Wachsmuth, *Comparison of optimality systems for the optimal control of the obstacle problem*, GAMM-Rep. **40** (2018), no. 4, 312 –338.
[26] A. Hehl, M. Mohammadi, I. Neitzel, and W. Wollner, *Optimizing fracture propagation using a phase-field approach*, SPP1962 Special Issue (2020 (to appear)).
[27] R. Herzog, C. Meyer, and G. Wachsmuth, *Integrability of displacement and stresses in linear and nonlinear elasticity with mixed boundary conditions*, J. Math. Anal. and Appl. **382** (2011), no. 2, 802 –813.
[28] P. Hild, A. Münch, and Y. Ousset, *On the active control of crack growth in elastic media*, Comput. Methods Appl. Mech. Eng. **198** (2008), 407–419.
[29] M. Hinze, R. Pinnau, M. Ulbrich, and S. Ulbrich, *Optimization with PDE constraints*, Math. Model., vol. 23, Springer, Dordrecht, 2009.
[30] A. D. Ioffe, *Necessary and sufficient conditions for a local minimium. III. second order conditions and augmented duality*, SIAM J. Control Optim. **17** (1979), no. 2, 266–288.

[31] A. M. Khludnev and G. Leugering, *Optimal control of cracks in elastic bodies with thin rigid inclusions*, ZAMM Z. Angew. Math. Mech. **91** (2011), no. 2, 125–137.

[32] D. Knees, R. Rossi, and C. Zanini, *A vanishing viscosity approach to a rate-independent damage model*, Math. Models Methods Appl. Sci. **23** (2013), no. 4, 565–616.

[33] K. Kunisch and D. Wachsmuth, *Sufficient optimality conditions and semi-smooth Newton methods for optimal control of stationary variational inequalities*, ESAIM Control Optim. Calc. Var. **2012** (18), 520 –547.

[34] G. Leugering, J. Sokolowski, and A. Zochowski, *Control of crack propagation by shape-topological optimization*, Discrete Contin. Dyn. Syst. **35** (2015), no. 6, 2625–2657.

[35] D. G Luenberger, *Optimization by vector space methods*, Wiley professional paperback series, Wiley, New York u.a., 1998.

[36] H. Meinlschmidt, C. Meyer, and S. Walther, *Optimal control of an abstract evolution variational inequality with application to homogenized plasticity*, J. Nonsmooth Anal. Optim. **1** (2020).

[37] C. Meyer, A. Rademacher, and W. Wollner, *Adaptive optimal control of the obstacle problem*, SIAM J. Sci. Comput. **37** (2015), no. 2, A918–A945.

[38] M. Mohammadi and W. Wollner, *A priori error estimates for a linearized fracture control problem*, Optim. Eng. (2020).

[39] I. Neitzel, T. Wick, and W. Wollner, *An optimal control problem governed by a regularized phase-field fracture propagation model*, SIAM J. Control Optim. **55** (2017), no. 4, 2271–2288.

[40] ———, *An optimal control problem governed by a regularized phase-field fracture propagation model. part II the regularization limit*, SIAM J. Control Optim. **57** (2019), no. 3, 1672–1690.

[41] R. Rannacher, *Probleme der Kontinuumsmechanik und ihre numerische Behandlung*, Heidelberg University Publishing, Heidelberg, Germany, 2017.

[42] A.-T. Rauls and S. Ulbrich, *On the characterization of generalized derivatives for the solution operator of the bilateral obstacle problem*, 2020. Preprint SPP1962-143.

[43] A.-T. Rauls and G. Wachsmuth, *Generalized derivatives for the solution operator of the obstacle problem*, Set Valued Var. Anal. **28** (2020), 259 –285.

[44] L. M. Susu, *Optimal control of a viscous two-field gradient damage model*, GAMM-Rep. **40** (2018), no. 4, 287 –311.

[45] F. Tröltzsch, *Optimal control of partial differential equations: theory, methods and applications*, Grad. Stud. Math., vol. 112, American Math. Soc, Providence, RI, 2010.

[46] E. Zeidler, *Nonlinear functional analysis and its applications*, Springer, New York, NY, 1985.