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A new elementary proof for M-stationarity under MPCC-GCQ for mathematical programs with complementarity constraints

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Abstract

It is known in the literature that local minimizers of mathematical programs with complementarity constraints (MPCCs) are so-called M-stationary points, if a very weak MPCC-tailored Guignard constraint qualification (called MPCC-GCQ) holds. In this paper we present a new elementary proof for this result. Our proof is significantly simpler than existing proofs and does not rely on deeper technical theory such as calculus rules for limiting normal cones. A crucial ingredient is a proof of a (to the best of our knowledge previously open) conjecture, which was formulated in a Diploma thesis by Schinabeck.

Keywords: Mathematical program with complementarity constraints, Mathematical program with equilibrium constraints, Necessary optimality conditions, M-stationarity, Guignard constraint qualification

MSC (2020): 90C33, 90C30

1 Introduction

We consider mathematical programs with complementarity constraints, or MPCCs for short, which are nonlinear optimization problems of the form

$$\begin{aligned} \min_x \quad & f(x) \\ \text{s.t.} \quad & g(x) \leq 0, \quad h(x) = 0, \\ & G(x) \geq 0, \quad H(x) \geq 0, \quad G(x)^\top H(x) = 0. \end{aligned} \tag{MPCC}$$

Here, $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^l$, $h : \mathbb{R}^n \to \mathbb{R}^m$, $G, H : \mathbb{R}^n \to \mathbb{R}^p$ are differentiable functions. In the literature, MPCCs are also often called mathematical programs with equilibrium constraints, or MPECs for short.

MPCCs have been studied extensively in the literature, both from a numerical and a theoretical perspective. It is well known that standard constraint qualifications are usually not satisfied for (MPCC). Therefore, one usually considers constraint qualifications and stationarity conditions that are tailored to MPCCs. One such (necessary, first-order) stationarity condition is strong stationarity, but there are examples of MPCCs where the data is linear, but the unique minimizer is not strongly stationary, see [Scheel, Scholtes, 2000, Example 3].

The next strongest stationarity condition for MPCCs in the literature is M-stationarity, which is defined in Definition 2.1. It has been shown in [Flegel, Kanzow, Outrata, 2006] that M-stationarity holds under MPCC-GCQ, see also [Flegel, Kanzow, 2006]. MPCC-GCQ, which is defined in Definition 2.3, is, to the best of our knowledge, the weakest constraint qualification that is used for MPCCs in the literature. These proofs for M-stationarity under MPCC-GCQ rely on the concept of so-called limiting normal cones. In particular, calculus rules for the limiting normal cone are used, which are based on deeper technical theory and require to verify the calmness of certain set-valued mappings. There are also proofs of M-stationarity using nonsmooth regularization methods, see [Kanzow, Schwartz, 2013]. However, they require significantly stronger constraint qualifications.

In this paper we want to present a new proof for M-stationarity under MPCC-GCQ which is elementary, i.e. we do not rely on advanced theory such as the properties of limiting normal cones. Our proof is significantly simpler than any existing proofs that we are aware of. The major novel contribution of this paper is a result which can be found in Lemma 3.2. This result was already conjectured in [Schinabeck, 2009, Section 4.4.2], and, to the best of our knowledge there has not been a proof of this conjecture so far. With the knowledge that Lemma 3.2 holds, the rest of the proof of M-stationarity under MPCC-GCQ will not be particularly surprising for readers familiar with the implications of MPCC-GCQ. For the convenience of the reader we give a self-contained presentation, which only requires familiarity with basic theory of nonlinear optimization.

We hope that the reader gains new insights into the structure of stationarity conditions for MPCCs and that this paper makes it easier to fully understand why M-stationarity holds under MPCC-GCQ.

The structure of this paper is as follows: In Section 2, we give the relevant definitions. Then we use MPCC-GCQ to construct various multipliers that satisfy an A-stationary system in Proposition 3.1. Afterwards, these multipliers are combined into a multiplier which is M-stationary with the help of Lemma 3.2. The main result is then stated in Theorem 3.3. Finally, we give a brief outlook and conclusion in Section 4.

2 Definitions

It will be convenient to work with the index sets

$$\begin{aligned}
I^l &:= \{1, \ldots, l\}, \\
I^m &:= \{1, \ldots, m\}, \\
I^p &:= \{1, \ldots, p\}, \\
I^g(\bar{x}) &:= \{i \in I^l \mid g_i(\bar{x}) = 0\}, \\
I^{+0}(\bar{x}) &:= \{i \in I^p \mid G_i(\bar{x}) > 0 \,\wedge\, H_i(\bar{x}) = 0\}, \\
I^{0+}(\bar{x}) &:= \{i \in I^p \mid G_i(\bar{x}) = 0 \,\wedge\, H_i(\bar{x}) > 0\}, \\
I^{00}(\bar{x}) &:= \{i \in I^p \mid G_i(\bar{x}) = 0 \,\wedge\, H_i(\bar{x}) = 0\},
\end{aligned}$$

where $\bar{x} \in \mathbb{R}^n$ is a feasible point of (MPCC). Note that $I^{+0}(\bar{x}), I^{0+}(\bar{x}), I^{00}(\bar{x})$ form a partition of I^p. We continue with the definition of M- and A-stationarity.

Definition 2.1. Let $\bar{x} \in \mathbb{R}^n$ be a feasible point of (MPCC). We call $\bar{x}$ an *M-stationary* point of (MPCC) if there exist multipliers $\bar{\lambda} \in \mathbb{R}^l$, $\bar{\eta} \in \mathbb{R}^m$, $\bar{\mu}, \bar{\nu} \in \mathbb{R}^p$ with

$$\nabla f(\bar{x}) + g'(\bar{x})^\top \bar{\lambda} + h'(\bar{x})^\top \bar{\eta} + G'(\bar{x})^\top \bar{\mu} + H'(\bar{x})^\top \bar{\nu} = 0, \tag{2.1a}$$

$$\bar{\lambda} \geq 0, \qquad \bar{\lambda}_i = 0 \quad \forall i \in I^l \setminus I^g(\bar{x}), \tag{2.1b}$$

$$\bar{\mu}_i = 0 \quad \forall i \in I^{+0}(\bar{x}), \tag{2.1c}$$

$$\bar{\nu}_i = 0 \quad \forall i \in I^{0+}(\bar{x}), \tag{2.1d}$$

$$(\bar{\mu}_i < 0 \wedge \bar{\nu}_i < 0) \vee \bar{\mu}_i \bar{\nu}_i = 0 \quad \forall i \in I^{00}(\bar{x}). \tag{2.1e}$$

If the multipliers $\bar{\lambda}, \bar{\eta}, \bar{\mu}, \bar{\nu}$ only satisfy (2.1a)–(2.1d) and $\bar{\mu}_i \leq 0 \vee \bar{\nu}_i \leq 0$ holds for all $i \in I^{00}(\bar{x})$, then $\bar{x}$ is called an *A-stationary* point of (MPCC).

Other stationarity concepts for (MPCC) can be found in [Ye, 2005, Definitions 2.2–2.7].

In preparation for the definition of MPCC-GCQ we introduce some additional concepts.

Definition 2.2. Let $\bar{x} \in \mathbb{R}^n$ be a feasible point of (MPCC).

(a) We define the *tangent cone* of (MPCC) at $\bar{x}$ via

$$\mathcal{T}(\bar{x}) := \left\{ d \in \mathbb{R}^n \;\middle|\; \begin{array}{r} \exists \{x_k\}_{k\in\mathbb{N}} \subset F,\, \exists \{t_k\}_{k\in\mathbb{N}} \subset (0,\infty) : \\ x_k \to \bar{x},\; t_k \downarrow 0,\; t_k^{-1}(x_k - \bar{x}) \to d \end{array} \right\},$$

where $F \subset \mathbb{R}^n$ denotes the feasible set of (MPCC).

(b) We define the *MPCC-linearized tangent cone* $\mathcal{T}^{\text{lin}}_{\text{MPCC}}(\bar{x}) \subset \mathbb{R}^n$ at $\bar{x}$ via

$$\mathcal{T}^{\text{lin}}_{\text{MPCC}}(\bar{x}) := \left\{ d \in \mathbb{R}^n \;\middle|\; \begin{array}{rl} \nabla g_i(\bar{x})^\top d \le 0 & \forall i \in I^g(\bar{x}), \\ \nabla h_i(\bar{x})^\top d = 0 & \forall i \in I^m, \\ \nabla G_i(\bar{x})^\top d = 0 & \forall i \in I^{0+}(\bar{x}), \\ \nabla H_i(\bar{x})^\top d = 0 & \forall i \in I^{+0}(\bar{x}), \\ \nabla G_i(\bar{x})^\top d \ge 0 & \forall i \in I^{00}(\bar{x}), \\ \nabla H_i(\bar{x})^\top d \ge 0 & \forall i \in I^{00}(\bar{x}), \\ (\nabla G_i(\bar{x})^\top d)(\nabla H_i(\bar{x})^\top d) = 0 & \forall i \in I^{00}(\bar{x}) \end{array} \right\}.$$

(c) For a set $C \subset \mathbb{R}^n$ its polar cone $C^\circ \subset \mathbb{R}^n$ is defined via

$$C^\circ := \{d \in \mathbb{R}^n \mid d^\top y \le 0 \quad \forall y \in C\}.$$

Note that, in general, $\mathcal{T}(\bar{x})$ and $\mathcal{T}^{\text{lin}}_{\text{MPCC}}(\bar{x})$ are nonconvex sets.

Now we are ready to give the definition of MPCC-GCQ, which can also be found in [Flegel, Kanzow, Outrata, 2006, (41)], where it is called MPEC-GCQ.

Definition 2.3. Let $\bar{x} \in \mathbb{R}^n$ be a feasible point of (MPCC). We say that $\bar{x}$ satisfies the *MPCC-tailored Guignard constraint qualification*, or *MPCC-GCQ*, if

$$\mathcal{T}(\bar{x})^\circ = \mathcal{T}^{\text{lin}}_{\text{MPCC}}(\bar{x})^\circ$$

holds. Additionally, if $\mathcal{T}(\bar{x}) = \mathcal{T}^{\text{lin}}_{\text{MPCC}}(\bar{x})$ holds then we say that $\bar{x}$ satisfies *MPCC-ACQ*.

Clearly, MPCC-ACQ implies MPCC-GCQ. We mention that there are also other stronger constraint qualifications (such as MPCC-MFCQ if g, h, G, H are continuously differentiable) which imply MPCC-ACQ or MPCC-GCQ and are easier to verify, see e.g. [Ye, 2005, Theorem 3.2]. In particular, we emphasize that MPCC-GCQ (and MPCC-ACQ) are satisfied at every feasible point of (MPCC) if the functions g, h, G, H are affine.

3 M-stationarity under MPCC-GCQ

We start with a proposition that generates several multipliers which satisfy a slightly stronger stationarity condition than A-stationarity. The result can also be obtained from the proof of [Flegel, Kanzow, 2005, Theorem 3.4], with the minor difference that we only require MPCC-GCQ and not MPCC-ACQ.

Proposition 3.1. Let $\bar{x} \in \mathbb{R}^n$ be a local minimizer of (MPCC) that satisfies MPCC-GCQ and let $\alpha \in \{1,2\}^p$ be given. Then there exist multipliers $\lambda^\alpha \in \mathbb{R}^l$, $\eta^\alpha \in \mathbb{R}^m$,

$\mu^\alpha, \nu^\alpha \in \mathbb{R}^p$ with

$$\nabla f(\bar{x}) + g'(\bar{x})^\top \lambda^\alpha + h'(\bar{x})^\top \eta^\alpha + G'(\bar{x})^\top \mu^\alpha + H'(\bar{x})^\top \nu^\alpha = 0, \tag{3.1a}$$

$$\lambda^\alpha \geq 0, \qquad \lambda_i^\alpha = 0 \quad \forall i \in I^l \setminus I^g(\bar{x}), \tag{3.1b}$$

$$\mu_i^\alpha = 0 \quad \forall i \in I^{+0}(\bar{x}), \tag{3.1c}$$

$$\nu_i^\alpha = 0 \quad \forall i \in I^{0+}(\bar{x}), \tag{3.1d}$$

$$\alpha_i = 1 \Rightarrow \mu_i^\alpha \leq 0 \quad \forall i \in I^{00}(\bar{x}), \tag{3.1e}$$

$$\alpha_i = 2 \Rightarrow \nu_i^\alpha \leq 0 \quad \forall i \in I^{00}(\bar{x}). \tag{3.1f}$$

Proof. With Definition 2.2 (a) and the fact that $\bar{x}$ is a local minimizer of (MPCC) it is easy to see that the condition

$$\nabla f(\bar{x})^\top d \geq 0 \qquad \forall\, d \in \mathcal{T}(\bar{x})$$

is satisfied. Using polar cones and MPCC-GCQ, we obtain

$$-\nabla f(\bar{x}) \in \mathcal{T}(\bar{x})^\circ = \mathcal{T}^{\text{lin}}_{\text{MPCC}}(\bar{x})^\circ \subset \mathcal{T}^{\text{lin}}_{\text{NLP}(\alpha)}(\bar{x})^\circ,$$

where the cone $\mathcal{T}^{\text{lin}}_{\text{NLP}(\alpha)}(\bar{x}) \subset \mathcal{T}^{\text{lin}}_{\text{MPCC}}(\bar{x})$ is defined via

$$\mathcal{T}^{\text{lin}}_{\text{NLP}(\alpha)}(\bar{x}) := \left\{ d \in \mathcal{T}^{\text{lin}}_{\text{MPCC}}(\bar{x}) \;\middle|\; \begin{array}{ll} \alpha_i = 1 \Rightarrow \nabla H_i(\bar{x})^\top d = 0 & \forall i \in I^{00}(\bar{x}), \\ \alpha_i = 2 \Rightarrow \nabla G_i(\bar{x})^\top d = 0 & \forall i \in I^{00}(\bar{x}) \end{array} \right\}.$$

Note that, unlike $\mathcal{T}^{\text{lin}}_{\text{MPCC}}(\bar{x})$, this is a convex and polyhedral cone. Thus, one can calculate its polar cone (e.g. using Farkas' Lemma), which results in

$$\mathcal{T}^{\text{lin}}_{\text{NLP}(\alpha)}(\bar{x})^\circ = \left\{ \begin{array}{l} \displaystyle\sum_{i \in I^g(\bar{x})} \lambda_i^\alpha \nabla g_i(\bar{x}) + \sum_{i \in I^m} \eta_i^\alpha \nabla h_i(\bar{x}) \\ \displaystyle\quad + \sum_{i \in I^{0+}(\bar{x}) \cup I^{00}(\bar{x})} \mu_i^\alpha \nabla G_i(\bar{x}) \\ \displaystyle\quad + \sum_{i \in I^{+0}(\bar{x}) \cup I^{00}(\bar{x})} \nu_i^\alpha \nabla H_i(\bar{x}) \end{array} \;\middle|\; \begin{array}{ll} \lambda_i^\alpha \geq 0, & i \in I^g(\bar{x}), \\ \eta_i^\alpha \in \mathbb{R}, & i \in I^m, \\ \mu_i^\alpha \in \mathbb{R}, & i \in I^{0+}(\bar{x}) \cup I^{00}(\bar{x}), \\ \nu_i^\alpha \in \mathbb{R}, & i \in I^{+0}(\bar{x}) \cup I^{00}(\bar{x}), \\ \mu_i^\alpha \leq 0, & \text{if } \alpha_i = 1,\, i \in I^{00}(\bar{x}), \\ \nu_i^\alpha \leq 0, & \text{if } \alpha_i = 2,\, i \in I^{00}(\bar{x}) \end{array} \right\}.$$

Then the result follows from $-\nabla f(\bar{x}) \in \mathcal{T}^{\text{lin}}_{\text{NLP}(\alpha)}(\bar{x})^\circ$ by setting the remaining components of the multipliers (i.e. λ_i^α for $i \in I^p \setminus I^g(\bar{x})$, μ_i^α for $i \in I^{+0}(\bar{x})$, ν_i^α for $i \in I^{0+}(\bar{x})$) to zero.

Clearly, if $\bar{x}$ satisfies (3.1) for some $\alpha \in \{1, 2\}^p$ and suitable multipliers, then $\bar{x}$ is an A-stationary point of (MPCC). However, the statement of Proposition 3.1 is stronger than A-stationarity, namely for each index in $I^{00}(\bar{x})$ we can choose whether μ_i^α or ν_i^α is non-positive. Note that (2.1a)–(2.1d) are already satisfied by all 2^p possible choices for the multipliers and any convex combination of these. Thus, the question naturally arises whether a convex combination of these multipliers can be found that also satisfies (2.1e). As the next result shows, this is indeed possible. The following lemma was already stated as a conjecture in [Schinabeck, 2009, Section 4.4.2]. To the best of our knowledge, this conjecture has not been proven before.

Lemma 3.2. Let $\hat{I} \subset I^p$ be an index set. Suppose that for all $\alpha \in \{1,2\}^p$ there exist points $(\mu^\alpha, \nu^\alpha) \in A^\alpha$, where

$$A^\alpha := \{(\mu, \nu) \in \mathbb{R}^{2p} \mid \mu_i \leq 0 \text{ if } \alpha_i = 1,\ \nu_i \leq 0 \text{ if } \alpha_i = 2 \quad \forall i \in \hat{I}\}.$$

Then we can find a point $(\bar{\mu}, \bar{\nu})$ in the set

$$B := \operatorname{conv}\{(\mu^\alpha, \nu^\alpha) \mid \alpha \in \{1,2\}^p\} \subset \mathbb{R}^{2p}$$

of convex combinations of these points, such that for all $i \in \hat{I}$ we have the condition

$$(\bar{\mu}_i < 0 \wedge \bar{\nu}_i < 0) \vee \bar{\mu}_i \bar{\nu}_i = 0. \tag{3.2}$$

Proof. Let us choose points $(\hat{\mu}^\alpha, \hat{\nu}^\alpha) \in B \cap A^\alpha$, $\bar{\mu}, \bar{\nu} \in \mathbb{R}^p$ and a vector $\beta \in \{1,2\}^p$ that satisfy

$$(\hat{\mu}^\alpha, \hat{\nu}^\alpha) \in \operatorname*{arg\,min}_{(\mu,\nu) \in B \cap A^\alpha} \|(\mu, \nu)\|_2^2 \qquad \forall \alpha \in \{1,2\}^p, \tag{3.3}$$

$$\beta \in \operatorname*{arg\,max}_{\alpha \in \{1,2\}^p} \|(\hat{\mu}^\alpha, \hat{\nu}^\alpha)\|_2^2, \tag{3.4}$$

$$(\bar{\mu}, \bar{\nu}) := (\hat{\mu}^\beta, \hat{\nu}^\beta) \in \mathbb{R}^{2p}. \tag{3.5}$$

Clearly, these choices are possible. Furthermore, we have $(\bar{\mu}, \bar{\nu}) \in B$, i.e. it is a convex combination as claimed.

Let $i \in \hat{I}$ be given. It remains to show that our choice for $(\bar{\mu}, \bar{\nu})$ satisfies (3.2). Without loss of generality we can assume that $\beta_i = 1$ holds (otherwise one would exchange the roles of μ and ν in the rest of the proof). Therefore, we have $\bar{\mu}_i \leq 0$ due to $(\bar{\mu}, \bar{\nu}) \in A^\beta$. Suppose that (3.2) is not satisfied, i.e. $\bar{\mu}_i < 0$ and $\bar{\nu}_i > 0$ hold. We define

$$\gamma \in \{1,2\}^p, \qquad \gamma_j := \begin{cases} 2 & \text{if } j = i, \\ \beta_j & \text{if } j \in I^p \setminus \{i\} \end{cases} \qquad \forall j \in I^p.$$

Due to $\bar{\mu}_i < 0$ we can choose $t \in (0,1)$ such that the convex combination

$$(\mu^t, \nu^t) := t(\hat{\mu}^\gamma, \hat{\nu}^\gamma) + (1-t)(\bar{\mu}, \bar{\nu}) \in \mathbb{R}^{2p}$$

still satisfies $\mu_i^t < 0$. Since $\gamma_j = \beta_j$ holds for $j \neq i$ we also have $(\mu^t, \nu^t) \in A^\beta$. However, $(\bar{\mu}, \bar{\nu}) \neq (\hat{\mu}^\gamma, \hat{\nu}^\gamma)$ due to $\hat{\nu}_i^\gamma \leq 0$, i.e. (μ^t, ν^t) is a strict convex combination. Thus, by also using (3.4), we have

$$\|(\mu^t, \nu^t)\|_2^2 < \max\{\|(\bar{\mu}, \bar{\nu})\|_2^2, \|(\hat{\mu}^\gamma, \hat{\nu}^\gamma)\|_2^2\} \leq \|(\hat{\mu}^\beta, \hat{\nu}^\beta)\|_2^2.$$

Due to $(\mu^t, \nu^t) \in B \cap A^\beta$ this is a contradiction to (3.3), which completes the proof.

We mention that it was recognized already in [Schinabeck, 2009, Section 4.4.2] that this lemma would significantly simplify the already existing proofs for M-stationarity.

A straightforward combination of Proposition 3.1 and Lemma 3.2 yields the desired M-stationarity result.

Theorem 3.3. Let $\bar{x} \in \mathbb{R}^n$ be a local minimizer of (MPCC) that satisfies MPCC-GCQ. Then $\bar{x}$ is an M-stationary point.

Proof. For all $\alpha \in \{1, 2\}^p$, let $(\lambda^\alpha, \eta^\alpha, \mu^\alpha, \nu^\alpha) \in \mathbb{R}^{l+m+2p}$ be the multipliers generated by Proposition 3.1. By applying Lemma 3.2 with $\hat{I} = I^{00}(\bar{x})$, we find a convex combination $(\bar{\lambda}, \bar{\eta}, \bar{\mu}, \bar{\nu}) \in \mathbb{R}^{l+m+2p}$ of these multipliers such that (2.1e) is satisfied. The conditions (2.1a)–(2.1d) follow from (3.1a)–(3.1d) by convexity.

4 Conclusion and outlook

We provided an new proof for M-stationarity of local minimizers of MPCCs under MPCC-GCQ. Although this result was already known, the new proof uses only basic and well-known tools from the theory for nonlinear programming. This new elementary proof for M-stationarity was enabled by providing a proof for a (to the best of our knowledge previously open) conjecture from [Schinabeck, 2009] in Lemma 3.2.

In the future, it would also be interesting to apply this approach to other problem classes from disjunctive programming and to investigate to what extend the ideas from this paper can be generalized.

In Sobolev or Lebesgue spaces, the limiting normal cone turned out to be not as effective as in finite dimensional spaces for obtaining stationarity conditions for complementarity-type optimization problems, see [Harder, Wachsmuth, 2018; Mehlitz, Wachsmuth, 2018]. Thus, it would be interesting to know whether the new elementary method from this paper can provide ideas for possible approaches for better stationarity conditions of complementarity-type optimization problems in Sobolev and Lebesgue spaces. However, it would not be trivial to transfer the method from finite-dimensional spaces to infinite-dimensional spaces.

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