Deutsche Forschungsgemeinschaft
Priority Programme 1962

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SPP 1962 | Non-smooth and Complementarity-based Distributed Parameter Systems: Simulation and Hierarchical Optimization

Preprint Number SPP1962-147

received on September 25, 2020

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Preprint

Non-optimality of conical parts for Newton's problem of minimal resistance in the class of convex bodies

Lev Lokutsievskiy · Gerd Wachsmuth · Mikhail Zelikin

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Abstract We consider Newton's problem of minimal resistance. Our main result shows that certain conical parts contained in the boundary of a convex body inhibit the optimality of the body. This is achieved by the investigation of a local variation of the conical part. We also consider the problem arising in the limit if the height goes to infinity. We apply the main result to certain bodies which are conjectured to being optimal in the literature and we show that they cannot be optimal.

Keywords Newton's problem of minimal resistance · Conical parts · Convex bodies

Mathematics Subject Classification (2010) 49K99 · 49Q10 · 52A15

1 Introduction

One of the first problems in calculus of variations is a least resistance problem posed by Newton in his *Principia*. A three dimensional body with base $\Omega \subset \mathbb{R}^2$ is travelling in negative z-direction. The upper boundary of the body is given by $\Omega \times \{0\}$, while the lower boundary is described by the graph of a function $u \colon \Omega \to [-M, 0]$, where $M > 0$ is the height of the body. The medium around the body is assumed to be very rare and under the assumption that each

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particle collides only once with the body, one arrives at the resistance

$$J(u) = \int_{\Omega} \frac{1}{|\nabla u|^2 + 1} \, \mathrm{d}x \wedge \mathrm{d}y,$$

see [1, 2]. In order to comply with the single-impact condition, one typically considers the convex situation, namely, $\Omega \subset \mathbb{R}^2$ is assumed to be convex and $u \colon \Omega \to [-M, 0]$ is convex as well. We denote the set of all such functions by $C_M \subset W^{1,2}_{\mathrm{loc}}(\Omega)$.

As we have mentioned, Newton obtained his resistance functional J under the assumption of a rare medium. Despite this fact, in the 20th century, it has been discovered (see [3, Chapter III], [4, §23]) that J also describes accurately enough the resistance of a convex body moving in dense media with hypersonic speed. Alternatively, the resistance for hypersonic speeds can be computed by the Buseman formula, which usually gives better accuracy for non-convex bodies, but is worse for convex ones [4, §23].

For Ω being the unit disc, Newton found an optimal solution among all convex bodies of revolution. Newton's solution has a very non-trivial peculiarity: its lateral boundary is strictly convex, but the lower part is a flat disc, and these parts adjoin each other by a corner of 45°. All standard facts about the problem can be found in a very well written survey [2].

Newton's result [5] was published in 1687, exactly $\frac{1}{3}$ of a millennium ago. Since that until the end of the 20th century, it was assumed that the Newton's body has minimal resistance among all convex bodies. Only in 1996, Guasoni (in his "Tesi di Laurea" [6] under the supervision of Buttazzo) found a "screwdriver" shape that has less resistance than the one found by Newton of the same base and height $M \geq 2$. An analytical argument for the non-optimality of Newton's solution is given in [7].

According to [1, Theorem 2.1], an optimal body exists in the class of convex bodies with given base and height. There are some analytical results on the structure of optimal bodies. Let Ω be the unit disc and let the convex function $u : \Omega \to [-M, 0]$ describe the shape of an optimal body for some given height $M > 0$. Then

- $|\nabla u(x, y)| \in \{0\} \cup [1; +\infty)$ for a.e. $(x, y) \in \Omega$, [2, Theorem 2.3];
- $\lim_{(x,y)\to\partial\Omega} u(x, y) = 0$, [8, Theorem 2];
- if ω is an open subset of Ω, then u is not strictly convex on ω, [9, Corollary 2].

The latter results also proves that Newton's body of revolution is not optimal in the class of all convex bodies with given base and height as it has a strictly convex lateral boundary. Moreover, this lack of strict convexity implies that the Euler-Lagrange equations cannot be used to solve the problem, cf. [2, Theorem 3.5].

There are several numerical results [10, 11], which give very good approximations of optimal bodies due to [12, Theorem 2].

In [13], the hypothesis of rotational symmetry was replaced by the less restrictive hypotheses of (i) mirror symmetry w.r.t. a vertical plane and (ii) developable structure of the side boundary. Let us remark that all existing aircraft and ships, to say nothing of living creatures, have such symmetry. We have obtained a remarkable formula that describes a curve in the plane of symmetry and proved that the convex hull of this curve and $\Omega \times \{0\}$ is locally optimal in the considered class of admissible bodies, see [13, Theorem 9.1].

The most astonishing fact concerning Newton's problem is that the exact shapes of optimal bodies in C_M are still unknown.

There were suggested a lot of different shapes as candidates that were considered as possible solutions to Newton's problem in the class of convex bodies, see [14, 11, 13]. Some of these profiles contain conical parts on their boundaries. The investigation of these conical parts allows us to prove non-optimality of all such bodies.

More precisely, the main contribution of this paper is to provide another optimality condition for the classical case $\Omega = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$: a local solution u cannot contain certain conical parts. Specifically, let the apex $(x_0, y_0) = (r_0 \cos\varphi_0, r_0 \sin\varphi_0) \in \operatorname{int}\Omega$ and $z_0 > 0$ be given. The (oblique circular) cone with vertex $(x_0, y_0, -z_0)$ is given by (the graph of) the function $v\colon \Omega \to \mathbb{R}$ defined via

$$v\big(\lambda\,(x_0, y_0) + (1-\lambda)\,(\sin\varphi, \cos\varphi)\big) = -\lambda\, z_0 \qquad \forall \varphi \in [0, 2\,\pi], \lambda \in [0, 1].$$

Further, for $\alpha < \beta < \alpha + 2\,\pi$ we define the (oblique) circular sector

$$S = \big\{\lambda\,(x_0, y_0) + (1-\lambda)\,\big(\sin\varphi, \cos\varphi\big) \in \mathbb{R}^2 \;\big|\; \varphi \in [\alpha, \beta], \lambda \in [0, 1]\big\}.$$

Then, our main results reads as follows: suppose that $u = v$ on S and that there exists $\varphi \in [\alpha, \beta]$ such that

$$\frac{3r_0^2 \sin^2(\varphi - \varphi_0) - [1 - r_0 \cos(\varphi - \varphi_0)]^2}{[1 - r_0 \cos(\varphi - \varphi_0)]^2 \{r_0^2 \sin^2(\varphi - \varphi_0) + [1 - r_0 \cos(\varphi - \varphi_0)]^2\}} < z_0^{-2}. \qquad (1)$$

Then u cannot be a local minimizer of J in the topology of $W^{1,\infty}(\Omega)$.

We also study what is happening in the limiting case $M \to \infty$. In order to pass to this limit, we do a rescaling $\hat{u} = u/M$. Observe that

$$J(u) = \int_\Omega \frac{1}{|\nabla u(x)|^2 + 1}\,\mathrm{d}x = M^{-2} \int_\Omega \frac{1}{|\nabla \hat{u}(x)|^2 + 1/M^2}\,\mathrm{d}x.$$

By neglecting the leading factor M^2 and formally passing to the limit, we arrive at the limiting problem

$$J_\infty(u) = \int_\Omega \frac{1}{|\nabla u(x)|^2}\,\mathrm{d}x$$

and this is made precise in Section 3 below. In particular, we show that as M goes to ∞, optimal bodies have at least one accumulation point and each such shape is optimal for the limiting problem.

It seems that this limiting problem was not studied so far. Our non-optimality result also extends to this infinite-height case.

2 Notation and Preliminaries

Let $\Omega \subset \mathbb{R}^n$ be a compact convex domain with nonempty interior, i.e., $\operatorname{int}\Omega \neq \emptyset$. For some fixed height $M > 0$, we define the class of functions[1]

$$C_M := \big\{u\colon \Omega \to [-M, 0] \;\big|\; u \text{ is convex and closed}\big\}.$$

Note that each $u \in C_M$ is locally Lipschitz in $\operatorname{int}\Omega$ and, therefore, differentiable a.e. Hence, we can define the objective $J\colon C_M \to \bar{\mathbb{R}}$ with $\bar{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ via

$$J(u) := \int_\Omega \frac{1}{|\nabla u|^2 + 1}\, \mathrm{d}x \wedge \mathrm{d}y \qquad \forall u \in C_M.$$

Now, Newton's problem of least resistance is given by

$$J(u) \to \min_{u \in C_M}. \tag{2}$$

The classical case considered by Newton uses the two-dimensional unit disc $\Omega := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$. In this case, the problem is rotationally symmetric. Under the additional condition that the solution is rotationally symmetric as well, Newton was able to solve the problem, see [2].

Buttazzo, Ferone and Kawohl proved in [1] that there exists an optimal solution for any $M > 0$ and any Ω (as above). This solution might not be unique. Indeed, on one hand, in the classical case, it was shown by [7], that Newton's rotationally symmetric body is not a solution in the class C_M. On the other hand, it is the unique solution among all bodies of revolution. Hence, any optimal solution in C_M cannot be rotationally symmetric. Since rotations of any solution to the classical problem are also solutions, a solution cannot be unique. Moreover, it is clear that the set of solutions depends on the height M.

Let us have a brief look into the existence result of [1]. We introduce the space

$$W^{1,p}_{\mathrm{loc}}(\Omega) := \big\{u\colon \Omega \to \mathbb{R} \;\big|\; u \in W^{1,p}(K) \text{ for all compact subsets } K \text{ of } \operatorname{int}\Omega\big\},$$

where $p \in [1, \infty]$ is arbitrary, and we say that $u_n \to u$ in $W^{1,p}_{\mathrm{loc}}(\Omega)$ if and only if $u_n \to u$ in $W^{1,p}(K)$ for all compact subsets K of $\operatorname{int}\Omega$. Then, we have the following result, see [1, Theorem 2.1 and Lemma 2.2].

Lemma 1. *For all $M > 0$ and any $p \in [1, \infty)$ we have $C_M \subset W^{1,p}_{\mathrm{loc}}(\Omega)$. The set C_M is sequentially compact in $W^{1,p}_{\mathrm{loc}}(\Omega)$, i.e., any sequence $(u_n) \subset C_M$ has a subsequence (u_{n_k}) with $u_{n_k} \to u$ in $W^{1,p}_{\mathrm{loc}}(\Omega)$ for some $u \in C_M$. Moreover, $u_{n_k} \to u$ everywhere and $\nabla u_{n_k} \to u$ a.e. in Ω. The functionals J and J_∞ are sequentially lower semicontinuous on $W^{1,p}_{\mathrm{loc}}(\Omega)$.*

With this lemma, the existence of minimizers of (2) follows from the direct method of calculus of variations, see [1, Theorem 2.1].

[1] We consider only closed (i.e., lower semicontinuous) convex function due to the following two reasons. First, $J(u) = J(\operatorname{cl} u)$ for any convex function u, and hence, $\operatorname{cl} u$ is a canonical representative for u in the Sobolev space $W^{1,1}_{\mathrm{loc}}(\Omega)$. Second, for closed convex functions, the mentioned result by Plakhov [8] can be stated in a very nice way: if $u \in C_M$ is optimal, then $u|_{\partial\Omega} = 0$.

3 The Limiting Case of Infinite Height

In this section, we study the limit of optimal solutions as $M \to \infty$.

It is easy to see that the minimum of any optimal solution in C_M is $-M$. Hence, if we want to find the limit shape, we need to reformulate the problem. Consider the following problem

$$J_M(u) := \int_\Omega \frac{1}{|\nabla u(x)|^2 + M^{-2}} \, \mathrm{d}x \to \min_{u \in C_1} . \tag{3}$$

Obviously, $J_M(u) = M^2 \, J(M\, u)$ and $u \in C_1$ if and only if $M\, u \in C_M$. Thus, if $u_M \in C_1$ is an optimal solution to problem (3) then $M u_M \in C_M$ is an optimal solution to problem (2) and vice versa. Solutions $\hat{u}_M$ are bounded in Ω, and we are interested in a limit (in some sense) of these solutions as $M \to \infty$.

Problem (3) is closely connected with the following problem with limit functional:

$$J_\infty(u) = \int_\Omega \frac{1}{|\nabla u(x)|^2} \, \mathrm{d}x \to \min_{u \in C_1} . \tag{4}$$

Again, the existence of minimizers follows from Lemma 1, see [1, Theorem 2.1].

First, let us show how minima in problems (3) and (4) are connected.

Theorem 2. *Let $p \in [1, \infty)$ be given and $(u_M)_{M>0}$ denote a family of solutions to problems (3). For every increasing sequence $(M_n)_{n\in\mathbb{N}}$ with $M_n \to \infty$, the sequence $(u_{M_n})_{n\in\mathbb{N}}$ possesses an accumulation point in $W^{1,p}_{\mathrm{loc}}(\Omega)$. Every such accumulation point is a solution to (4). Moreover,*

$$\lim_{M\to\infty} J_M(u_M) = \min_{u \in C_1} J_\infty(u) < \infty.$$

Proof. Due to $u_M \in C_1$ for all $M > 0$, Lemma 1 implies the claimed existence of accumulation points. Now, for any sequence (u_{M_n}) with $M_n \to \infty$ and $u_{M_n} \to \hat{u}$ in $W^{1,p}_{\mathrm{loc}}(\Omega)$, we have (along a subsequence) $\nabla u_{M_n} \to \nabla \hat{u}$ a.e. in Ω. Hence, Fatou's lemma implies

$$J_\infty(\hat{u}) = \int_\Omega \frac{1}{|\nabla \hat{u}|^2} \, \mathrm{d}x \le \liminf_{n\to\infty} \int_\Omega \frac{1}{M_n^{-2} + |\nabla u_{M_n}|^2} \, \mathrm{d}x = \liminf_{n\to\infty} J_{M_n}(u_{M_n})$$

On the other hand, we trivially have $J_M(u) \le J_\infty(u)$ for all $M > 0$ and $u \in C_1$. Hence, the optimality of u_{M_n} implies

$$\forall u \in C_1 \quad J_\infty(\hat{u}) \le \liminf_{n\to\infty} J_{M_n}(u_{M_n}) \le \liminf_{n\to\infty} J_{M_n}(u) \le J_\infty(u).$$

This shows that $\hat{u}$ is a solution to (4).

From $J_M(u) \ge J_{M'}(u)$ for $M \ge M'$, we get that $\inf_{u\in C_1} J_M(u)$ is monotonically increasing in M. Hence,

$$\lim_{M\to\infty} J_M(u_M) = \lim_{M\to\infty} \inf_{u\in C_1} J_M(u) \le \inf_{u\in C_1} J_\infty(u) = J_\infty(\hat{u}).$$

It remains to prove $J_\infty(\hat{u}) < \infty$. Without loss of generality $0 \in \operatorname{int} \Omega$. Consider $u(x) = -1 + |x|/R$ where $R = \max_{x\in\Omega} \operatorname{dist}(x, 0)$. Obviously, $u \in C_1$ and $|\nabla u(x)| \equiv 1/R$ (except for $x = 0$). Hence, $J_\infty(\hat{u}) \le J_\infty(u) = R^2 \operatorname{area}(\Omega) < \infty$, since Ω is compact. □

In [13], an important subclass $E_M \subset C_M$ for the classical case $\Omega = \{x^2 + y^2 \leq 1\} \subset \mathbb{R}^2$ is considered. The subclass E_M consists of functions being a convex envelope of δ_Ω and a convex curve lying in a vertical plane of symmetry (see Section 4 in [13] for details). In [13], a family of functions $\tilde{u}_M \in E_M$ of special form is constructed. Moreover, it is analytically proved that $\tilde{u}_M$ is a local minimum for large enough M in E_M w.r.t. a certain class of variations, see [13, Theorem 9.1]. It is known that the resistances of analytically found $\tilde{u}_M \in E_M$ and numerically found optimal solution $\hat{u}_M$ in C_M (see [11, 10]) coincide up to 1% for $M = 1.5$. In this paper, we will present a new result on optimality of certain conical parts of the body side boundary, which allows us investigate the question whether $\tilde{u}_M \in E_M$ are optimal in C_M or not. On the first glance, they seems to be not optimal, since the numerical results are accurate enough and give a slightly better values of the resistance functional. But the following question is much more interesting: does the family $\tilde{u}_M \in E_M$ is at least asymptotically optimal in C_M for J (see conclusion section in [13]). This question is equivalent to the following: does the family $M^{-1}\tilde{u}_M$ is asymptotically optimal in C_1 for J_M. Recall that a family (u_M) is called asymptotically optimal for functional J_M as $M \to \infty$ if

$$\lim_{M\to\infty} \frac{J_M(u_M)}{\inf_{u\in C_1} J_M(u)} = 1.$$

The following proposition gives a simple way to work with asymptotically optimal families, it can be proved analogously to Theorem 2.

Proposition 3. *Let $p \in [1, \infty)$ be given and consider an asymptotically optimal family $u_M \in C_1$ for J_M as $M \to \infty$. Then there exists a sequence $M_k \to \infty$ as $k \to \infty$ and $u_\infty \in C_1$, $u_{M_k} \to u_\infty$ in $W^{1,p}_{\mathrm{loc}}(\Omega)$ and u_∞ is optimal in C_1 for limit functional J_∞.*

This proposition gives us a tool to check if a certain family of bodies is asymptotically optimal. Together with results in the next section it allows us to investigate the family found in [13].

4 Non-optimality of Conical parts

In this section, we will prove a non-optimality result for certain conical parts included in the boundary of the body in the classical situation of a circular base $\Omega = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$. In other words, we will prove that the boundary of an optimal body cannot have certain conical parts.

We will write $\delta = M^{-2}$ for short. Hence, $\delta \geq 0$, and the case $\delta = 0$ corresponds to $M = \infty$. Therefore,

$$J_M(u) = \int_\Omega \frac{1}{|\nabla u(x)|^2 + \delta}\, \mathrm{d}x,$$

and the function u is normalized, i.e., $-1 \leq u \leq 0$.

We start by considering a simple situation, in which the entire body is just an oblique circular cone. The base is given by $\Omega \times \{0\}$ and the apex is given by the point $P_0 = (x_0, y_0, -1)$ with $(x_0, y_0) \in \operatorname{int} \Omega$. We take a different point $(x_1, y_1) \in \operatorname{int} \Omega$. We further take some height $M_1 > 0$, such that $(x_1, y_1, -M_1)$ lies exactly on the boundary of the cone. Now, for $\varepsilon > 0$ we consider the perturbed point $P_1 = (x_1, y_1, -m)$ with $m = M_1 + \varepsilon^2 (1 - M_1)$. The perturbed body is given by the convex hull of the base $\Omega \times \{0\}$ and the points P_0 and P_1, see Fig. 1. In the following, we derive an expansion formula of the resistance of

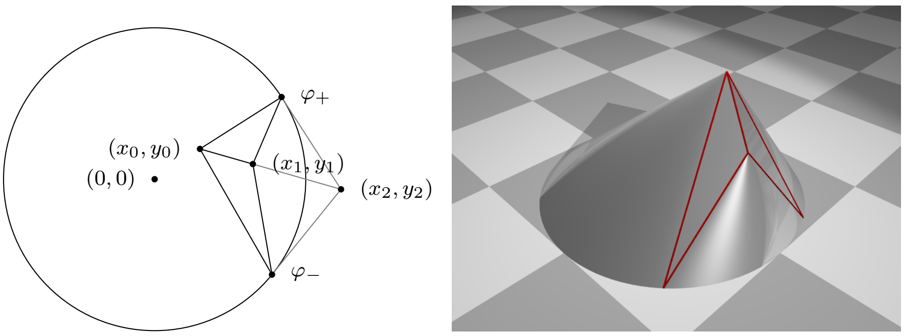

Fig. 1 Visualization of the perturbed surface.

the perturbed body in terms of the parameter ε. Note that the original cone corresponds to $\varepsilon = 0$.

Since the resistance does not change under rotations and reflections, we can assume without loss of generality, that the line through the points (x_0, y_0) and (x_1, y_1) also contains the point $(1, 0)$ and that $y_0 > 0$.

The line $P_0 P_1$ intersects the horizontal plane $\{z = 0\}$ at the point

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \frac{m}{m-1} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \frac{1}{1-m} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}.$$

The perturbed body can be described by the four parameters x_0, y_0, $M_1 \in (0, 1)$ and $\varepsilon > 0$, since the point P_1 is given by

$$x_1 = 1 - M_1 + x_0 M_1 \quad \text{and} \quad y_1 = M_1 y_0.$$

Plugging this into the above equation, we find

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \frac{1}{1-\varepsilon^2} \begin{pmatrix} 1 - \varepsilon^2 x_0 \\ -\varepsilon^2 y_0 \end{pmatrix}.$$

Next, we write the point P_2 in polar coordinates, i.e., $(x_2, y_2) = r_2 (\cos\theta_2, \sin\theta_2)$ with

$$r_2^2 = \frac{1 - 2\varepsilon^2 x_0 + \varepsilon^4 (x_0^2 + y_0^2)}{(1-\varepsilon^2)^2} \quad \text{and} \quad \theta_2 = -\arctan \frac{\varepsilon^2 y_0}{1 - \varepsilon^2 x_0},$$

where we used $y_2 < 0$. We compute some parameters to describe the structure of the perturbed body. The circle $\partial\Omega$ contains two important points $(\cos\varphi_\pm, \sin\varphi_\pm)$, where $\pm\varphi_\pm > 0$ and $\varphi_\pm \to 0$ as $\varepsilon \to +0$. These are the tangent points of the tangent lines to the unit disc passing through (x_2, y_2). The lateral boundary of the body consists of the following parts:

1. A big conic surface with apex $(x_0, y_0, -1)$ and boundary arc $(\cos\varphi, \sin\varphi, 0)$ for $\varphi \in [\varphi_+, \varphi_- + 2\pi]$. Let us compute the total resistance of this surface. We parametrize this part of the boundary via

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (1-\lambda)x_0 + \lambda\cos\varphi \\ (1-\lambda)y_0 + \lambda\sin\varphi \\ -(1-\lambda) \end{pmatrix}, \qquad \lambda \in [0,1], \varphi \in [\varphi_+, \varphi_- + 2\pi].$$

Note that a normal vector of this surface is given by

$$n = \big(\cos\varphi \;\; \sin\varphi \;\; x_0\cos\varphi + y_0\sin\varphi - 1\big)^\top,$$

hence we have $|\nabla u(x,y)| = 1/(1 - x_0\cos\varphi - y_0\sin\varphi)$, where (x, y) is linked with (λ, φ) via the above parametrization. For the area of the surface element, we get

$$\mathrm{d}x \wedge \mathrm{d}y = \lambda\,(1 - x_0\cos\varphi - y_0\sin\varphi)\,\mathrm{d}\lambda \wedge \mathrm{d}\varphi.$$

Hence, the total resistance is given by

$$\begin{aligned} R_0 &= \int_0^1 \lambda\,\mathrm{d}\lambda \left(\int_0^{2\pi} - \int_{\varphi_-}^{\varphi_+}\right) \frac{1 - x_0\cos\varphi - y_0\sin\varphi}{(1 - x_0\cos\varphi - y_0\sin\varphi)^{-2} + \delta}\,\mathrm{d}\varphi \\ &= \frac{1}{2}\left(\int_0^{2\pi} - \int_{\varphi_-}^{\varphi_+}\right) \frac{(1 - x_0\cos\varphi - y_0\sin\varphi)^3}{1 + \delta(1 - x_0\cos\varphi - y_0\sin\varphi)^2}\,\mathrm{d}\varphi. \end{aligned}$$

2. A small conic surface consisting of the apex $(x_1, y_1, -m)$ and the boundary arc $(\cos\varphi, \sin\varphi, 0)$ for $\varphi \in [\varphi_-, \varphi_+]$. Similarly, we arrive at

$$R_1 = \frac{1}{2}\int_{\varphi_-}^{\varphi_+} \frac{(1 - x_1\cos\varphi - y_1\sin\varphi)^3}{m^2 + \delta(1 - x_1\cos\varphi - y_1\sin\varphi)^2}\,\mathrm{d}\varphi.$$

3. Two triangles with vertices $(x_0, y_0, -1)$, $(x_1, y_1, -m)$ and $(\cos\varphi_\pm, \sin\varphi_\pm, 0)$. On these triangles we have $\nabla u_\pm = 1/(1 - x_0\cos\varphi_\pm - y_0\sin\varphi_\pm)$. The areas of their projections onto the plane $\{z = 0\}$ are

$$S_\pm = \pm\frac{1}{2}\det\begin{bmatrix} \cos\varphi_\pm - x_0 & \cos\varphi_\pm - x_1 \\ \sin\varphi_\pm - y_0 & \sin\varphi_\pm - y_1 \end{bmatrix}.$$

Hence, the total resistance of the perturbed body is given by the expression

$$\mathfrak{R}(\varepsilon) = R_0 + R_1 + \frac{S_+}{|\nabla u_+|^2 + \delta} + \frac{S_-}{|\nabla u_-|^2 + \delta}.$$

In what follows, we will derive an asymptotic expansion of R as $\varepsilon \searrow 0$. Note that the resistance of the unperturbed body is given by

$$\mathfrak{R}(0) = R_0(0) = \frac{1}{2}\int_0^{2\pi} \frac{(1 - x_0\cos\varphi - y_0\sin\varphi)^3}{1 + \delta(1 - x_0\cos\varphi - y_0\sin\varphi)^2}\,\mathrm{d}\varphi.$$

It is easy to see that

$$\varphi_\pm = \theta_2 \pm \arccos\frac{1}{r_2} = -\arctan\frac{\varepsilon^2 y_0}{1 - \varepsilon^2 x_0} \pm \arccos\frac{1 - \varepsilon^2}{\sqrt{1 - 2\varepsilon^2 x_0 + \varepsilon^4(x_0^2 + y_0^2)}}.$$

This right-hand side cannot be used to obtain an expansion of $\varphi_\pm$, since the argument of arccos goes to 1 as $\varepsilon \searrow 0$. Nonetheless, by using the addition theorems for cosine and sine, a straightforward computation gives

$$\cos\varphi_\pm = \frac{(1 - x_0\varepsilon^2)(1 - \varepsilon^2) \pm y_0\varepsilon^3\sqrt{2(1 - x_0) - (1 - x_0^2 - y_0^2)\varepsilon^2}}{1 - 2\varepsilon^2 x_0 + \varepsilon^4(x_0^2 + y_0^2)},$$

$$\sin\varphi_\pm = \frac{\pm(1 - x_0\varepsilon^2)\varepsilon\sqrt{2(1 - x_0) - (1 - x_0^2 - y_0^2)\varepsilon^2} - y_0(1 - \varepsilon^2)\varepsilon^2}{1 - 2\varepsilon^2 x_0 + \varepsilon^4(x_0^2 + y_0^2)}.$$

Using the last formula, we see that $\pm\varphi_\pm = \varepsilon\sqrt{2\,(1 - x_0)} + O(\varepsilon^2)$ as $\varepsilon \to +0$. Note that $\varphi_\pm$ were initially defined for $\varepsilon \geq 0$. However, they are analytic functions of $\varepsilon \geq 0$ as arcsin is analytic. Hence, we are able to extend their domains for $\varepsilon < 0$ by analyticity. Both φ_+ and φ_- become analytic functions of ε around 0. Moreover, $\varphi_+(-\varepsilon) = \varphi_-(\varepsilon)$

First, let us compute expansions for the integrals appearing in R_0 and R_1. Using the Leibniz integral rule, we arrive at

$$\frac{1}{2}\int_{\varphi_-(\varepsilon)}^{\varphi_+(\varepsilon)} \frac{(1 - x_0\cos\varphi - y_0\sin\varphi)^3}{1 + \delta(1 - x_0\cos\varphi - y_0\sin\varphi)^2}\,\mathrm{d}\varphi = \frac{\sqrt{2}(1 - x_0)^{7/2}}{1 + \delta(1 - x_0)^2}\varepsilon + O(\varepsilon^2). \quad (5)$$

Similarly, using $m = M_1 + \varepsilon^2(1 - M_1)$, the following expansion can be computed by converting the fraction under the integral into a Taylor series

$$\begin{aligned}
&\frac{1}{2}\int_{\varphi_-(\varepsilon)}^{\varphi_+(\varepsilon)} \frac{(1 - x_1\cos\varphi - y_1\sin\varphi)^3}{m^2 + \delta(1 - x_1\cos\varphi - y_1\sin\varphi)^2}\,\mathrm{d}\varphi \\
&\quad = \frac{1}{2}\int_{\varphi_-(\varepsilon)}^{\varphi_+(\varepsilon)} \frac{(1 - x_1\cos\varphi - y_1\sin\varphi)^3}{M_1^2 + \delta(1 - x_1\cos\varphi - y_1\sin\varphi)^2}\,\mathrm{d}\varphi + O(\varepsilon^2) \qquad (6)\\
&\quad = \frac{\sqrt{2}(1 - x_0)^{1/2}(1 - x_1)^3}{M_1^2 + \delta(1 - x_1)^2} + O(\varepsilon^2) = \frac{M_1\sqrt{2}(1 - x_0)^{7/2}}{1 + \delta(1 - x_0)^2}\varepsilon + O(\varepsilon^2).
\end{aligned}$$

In the last step, we used $1 - x_1 = M_1(1 - x_0)$. Moreover, both integrals are odd analytic function of ε, since $\varphi_+(-\varepsilon) = \varphi_-(\varepsilon)$ and the integrand in (6) is an even function w.r.t. ε. Hence, the remainder terms in (5) and (6) are actually $O(\varepsilon^3)$.

Second, we consider the triangles. Using again $\varphi_+(-\varepsilon) = \varphi_-(\varepsilon)$, we have $\nabla u_+(-\varepsilon) = \nabla u_-(\varepsilon)$ and $S_+(-\varepsilon) = -S_-(\varepsilon)$. Hence, $\frac{S_+}{|\nabla u_+|^2+\delta} + \frac{S_-}{|\nabla u_-|^2+\delta}$ is an odd function of ε. To expand $\nabla u_\pm = 1/(1 - x_0 \cos\varphi_\pm - y_0 \sin\varphi_\pm)$, we use

$$\begin{aligned} 1 - x_0 \cos\varphi_\pm - y_0 \sin\varphi_\pm \\ &= \frac{1 - x_0 \mp y_0\varepsilon\sqrt{2(1-x_0) - (1 - x_0^2 - y_0^2)\varepsilon^2} + (x_0^2 + y_0^2 - x_0)\varepsilon^2}{1 - 2\varepsilon^2 x_0 + \varepsilon^4(x_0^2 + y_0^2)} \\ &= 1 - x_0 \mp y_0\sqrt{2(1-x_0)} + O(\varepsilon^2) \end{aligned} \tag{7}$$

and

$$S_\pm = \frac{1}{2}(1 - M_1)(1 - x_0)\sqrt{2(1-x_0)}\,\varepsilon + O(\varepsilon^2). \tag{8}$$

Thus,

$$\frac{S_\pm}{|\nabla u_\pm|^2 + \delta} = \frac{(1 - M_1)(1 - x_0)^{7/2}}{\sqrt{2}(\delta(1 - x_0)^2 + 1)}\varepsilon + O(\varepsilon^2),$$

and

$$\frac{S_+}{|\nabla u_+|^2 + \delta} + \frac{S_-}{|\nabla u_-|^2 + \delta} = \frac{\sqrt{2}(1 - M_1)(1 - x_0)^{7/2}}{\delta(1 - x_0)^2 + 1}\varepsilon + O(\varepsilon^3). \tag{9}$$

By combining (5), (6) and (9), we have

$$\mathfrak{R}(\varepsilon) - \mathfrak{R}(0) = O(\varepsilon^3).$$

Hence, a first-order Taylor expansion of $\mathfrak{R}$ does not yield enough information and we have to use a higher order Taylor expansion. As we mentioned, $\mathfrak{R}(\varepsilon)$ is odd analytic in ε. Thus, also the second-order term vanishes and the third-order term can be computed in a similar way by expanding (5)–(9) up to the ε^3 terms. We arrive at

$$\begin{aligned} \frac{\mathfrak{R}(\varepsilon) - \mathfrak{R}(0)}{(1 - x_0)^{5/2}} \\ &= \frac{4(1 - M_1)\sqrt{2}}{3(1 + \delta(1 - x_0)^2)^3}\left[3y_0^2 - (1 - x_0)^2 - \delta(1 - x_0)^2((1 - x_0)^2 + y_0^2)\right]\varepsilon^3 \\ &\quad + O(\varepsilon^5). \end{aligned}$$

Thereby, since $\varepsilon > 0$, we obtain that the sign of the variation of the resistance coincides with the sign of the expression

$$3y_0^2 - (1 - x_0)^2 - \delta(1 - x_0)^2((1 - x_0)^2 + y_0^2),$$

in case that this expression is not zero. It is interesting to note that the parameter M_1 does not appear. Recall that we were assuming $x_2 = 1$ and $y_2 = 0$. For an arbitrary case, we must rotate the body in such a way that the point (x_2, y_2) will coincide with $(1, 0)$.

Theorem 4. *Let $u \in C_1$ and $\delta \geq 0$. Suppose that u contains a conical part made up by the convex hull of a vertex $(x_0, y_0, -z_0)$ ($z_0 > 0$ and $(x_0, y_0) = (r_0 \cos\varphi_0, r_0 \sin\varphi_0)$ with $0 < r_0 < 1$) and an arc $(\cos\varphi, \sin\varphi, 0) \in \partial\Omega$ for $\varphi \in [\alpha, \beta]$ with $\alpha < \beta$. If there exists $\varphi \in [\alpha, \beta]$ such that*

$$\frac{3r_0^2 \sin^2(\varphi - \varphi_0) - [1 - r_0 \cos(\varphi - \varphi_0)]^2}{[1 - r_0 \cos(\varphi - \varphi_0)]^2 [1 + r_0^2 - 2r_0 \cos(\varphi - \varphi_0)]} < \delta z_0^{-2}. \tag{10}$$

Then u is not optimal for J_M with $M = \delta^{-1/2}$ for $\delta > 0$ and $M = \infty$ for $\delta = 0$

Proof. The left-hand side of (10) is continuous w.r.t. φ. Thus, w.l.o.g., we suppose $\varphi \in (\alpha, \beta)$. In order to apply the above arguments, we rotate and rescale the function u via $\tilde{u}(x, y) = u(x \cos\varphi + y \sin\varphi, -x \sin\varphi + y \cos\varphi)/z_0$. Then, $J_M(u) = z_0^{-2} J_{z_0 M}(\tilde{u})$. The function $\tilde{u}$ contains a conical part made up by the apex $(X_0, Y_0, -1)$ with

$$\begin{aligned} X_0 &= \quad x_0 \cos\varphi + y_0 \sin\varphi = r_0 \cos(\varphi - \varphi_0); \\ Y_0 &= -x_0 \sin\varphi + y_0 \cos\varphi = r_0 \sin(\varphi - \varphi_0); \end{aligned}$$

and an arc on the unit circle $\partial\Omega \times \{0\}$ containing point $(1, 0, 0)$ in its interior. Hence, applying the variation described in the beginning of the present section to $\tilde{u}$, we obtain that the change of the cost functional $J_{z_0 M}(\tilde{u})$ has the same sign as

$$3Y_0^2 - (1 - X_0)^2 - \tilde{\delta}(1 - X_0)^2((1 - X_0)^2 + Y_0^2)$$

where $\tilde{\delta} = (M z_0)^{-2} = \delta\, z_0^{-2}$. Due to (10), the variation has a negative sign. Hence, $\tilde{u}$ is not optimal for $J_{z_0\, M}$ and, consequently, u is not optimal for J_M. □

We denote the left-hand side of inequality (10) by $Z_0(r_0, \Delta\varphi)$, where $\Delta\varphi = \varphi - \varphi_0$. In case $\delta = 0$, non-optimality occurs if $Z_0(r_0, \Delta\varphi) < 0$. In case $\delta > 0$, the condition (10) is equivalent to

$$M z_0 < \begin{cases} \infty & \text{if } Z_0(r_0, \Delta\varphi) \leq 0 \\ Z_0(r_0, \Delta\varphi)^{-1/2} & \text{if } Z_0(r_0, \Delta\varphi) > 0. \end{cases}$$

In Fig. 2, we plotted some level sets of $Z_0(r_0, \Delta\varphi)^{-1/2}$ and the level set $Z_0(r_0, \Delta\varphi) = 0$ (labeled with ∞) in the polar coordinates $(r_0, \Delta\varphi)$.

Theorem 4 applies to the rescaled version of Newton's problem. For later reference, we also give a formulation which can be directly applied to the original problem (2).

Corollary 5. *Let $M \geq 0$ and $u \in C_M$ be given. Suppose that u contains a conical part made up by the convex hull of a vertex $(x_0, y_0, -z_0)$ ($z_0 > 0$ and $(x_0, y_0) = (r_0 \cos\varphi_0, r_0 \sin\varphi_0)$ with $0 < r_0 < 1$) and a boundary arc $(\cos\varphi, \sin\varphi, 0) \in \partial\Omega$ for $\varphi \in [\alpha, \beta]$ with $\alpha < \beta$. If there exists $\varphi \in [\alpha, \beta]$ such that*

$$\frac{3r_0^2 \sin^2(\varphi - \varphi_0) - [1 - r_0 \cos(\varphi - \varphi_0)]^2}{[1 - r_0 \cos(\varphi - \varphi_0)]^2 [1 + r_0^2 - 2r_0 \cos(\varphi - \varphi_0)]} < z_0^{-2}. \tag{11}$$

Then u is not optimal for J.

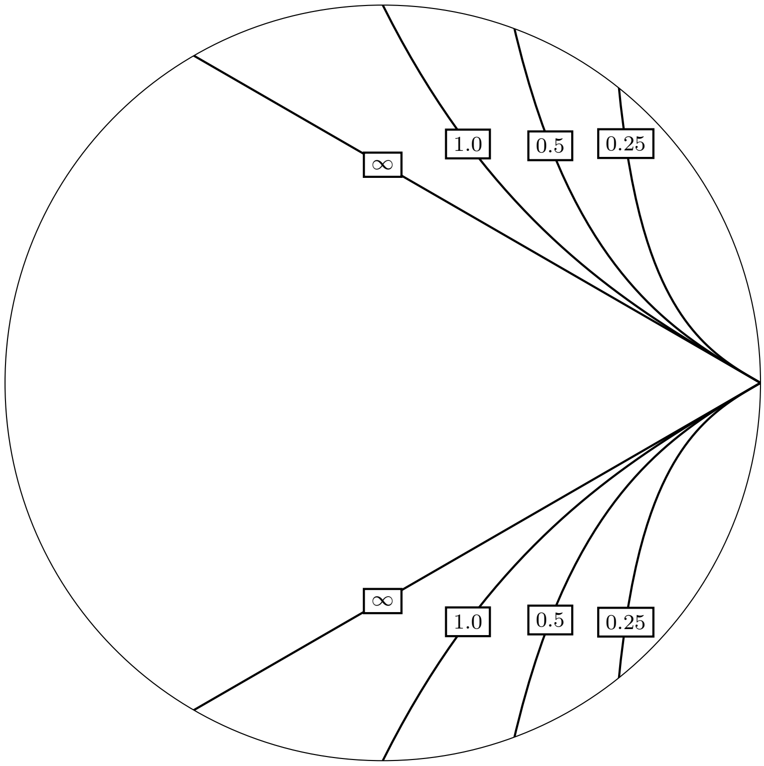

Fig. 2 This shows some level sets of the function $Z_0(r_0, \Delta\varphi)^{-1/2}$, see (10), in the polar coordinate system $(r_0, \Delta\varphi)$.

5 Non-optimality in the class of all convex function of suggested solutions in the literature

In this section, we apply Corollary 5 to some conjectured solutions. In particular, we address the contributions [14, 11, 13].

5.1 Conjectured solutions by Lachand-Robert and Peletier (2001)

We proceed in chronological order and start with the bodies given in [14]. Therein, the authors studied Newton's problem in a restricted class of functions and obtained bodies which are the convex hull of $\Omega \times \{0\} \cup N_0 \times \{-M\}$, where $N_0 \subset \mathbb{R}^2$ is a regular polygon centered at 0. We note that the (global) non-optimality of these bodies was already observed in [10, 11] via the comparison with the numerical solutions. We will check that the (local) non-optimality also follows from Corollary 5. Let us assume that N_0 is a regular polygon with $k \geq 2$ vertices and we rotate N_0 such that one vertex is given by $(x_0, 0)$ for some $x_0 \in (0, 1)$. Then, it is clear that the body contains a conical part with

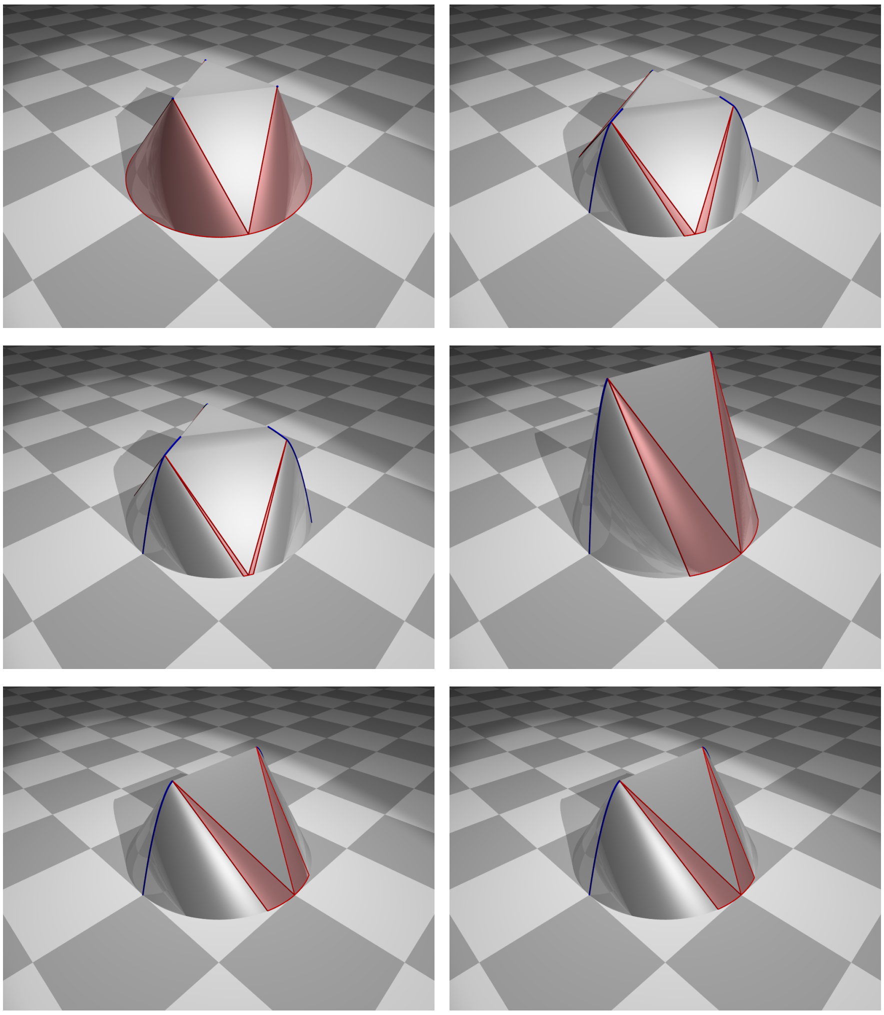

Fig. 3 Illustration of conjectured solutions (shown upside down) containing non-optimal conical parts (highlighted red): [14] with $M = 1.0$ (top left), [11] with $(M, k) = (0.9, 3)$ (top right) and with $(M, k) = (1.0, 3)$ (middle left), [13] with $M = 1.5$ (middle right), $M = 5.0$ (bottom left), $M = \infty$ (bottom right). The bodies in the bottom row are rescaled to height 1.0, cf. (3). All the bodies are constructed as the convex hull of the blue points and the base $\Omega \times \{0\}$.

parameters

$$(x_0, y_0, z_0) = (x_0, 0, -M) = (r_0 \cos\varphi_0, r_0 \sin\varphi_0, -M) \qquad r_0 = x_0,$$
$$\varphi_0 = 0, \qquad \beta = -\alpha = \pi/k.$$

The body with $M = 1.0$ is shown in Fig. 3 (top left). Now, it is easy to check that the left-hand side of inequality (11) is negative for $\varphi = 0 \in (\alpha, \beta)$ and, therefore, (11) holds true. Hence, the bodies cannot be optimal for any value of $M > 0$ and $k \geq 2$.

5.2 Conjectured solutions by Wachsmuth (2014)

Next, we investigate the structural conjecture from [11, Section 3] where the author has supposed that optimal bodies for a height $M \in (0, \bar{M})$, with $\bar{M} \in (1.4, 1.5)$, have the following structure. There exists $k \in \mathbb{N}$, $k \geq 3$, and a convex function $g\colon [0,1] \to [-M, 0]$, $g(0) = -M$, $g(1) = 0$, such that the optimal body is the convex hull of the set

$$\partial\Omega \times \{0\} \cup \{(r\,\cos(2\,i\,\varphi), r\,\sin(2\,i\,\varphi), g(r)),\ i = 0, \ldots, k-1,\ r \in [0,1]\},$$

where $\varphi = \pi/k$. Examples with $M = 1.0$ and $M = 0.9$ are depicted in Fig. 3 (top right and middle left). The extremal lines

$$\{(r\,\cos(2\,i\,\varphi), r\,\sin(2\,i\,\varphi), g(r)),\ i = 0, \ldots, k-1,\ r \in [0,1]\}$$

are highlighted in blue. Under some natural assumptions on g, the problem becomes a one-dimensional problem of calculus of variations and can be solved for g by the corresponding Euler-Lagrange equations. The obtained class of solutions contains a conical part (denoted by Region II therein), see [11, Figure 6] and Fig. 3 (middle left). However, for the results presented in [11, Table 3] (reproduced and extended in Table 1), only the solution corresponding to $M = 1.0$ (the height parameter is denoted by L in [11]) and symmetry parameter $k = 3$ (denoted by m in [11]) contains this conical part and for all other presented solutions, this conical part vanishes. Moreover, it can be checked that Corollary 5 applies to this solution with $M = 1.0$ and $k = 3$. Hence, the structural conjecture of [11] cannot be true for this height $M = 1.0$.

The (non-optimal) body from [11, Section 3] with $(M, k) = (1.0, 3)$ is displayed in Fig. 3 (middle left). Note that the conical parts are rather small. We believe that the non-optimality of this body is (informally speaking) only due to these small conical parts. Therefore, we expect that the objective value can be improved only by a small amount and this seems to be hard to achieve via numerical methods. We also show the body corresponding to $(M, k) = (0.9, 3)$ (top right) which has a larger conical part.

In Table 1, we present an updated[2] and completed version of [11, Table 3]. In this table, we underlined the best solution in each row, i.e. for each height parameter M. Note that for $M = 1.5$ a better solution was obtained numerically in [11, Section 2] whereas for $M \leq 0.3$ a structured solution with $k = 9$ produces better values than the solutions given in the table. Hence, we do not underline solutions in the lines corresponding to $M = 1.5$ and $M \leq 0.3$.

For each solution presented in Table 1, we checked whether this solution contains a conical part (indicated by "C") and whether Corollary 5 applies to

[2] There is one significant difference to [11, Table 3]: the given objective value corresponding to $M = 1.0$ and $k = 3$ was suboptimal and has been corrected in Table 1.

$M \setminus k$	3	4	5	6	7	8
1.5	0.6999489	0.7123013	0.7218282	0.7285828	0.7334158	0.7369567
1.4	0.7677364	0.7792767	0.7887148	0.7955246	0.8004380	0.8040548
1.3	0.8441426	0.8544163	0.8635882	0.8703656	0.8753090	0.8789708
1.2	0.9303614	0.9387924	0.9474690	0.9540986	0.9590059	0.9626708
1.1	1.0277294	1.0335975	1.0414890	1.0478237	1.0526095	1.0562233
1.0	1.1377294 CN	1.1401510	1.1468980	1.1527533	1.1573095	1.1608036
0.9	1.2619895 CN	1.2599052	1.2650665	1.2702151	1.2744081	1.2776964
0.8	1.4022724 CN	1.3944389	1.3974884	1.4016551	1.4053219	1.4082995
0.7	1.5604532 CN	1.5454605	1.5457807	1.5486392	1.5515845	1.5541263
0.6	1.7392117 CN	1.7147979	1.7116824	1.7128535	1.7148496	1.7168060
0.5	1.9438498 CN	1.9043842	1.8970564	1.8961062	1.8968909	1.8980899
0.4	2.1775037 CN	2.1176695 C	2.1038722	2.1003292	2.0996063	2.0998537
0.3	2.4390875 CN	2.3611648 CN	2.3354683 C	2.3276201 C	2.3250270	2.3241056
0.2	2.7171218 CN	2.6385156 CN	2.5993030 C	2.5828003 C	2.5760193 C	2.5731257 C
0.1	2.9786691 CN	2.9338815 CN	2.8990387 CN	2.8754404 CN	2.8612811 C	2.8533036 C

Table 1 Conjectured optimal values using the conjecture from [11, Section 3]. A "C" indicates that this solution contains a conical part. For the solutions marked by "N", Corollary 5 applies and, therefore, those solutions cannot be optimal in the class of all convex bodies C_M. The minimal (conjectured) solutions for each M are underlined.

$M \setminus k$	3	4
1.10	1.0277294	1.0335975
1.09	1.0381352 C	1.0437014
1.08	1.0486688 CN	1.0539240
1.07	1.0593317 CN	1.0642667
1.06	1.0701256 CN	1.0747311
1.05	1.0810520 CN	1.0853184
1.04	1.0921125 CN	1.0960303
1.03	1.1033089 CN	1.1068681
1.02	1.1146427 CN	1.1178333
1.01	1.1261157 CN	1.1289274
1.00	1.1377294 CN	1.1401510

$M \setminus k$	3	4
1.00	1.1377294 CN	1.1401510
0.99	1.1494856 CN	1.1515072
0.98	1.1613861 CN	1.1629969
0.97	1.1734324 CN	1.1746215
0.96	1.1856264 CN	1.1863837
0.95	1.1979697 CN	1.1982830
0.94	1.2104642 CN	1.2103218
0.93	1.2231117 CN	1.2225019
0.92	1.2359138 CN	1.2348243
0.91	1.2488725 CN	1.2472919
0.90	1.2619895 CN	1.2599052

Table 2 Similar as Table 1, but for different values of M and k.

this conical part and provides the non-optimality (indicated by "N"). For each fixed k it seems that conical parts appear for small values of M (depending on k) and that, eventually, this conical part becomes non-optimal. However, for $k \geq 4$ the non-optimality appears only for "very small" values of M and for these values, $k + 1$ provides a better solution. Hence, for $M \leq 0.9$ (and, therefore, $k \geq 4$) we cannot apply Corollary 5 and we cannot disprove the conjecture of [11, Section 3]. For M between 1.0 and 1.4 the situation is different. Here, the best results (according to the structural conjecture of [11, Section 3]) are obtained by $k = 3$ and these contain non-optimal conical parts for heights M that are smaller than approximately 1.0. In particular, we can apply Corollary 5 for the height $M = 1.0$ and therefore, the conjecture of [11, Section 3] is disproved for this value. For M bigger than 1.1, the solutions with $k = 3$ do not contain conical parts and therefore, we cannot disprove the conjecture for M between 1.1 and 1.4.

In Table 2, we list some more values for $M \in [0.9, 1.1]$ and $k \in \{3, 4\}$. This table suggests the following observations:

– For $M \geq 1.09$, the bodies conjectured in [11, Section 3] might be optimal since these bodies do not contain conical parts or their conical parts do not satisfy Corollary 5.
– For $M \in [0.95, 1.08]$, the conjectured optimal bodies contain a non-optimal conical part and, therefore, the structural conjecture of [11, Section 3] is disproved for these values of M.
– For $M \in [0.90, 0.94]$, our non-optimality result does not apply to the conjectured bodies with $k = 4$ and these bodies possess better values than those with $k = 3$. However, the bodies with symmetry parameter $k = 3$ are not locally optimal in C_M by Corollary 5. It is also clear that our variation from Section 4 can be modified to produce bodies with a threefold symmetry which possess smaller objective values than those indicated in Table 2 for $k = 3$. In particular, these improved values could be smaller than the corresponding values with $k = 4$ from Table 2 and this would disprove the structural conjecture of [11] for some values of M around 0.94. This is subject to future research.

To summarize, Corollary 5 can be used to disprove the conjectured bodies from [11, Section 3] at least for $M \in [0.95, 1.08]$, see Table 2. It does not apply for $M \leq 0.94$ and $M \in [1.09, 1.4]$, and for these values, the conjecture might be true.

5.3 Conjectured solutions by Lokutsievskiy and Zelikin (2020)

In [13], the authors study the class E_M of convex bodies of height M which can be written as the convex hull of the union of the base $\Omega \times \{0\}$ and of a convex curve $z = v^*(x_1)$ in the plane $\{x_2 = 0\}$ (we keep notations of [13], and v^* denotes the Legendre-Young-Fenchel transform of a convex function v). We note that this approach is similar to Section 5.2 with $k = 2$. The authors proved local optimality of such bodies in the corresponding class (see [13, Theorem 9.1]).

In this paper, there is a table with numerically found parameters of the locally optimal curve v^* for some different values of the height M (see [13, Table 1]). The solution v^* has a horizontal line segment in the front of the body, since v has a corner at 0. It can be checked that all the bodies from the table contain a conical part. Exemplarily, we have shown the bodies corresponding to $M = 1.5$ in Fig. 3 (middle right) and to $M = 5.0$ (rescaled to height 1.0, bottom left). The conical part is given by the vertex $(x_0, y_0, z_0) = (v'(+0), 0, -M)$ and the arc with angles $[\alpha, \beta] = [\pi/2 - \varepsilon, \pi/2]$ for some[3] $\varepsilon > 0$. Now, it is easily checked that inequality (11) from Corollary 5 is fulfilled for $r_0 = v'(+0)$, $\varphi_0 = 0$, $\varphi = \beta = \pi/2$ and $z_0 = M$. Therefore, this conical part is always non-optimal in the class C_M of all convex bodies. Hence, it seems that the optimal bodies in the class E_M are never optimal in C_M.

[3] Using the notation from [13], $\varepsilon = \arcsin[r(p_0)/(M + v'(0)r(p_0))]$, e.g. $\varepsilon \approx 0.574610$ for $M = 1.5$ and $\varepsilon \approx 0.330507$ for $M = 5.0$.

A similar approach can be used for the limiting problem of minimizing J_∞ in the class E_1. Using the same strategy, one obtains the values (with the notation of [13])

$$p_0 \approx 3.167203701258, \qquad r(p_0) \approx 0.3451623687826,$$
$$v'(+0) \approx 0.5300674211893, \qquad J_\infty(u) \approx 2.140225047120.$$

The corresponding body is shown in Fig. 3 (bottom right). Again, this body has a conical part (with $\varepsilon \approx 0.296085$), which is non-optimal by Corollary 5. Thus, the minimizers of J_∞ in C_1 do not belong to E_1. Hence, the family found in [13] cannot be asymptotically optimal for J in C_M by Proposition 3 despite the fact that it is locally optimal in E_M under the corresponding variations by [13, Theorem 9.1].

Acknowledgement

Gerd Wachsmuth acknowledges fruitful discussions with Luca Landwehrjohann back in 2017 which spawned the seeds for some of the ideas in the present paper.

The work of Lev Lokutsievskiy was performed at the Steklov International Mathematical Center and supported by the Ministry of Science and Higher Education of the Russian Federation (agreement no. 075-15-2019-1614). The work of Gerd Wachsmuth was partially supported by the DFG Grant *Approximation of Non-Smooth Optimal Convex Shapes with Applications in Optimal Insulation and Minimal Resistance* (Grant No. WA 3636/5-2) within the Priority Program SPP 1962 (Non-smooth and Complementarity-based Distributed Parameter Systems: Simulation and Hierarchical Optimization). The work of Mikhail Zelikin was supported by Russian Foundation for Basic Research under grant 20-01-00469.

References

[1] Giuseppe Buttazzo, Vincenzo Ferone, and Bernhard Kawohl. "Minimum Problems over Sets of Concave Functions and Related Questions". In: *Mathematische Nachrichten* 173.1 (1995), pp. 71–89. ISSN: 1522-2616. DOI: 10.1002/mana.19951730106.

[2] Giuseppe Buttazzo. "A survey on the Newton problem of optimal profiles". In: *Variational Analysis and Aerospace Engineering*. Springer New York, 2009, pp. 33–48. DOI: 10.1007/978-0-387-95857-6_3.

[3] Wallace D. Hayes and Ronald F. Probstein. *Hypersonic Flow Theory*. 3rd ed. Academic Press, 1964.

[4] Черный Г.Г. *Газовая динамика*. Москва: НАУКА, 1988. ISBN: 5-02-013814-2.

[5] Isaac Newton. *Philosophiæ Naturalis Principia Mathematica*. 1687.

[6] Paolo Guasoni. «Problemi di ottimizzazione di forma su classi di insiemi convessi». Italiano. Tesi di Laurea. Universita di Pisa, 1996. URL: http://cvgmt.sns.it/paper/1146/.

[7] Friedemann Brock, Vincenzo Ferone, and Bernhard Kawohl. "A symmetry problem in the calculus of variations". In: *Calculus of Variations and Partial Differential Equations* 4.6 (1996), pp. 593–599. DOI: 10.1007/bf01261764.

[8] Alexander Plakhov. *A note on Newton's problem of minimal resistance for convex bodies*. 2019. arXiv: 1908.01042.

[9] Thomas Lachand-Robert and M. A. Peletier. "An example of non-convex minimization and an application to Newton's problem of the body of least resistance". en. In: *Annales de l'I.H.P. Analyse non linéaire* 18.2 (2001), pp. 179–198. URL: http://www.numdam.org/item/AIHPC_2001__18_2_179_0.

[10] Thomas Lachand-Robert and Édouard Oudet. "Minimizing within Convex Bodies Using a Convex Hull Method". In: *SIAM Journal on Optimization,* 16.2 (2005), pp. 368–379. DOI: 10.1137/040608039.

[11] Gerd Wachsmuth. "The numerical solution of Newton's problem of least resistance". In: *Mathematical Programming* 147.1 (2014), pp. 331–350. DOI: 10.1007/s10107-014-0756-2.

[12] Lev V. Lokutsievskiy and Mikhail I. Zelikin. "Hessian Measures in the Aerodynamic Newton Problem". In: *Dyn Control Syst* 24 (2018), pp. 475–495. DOI: 10.1007/s10883-018-9395-x.

[13] Lev V. Lokutsievskiy and Mikhail I. Zelikin. "The analytical solution of Newton's aerodynamic problem in the class of bodies with vertical plane of symmetry and developable side boundary". In: *ESAIM: COCV* 26 (2020). DOI: 10.1051/cocv/2019064.

[14] Thomas Lachand-Robert and Mark A. Peletier. "Newton's Problem of the Body of Minimal Resistance in the Class of Convex Developable Functions". In: *Mathematische Nachrichten* 226.1 (2001), pp. 153–176. DOI: 10.1002/1522-2616(200106)226:1<153::aid-mana153>3.0.co;2-2.