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Anne-Therese Rauls, Stefan Ulbrich

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SPP1962 at Weierstrass Institute for Applied Analysis and Stochastics (WIAS)
Leibniz Institute in the Forschungsverbund Berlin e.V.
Mohrenstraße 39, 10117 Berlin, Germany
E-Mail: spp1962@wias-berlin.de

World Wide Web: http://spp1962.wias-berlin.de/

On the Characterization of Generalized Derivatives for the Solution Operator of the Bilateral Obstacle Problem

Anne-Therese Rauls [*], Stefan Ulbrich [†]

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Abstract

We consider optimal control problems for a wide class of bilateral obstacle problems where the control appears in a possibly nonlinear source term. The non-differentiability of the solution operator poses the main challenge for the application of efficient optimization methods and the characterization of Bouligand generalized derivatives of the solution operator is essential for their theoretical foundation and numerical realization. In this paper, we derive specific elements of the Bouligand generalized differential if the control operator satisfies natural monotonicity properties. We construct monotone sequences of controls where the solution operator is Gâteaux differentiable and characterize the corresponding limit element of the Bouligand generalized differential as being the solution operator of a Dirichlet problem on a quasi-open domain. In contrast to a similar recent result for the unilateral obstacle problem [RU19], we have to deal with an opposite monotonic behavior of the active and strictly active sets corresponding to the upper and lower obstacle. Moreover, the residual is no longer a nonnegative functional on H^{-1} and its representation as the difference of two nonnegative Radon measures requires special care. This necessitates new proof techniques that yield two elements of the Bouligand generalized differential. Also for the unilateral case we obtain an additional element to that derived in [RU19].

Key words— bilateral obstacle problem, variational inequalities, optimal control, nonsmooth optimization, generalized derivatives, Bouligand generalized differential

AMS subject classifications— 47J20, 49J40, 49K40, 49J52, 58C20, 58E35

1 Introduction

We consider the optimal control of bilateral obstacle problems

$$\text{Find } y \in K_\psi^\varphi : \quad \langle Ly - f(u), z - y\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \geq 0 \quad \forall\, z \in K_\psi^\varphi, \tag{1}$$

[*]Departments of Mathematics, TU Darmstadt, Dolivostr. 15, 64293 Darmstadt, Germany, anne-therese.rauls@tu-darmstadt.de

[†]Departments of Mathematics, TU Darmstadt, Dolivostr. 15, 64293 Darmstadt, Germany, stefan.ulbrich@tu-darmstadt.de

with

$$K_\psi^\varphi := \{z \in H_0^1(\Omega) : \psi \le z \le \varphi\}.$$

Here, $\Omega \subseteq \mathbb{R}^d$ is an open, bounded set, $L \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$ is a coercive and strictly T-monotone operator. Here, strict T-monotonicity means that

$$\langle L(y-z), (y-z)_+\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} > 0$$

for all $y, z \in H_0^1(\Omega)$ with $(y-z)_+ = \sup(0, y-z) \neq 0$, see [Rod87, p. 105]. Furthermore, $f\colon U \to H^{-1}(\Omega)$ is a continuously differentiable and monotone operator on a partially ordered Banach space U. The detailed assumptions on f and U will be given below in Assumption 1.2. We assume that the obstacles $\psi, \varphi \in H^1(\Omega)$ are such that K_ψ^φ is nonempty. In addition, for some results in this paper we require the following assumption.

Assumption 1.1
We consider lower and upper obstacles $\psi, \varphi \in H^1(\Omega) \cap L^\infty(\Omega)$ and assume there is $\delta > 0$ such that $\varphi - \psi \ge \delta$ holds a.e. in Ω.

It is well known, see for example [Bar84,KS00], that for each $u \in U$, the variational inequality (1) has a unique solution and the solution operator $S = S_\psi^\varphi\colon U \to H_0^1(\Omega)$ is locally Lipschitz continuous.

The optimal control of obstacle problems and elliptic variational inequalities has been studied by many authors, see for example [Bar84,Ber97,BL04,Fri88,IK00,HW18, HK11,KKT03,KW12,MRW15,Mig76,MP84,RU19,RW19,SW13]. By using penalization, relaxation or regularization approaches, optimality conditions have been derived in [Bar84,Ber97,BL04,MP84,IK00,HK11], where [BL04] considers the obstacles for the bilateral case as controls. Numerical solution methods based on these techniques have been developed in [IK00,HK11,KKT03,KW12,MRW15,SW13]. Other approaches consider directly the nonsmooth solution operator. Different optimality systems for the optimal control of the obstacle problem are compared in [HW18]. For the application of nonsmooth optimization methods like bundle methods, the knowledge of at least one element in the generalized differential of the objective function is required. For the finite dimensional obstacle problems, a characterization of the whole Clarke subdifferential of the reduced objective function was obtained in [HR86]. The directional differentiability of solution operators of elliptic variational inequalities and a variational inequality for the directional derivative have been obtained in [Har77, Mig76], see also [CW19]. The resulting structure of Gâteaux derivatives in points of differentiability was used in [RW19] to characterize the full Bouligand generalized differential for the solution operator of the unilateral obstacle problem with distributed control $f(u) = u \in H^{-1}(\Omega)$. Subsequently, for Lipschitz continuous, continuously differentiable, and monotone control operators $f : U \to H^{-1}(\Omega)$ on a partially ordered Banach space U, an element of the Bouligand generalized differential for the solution operator of the unilateral obstacle problem has been characterized in [RU19] as the solution operator of a variational equation on the inactive set. This forms an analytical foundation to develop error estimators for the numerical computation of subgradients and to apply inexact bundle methods in Hilbert space, see for example [HU19].

In this paper, we derive, based on a characterization of the Gâteaux derivative in points of differentiability, two elements of the Bouligand generalized differential for the solution operator $S\colon U \to H_0^1(\Omega)$ of the bilateral obstacle problem (1) in points of nonsmoothness. To this end, we require the following monotonicity property of the control operator $f\colon U \to H^{-1}(\Omega)$.

Assumption 1.2
We assume that the operator $f\colon U \to H^{-1}(\Omega)$ entering the variational inequality (1) lives on a Banach space U fulfilling the following properties. Let $(V, \geq_V)$ be a partially

ordered Banach space such that the positive cone $\mathcal{P} = \{v \in V : v \geq_V 0\}$ has nonempty interior. Suppose $(U, \geq_U)$ is a separable partially ordered Banach space such that V is embedded into U. We assume that the embedding $\iota\colon V \to U$ is continuous, dense and compatible with the order structures in V and U; i.e., for $v \in V$ with $v \geq_V 0$ we have $\iota(v) \geq_U 0$ in U.

The operator $f\colon U \to H^{-1}(\Omega)$ is assumed to be continuously differentiable and monotone in the sense that $u \geq_U v$ implies $f(u) \geq f(v)$ in $H^{-1}(\Omega)$.

With the help of a generalization of Rademacher's theorem to infinite dimension, see Theorem 6.1, the assumption allows us to construct for each $u \in U$ a monotone increasing (or monotone decreasing) sequence $(u_n)_{n\in\mathbb{N}} \subset U$ converging to u, such that the locally Lipschitz continuous solution operator S is Gâteaux differentiable in u_n. In particular, the assumptions on U are fulfilled for $U = L^2(\Omega)$, $U = H^{-1}(\Omega)$ and $U = \mathbb{R}^k$.

Our strategy for computing an element of the Bouligand generalized differential is as follows. Let $u \in U$ and denote by $I(u)$ and $A(u) := A_\psi(u) \cup A^\varphi(u)$ the inactive and active set of the solution $S(u)$, respectively. By representing the residual $LS(u) - f(u) \in H^{-1}(\Omega)$ as the difference $\xi_\psi - \xi^\varphi$ of two nonnegative Radon measures (Theorem 3.3), we can define the strictly active set $A_s(u) := A^s_\psi(u) \cup A^\varphi_s(u)$ using the measures ξ_ψ and ξ^φ.

If the solution operator S is Gâteaux differentiable in u, then $S'(u)$ can be obtained as the solution operator of a variational equation on $H^1_0(D(u))$, where $D(u)$ can be chosen as any quasi-open subset of Ω satisfying $I(u) \subseteq D(u) \subseteq \Omega \setminus A_s(u)$, see Theorem 3.8.

If the solution operator S is nonsmooth in u, then we can construct a monotone increasing sequence $(u_n)_{n\in\mathbb{N}} \subset U$, where S is Gâteaux differentiable, converging to u and $S'(u_n)$ can thus be represented as the solution operator of a variational equation on $H^1_0(D(u_n))$ with $D(u_n) = I(u_n) \cup (A^\varphi(u_n) \setminus A^\varphi_s(u_n))$. By using monotonicity properties of the sequence of sets $(A^\varphi_s(u_n))_{n\in\mathbb{N}}$ and $(A_\psi(u_n))_{n\in\mathbb{N}}$ we show that $(H^1_0(D(u_n)))_{n\in\mathbb{N}}$ converges in the sense of Mosco to $H^1_0(D(u))$, cf. Lemma 5.3, and stability properties of variational equations yield that $(S'(u_n))_{n\in\mathbb{N}}$ converges in the strong operator topology to an element in the Bouligand generalized differential of S at u, which can be characterized as the solution operator of a variational equation on $H^1_0(D(u))$, see Theorem 6.2. Working with a monotone decreasing sequence $(u_n)_{n\in\mathbb{N}} \subset U$ yields another generalized derivative.

In [RU19], a similar approach has been used for the unilateral obstacle problem. The analysis for the bilateral obstacle problem is more involved and requires new proof techniques for several reasons. First of all, to the best of our knowledge, the decomposition of $LS(u) - f(u) \in H^{-1}(\Omega)$ as the difference $\xi_\psi - \xi^\varphi$ of two nonnegative Radon measures has not been established so far and requires care. This representation is, in general, only valid on $H^1_0(\Omega) \cap L^\infty(\Omega)$, see Theorem 3.3, or, alternatively, if the distance of the active sets $A^\varphi(u)$ and $A_\psi(u)$ vanishes (Lemma 3.4), which we demonstrate by giving a counter example in Example 3.5. Next, while for $(u_n)_{n\in\mathbb{N}}$ increasing the choice $D(u_n) = I(u_n)$ is possible and monotone increasing in the unilateral case, the sets $D(u_n) = I(u_n) \cup (A^\varphi(u_n) \setminus A^\varphi_s(u_n))$ do not enjoy monotonicity properties and require the more difficult study of monotonicty properties of the strictly active sets $A^\varphi_s(u_n)$, see Lemma 4.5. Also in the unilateral case, the analysis in this paper yields an additional element of the Bouligand generalized differential to that derived in [RU19].

If L is induced by a symmetric coercive bilinear form, then it is well known that (1) are first order optimality conditions of the problem

$$\min_{y \in H^1_0(\Omega)} \langle \tfrac{1}{2} Ly - f(u), y \rangle_{H^{-1}(\Omega), H^1_0(\Omega)} \quad \text{subject to } y \in K^\varphi_\psi.$$

Then the residual $LS(u) - f(u) \in H^{-1}(\Omega)$ corresponds to the Lagrange multiplier and our careful study of its representation as the difference $\xi_\psi - \xi^\varphi$ of two nonnegative Radon measures gives detailed insights into the structure of the Lagrange multiplier. This might be helpful also in other contexts, for example the design and analysis of efficient solution methods for (1).

The paper is organized as follows. Section 2 recalls some basic definitions and results on capacity theory, Sobolev spaces on quasi-open sets and generalized derivatives. In Section 3, monotonicity and differentiability properties of the solution operator are analyzed. The well known variational inequality for the directional derivative is stated and the structure of the critical cone is studied. To this end, a representation of the residual $LS(u) - f(u)$ as the difference $\xi_\psi - \xi^\varphi$ of two nonnegative Radon measures is derived and the strongly active sets are defined based on these measures. A counterexample shows that this representation is, in general, only valid on $H_0^1(\Omega) \cap L^\infty(\Omega)$, which requires some care in the sequel. Moreover, a representation of $S'(u)$ is derived if u is a point of Gâteaux differentiability. In Section 4 the monotonicity of the strictly active and active sets $A_s^\varphi(u_n)$ and $A_\psi(u_n)$ is analyzed for monotone increasing control sequences (u_n) (respectively, of $A_\psi^s(u_n)$ and $A^\varphi(u_n)$ for monotone decreasing $(u_n)_{n\in\mathbb{N}}$). This is used in Section 5 to show the Mosco convergence of $(H_0^1(I(u_n) \cup (A^\varphi(u_n) \setminus A_s^\varphi(u_n))))_{n\in\mathbb{N}}$ and $(H_0^1(I(u_n) \cup (A_\psi(u_n) \setminus A_\psi^s(u_n))))_{n\in\mathbb{N}}$, respectively. Section 6 uses now stability properties of variational equalities under Mosco convergence to characterize elements of the Bouligand generalized differential. Finally, an adjoint representation of corresponding Clarke subgradients for an objective functional is derived.

2 Fundamental Definitions and Results

Denote by $C_c(\Omega)$ the space of continuous functions on Ω with compact support contained in Ω and by $C_c^\infty(\Omega)$ the subspace of infinitely differentiable functions. Furthermore, we denote by $H^1(\Omega)$ the space

$$H^1(\Omega) := \left\{ z \in L^2(\Omega) : \tfrac{\partial z}{\partial x_i} \in L^2(\Omega), i = 1, \dots, d \right\},$$

where $\frac{\partial z}{\partial x_i}$ is to be understood in the distributional sense. $H^1(\Omega)$ is equipped with the norm

$$\|z\|_{H^1(\Omega)} = \int_\Omega \left(z^2 + \sum_{i=1}^d \left(\frac{\partial z}{\partial x_i} \right)^2 d\lambda^d \right)^{1/2}.$$

In this paper, we work with the space $H_0^1(\Omega)$ and we define it as the completion of $C_c^\infty(\Omega)$ in $H^1(\Omega)$. On $H_0^1(\Omega)$ we consider the norm $\|z\|_{H_0^1(\Omega)} := \|\nabla z\|_{L^2(\Omega)}$. The dual space of $H_0^1(\Omega)$ is denoted by $H^{-1}(\Omega)$ and if $\mu \in H^{-1}(\Omega)$ and $z \in H_0^1(\Omega)$ we use the notation $\langle \mu, z \rangle$ for the dual pairing.

Note that $z \in H_0^1(\Omega)$ can be extended by zero to an element of $z \in H^1(\mathbb{R}^d)$, since the zero extension of the approximating sequence in $C_c^\infty(\Omega)$ is a Cauchy sequence in $H^1(\mathbb{R}^d)$.

We denote by $H^1(\Omega)_+$, respectively by $H_0^1(\Omega)_+$, the respective subsets of nonnegative elements. For $u \in H_0^1(\Omega)$ and $v \in H^1(\Omega)_+$ we often use that $\min(u, v) \in H_0^1(\Omega)$. We use the notation $u_+ := \max(0, u)$, $u_- = -\min(0, u)$, for the positive and negative part of $u \in L^2(\Omega)$ and have $u = u_+ - u_-$. Furthermore, for $n \in \mathbb{N}$ and $u \in H_0^1(\Omega)$, $u_n := \max(-n, \min(u, n))$ is in $H_0^1(\Omega) \cap L^\infty(\Omega)$ and it holds $u_n \to u$ in $H_0^1(\Omega)$, which can be seen by application of Lebesgue's dominated convergence theorem.

2.1 Capacity Theory

We quickly recall and clarify the definitions and concepts related to capacity theory that we consider in this paper. For the definitions, see also [ABM14, Sect. 5.82, 5.83], [DZ11, Def. 6.2], [KM92].

Definition 2.1. 1. For a set $E \subset \Omega$ we define the capacity of E in Ω by

$$\operatorname{cap}(E) := \inf\{\|z\|^2_{H^1_0(\Omega)} : z \in H^1_0(\Omega), z \geq 1 \text{ a.e. in a neighborhood of } E\}. \quad (2)$$

If a property holds on a set $E \subseteq \Omega$ except on a subset of capacity zero, we say that this property holds quasi-everywhere (q.e.) on E.

2. We call a set $O \subseteq \Omega$ quasi-open if for all $\varepsilon > 0$ there is an open set $O_\varepsilon \subseteq \Omega$ such that $O \cup O_\varepsilon$ is open and $\operatorname{cap}(O_\varepsilon) < \varepsilon$. A set $A \subseteq \Omega$ is quasi-closed if the complement in Ω is quasi-open.
3. Let $v\colon \Omega \to \mathbb{R}$ be a function. Then v is quasi-continuous if for all $\varepsilon > 0$ there exists an open set $O_\varepsilon \subseteq \Omega$ such that $v|_{\Omega\setminus O_\varepsilon}$ is continuous and $\operatorname{cap}(O_\varepsilon) < \varepsilon$.
4. Let $E \subseteq \Omega$ be a set. A family $(O_i)_{i\in I}$ of quasi-open subsets of Ω is called a quasi-covering of E if there is a countable subfamily $(O_{i_n})_{n\in\mathbb{N}}$ satisfying $\operatorname{cap}\left(E \setminus \bigcup_{n\in\mathbb{N}} O_{i_n}\right) = 0$.
5. Let $O \subseteq \Omega$ be quasi-open. Then we define

$$H^1_0(O) := \{z \in H^1_0(\Omega) : z = 0 \text{ q.e. on } \Omega \setminus O\}.$$

Remark 2.2. If $O \subseteq \Omega$ is open, then the definition of $H^1_0(O)$ in Definition 2.1 coincides with the classical definition of $H^1_0(O)$ that we also use in this paper, see e.g. [AH96, Thm. 9.1.3]. Moreover, the definition of $H^1_0(O)$ for $O \subseteq \Omega$ quasi-open coincides with the definition

$$H^1_0(O) = \bigcap\{H^1_0(G) : O \subseteq G \subseteq \Omega, G \text{ open}\}$$

given in [KM92].

We could also define a capacity Cap for subsets of $\mathbb{R}^d$ by testing with $H^1(\mathbb{R}^d)$-elements in (2) and by considering the infimum over the squared $H^1(\mathbb{R}^d)$-norms. Then $H^1_0(U) = \{v \in H^1(\mathbb{R}^d) : v = 0 \text{ Cap-q.e. outside } U\}$ for Cap-quasi-open subsets U of $\mathbb{R}^d$. For $U \subseteq \Omega$, U is Cap-quasi-open if and only if it is quasi-open and the definitions of $H^1_0(U)$ coincide.

Lemma 2.3

1. *Let $v \in H^1(\Omega)$. Then v has a quasi-continuous representative $\tilde{v}$. If $\tilde{w}$ is another quasi-continuous representative of v, then $\tilde{v} = \tilde{w}$ up to a set of capacity zero.*
2. *Suppose $(v_n)_{n\in\mathbb{N}}, v \subseteq H^1_0(\Omega)$ and $v_n \to v$ in $H^1_0(\Omega)$. Then there is a subsequence $(v_{n_k})_{k\in\mathbb{N}}$ such that $\tilde{v}_{n_k} \to \tilde{v}$ pointwise q.e. for the quasi-continuous representatives.*
3. *Let $O \subseteq \Omega$ be a quasi-open set, let $z \in H^1_0(O)$ and assume that $(O_i)_{i\in I}$ is a quasi-covering of O. Then there exists a sequence $(z_n)_{n\in\mathbb{N}} \subseteq H^1_0(O)$ such that $z_n \to z$ and such that each z_n is a finite sum of functions from $\bigcup_{i\in I} H^1_0(O_i)$. If $(O_n)_{n\in\mathbb{N}}$ is a quasi-covering that is increasing in n, then we find a sequence $(z_n)_{n\in\mathbb{N}}$ converging to z such that $z_n \in H^1_0(O_n)$. If $z \in H^1_0(\Omega)_+$, then, w.l.o.g., $(z_n)_{n\in\mathbb{N}} \subset H^1_0(\Omega)_+$.*
4. *Assume $O \subseteq \Omega$ is quasi-open and suppose $v\colon \Omega \to \mathbb{R}$ is quasi-continuous. Then, $v \geq 0$ a.e. on O if and only if $v \geq 0$ q.e. on O.*
5. *Assume $O \subseteq \Omega$ is a quasi-open set. Then there exists $v \in H^1_0(\Omega)_+$ with $\{\tilde{v} > 0\} = O$ up to a set of zero capacity.*

Proof. The first statement can be found, e.g., in [DZ11, Chap. 6, Thm. 6.1] or [HKM93, Thm. 4.4], the second in [BS00, Lem. 6.52]. The first part of the third statement can be obtained by combining [KM92, Lem. 2.4 and Lem. 2.10]. For the second part of the third statement use that $(O_n)_{n\in\mathbb{N}}$ is increasing and use the first part. In case $z \geq 0$, the statement in [KM92, Lem. 2.4] and the proof of [KM92, Lem. 2.10] imply that we can choose $(z_n)_{n\in\mathbb{N}} \subseteq H_0^1(\Omega)_+$. We refer to [Wac14, Lem. 2.3] for the fourth statement and to [Vel15, Prop. 2.3.14], [HW18, Lem. 3.6] for the last statement. □

Remark 2.4. Let $v\colon \Omega \to \mathbb{R}$ be a quasi-continuous function. Then the set $\{v > 0\}$ is quasi-open and the set $\{v \geq 0\}$ is quasi-closed. If $v \in H^1(\Omega)$, then by $\{v > 0\}, \{v \geq 0\}$ we always mean the sets $\{\tilde{v} > 0\}, \{\tilde{v} \geq 0\}$, where $\tilde{v}$ is a quasi-continuous representative. Thus, these sets are quasi-open, respectively quasi-closed, and determined up to a set of capacity zero.

Throughout the rest of the paper, when considering set equations or inclusions for subsets of Ω, they have to be understood to hold up to an exceptional set of capacity zero.

2.2 Generalized Derivatives

We consider the following generalized differential for the solution operator of (1).

Definition 2.5. Consider a separable Banach space X and a Hilbert space Y. Assume that $T\colon X \to Y$ is a locally Lipschitz continuous operator. The set of Bouligand generalized derivatives in $x \in X$ is defined as

$$\partial T(x) := \{\Xi \in \mathcal{L}(X,Y) : \exists (x_n)_{n\in\mathbb{N}} \subset D_T \text{ with } x_n \to x \tag{3}$$

$$\text{and } T'(x_n) \to \Xi \text{ in the weak operator topology}\}, \tag{4}$$

for $D_T := \{x \in X : T \text{ is Gâteaux diff. in } x \text{ with Gâteaux derivative } T'(x)\}$.

For infinite dimensional spaces, there are several choices of topologies in X and Y. Combinations of strong and weak topologies in X and Y lead to four possible definitions that do not coincide, in general. The four versions are defined, e.g. in [CMWC18, Def. 3.1], [RW19, Def. 2.10], and characterized for the solution operators of a nonsmooth semilinear elliptic equation and the unilateral obstacle problem with distributed controls, respectively. The set $\partial T(x)$ as defined in Definition 2.5 is always nonempty as a consequence of Rademacher's theorem for locally Lipschitz continuous mappings. Nevertheless, the generalized derivatives we construct in this paper are also contained in the generalized differential that can be obtained by replacing the weak operator topology in (3) by the strong operator topology. A priori it is not clear that this set is nonempty.

Remark 2.6. Let $J\colon H_0^1(\Omega) \times U \to \mathbb{R}$ be a continuously differentiable objective function. Denote by $S\colon U \to H_0^1(\Omega)$ the solution operator of (1). We use the notation $\hat{J} := J(S(\cdot), \cdot)$ for the reduced objective function and denote by $\partial_C \hat{J}$ Clarke's generalized differential of $\hat{J}\colon U \to \mathbb{R}$. Let $u \in U$ be arbitrary. Then the set inclusion

$$\{\Xi^* J_y(S(u), u) + J_u(S(u), u) : \Xi \in \partial S(u)\} \subseteq \partial \hat{J}(u) \subseteq \partial_C \hat{J}(u)$$

holds.

3 Properties of the Solution Operator

In this section, we collect properties of the solution operator S of (1).

3.1 Monotonicity Properties of the Solution Operator

We state two lemmata on monotonicity properties of S. The next lemma summarizes the monotonicity of S with respect to the elements in U. The proof is similar to the proof of [Rod87, Sect. 4:5, Thm. 5.1] where the property is shown for the unilateral obstacle problem.

Lemma 3.1
Let $u_1, u_2 \in U$ with $u_1 \geq u_2$. Then $S(u_1) \geq S(u_2)$ a.e. on Ω.

Proof. For $i = 1, 2$, we set $y_i := S(u_i)$. We test with $z_1 = y_1 + (y_2 - y_1)_+$ and $z_2 = y_2 - (y_2 - y_1)_+$, respectively, and obtain

$$\langle Ly_1 - f(u_1), z_1 - y_1 \rangle = \langle Ly_1 - f(u_1), (y_2 - y_1)_+ \rangle \geq 0$$

and

$$\langle Ly_2 - f(u_2), z_2 - y_2 \rangle = \langle Ly_2 - f(u_2), -(y_2 - y_1)_+ \rangle \geq 0.$$

Adding up both inequalities we obtain

$$\langle Ly_1 - Ly_2, (y_2 - y_1)_+ \rangle \geq \langle f(u_1) - f(u_2), (y_2 - y_1)_+ \rangle \geq 0.$$

By T-monotonicity, we have $(y_2 - y_1)_+ = 0$, i.e., $y_1 \geq y_2$. □

The following lemma establishes monotonicity properties of S with respect to one of the obstacles.

Lemma 3.2
Let $\psi_i \in H^1(\Omega)$, $i = 1, 2$, such that $K^{\varphi}_{\psi_i}$ are nonempty, and denote by y_i the corresponding solutions of (1) for fixed $u \in U$. Then $\psi_1 \leq \psi_2$ implies $y_1 \leq y_2$.

Proof. We want to show that $(y_1 - y_2)_+ = 0$. Use $z_1 = \min(y_1, y_2)$ and $z_2 = \max(y_1, y_2)$ as test functions. Then $z_1 - y_1 = \min(0, y_2 - y_1) = -(y_1 - y_2)_+$, $z_2 - y_2 = \max(0, y_1 - y_2) = (y_1 - y_2)_+$ and thus

$$\langle Ly_1 - f(u), -(y_1 - y_2)_+ \rangle \geq 0, \quad \langle Ly_2 - f(u), (y_1 - y_2)_+ \rangle \geq 0.$$

Adding the inequalities yields

$$\langle L(y_1 - y_2), (y_1 - y_2)_+ \rangle \leq 0$$

and T-monotonicity implies $(y_1 - y_2)_+ = 0$. □

3.2 Differentiability Properties of the Solution Operator

We distinguish the following subsets of Ω for a fixed element $u \in U$ that result from the solution $S(u)$ of (1). By

$$A(u) := \{\omega \in \Omega : S(u)(\omega) = \psi(\omega) \text{ or } S(u)(\omega) = \varphi(\omega)\}$$

we denote the active set. We also distinguish the active sets with respect to ψ and φ, i.e., we define

$$A_\psi(u) := \{\omega \in \Omega : S(u)(\omega) = \psi(\omega)\} \quad \text{and} \quad A^\varphi(u) := \{\omega \in \Omega : S(u)(\omega) = \varphi(\omega)\}.$$

Note that $A(u) = A_\psi(u) \cup A^\varphi(u)$ and that all these sets are quasi-closed sets that are determined up to a set of capacity zero, since we consider quasi-continuous representatives of $S(u), \psi, \varphi \in H^1(\Omega)$ in the definition of the active sets. We denote by $I(u)$ the inactive set, i.e. the complement of $A(u)$ in Ω and by $I_\psi(u) := \Omega \setminus A_\psi(u)$, respectively $I^\varphi(u) := \Omega \setminus A^\varphi(u)$, the inactive sets with respect to the two obstacles.

Let $u, h \in U$. It can be shown that the directional derivative $S'(u;h)$ of the solution operator of variational inequality (1) is given by the solution of

$$\begin{aligned} &\text{Find } \xi \in (LS(u) - f(u))^\perp \cap T_{K_\psi^\varphi}(S(u)) : \\ &\langle L\xi - f'(u;h), z - \xi \rangle \geq 0 \quad \forall z \in (LS(u) - f(u))^\perp \cap T_{K_\psi^\varphi}(S(u)). \end{aligned} \tag{5}$$

Here, $T_{K_\psi^\varphi}(S(u))$ is the tangent cone of K_ψ^φ at $S(u)$, i.e., the closed conic hull of $K_\psi^\varphi - S(u)$. When f is the identity operator on $H^{-1}(\Omega)$, the variational inequality (5) follows, e.g., from [Mig76, Thm. 3.3]. Since S is locally Lipschitz continuous, S is even directionally differentiable in the sense of Hadamard, see [BS00, Prop. 2.49]. When considering a general operator $f \colon U \to H^{-1}(\Omega)$ fulfilling our assumptions, (5) can be obtained using the chain rule for Hadamard directionally differentiable maps, see e.g. [BS00, Prop. 2.47].

By [Mig76, Lem. 3.4], we have

$$T_{K_\psi^\varphi}(S(u)) = \{z \in H_0^1(\Omega) : z \geq 0 \text{ q.e. on } A_\psi(u), z \leq 0 \text{ q.e. on } A^\varphi(u)\}. \tag{6}$$

3.2.1 Analysis of the Critical Cone

As in the case with a single obstacle, we want to find a suitable characterization of the critical cone $(LS(u) - f(u))^\perp \cap T_{K_\psi^\varphi}(S(u))$. Note that in the case with a single lower obstacle such a characterization is given by $\{z \in H_0^1(\Omega) : z \geq 0 \text{ q.e. on } A(u), z = 0 \text{ q.e. on } A_s(u)\}$, see [Wac14, Lem. 3.1]. Here, $A_s(u)$ is the strictly active set, which can also be characterized as in [Wac14, App. A].

A crucial difference to the case with only one obstacle is that $LS(u) - f(u)$ is not a nonnegative functional and thus cannot be identified with a positive measure. Instead, we will see that, in some cases, it can be identified with the difference of two nonnegative Radon measures. In general, i.e., when the active sets $A_\psi(u)$ and $A^\varphi(u)$ do not have a positive distance, $LS(u) - f(u)$ acts as the difference of two measures on all elements of $H_0^1(\Omega) \cap L^\infty(\Omega)$, but the characterization does not carry over to unbounded elements of $H_0^1(\Omega)$, see Example 3.5.

We define the set of nonnegative Radon measures $M_+(\Omega)$ on Ω as

$$\begin{aligned} M_+(\Omega) = \{\mu : \mu \text{ is the completion of a nonnegative,} \\ \text{regular, locally finite Borel measure on } \Omega\}. \end{aligned}$$

Hence, $\mu \in M_+(\Omega)$ is defined on the smallest sigma algebra containing the Borel sigma algebra and all subsets $M \subset B$ of Borel sets B with $\mu(B) = 0$.

Theorem 3.3
Assume the obstacles ψ, φ satisfy Assumption 1.1.

1. *Then $LS(u) - f(u) \in H^{-1}(\Omega)$ acts as the difference $\xi_\psi - \xi^\varphi$ of nonnegative Radon measures $\xi_\psi, \xi^\varphi \in M_+(\Omega)$ on all elements of $H_0^1(\Omega) \cap C_c(\Omega)$, i.e.,*

$$\langle LS(u) - f(u), w \rangle = \int_\Omega w \, d\xi_\psi - \int_\Omega w \, d\xi^\varphi \tag{7}$$

   holds for all $w \in H_0^1(\Omega) \cap C_c(\Omega)$.

2. *Let $A \subseteq \Omega$ be an arbitrary set. Then $\operatorname{cap}(A) = 0$ implies $\xi_\psi(A) = \xi^\varphi(A) = 0$.*
3. *The characterization (7) carries over to all $w \in H_0^1(\Omega) \cap L^\infty(\Omega)$. In particular, the quasi-continuous representatives of w are ξ_ψ- and ξ^φ-integrable.*

4. *Furthermore, it holds* $S(u) = \psi$ ξ_ψ*-a.e. on* Ω *and* $S(u) = \varphi$ ξ^φ*-a.e. on* Ω*, i.e.,* $\xi_\psi(I_\psi(u)) = 0$ *and* $\xi^\varphi(I^\varphi(u)) = 0$.
5. *Assume* $w \in H_0^1(\Omega) \cap L^1(\xi_\psi)$*. Then we have* $w \in L^1(\xi^\varphi)$ *and (7) holds for* w*. The opposite statement with exchanged roles of* ξ_ψ *and* ξ^φ *is also true.*

Proof. ad 1.: We define

$$v := \frac{S(u) - \psi}{\varphi - \psi}.$$

By the assumptions on ψ and φ, we have $v \in H^1(\Omega) \cap L^\infty(\Omega)$.

Now, for $w \in H_0^1(\Omega) \cap L^\infty(\Omega)$, we have $v\,w$, $(1 - v)\,w \in H_0^1(\Omega)$ and we write

$$\langle LS(u) - f(u), w\rangle = \langle LS(u) - f(u), (1 - v)\,w\rangle + \langle LS(u) - f(u), v\,w\rangle.$$

Thus, we have

$$\langle LS(u) - f(u), w\rangle = \tilde{\xi}_\psi(w) - \tilde{\xi}^\varphi(w),$$

where $\tilde{\xi}_\psi$, $\tilde{\xi}^\varphi$ are defined by

$$\tilde{\xi}_\psi : w \mapsto \langle LS(u) - f(u), (1 - v)\,w\rangle, \quad \tilde{\xi}^\varphi : w \mapsto \langle LS(u) - f(u), -v\,w\rangle.$$

Note that $\tilde{\xi}_\psi$, $\tilde{\xi}^\varphi$ are nonnegative linear forms on $H_0^1(\Omega) \cap L^\infty(\Omega)$.

To see this, assume $w \in H_0^1(\Omega)_+ \cap L^\infty(\Omega)$ and let first $\|w\|_{L^\infty} \leq \delta$. By definition of v, we have $-v\,w + S(u) \in K_\psi^\varphi$ and therefore

$$\tilde{\xi}^\varphi(w) = \langle LS(u) - f(u), -v\,w + S(u) - S(u)\rangle \geq 0. \tag{8}$$

Since $\tilde{\xi}^\varphi$ is linear, (8) holds for all $w \in H_0^1(\Omega)_+ \cap L^\infty(\Omega)$. In a similar fashion, we can show that $\tilde{\xi}_\psi$ is nonnegative on $H_0^1(\Omega) \cap L^\infty(\Omega)$.

In particular, $\tilde{\xi}_\psi$, $\tilde{\xi}^\varphi$ are nonnegative linear forms on $H_0^1(\Omega) \cap C_c(\Omega)$. By [BS00, Lem. 6.53], $\tilde{\xi}_\psi$ and $\tilde{\xi}^\varphi$ have unique nonnegative continuous extensions over $C_c(\Omega)$, also denoted $\tilde{\xi}_\psi$, respectively $\tilde{\xi}^\varphi$. Moreover, by [BS00, Thm. 6.54], there are unique nonnegative, regular, locally finite Borel measures ξ_ψ, ξ^φ such that

$$\tilde{\xi}_\psi(w) = \int_\Omega w\,d\xi_\psi \quad \text{and} \quad \tilde{\xi}^\varphi(w) = \int_\Omega w\,d\xi^\varphi$$

holds for all $w \in C_c(\Omega)$. By completion, we obtain ξ_ψ, $\xi^\varphi \in M_+(\Omega)$.

ad 2.: Now, we modify the proof of [BS00, Lem. 6.55] to show that for a set $A \subset \Omega$, $\operatorname{cap}(A) = 0$ implies $\xi_\psi(A) = \xi^\varphi(A) = 0$. W.l.o.g. we show the statement for ξ^φ.

Let $(\varepsilon_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}_+$ be a sequence with $\varepsilon_n \to 0$ as $n \to \infty$. Fix $n \in \mathbb{N}$. Then we find an open superset A_n of A in Ω with $\operatorname{cap}(A_n) < \varepsilon_n/2$. Furthermore, by [HW18, Lem. 3.4], there is $u_n \in H_0^1(\Omega)_+$ satisfying $u_n = 1$ q.e. on A_n as well as $\|u_n\|^2_{H_0^1(\Omega)} = \operatorname{cap}(A_n) < \varepsilon_n/2$. Moreover, we have $u_n \in L^\infty(\Omega)$, since $\min(z, 1) \in H_0^1(\Omega)$ satisfies $\|\min(z, 1)\|_{H_0^1(\Omega)} \leq \|z\|_{H_0^1(\Omega)}$ for $z \in H_0^1(\Omega)$. By regularity of ξ^φ, we can find a compact set $K_n \subseteq A_n$ satisfying $\xi^\varphi(A_n) \leq \xi^\varphi(K_n) + \varepsilon_n$. Using a smooth version of Urysohn's lemma, there exists a smooth function g_n with values in $[0, 1]$ satisfying $g_n = 1$ on K_n and with compact support in A_n. Then we have $g_n \leq u_n$ and, furthermore, $\xi^\varphi(A_n) \leq \xi^\varphi(K_n) + \varepsilon_n \leq \int_\Omega g_n\,d\xi^\varphi + \varepsilon_n$.

Now, we conclude

$$\begin{aligned}
\xi^\varphi(A_n) &\le \int_\Omega g_n \, d\xi^\varphi + \varepsilon_n \\
&= \langle LS(u) - f(u), -v\, g_n \rangle + \varepsilon_n \\
&\le \langle LS(u) - f(u), -v\, u_n \rangle + \varepsilon_n \\
&\le \|LS(u) - f(u)\|_{H^{-1}(\Omega)} \|v\, u_n\|_{H_0^1(\Omega)} + \varepsilon_n \\
&\le \|LS(u) - f(u)\|_{H^{-1}(\Omega)} (\|v \nabla u_n\|_{L^2(\Omega)} + \|\nabla v u_n\|_{L^2(\Omega)}) + \varepsilon_n.
\end{aligned}$$

As $n \to \infty$, the term $\|v \nabla u_n\|_{L^2(\Omega)}$ tends to zero, since $v \in L^\infty(\Omega)$ and $\|u_n\|_{H_0^1(\Omega)} \to 0$. Moreover, $\|\nabla v u_n\|_{L^2(\Omega)}$ tends to zero aswell, since $|\nabla v\, u_n| \le |\nabla v|$, $\nabla v \in L^2(\Omega)$ and $\nabla v\, u_n \to 0$ pointwise q.e. for a subsequence and thus pointwise λ^d-a.e. The convergence (for a subsequence) follows from Lebesgue's dominated convergence theorem. Now, $\bigcap_{n \in \mathbb{N}} A_n$ is Borel measurable and

$$\xi^\varphi\left(\bigcap_{n \in \mathbb{N}} A_n\right) = 0.$$

Since $A \subseteq \bigcap_{n \in \mathbb{N}} A_n$, we conclude $\xi^\varphi(A) = 0$.

ad 3.: Now, we argue in a similar fashion as in [BS00, Lem. 6.56] to show that each $w \in H_0^1(\Omega) \cap L^\infty(\Omega)$ satisfies

$$\langle LS(u) - f(u), w \rangle = \int_\Omega w \, d\xi_\psi - \int_\Omega w \, d\xi^\varphi.$$

Let $w \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Then we find $(\bar{w}_n)_{n \in \mathbb{N}} \subseteq C_c^\infty(\Omega)$ with $\bar{w}_n \to w$ in $H_0^1(\Omega)$. Defining $w_n := \max(-\|w\|_{L^\infty}, \min(\bar{w}_n, \|w\|_{L^\infty}))$ we have $w_n \in H_0^1(\Omega) \cap C_c(\Omega)$ and $w_n \to w$ in $H_0^1(\Omega)$.

We have $\nabla v\, |w_n - w| \to 0$ in $L^2(\Omega)$ by Lebesgue's dominated convergence theorem, since $2\|w\|_{L^\infty} |\nabla v|$ is an integrable majorant, and since $w_n - w \to 0$ pointwise q.e. (after choosing a subsequence) and thus pointwise a.e. W.l.o.g., we assume $LS(u) - f(u) \neq 0$. Let $\varepsilon > 0$ and let N_0 be such that $\|w_n - w_m\|_{H_0^1(\Omega)} < \frac{\varepsilon}{2\|LS(u) - f(u)\|_{H^{-1}(\Omega)}}$ and such that

$$\begin{aligned}
&\|\nabla v\, |w_n - w_m|\|_{L^2(\Omega)} \\
&\le \|\nabla v\, |w_n - w|\|_{L^2(\Omega)} + \|\nabla v\, |w_m - w|\|_{L^2(\Omega)} \\
&< \frac{\varepsilon}{2\|LS(u) - f(u)\|_{H^{-1}(\Omega)}}
\end{aligned}$$

for all $n, m \ge N_0$. Assume $n, m \ge N_0$, then we have

$$\begin{aligned}
&\|w_n - w_m\|_{L^1(\xi^\varphi)} = \int_\Omega |w_n - w_m| \, d\xi^\varphi = \tilde{\xi}^\varphi(|w_n - w_m|) \\
&= \langle LS(u) - f(u), -v\, |w_n - w_m| \rangle \\
&\le \|LS(u) - f(u)\|_{H^{-1}(\Omega)} \| - v\, |w_n - w_m| \|_{H_0^1(\Omega)} \\
&\le \|LS(u) - f(u)\|_{H^{-1}(\Omega)} (\|w_n - w_m\|_{H_0^1(\Omega)} + \|\nabla v\, |w_n - w_m|\|_{L^2(\Omega)}) \\
&< \varepsilon.
\end{aligned} \tag{9}$$

This means, $(w_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $L^1(\xi^\varphi)$. In a similar fashion, we can also show that $(w_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $L^1(\xi_\psi)$.

As shown in part 2., the convergence $w_n \to w$ pointwise q.e. implies the convergence $w_n \to w$ pointwise ξ_ψ- and ξ^φ-a.e., respectively. This means, the quasi-continuous representative of w is measurable. Furthermore, this representative has to be a representative of the limit of $(w_n)_{n\in\mathbb{N}}$ in $L^1(\xi_\psi)$, respectively $L^1(\xi^\varphi)$, since convergent sequences in $L^1(\xi_\psi)$ and $L^1(\xi^\varphi)$ possess pointwise ξ_ψ-, respectively ξ^φ-convergent subsequences converging to the limit.

This implies

$$\tilde{\xi}_\psi(w) = \langle LS(u) - f(u), (1-v)w\rangle = \int_\Omega w\, d\xi_\psi$$

and

$$\tilde{\xi}^\varphi(w) = \langle LS(u) - f(u), -vw\rangle = \int_\Omega w\, d\xi^\varphi$$

for all $w \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

ad 4.: This part of the proof is similar as the proof of [Wac14, Prop. 2.5]. We consider a smooth cut-off function $\chi \in C_c^\infty(\Omega)$ with $0 \le \chi \le 1$ and $\chi = 1$ on a compact set $C \subset \Omega$. We define $w := \chi\,[(1-v)\,\psi + v\,S(u)] + (1-\chi)\,S(u)$ and obtain $w \in K_\psi^\varphi$. This implies

$$\begin{aligned} 0 &\le \langle LS(u) - f(u), w - S(u)\rangle \\ &= \langle LS(u) - f(u), \chi\,(1-v)\,\psi + \chi\,v\,S(u) - \chi\,S(u)\rangle \\ &= \langle LS(u) - f(u), (1-v)\,\chi\,(\psi - S(u))\rangle \\ &= \int_\Omega \chi\,(\psi - S(u))\,d\xi_\psi. \end{aligned}$$

Since $\chi\,(\psi - S(u)) \le 0$ q.e. on Ω, and thus ξ_ψ-a.e., we conclude $S(u) = \psi$ ξ_ψ-a.e. on C. Covering Ω with countably many compact subsets, we infer $S(u) - \psi = 0$ ξ_ψ-a.e. on Ω. Similarly, we can show $\varphi - S(u) = 0$ ξ^φ-a.e. on Ω.

ad 5.: Assume $w \in H_0^1(\Omega) \cap L^1(\xi_\psi)$. We approximate w in $H_0^1(\Omega)$ by $(w_n)_{n\in\mathbb{N}}$ defined via $w_n := \max(-n, \min(n, w))$. Then we have $w_n \to w$ in $H_0^1(\Omega)$ and $w_n \to w$ pointwise ξ_ψ-a.e. Since $|w_n| \le |w|$ and since $w \in L^1(\xi_\psi)$, we apply Lebesgue's dominated convergence theorem and obtain $w_n \to w$ in $L^1(\xi_\psi)$.

From

$$\begin{aligned} &\|w_n - w_m\|_{L^1(\xi^\varphi)} = \int_\Omega |w_n - w_m|\,d\xi^\varphi \\ &= \int_\Omega |w_n - w_m|\,d\xi_\psi - \langle LS(u) - f(u), |w_n - w_m|\rangle \\ &\le \|w_n - w_m\|_{L^1(\xi_\psi)} + \|LS(u) - f(u)\|_{H^{-1}(\Omega)}\|w_n - w_m\|_{H_0^1(\Omega)} \end{aligned}$$

it follows that $(w_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^1(\xi^\varphi)$ and we can again conclude that $w_n \to w$ in $L^1(\xi^\varphi)$.

From the representation

$$\langle LS(u) - f(u), w_n\rangle = \int_\Omega w_n\,d\xi_\psi - \int_\Omega w_n\,d\xi^\varphi$$

for all $n \in \mathbb{N}$ we conclude since $w_n \to w$ in $H_0^1(\Omega)$, $L^1(\xi_\psi)$ and $L^1(\xi^\varphi)$

$$\langle LS(u) - f(u), w\rangle = \int_\Omega w\,d\xi_\psi - \int_\Omega w_n\,d\xi^\varphi.$$

The opposite statement follows similarly. □

Lemma 3.4
Let Assumption 1.1 be satisfied and suppose $A_\psi(u)$ *and* $A^\varphi(u)$ *have a positive distance, i.e., there is a constant* $C > 0$ *such that* $\inf_{x\in A^\varphi(u)} |y - x| \geq C$ *for all* $y \in A_\psi(u)$*. Then, with* $\xi_\psi, \xi^\varphi \in M_+(\Omega)$ *as in Theorem 3.3, it holds* $H_0^1(\Omega) \subset L^1(\xi_\psi) \cap L^1(\xi^\varphi)$ *and*

$$\langle LS(u) - f(u), w\rangle = \int_\Omega w \, d\xi_\psi - \int_\Omega w \, d\xi^\varphi$$

for all $w \in H_0^1(\Omega)$*.*

Proof. Since the active sets have a positive distance, we can construct an element v_2 satisfying $v_2 \in C^\infty(\mathbb{R}^d)$ as well as $v_2 = 1$ on $A_\psi(u)$ and $v_2 = 0$ outside $A_\psi(u) + B_{C/2}$.

Since v_2 is smooth, we have $v_2 w, (v_2 - 1)w \in H_0^1(\Omega)$ for all $w \in H_0^1(\Omega)$ and we define the functionals

$$\tilde{\xi}_\psi^2 \colon w \mapsto \langle LS(u) - f(u), v_2 \, w\rangle, \quad \tilde{\xi}_2^\varphi \colon w \mapsto \langle LS(u) - f(u), (v_2 - 1) \, w\rangle.$$

on $H_0^1(\Omega)$. Since v_2 is in $C^\infty(\mathbb{R}^d)$, we can show that $\tilde{\xi}_\psi^2$ and $\tilde{\xi}_2^\varphi$ are bounded linear functionals on $H_0^1(\Omega)$. Moreover, we have

$$\langle LS(u) - f(u), w\rangle = \langle \tilde{\xi}_\psi^2, w\rangle - \langle \tilde{\xi}_2^\varphi, w\rangle$$

for all $w \in H_0^1(\Omega)$.

Assume first w is in $H_0^1(\Omega) \cap L^\infty(\Omega)$. Then we have

$$\begin{aligned}\int_\Omega w \, d\xi_\psi &= \int_\Omega v_2 \, w \, d\xi_\psi \\ &= \int_\Omega v_2 \, w \, d\xi_\psi - \int_\Omega v_2 \, w \, d\xi^\varphi \\ &= \langle LS(u) - f(u), v_2 \, w\rangle \\ &= \langle \tilde{\xi}_\psi^2, w\rangle.\end{aligned}$$

Here, the first equation holds since $\xi_\psi(I_\psi(u)) = 0$ and $w = v_2 \, w$ on $A_\psi(u)$, see Theorem 3.3. Similarly, the second equation holds since $v_2 \, w = 0$ ξ^φ-a.e. on Ω.

Let $w \in H_0^1(\Omega)$. Now we have $\max(-n, \min(w, n)) \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and $w_n \to w$ in $H_0^1(\Omega)$. Furthermore,

$$\|w_n - w_m\|_{L^1(\xi_\psi)} = \int_\Omega |w_n - w_m| \, d\xi_\psi = \langle \tilde{\xi}_\psi^2, |w_n - w_m|\rangle \leq \|\tilde{\xi}_\psi^2\| \| |w_n - w_m| \|.$$

Thus, $(w_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^1(\xi_\psi)$. Since $w_n \to w$ pointwise q.e. and thus ξ_ψ-a.e., the quasi-continuous representative of w is measurable and we have $w_n \to w$ in $L^1(\xi_\psi)$. Arguing for ξ^φ in a similar fashion, we obtain that

$$\langle LS(u) - f(u), w\rangle = \langle \tilde{\xi}_\psi^2, w\rangle - \langle \tilde{\xi}_2^\varphi, w\rangle = \int_\Omega w \, d\xi_\psi - \int_\Omega w \, d\xi^\varphi. \qquad \square$$

The following example shows that, in general, i.e., when the active sets do not have a positive distance, the characterization of the functional $LS(u) - f(u)$ as the difference of the two measures ξ_ψ and ξ^φ does not need to apply for all possible arguments in $H_0^1(\Omega)$.

Example 3.5. For $d = 2$ and for $0 < \beta < \frac{1}{2}$, consider the function

$$y(x) = \sin((-\ln(|x|))^\beta), \quad x \in \Omega := B_\rho(0), \quad \rho = \exp(-\pi^{1/\beta}). \tag{10}$$

Then $y|_{\partial B_\rho(0)} = \sin(\pi) = 0$ and $y \in H_0^1(\Omega) \cap L^\infty(\Omega)$, since

$$y_{x_i}(x)^2 = \frac{4\beta^2(1/2)^{2\beta} x_i^2 \cos((-1/2\ln(|x|^2))^\beta)^2}{|x|^4(-\ln(|x|^2))^{2-2\beta}} \le \frac{4}{|x|^2(-\ln(|x|^2))^{2-2\beta}},$$

$$\int_{B_\rho(0)} y_{x_i}(x)^2\,dx \le \int_0^{1/2} \frac{8r\pi}{r^2(-\ln(r^2))^{2-2\beta}}\,dr = \frac{8\pi 2^{2\beta-2}\ln(2)^{2\beta-1}}{1-2\beta}.$$

Now, set $\psi(x) = \min\left(-\frac{1}{2}, y(x)\right)$, $\varphi(x) = \max\left(\frac{1}{2}, y(x)\right)$, and

$$A_\psi = \{x \in \Omega : y(x) = \psi(x)\}, \quad A^\varphi = \{x \in \Omega : y(x) = \varphi(x)\}.$$

Hence, the above choice of A_ψ and A_φ admits a function $y \in H_0^1(\Omega)$ taking the values ψ and φ on the active sets.

We have

$$(-\ln(r(t)))^\beta = t \quad \iff \quad r(t) = \exp(-t^{1/\beta})$$

and, for $k \in \mathbb{N}$, we set $r_k^- := r(2k\pi - \pi/2) > r(2k\pi + \pi/2) =: r_k^+$. This choice implies $y(r_k^\pm(\cos t, \sin t)) = \pm 1$.

Now, let $\omega_k > 0$ be weights (that will be adjusted below), with $\sum_{k=1}^\infty \omega_k^2 < \infty$ and consider the functional

$$\langle \mu, w\rangle := \sum_{k=1}^\infty \frac{\omega_k}{\sqrt{\ln(r_k^-/r_k^+)}} \int_0^{2\pi} (w(r_k^-(\cos t, \sin t)) - w(r_k^+(\cos t, \sin t))\,dt$$

for $w \in H_0^1(\Omega)$. Then we have $\mu \in H^{-1}(\Omega)$, since

$$\begin{aligned}
|\langle \mu, w\rangle| &\le \sum_{k=1}^\infty \frac{\omega_k}{\sqrt{\ln(r_k^-/r_k^+)}} \int_0^{2\pi}\int_{r_k^+}^{r_k^-} |\nabla w(r(\cos t, \sin t))|\sqrt{r}\frac{1}{\sqrt{r}}\,dr\,dt \\
&\le \sum_{k=1}^\infty \frac{\omega_k}{\sqrt{\ln(r_k^-/r_k^+)}} \|\nabla w\|_{L^2(B_{r_k^-}(0)\setminus B_{r_k^+}(0))} \left(\int_0^{2\pi}\int_{r_k^+}^{r_k^-} \frac{1}{r}\,dr\,dt\right)^{\frac{1}{2}} \\
&= \sqrt{2\pi}\sum_{k=1}^\infty \omega_k \|\nabla w\|_{L^2(B_{r_k^-}(0)\setminus B_{r_k^+}(0))} \\
&\le \sqrt{2\pi}\left(\sum_{k=1}^\infty \omega_k^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^\infty \|\nabla w\|^2_{L^2(B_{r_k^-}(0)\setminus B_{r_k^+}(0))}\right)^{\frac{1}{2}} \\
&\le \sqrt{2\pi}\left(\sum_{k=1}^\infty \omega_k^2\right)^{\frac{1}{2}} \|\nabla w\|_{L^2}.
\end{aligned}$$

Here, we have used that, for all $k \in \mathbb{N}$, the sets $B_{r_k^-}(0)\setminus B_{r_k^+}(0)$ are disjoint. Moreover, for $w \in H_0^1(\Omega) \cap L^\infty(\Omega)$, we have for any quasi-continuous representative $\tilde{w}$

$$\langle \mu, w\rangle = \int_\Omega \tilde{w}\,d\xi_\psi - \int_\Omega \tilde{w}\,d\xi^\varphi, \tag{11}$$

where ξ_ψ, ξ^φ are nonnegative finite measures with support in A_ψ and A^φ, respectively. In fact, to show that ξ^φ is a finite measure, we observe that

$$\ln(r_k^-/r_k^+) = (2k\pi + \pi/2)^{1/\beta} - (2k\pi - \pi/2)^{1/\beta} \begin{cases} \ge (2k\pi - \pi/2)^{1/\beta - 1}\pi/\beta, \\ \le (2k\pi + \pi/2)^{1/\beta-1}\pi/\beta. \end{cases} \tag{12}$$

Hence, for any $w \in C(\bar{\Omega})$ with $0 \le w \le 1$

$$0 \le \int_\Omega w\, d\xi^\varphi \le \sum_{k=1}^\infty \frac{\omega_k}{\sqrt{\ln(r_k^-/r_k^+)}} \int_0^{2\pi} 1\, dt \le 2\pi \left(\sum_{k=1}^\infty \omega_k^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^\infty \frac{1}{\ln(r_k^-/r_k^+)}\right)^{\frac{1}{2}}$$

$$\le 2\pi \left(\sum_{k=1}^\infty \omega_k^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^\infty \frac{1}{\pi/\beta(2k\pi - \pi/2)^{1/\beta-1}}\right)^{\frac{1}{2}} \le C$$

with a constant $C > 0$, since $\beta < 1/2$. The same argument shows that ξ^φ is a bounded functional on $H_0^1(\Omega) \cap L^\infty(\Omega)$ w.r.t. $\|\cdot\|_{H_0^1(\Omega)} + \|\cdot\|_{L^\infty(\Omega)}$.

Now, consider the unbounded function $w(x) = (-\ln(|x|))^\beta - \pi$. Then since $w_{x_i}^2 \le \frac{4}{|x|^2(-\ln(|x|^2))^{2-2\beta}}$, we have $w \in H_0^1(\Omega)$ as above. With $\omega_k = k^{-1/2-\varepsilon}$ and by using (12), we obtain the estimate

$$\int_\Omega w\, d\xi^\varphi = \sum_{k=1}^\infty \frac{\omega_k}{\sqrt{\ln(r_k^-/r_k^+)}} \int_0^{2\pi} (2k\pi + \pi/2 - \pi)\, dt$$

$$\ge \sum_{k=1}^\infty \frac{2\pi(2k\pi - \pi/2)}{k^{1/2+\varepsilon}(2k\pi + \pi/2)^{1/(2\beta)-1/2}} \ge \sum_{k=1}^\infty \frac{C}{k^{1/(2\beta)+\varepsilon-1}} = \infty$$

with a constant $C > 0$ for $\beta = 1/2 - \varepsilon$ and $\varepsilon > 0$ small.

Finally, the function y in (10) satisfies the bilateral obstacle problem (1) with $f(u) := Ly - \mu$, since y is obviously feasible and $Ly - f(u) = \mu$ imply together with (11) that (1) holds.

In the following lemma, we find a characterization of the critical cone. In parts, the proof of the second statement follows similar lines as the proof of [Wac14, Lem. 3.1].

Lemma 3.6
Let Assumption 1.1 be satisfied, let $\xi_\psi, \xi^\varphi \in M_+(\Omega)$ be as in Theorem 3.3 and let $u \in U$ be arbitrary.

1. *Assume $z \in (LS(u) - f(u))^\perp \cap T_{K_\psi^\varphi}(S(u))$. If z is not integrable with respect to ξ_ψ, ξ^φ, then there is a sequence $(z_n)_{n\in\mathbb{N}} \subset (LS(u) - f(u))^\perp \cap T_{K_\psi^\varphi}(S(u))$ with $z_n \to z$ in $H_0^1(\Omega)$ such that $(z_n)_{n\in\mathbb{N}} \subset H_0^1(\Omega) \cap L^\infty(\Omega) \subset L^1(\xi_\psi) \cap L^1(\xi^\varphi)$.*
2. *There exist quasi-closed sets $A_\psi^s(u) \subseteq A_\psi(u)$ and $A_s^\varphi(u) \subseteq A^\varphi(u)$ such that we have the set equality*

$$\begin{aligned}&(LS(u) - f(u))^\perp \cap T_{K_\psi^\varphi}(S(u)) \\ &= \big\{z \in H_0^1(\Omega) : z \ge 0 \text{ q.e. on } A_\psi(u), z \le 0 \text{ q.e. on } A^\varphi(u), \\ &\qquad\qquad z = 0 \text{ q.e. on } A_\psi^s(u) \cup A_s^\varphi(u)\big\}.\end{aligned}$$

Furthermore, each $z \in (LS(u) - f(u))^\perp \cap T_{K_\psi^\varphi}(S(u))$ is automatically integrable with respect to ξ_ψ and with respect to ξ^φ.

Proof. ad 1.: Let $z \in (LS(u) - f(u))^\perp \cap T_{K_\psi^\varphi}(S(u))$ be given. Note that by (6), z satisfies $z \ge 0$ q.e. on $A_\psi(u)$ and $z \le 0$ q.e. on $A^\varphi(u)$.

Suppose z is not integrable with respect to ξ_ψ and with respect to ξ^φ. This means,

$$\int_\Omega z\, d\xi_\psi = \int_\Omega z\, d\xi^\varphi = \infty.$$

Let $(t_\psi^n)_{n\in\mathbb{N}} \subseteq \mathbb{R}_+$ be a sequence with $t_\psi^n \to \infty$ as $n \to \infty$.

Let $n \in \mathbb{N}$. Then, we can find $t_n^\varphi \in \mathbb{R}$ such that

$$\begin{aligned}\int_{A_\psi(u)} \max(-t_n^\varphi, \min(z, t_\psi^n))\, d\xi_\psi &= \int_{A_\psi(u)} \min(z, t_\psi^n)\, d\xi_\psi \\ &= \int_{A^\varphi(u)} \max(-t_n^\varphi, z)\, d\xi^\varphi \\ &= \int_{A^\varphi(u)} \max(-t_n^\varphi, \min(z, t_\psi^n))\, d\xi^\varphi .\end{aligned} \tag{13}$$

To see this, observe first that since $\max(-t_n^\varphi, \min(z, t_\psi^n)) \in H_0^1(\Omega) \cap L^\infty(\Omega)$, the integrals are finite, and since the map

$$\mathbb{R} \ni t \mapsto \int_{A^\varphi(u)} \max(-t, z)\, d\xi^\varphi \in \mathbb{R},$$

is continuous, which can be seen using Lebesgue's dominated convergence theorem, we can apply the intermediate value theorem to obtain the statement in (13). Then

$$z_n := \max(-t_n^\varphi, \min(z, t_\psi^n))$$

is in $L^\infty(\Omega) \cap H_0^1(\Omega)$, satisfies $z_n \geq 0$ q.e. on $A_\psi(u)$, $z_n \leq 0$ q.e. on $A^\varphi(u)$ and $z_n \to z$ in $H_0^1(\Omega)$ as $n \to \infty$. Furthermore, by (13), we have

$$\langle LS(u) - f(u), z_n \rangle = \int_{A_\psi(u)} z_n\, d\xi_\psi - \int_{A^\varphi(u)} z_n\, d\xi^\varphi = 0.$$

for all $n \in \mathbb{N}$.

ad 2.: This part of the proof is partially based on the proof of [Wac14, Lem. 3.1]. Assume $z \in H_0^1(\Omega)$ is integrable with respect to ξ_ψ and ξ^φ and assume $z \geq 0$ q.e. on $A_\psi(u)$ and $z \leq 0$ q.e. on $A^\varphi(u)$. Note that $z \in H_0^1(\Omega) \cap L^1(\xi_\psi)$ already implies $z \in L^1(\xi^\varphi)$ and vice versa, by the last statement in Theorem 3.3. We have

$$\langle LS(u) - f(u), z \rangle = \int_\Omega z\, d\xi_\psi - \int_\Omega z\, d\xi^\varphi = \int_{A_\psi(u)} z\, d\xi_\psi - \int_{A^\varphi(u)} z\, d\xi^\varphi,$$

as $\xi_\psi(I_\psi(u)) = 0$ and $\xi^\varphi(I^\varphi(u)) = 0$. Since $z \geq 0$ ξ_ψ-a.e. on $A_\psi(u)$ and $z \leq 0$ ξ^φ-a.e. on $A^\varphi(u)$, see Theorem 3.3, we conclude that

$$\langle LS(u) - f(u), z \rangle = \int_{A_\psi(u)} z\, d\xi_\psi - \int_{A^\varphi(u)} z\, d\xi^\varphi = 0$$

is equivalent to $z = 0$ ξ_ψ-a.e. on $A_\psi(u)$ and $z = 0$ ξ^φ-a.e. on $A^\varphi(u)$. Using that $\xi_\psi(I_\psi(u)) = 0$ and $\xi^\varphi(I^\varphi(u)) = 0$, we can see that this means $z = 0$ ξ_ψ- and ξ^φ-a.e. on Ω.

Assume $z \in (LS(u) - f(u))^\perp \cap T_{K_\psi^\varphi}(S(u))$ is not integrable with respect to ξ_ψ and ξ^φ. Then, by the first part of the lemma and by what we have shown now, z can be approximated by a sequence $(z_n)_{n\in\mathbb{N}} \subset L^1(\xi_\psi) \cap L^1(\xi^\varphi)$ and the members of this sequence satisfy $z_n = 0$ ξ_ψ- and ξ^φ-a.e. on Ω. Since $z_n \to z$ pointwise q.e. (after choosing a subsequence) and thus pointwise ξ_ψ- and ξ^φ-a.e., we have, in particular, $z = 0$ ξ_ψ- and ξ^φ-a.e. Thus, z is actually integrable with respect to the two measures.

We have shown that

$$\begin{aligned}&(LS(u) - f(u))^\perp \cap T_{K_\psi^\varphi}(S(u)) \\ &= \big\{ z \in H_0^1(\Omega) : z \geq 0 \text{ q.e. on } A_\psi(u), z \leq 0 \text{ q.e. on } A^\varphi(u), \\ &\qquad z = 0\ \xi_\psi\text{- and } \xi^\varphi\text{-a.e.} \big\}.\end{aligned}$$

By [Sto93, Thm. 1], there exist quasi-closed sets $A^s_\psi(u)$ and $A^\varphi_s(u)$ such that

$$\{z \in H^1_0(\Omega) : z = 0 \ \xi_\psi\text{-a.e.}\} = \{z \in H^1_0(\Omega) : z = 0 \text{ q.e. on } A^s_\psi(u)\} \tag{14}$$

and

$$\{z \in H^1_0(\Omega) : z = 0 \ \xi^\varphi\text{-a.e.}\} = \{z \in H^1_0(\Omega) : z = 0 \text{ q.e. on } A^\varphi_s\}. \tag{15}$$

We have $S(u) - \psi = 0$ ξ_ψ-a.e. and thus $S(u) - \psi = 0$ q.e. on $A^s_\psi(u)$, which implies $\text{cap}(A^s_\psi(u) \setminus A_\psi(u)) = 0$. This means, w.l.o.g. we can replace $A^s_\psi(u)$ in (14) by $A^s_\psi(u) \cap A_\psi(u)$. The same arguments apply to show $A^\varphi_s(u) \subseteq A^\varphi(u)$. □

Corollary 3.7
Under the assumptions of Lemma 3.6, for $w \in H^1_0(\Omega)$ it holds $w = 0$ q.e. on $A^s_\psi(u)$, respectively $w = 0$ q.e. on $A^\varphi_s(u)$ if and only if $w = 0$ ξ_ψ-a.e., respectively $w = 0$ ξ^φ-a.e. In both cases we have $w \in L^1(\xi_\psi) \cap L^1(\xi^\varphi)$ and $\langle LS(u) - f(u), w\rangle = \int_\Omega w \, d\xi_\psi - \int_\Omega w \, d\xi^\varphi$.

Proof. The equivalence is implied by the proof of Lemma 3.6, see (14) and (15). The statements $w \in L^1(\xi_\psi) \cap L^1(\xi^\varphi)$ and $\langle LS(u) - f(u), w\rangle = \int_\Omega w \, d\xi_\psi - \int_\Omega w \, d\xi^\varphi$ follow from Theorem 3.3. □

In the following sections, we also write $A_s(u) := A^s_\psi(u) \cup A^\varphi_s(u)$ for the strictly active set with respect to both obstacles, we have $A_s(u) \subseteq A(u)$.

Moreover, we will use the notation $A^w_\psi(u) := A_\psi(u) \setminus A^s_\psi(u)$ for the weakly active set with respect to the lower obstacle ψ and $A^\varphi_w(u) := A^\varphi(u) \setminus A^\varphi_s(u)$ for the weakly active set with respect to the upper obstacle φ. For the sake of completeness, we also introduce the notation $A_w(u) := A^w_\psi(u) \cup A^\varphi_w(u)$ for the weakly active set with respect to upper and lower obstacle.

3.2.2 Gâteaux Differentiability of the Solution Operator

As in the case of unilateral obstacle problems, in points u where S is Gâteaux differentiable, we can replace the critical cone in the characterization of the directional derivative by the largest linear subset contained in the critical cone, and by the linear hull of the critical cone, respectively. Both versions yield a characterization of the Gâteaux derivative. The reasoning for these facts in the case of the unilateral obstacle problem can be found in [RU19, Lem. 3.7].

For the bilateral case, characterizations of the Gâteaux derivative are summarized in the following theorem.

Theorem 3.8
Assume the obstacles ψ, φ satisfy Assumption 1.1. Suppose S is Gâteaux differentiable in $u \in U$. Let $h \in U$. Then $S'(u; h)$ is determined by the solution of the variational equation

$$\text{Find } \xi \in H^1_0(D(u)) : \quad \langle L\xi - f'(u; h), z\rangle = 0 \quad \forall\, z \in H^1_0(D(u)) \tag{16}$$

and $D(u)$ can be chosen as any quasi-open subset of Ω fulfilling $I(u) \subseteq D(u) \subseteq \Omega \setminus A_s(u)$.

Proof. Assume S is Gâteaux differentiable in $u \in U$. Then the map $S'(u; \cdot)$ is linear and the image is a linear subspace of $H^1_0(\Omega)$. By the characterization in (5), the image of $S'(u; \cdot)$ lies in a linear subspace of the critical cone. The structure of the critical cone established in Lemma 3.6 implies that $S'(u; h) \subseteq H^1_0(I(u))$, since $H^1_0(I(u))$ is the largest linear subset contained in the critical cone. Now $S'(u; h)$ solves the variational equation (16) with $D(u) = I(u)$, since $H^1_0(D(U))$ is a linear subspace.

Obviously, the image of $S'(u;\cdot)$ is also contained in the linear hull of the critical cone, the set $H_0^1(\Omega \setminus A_s(u))$. Assume z is in the critical cone and $h \in U$ is arbitrary. Then

$$\langle LS'(u;-h) - f'(u;-h), z - S'(u;-h)\rangle \geq 0,$$

which implies

$$\langle LS'(u;h) - f'(u;h), -z - S'(u;h)\rangle \geq 0.$$

Thus, by linearity arguments we obtain that we can also use test functions from the negative critical cone. Let $z \in H_0^1(\Omega \setminus A_s(u))$ be arbitrary. Since the two sets $\Omega \setminus (A_s(u) \cup A_\psi(u))$ and $\Omega \setminus (A_s(u) \cup A^\varphi(u))$ are a quasi-covering of $\Omega \setminus A_s(u)$, we can find a sequence $(z_\psi^n + z_n^\varphi)_{n\in\mathbb{N}}$ converging to z and fulfilling $z_\psi^n \in H_0^1(\Omega \setminus (A_s(u) \cup A^\varphi(u)))$ and $z_n^\varphi \in H_0^1(\Omega \setminus (A_s(u) \cup A_\psi(u)))$, see Lemma 2.3. Considering positive and negative parts we write

$$z_n^\varphi = z_n^{\varphi,+} - z_n^{\varphi,-} \quad \text{ and } z_\psi^n = z_{\psi,+}^n - z_{\psi,-}^n.$$

This implies that $z_{\psi,+}^n$, $z_{\psi,-}^n$, $-z_n^{\varphi,+}$ and $-z_n^{\varphi,-}$ are elements of the critical cone. This shows

$$\langle LS'(u;h) - f'(u;h), z_\psi^n + z_n^\varphi - S'(u;h)\rangle \geq 0$$

for all $n \in \mathbb{N}$. Taking the limit and observing that $H_0^1(\Omega \setminus A_s(u))$ is a linear subspace we obtain

$$\langle LS'(u;h) - f'(u;h), z\rangle = 0$$

for all $z \in H_0^1(\Omega \setminus A_s(u))$.

Thus, since (16) is a characterization of the Gâteaux derivative for $D(u) = I(u)$ and for $D(u) = \Omega \setminus A_s(u)$, each set $H_0^1(D(u))$ with $I(u) \subseteq D(u) \subseteq \Omega \setminus A_s(u)$ also yields a characterization for the Gâteaux derivative. □

4 Monotonicity of the Active and Strictly Active Sets

In this subsection, the monotonicity of the active and strictly active sets is studied. Starting with this section, we specify our notation and write S_ψ^φ instead of S for the solution operator of (1).

The monotonicity of the active sets is a direct consequence of Lemma 3.1.

Lemma 4.1
Let $u_1, u_2 \in U$ satisfy $u_1 \geq u_2$. Then

1. $A_\psi(u_1) \subseteq A_\psi(u_2)$,
2. $A^\varphi(u_1) \supseteq A^\varphi(u_2)$.

Proof. By Lemma 3.1, we have $S_\psi^\varphi(u_1) \geq S_\psi^\varphi(u_2)$ a.e. and by Lemma 2.3 also q.e. in Ω. This implies the statements. □

Lemma 4.2
Suppose the conditions of Assumption 1.1 are satisfied. Let $u \in U$ and let $v \in H_0^1(\Omega)_+$ such that $\{v > 0\} \subseteq \Omega \setminus A_\psi^s(u)$. Then $S_\psi^\varphi(u) = S_{\psi-v}^\varphi(u)$.

Proof. Obviously, $S_\psi^\varphi(u) \geq \psi - v$. Now, let $z \in K_{\psi-v}^\varphi$ be arbitrary. We need to show that

$$\langle LS_\psi^\varphi(u) - f(u), z - S_\psi^\varphi(u)\rangle \geq 0.$$

Now, $z = \max(z,\psi)+\min(z-\psi,0) =: z_1+z_2$, where $z_1 \in K_\psi^\varphi$ and $z_2 \in H_0^1(\Omega\backslash A_\psi^s(u))_-$. Thus

$$\langle LS_\psi^\varphi(u) - f(u), z_1 - S_\psi^\varphi(u)\rangle \geq 0$$

and moreover, since $z_2 = 0$ q.e. on $A_\psi^s(u)$ and thus ξ_ψ-a.e., see Theorem 3.3, we have by Corollary 3.7

$$\langle LS_\psi^\varphi(u) - f(u), z_2\rangle = \int_{A_\psi} z_2 \, d\xi_\psi - \int_{A^\varphi} z_2 \, d\xi^\varphi = -\int_{A^\varphi} z_2 \, d\xi^\varphi \geq 0. \qquad \square$$

Lemma 4.3
Let $\mu \in H^{-1}(\Omega)$. Denote by T_ψ^φ the solution operator of the bilateral obstacle problem on $H^{-1}(\Omega)$, i.e., $(T_\psi^\varphi \circ f)(\cdot) = S_\psi^\varphi(\cdot)$ for $f\colon U \to H^{-1}(\Omega)$ fulfilling our assumptions. Then we have $-T_\psi^\varphi(\mu) = T_{-\varphi}^{-\psi}(-\mu)$.

Moreover, $\tilde{A}_\psi(\mu) = \tilde{A}^{-\psi}(-\mu)$ and $\tilde{A}^\varphi(\mu) = \tilde{A}_{-\varphi}(-\mu)$. Here, $\tilde{A}_\psi(\mu), \tilde{A}^\varphi(\mu)$ denote the respective active sets for $T_\psi^\varphi(\mu)$ and, vice versa, $\tilde{A}_{-\varphi}(\mu) = \{\omega \in \Omega : T_{-\varphi}^{-\psi}(\mu)(\omega) = -\varphi(\omega)\}$, $\tilde{A}^{-\psi}(\mu) = \{\omega \in \Omega : T_{-\varphi}^{-\psi}(\mu)(\omega) = -\psi(\omega)\}$. Furthermore, if the conditions of Assumption 1.1 are fulfilled, we have $\tilde{A}_\psi^s(\mu) = \tilde{A}_s^{-\psi}(-\mu)$ and $\tilde{A}_s^\varphi(\mu) = \tilde{A}_{-\varphi}^s(-\mu)$. Here, $\tilde{A}_\psi^s(\mu), \tilde{A}_s^\varphi(\mu)$ denote the strictly active sets for $T_\psi^\varphi(\mu)$ and $\tilde{A}_s^{-\psi}(\mu), \tilde{A}_{-\varphi}^s(\mu)$ denote the strictly active sets for $T_{-\varphi}^{-\psi}(\mu)$.

Proof. First, let us note that for $z \in H_0^1(\Omega)$ it holds $\psi \leq z \leq \varphi$ if and only if $-\varphi \leq -z \leq -\psi$ and this implies

$$-K_\psi^\varphi = K_{-\varphi}^{-\psi}.$$

Thus, $-T_\psi^\varphi(\mu) \in K_{-\varphi}^{-\psi}$. Let now $z \in K_{-\varphi}^{-\psi}$ be arbitrary. Then we have

$$\begin{aligned}
&\langle L(-T_\psi^\varphi(\mu)) + \mu, z - (-T_\psi^\varphi(\mu))\rangle \\
&= -\langle LT_\psi^\varphi(\mu) - \mu, z - (-T_\psi^\varphi(\mu))\rangle \\
&= \langle LT_\psi^\varphi(\mu) - \mu, -z - T_\psi^\varphi(\mu)\rangle \geq 0.
\end{aligned}$$

This implies $-T_\psi^\varphi(\mu) = T_{-\varphi}^{-\psi}(-\mu)$.

Now, it holds $T_{-\varphi}^{-\psi}(-\mu)(\omega) = -\psi(\omega)$ if and only if $T_\psi^\varphi(\mu)(\omega) = \psi(\omega)$ and we have $T_{-\varphi}^{-\psi}(-\mu)(\omega) = -\varphi(\omega)$ if and only if $T_\psi^\varphi(\mu)(\omega) = \varphi(\omega)$, thus $\tilde{A}_\psi(\mu) = \tilde{A}^{-\psi}(-\mu)$ and $\tilde{A}^\varphi(\mu) = \tilde{A}_{-\varphi}(-\mu)$.

We have

$$LT_\psi^\varphi(\mu) - \mu = -(LT_{-\varphi}^{-\psi}(-\mu) - (-\mu)).$$

This shows the statements for the strictly active sets. $\square$

Remark 4.4. If $f\colon U \to H^{-1}(\Omega)$ satisfies $f(-u) = -f(u)$ for all $u \in U$, then $-S_\psi^\varphi(u) = S_{-\psi}^{-\varphi}(-u)$.

Now, we check the monotonicity of the strictly active sets.

Lemma 4.5
Let the requirements of Assumption 1.1 be satisfied and let $u_1 \geq u_2$. Then it follows

1. $A^s_\psi(u_1) \subseteq A^s_\psi(u_2)$,
2. $A^\varphi_s(u_1) \supseteq A^\varphi_s(u_2)$.

Proof. ad 1.: As in Lemma 4.3, denote by T^φ_ψ the solution operator of the bilateral obstacle problem on $H^{-1}(\Omega)$, i.e., $(T^\varphi_\psi \circ f)(u) = S^\varphi_\psi(u)$.

Let $\mathcal{U} = \{S^\varphi_\psi(u_1) - \psi < (\varphi - \psi)/2\}$. Then $\mathcal{U}$ is quasi-open and contained in $I^\varphi(u_1)$. Fix $v \in H^1_0(\mathcal{U})_+$ satisfying $\{v > 0\} = \mathcal{U} \setminus A^s_\psi(u_2)$, $v < (\varphi - \psi)/2$, see Lemma 2.3. Let $y_v(t) = S^\varphi_{\psi - tv}(u_1)$, $t \in [0, 1]$, and denote $\bar{y}_v(t) := y_v(t) + tv$. Then

$$\begin{aligned} &\langle Ly_v(t) - f(u_1), z - y_v(t)\rangle \geq 0 \quad \forall\, z \in K^\varphi_{\psi - tv} \\ \Longleftrightarrow\ &\langle L\bar{y}_v(t) - f(u_1) - tLv, \bar{z} - \bar{y}_v(t)\rangle \geq 0 \quad \forall\, \bar{z} \in K^{\varphi + tv}_\psi \\ \Longrightarrow\ &\langle L\bar{y}_v(t) - f(u_1) - tLv, \bar{z} - \bar{y}_v(t)\rangle \geq 0 \quad \forall\, \bar{z} \in K^\varphi_\psi. \end{aligned}$$

Now $\psi \leq \bar{y}_v(t) = y_v(t) + tv \leq y_v(0) + tv \leq \varphi$ by Lemma 3.2 and the definition of $\mathcal{U}$ and v.

Hence, $\bar{y}_v(t) \in K^\varphi_\psi$ and the last line shows that $y_v(t) = T^\varphi_\psi(T(tv)) - tv$ with $T\colon H^1_0(\Omega) \to H^{-1}(\Omega), v \mapsto f(u_1) + Lv$. Since T^φ_ψ is directionally differentiable in the Hadamard sense, we can apply the chain rule for the directional derivatives and obtain

$$y'_v(0; 1) = (T^\varphi_\psi)'(T(0); T'(0; v)) - v = (T^\varphi_\psi)'(f(u_1); Lv) - v.$$

Since $(T^\varphi_\psi)'(f(u_1); T'(0; v))$ is 0 q.e. on the strictly active set, compare Section 3.2 and, in particular, Lemma 3.6, we have $y'_v(0; 1) < 0$ q.e. on $A^s_\psi(u_1) \cap \{v > 0\}$.

Thus, by reducing the lower obstacle on a subset of $A^s_\psi(u_1)$ the solution with respect to the new obstacle will drop on this set.

Now, we show the statement of the lemma by contradiction. Therefore, assume the set $W \subseteq \Omega$ is a set of positive capacity which is (lower) weakly active for u_2 and (lower) strictly active for u_1, i.e., $W \subseteq A^s_\psi(u_1) \subseteq A_\psi(u_2)$ and $\mathrm{cap}(W \cap A^s_\psi(u_2)) = 0$. Then $\mathcal{U} = \{S^\varphi_\psi(u_1) - \psi < (\varphi - \psi)/2\}$ as above is a quasi-open neighborhood of W contained in $I^\varphi(u_1)$.

Let as above $v \in H^1_0(\mathcal{U})_+$ satisfy $\{v > 0\} = \mathcal{U} \setminus A^s_\psi(u_2)$. Then, Lemma 4.2 yields

$$S^\varphi_{\psi - v}(u_2) = S^\varphi_\psi(u_2) \tag{17}$$

and on W we have

$$S^\varphi_{\psi - v}(u_1)|_W < S^\varphi_\psi(u_1)|_W = S^\varphi_\psi(u_2)|_W \tag{18}$$

by the structure of the directional derivative with respect to the obstacle. Putting Eqs. (17) and (18) together, we see that

$$S^\varphi_{\psi - v}(u_2) > S^\varphi_{\psi - v}(u_1)$$

on W. On the other hand, $S^\varphi_{\psi - v}(u_1) \geq S^\varphi_{\psi - v}(u_2)$ since $u_1 \geq u_2$. Thus, such a set W cannot exist and we conclude $A^s_\psi(u_1) \subseteq A^s_\psi(u_2)$.

ad 2.: By Lemma 4.3, we have $A^\varphi_s(u_i) = \tilde{A}^\varphi_s(f(u_i)) = \tilde{A}^s_{-\varphi}(-f(u_i))$ for $i = 1, 2$, where we use a similar notation as in Lemma 4.3. Now, the first part of the lemma implies the statement, since

$$A^\varphi_s(u_1) = \tilde{A}^s_{-\varphi}(-f(u_1)) \supseteq \tilde{A}^s_{-\varphi}(-(f(u_2)) = A^\varphi_s(u_2). \qquad \square$$

5 Mosco Convergence

For the rest of the paper, we use again the notation S for the solution operator of (1).

The following definition goes back to [Mos69]. In this form, the definition can be found, e.g., in [Rod87, Ch. 4:4].

Definition 5.1. We say that a sequence $(C_n)_{n\in\mathbb{N}}$ of nonempty, closed, convex subsets of a Banach space X converges to a set $C \subset X$ in the sense of Mosco if the following two conditions hold.

1. For each $x \in C$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n \in C_n$ holds for every $n \in \mathbb{N}$ and such that $x_n \to x$ in X.
2. For each subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of a sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ fulfilling $x_n \in C_n$ for all $n \in \mathbb{N}$ such that for some $x \in X$ we have $x_{n_k} \rightharpoonup x$ in X, the weak limit x is in C.

Based on this definition, the following result on convergence of solutions of variational inequalities can be established. It is taken from [Rod87, Thm. 4.1], see also [Mos69, Prop. 35.].

Lemma 5.2
Assume $L \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$ is coercive and let C_n and C be nonempty closed, convex subsets of $H_0^1(\Omega)$, $n \in \mathbb{N}$, such that $C_n \to C$ in the sense of Mosco. Furthermore, let $(h_n)_{n\in\mathbb{N}}, h \subset H^{-1}(\Omega)$ with $h_n \to h$. Then the solutions of

$$\text{Find } \xi_n \in C_n : \quad \langle L\xi_n - h_n, z - \xi_n\rangle \geq 0 \quad \forall\, z \in C_n$$

converge to the solution of

$$\text{Find } \xi \in C : \quad \langle L\xi - h, z - \xi\rangle \geq 0 \quad \forall\, z \in C.$$

Based on this tool, in order to obtain the convergence of the Gâteaux derivatives, which can be characterized as solutions to variational equations, see Theorem 3.8, and in order to characterize the limit, we establish the Mosco convergence of the sets $H_0^1(D(u_n))$. Depending on either the choice $D(u_n) = I(u_n) \cup A_w^\varphi(u_n)$ or $D(u_n) = I(u_n) \cup A_\psi^w(u_n)$, i.e., depending on whether we focus on the inactive set or the complement of the strictly active set with respect to either the upper or the lower obstacle, sequences with different monotone behavior have to be considered.

Lemma 5.3
Suppose Assumption 1.1 is satisfied.

1. *Let $(u_n)_{n\in\mathbb{N}} \subseteq U$ be an increasing sequence with $u_n \to u$. Then $H_0^1(I(u_n) \cup A_w^\varphi(u_n)) \to H_0^1(I(u) \cup A_w^\varphi(u))$ in the sense of Mosco.*
2. *Let $(u_n)_{n\in\mathbb{N}} \subseteq U$ be a decreasing sequence with $u_n \to u$. Then $H_0^1(I(u_n) \cup A_\psi^w(u_n)) \to H_0^1(I(u) \cup A_\psi^w(u))$ in the sense of Mosco.*

Proof. ad 1.: Assume $v \in H_0^1(I(u) \cup A_w^\varphi(u))$ and w.l.o.g. $v \geq 0$. We can rewrite the function space as

$$\begin{aligned} &H_0^1(I(u) \cup A_w^\varphi(u)) \\ &= \{z \in H_0^1(\Omega) : z = 0 \text{ q.e. on } A_\psi(u) \text{ and } z = 0 \text{ q.e. on } A_s^\varphi(u)\}. \end{aligned}$$

Since $A_s^\varphi(u_n) \subseteq A_s^\varphi(u)$ for all $n \in \mathbb{N}$, see Lemma 4.5, it holds $v = 0$ q.e. on $A_s^\varphi(u_n)$ for all $n \in \mathbb{N}$. Since $S(u_n) \to S(u)$ in $H_0^1(\Omega)$ by continuity of S, we have $S(u_n) \to S(u)$ for a subsequence pointwise quasi everywhere, see Lemma 2.3. This means

$$\operatorname{cap}\left(I_\psi(u) \setminus \bigcup_{k\in\mathbb{N}} I_\psi(u_k)\right) = 0,$$

i.e., $(I_\psi(u_n))_{n\in\mathbb{N}}$ is a quasi-covering of $I_\psi(u)$, which is increasing in n. We can therefore find nonnegative $v_n \in H_0^1(\Omega)$ with $v_n \to v$ and $v_n = 0$ q.e. on $A_\psi(u_n)$, see Lemma 2.3. By setting

$$z_n := \min(v_n, v)$$

we have $z_n \in H_0^1(I(u_n) \cup A_w^\varphi(u_n))$ for all $n \in \mathbb{N}$ as well as $z_n \to v$.

Let $v_n \in H_0^1(I(u_n) \cup A_w^\varphi(u_n))$ for all $n \in \mathbb{N}$. Assume there is a subsequence $(v_{n_k})_{k\in\mathbb{N}}$ with $v_{n_k} \rightharpoonup v$ for some $v \in H_0^1(\Omega)$ as $k \to \infty$. Since $A_\psi(u) \subseteq A_\psi(u_n)$ for all $n \in \mathbb{N}$, we conclude $v \in H_0^1(I_\psi(u))$ by Mazur's lemma. By Corollary 3.7 and Theorem 3.3, from $v_n = 0$ q.e. on $A_s^\varphi(u_n)$ and $v_n = 0$ q.e. on $A_\psi(u_n)$ it follows

$$\langle LS(u_n) - f(u_n), |v_n|\rangle = \int_\Omega |v_n|\, d\xi_\psi^n - \int_\Omega |v_n|\, d\xi_n^\varphi = 0$$

for all $n \in \mathbb{N}$. From $v_{n_k} \rightharpoonup v$ in $H_0^1(\Omega)$ we conclude $|v_{n_k}| \rightharpoonup |v|$ in $H_0^1(\Omega)$. (To see this, one can use the compact embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ and the estimate $\||z|\|_{H_0^1(\Omega)} \leq \|z\|_{H_0^1(\Omega)}$ for all $z \in H_0^1(\Omega)$, see also [KS00, Cor. A.5].) Since also $LS(u_{n_k}) - f(u_{n_k}) \to LS(u) - f(u)$ in $H^{-1}(\Omega)$ we conclude

$$0 = \langle LS(u) - f(u), |v|\rangle = -\int_\Omega |v|\, d\xi^\varphi$$

since $v = 0$ q.e. on $A_\psi(u)$. Finally, $v = 0$ ξ^φ-a.e. on Ω and thus $v = 0$ q.e. on $A_s^\varphi(u)$, see Corollary 3.7.

ad 2.: Again, this part of the lemma follows from the first part of the lemma combined with Lemma 4.3. □

6 Generalized Derivatives for the Bilateral Obstacle Problem

In this section, we will find a characterization of two generalized derivatives for the solution operator S of (1). To establish this result, we impose the monotonicity assumption Assumption 1.2 on U and f stated in the introduction.

As already indicated in the introduction, the assumptions posed on the positive cone in V will ensure that we can construct monotone convergent sequences in U where the Gâteaux differentiability of the locally Lipschitz continuous solution operator S can be guaranteed. The tool that is used is the following generalization of Rademacher's theorem to infinite dimensions, see e.g. [Aro76, Ch. II, Sect.2, Thm. 1], [BL00, Thm. 6.42]. If the space X in Theorem 6.1 is additionally a Hilbert space, a version can be found in [Mig76, Thm. 1.2].

Theorem 6.1
Assume $T\colon X \to Y$ is locally Lipschitz continuous from a separable Banach space X to a Hilbert space Y. Then the set D_T of points where T is Gâteaux differentiable is a dense subset of X.

In [Aro76], the map T is Lipschitz continuous and defined on an open subset of X. By considering neighborhoods of points separately, the formulation as in Theorem 6.1 can be obtained.

Now, we can formulate the main theorem of this paper.

Theorem 6.2
Suppose that Assumption 1.1 and Assumption 1.2 are satisfied and let $u \in U$ be arbitrary. Then a Bouligand generalized derivative for S in u is given by the operator $\Xi(u;\cdot) \in \mathcal{L}(U, H_0^1(\Omega))$, where $\Xi(u;h)$ is the unique solution of the variational equation

$$\textit{Find } \xi \in H_0^1(D(u)) : \quad \langle L\xi - f'(u;h), z\rangle = 0 \quad \forall\, z \in H_0^1(D(u)). \tag{19}$$

Here, the sets

$$D(u) := I(u) \cup A_w^\varphi(u) \qquad \textit{and} \qquad D(u) := I(u) \cup A_\psi^w(u)$$

can be chosen and result in generally different generalized derivatives.

Proof. The proof of [RU19, Prop. 5.5] implies that we can find an increasing, respectively, decreasing, sequence $(u_n)_{n\in\mathbb{N}} \subseteq U$ such that S is Gâteaux differentiable in each u_n and such that u_n converges to u. Here, Theorem 6.1 is used.

Let us first assume $(u_n)_{n\in\mathbb{N}}$ is an increasing sequence with these properties. Let $h \in U$ be arbitrary. By Theorem 3.8, for each $n \in \mathbb{N}$, $S'(u_n;h)$ can be written as the solution of the variational equation

$$\text{Find } \xi_n \in H_0^1(D(u_n)) : \quad \langle L\xi_n - f'(u_n;h), z\rangle = 0 \quad \forall\, z \in H_0^1(D(u_n)) \tag{20}$$

with the choice $D(u_n) = I(u_n) \cup A_w^\varphi(u_n)$.

By Lemma 5.3, we conclude

$$H_0^1(I(u_n) \cup A_w^\varphi(u_n)) \to H_0^1(I(u) \cup A_w^\varphi(u))$$

in the sense of Mosco. Now, Lemma 5.2 implies that $(S'(u_n;h))_{n\in\mathbb{N}}$ converges to the solution of (19) with $D(u) = I(u) \cup A_w^\varphi(u)$. By definition, the resulting operator is a generalized derivative for S.

When considering a decreasing sequence $(u_n)_{n\in\mathbb{N}}$, we use the representation of $S'(u_n;h)$ as the solution of the variational equation (20) with the choice $D(u_n) = I(u_n) \cup A_\psi^w(u_n)$ and obtain the respective Mosco convergence, and thus the convergence of $S'(u_n)$ to the solution operator of (19) with $D(u) = I(u) \cup A_\psi^w(u)$ from the second part of Lemma 5.3. $\square$

6.1 Adjoint Representation of Clarke Subgradients

As in the unilateral case, see [RU19, Thm. 5.7], we can find an adjoint representation for the subgradient of a reduced objective function.

Assume $J\colon H_0^1(\Omega) \times U \to \mathbb{R}$ is a continuously differentiable objective function. We consider the optimization problem

$$\begin{aligned} &\min_{y,u} J(y,u) \\ &\text{subject to } y \in K_\psi^\varphi, \quad \langle Ly - f(u), z - y\rangle \geq 0 \quad \forall\, z \in K_\psi^\varphi. \end{aligned}$$

We present a formula for two generalized derivatives contained in Clarke's generalized differential that can be obtained for the reduced objective function

$$\hat{J}(u) := J(S(u), u)$$

in an arbitrary point $u \in U$.

Corollary 6.3
Suppose that Assumption 1.1 and Assumption 1.2 are satisfied and let $u \in U$ be arbitrary. Denote by q the unique solution of the variational equation

$$\textit{Find } q \in H_0^1(D(u)), \quad \langle L^* q, v\rangle = \langle J_y(S(u),u), v\rangle \quad \textit{for all } v \in H_0^1(D(u)). \tag{21}$$

Then the element

$$f'(u)^*q + J_u(S(u), u)$$

is in Clarke's generalized differential $\partial_C \hat{J}(u)$*. In (21), the respective sets*

$$D(u) := I(u) \cup A_w^\varphi(u) \quad \text{and} \quad D(u) := I(u) \cup A_\psi^w(u)$$

can be chosen and result in a particular generalized derivative.

Proof. The proof is similar to the proof of [RU19, Thm. 5.4].

Since L^* is coercive, (21) has a unique solution. As stated in Remark 2.6, we have

$$\partial_C \hat{J}(u) \ni \Xi^* J_y(S(u), u) + J_u(S(u), u) \tag{22}$$

for all $\Xi \in \partial S(u)$.

Assume that q solves (21) for $D(u) = I(u) \cup A_w^\varphi(u)$, respectively $D(u) = I(u) \cup A_\psi^w(u)$. For $h \in U$, denote by $\Xi(u; h)$ the solution to (19). Now, we have

$$\begin{aligned} \langle f'(u)^* q, w\rangle_{U^*,U} &= \langle f'(u; w), q\rangle \\ &\overset{(19)}{=} \langle L^* q, \Xi(u; w)\rangle \\ &\overset{(21)}{=} \langle J_y(S(u), u), \Xi(u; w)\rangle \\ &= \langle \Xi(u; \cdot)^* J_y(s(u), u), w\rangle_{U^*,U} \end{aligned}$$

for all $w \in U$. Since $\Xi(u; \cdot) \in \partial S(u)$, the statement follows from (22). □

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