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*Optimal Control of Perfect Plasticity Part II:  
Displacement Tracking*

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1                   **OPTIMAL CONTROL OF PERFECT PLASTICITY**  
2                   **PART II: DISPLACEMENT TRACKING\***

3                   CHRISTIAN MEYER<sup>†</sup> AND STEPHAN WALTHER<sup>†</sup>

4       **Abstract.** The paper is concerned with an optimal control problem governed by the rate-  
5 independent system of quasi-static perfect elasto-plasticity. The objective is optimize the displace-  
6 ment field in the domain occupied by the body by means of prescribed Dirichlet boundary data,  
7 which serve as control variables. The arising optimization problem is nonsmooth for several reasons,  
8 in particular, since the control-to-state mapping is not single-valued. We therefore apply a Yosida  
9 regularization to obtain a single-valued control-to-state operator. Beside the existence of optimal  
10 solutions, their approximation by means of this regularization approach is the main subject of this  
11 work. It turns out that a so-called reverse approximation guaranteeing the existence of a suitable  
12 recovery sequence can only be shown under an additional smoothness assumption on at least one  
13 optimal solution.

14       **Key words.** Optimal control of variational inequalities, perfect plasticity, rate-independent  
15 systems, Yosida regularization, reverse approximation

16       **AMS subject classifications.** 49J20, 49J40, 74C05

17       **1. Introduction.** In this paper, we investigate the following optimal control  
18 problem governed by the equations of *quasi-static perfect plasticity* at small strain:

$$\begin{cases}
 \min & J(u, u_D) := \Psi(u) + \frac{\alpha}{2} \|u_D\|_{H^1(0,T;H^2(\Omega;\mathbb{R}^n))}^2 \\
 \text{s.t.} & -\operatorname{div} \sigma = 0 & \text{in } \Omega, \\
 & \sigma = \mathbb{C}(\nabla^s u - z) & \text{in } \Omega, \\
 & \dot{z} \in \partial I_{\mathcal{K}(\Omega)}(\sigma) & \text{in } \Omega, \\
 & u = u_D & \text{on } \Gamma_D, \\
 & \sigma \nu = 0 & \text{on } \Gamma_N, \\
 & u(0) = u_0, \quad \sigma(0) = \sigma_0 & \text{in } \Omega, \\
 \text{and} & u_D(0) = u_0 & \text{on } \Gamma_D.
 \end{cases}
 \tag{P}$$

19       Herein,  $u : (0, T) \times \Omega \rightarrow \mathbb{R}^n$ ,  $n = 2, 3$ , is the displacement field, while  $\sigma, z : (0, T) \times \Omega \rightarrow$   
20  $\mathbb{R}^{n \times n}$  are the stress tensor and the plastic strain. The boundary of  $\Omega$  is split in two  
21 disjoint parts  $\Gamma_D$  and  $\Gamma_N$  with outward unit normal  $\nu$ . Moreover,  $\mathbb{C}$  is the elasticity  
22 tensor and  $\mathcal{K}(\Omega)$  denotes the set of feasible stresses. The initial data  $u_0$  and  $\sigma_0$  are  
23 given and fixed. The Dirichlet data  $u_D$  represent the control variable and  $\alpha > 0$  a fixed  
24 Tikhonov regularization parameter. The objective  $\Psi$  only contains the displacement  
25 field. Objectives involving the stress are considered in a companion paper [21]. This is  
26 the reason for calling (P) *displacement tracking problem*. A mathematically rigorous  
27 version of (P) involving the function spaces and a rigorous notion of solutions for the  
28 state equation will be formulated in section 3 and 4 below. The precise assumptions  
29

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30 on the data are given in [section 2](#). Regarding to a more detailed description of the  
 31 plasticity model, we refer to [\[25\]](#) and the references therein.

32 Some words concerning our choice of the control variable are in order: In general,  
 33 Dirichlet control problems provide particular difficulties due to regularity issues, when  
 34 control functions in  $L^2(\partial\Omega)$  are considered, see e.g. [\[18\]](#). Nonetheless, we consider the  
 35 Dirichlet displacement as control variables instead of distributed loads or forces on  
 36 the Neumann boundary due to the *safe load condition*. It is well known that the  
 37 existence of solutions for the perfect plasticity system can only be shown under this  
 38 additional condition (see e.g. [\[30, 8\]](#)), which would lead to rather complex control  
 39 constraints and it is a completely open question how to incorporate these constraints  
 40 in the analysis of [\(P\)](#). For this reason, we focus on the Dirichlet control problem.  
 41 A possible realization of these controls by means of an additional linear elasticity  
 42 equation avoiding the  $H^2$ -norm in the objective is elaborated in the companion paper  
 43 [\[21\]](#).

44 Beside the safe-load condition, problem [\(P\)](#) exhibits several additional particular  
 45 challenges. First of all, it is obviously nonsmooth due to the convex subdifferen-  
 46 tial appearing in the state equation. Moreover, the state equation is in general not  
 47 uniquely solvable and its solutions significantly lack regularity, see [\[30, 8\]](#). Therefore,  
 48 there is no single-valued control-to-state mapping and [\(P\)](#) should rather be regarded  
 49 as an optimization problem in Banach space rather than an optimal control problem.  
 50 Beside the existence of optimal solutions, our main goal is to approximate [\(P\)](#) via  
 51 replacing  $\partial I_{\mathcal{K}(\Omega)}$  by its Yosida regularization. This is of course a classical procedure  
 52 and, in order to show that the approximation works, i.e., that optimal solutions of  
 53 the regularized problems converge to solutions of [\(P\)](#) (in a certain topology), the  
 54 following steps have to be performed:

- 55 1. The existence of (weak) accumulation points of sequences of optimal solutions  
 56 of the regularized problems have to be verified.
- 57 2. Weak limits have to be feasible for the original problem [\(P\)](#).
- 58 3. In order to show the optimality of the weak limit, one has to construct a  
 59 *recovery sequence* for at least one optimal solution of the original problem.

60 The last item is also known as *reverse approximation* and might become a challenging  
 61 task in the context of optimization of rate-independent systems, see [\[22\]](#). This also  
 62 happens to be the case here: In contrast to the perfect plasticity system, its regu-  
 63 larized counterpart admits a unique solution with full regularity. It is therefore very  
 64 unlikely that one can approximate *every* solution of the perfect plasticity system by  
 65 means of regularization and indeed, as classical examples demonstrate, this is in fact  
 66 not true, see e.g. [\[30\]](#) and [Example 3.10](#) below. However, in the context of optimal  
 67 control and optimization, respectively, we have the control as an additional variable  
 68 at hand and, in order to construct a recovery sequence, we have to find a sequence  
 69 of *tuples of state and control* feasible for the regularized problems so that the asso-  
 70 ciated objective function values converge to the optimal value of [\(P\)](#). This leads to  
 71 much more flexibility in the construction of recovery sequences, provided that the set  
 72 of controls is sufficiently rich. Unfortunately, this is not the case for our Dirichlet  
 73 control and we need an additional control variable in terms of *distributed loads* for the  
 74 construction of a recovery sequence. The idea is thus to introduce an additional load  
 75 in the balance of momentum of the regularized problems and to drive this load to  
 76 zero for vanishing regularization parameter. Our regularization procedure therefore  
 77 does not only replace the convex subdifferential by its Yosida regularization, but also  
 78 introduces a new additional control variable. To the best of our knowledge, this is a  
 79 completely new idea.

80 Nevertheless, even with this additional control variable, we are only able to con-  
 81 struct a recovery sequence under a fairly restrictive assumption. This assumption  
 82 is caused by additional smoothness constraints as part of the regularized optimal  
 83 control problems, which in turn are needed to pass to the limit in the regularized  
 84 plasticity system, when the regularization parameter is driven to zero. If we assume  
 85 that at least one optimal solution of the original (i.e., unregularized) optimization  
 86 problem admits an admittedly high regularity, then we are able to construct a re-  
 87 covery sequence for this particular solution, which meets the smoothness constraints  
 88 and is therefore feasible for the regularized optimal control problems. We thus obtain  
 89 the desired approximation result under the assumption that there exists at least one  
 90 “smooth” solution of (P).

91 Let us put our work into perspective: Quasi-static perfect plasticity is a rate-  
 92 independent system. Optimization and optimal control of such systems have been  
 93 considered by various authors and we only refer to [4, 5, 1, 6, 7, 29, 24, 2, 14] and the  
 94 references therein. Albeit still nonsmooth, optimization problems of this type sub-  
 95 stantially simplify, if the energy underlying the rate-independent system is uniformly  
 96 convex. In quasi-static plasticity, this is the case, if hardening is present. In this  
 97 case, the plasticity system admits a unique solution in the energy space, which makes  
 98 the construction of recovery sequences almost trivial. Nevertheless, the derivation of  
 99 optimality conditions is still an intricate issue, see [32, 33, 34]. While all contributions  
 100 mentioned so far deal with uniformly convex energies, the literature becomes rather  
 101 scarce, when it comes to energies that lack strict convexity. In [26, 28, 11, 10] the  
 102 existence of optimal solutions for problems with non-convex energies are shown. To  
 103 the best of our knowledge, the approximation of such problems has only been investi-  
 104 gated in [22, 27], where a time-discretization instead of a regularization is considered.  
 105 The approximation via discretization can however be hardly compared to our situ-  
 106 ation, since the discrete rate-independent systems are still not uniquely solvable so  
 107 that there is still no (discrete) control-to-state map in contrast to the regularized set-  
 108 ting. Therefore, the discrete optimization problems are still all but straight forward  
 109 to solve, whereas the regularized optimal control problems are amenable for standard  
 110 adjoint-based optimization methods.

111 The paper is organized as follows: After introducing our notation and standard  
 112 assumptions in section 2, we introduce a rigorous notion of solution to the perfect  
 113 plasticity system and recall the known results concerning the existence of solutions  
 114 and the lack of uniqueness in section 3. Then, section 4 is devoted to the existence of  
 115 at least one (globally) optimal solution of (P). In section 5, we lay the foundations  
 116 for our reverse approximation argument for the construction of a recovery sequence,  
 117 which is a basic ingredient for our main result in Theorem 6.3. The last section 6  
 118 covers this result and shows that solutions of (P) can indeed be approximated via  
 119 Yosida regularization provided the mentioned regularity assumption is fulfilled.

120 **2. Notation and Standing Assumptions.** We start with a short introduction  
 121 in the notation used throughout the paper and in parallel list our standing assump-  
 122 tions. The latter are tacitly assumed for the rest of the paper without mentioning  
 123 them every time.

124 *General notation.* Given two vector spaces  $X$  and  $Y$ , we denote the space of  
 125 linear and continuous functions from  $X$  into  $Y$  by  $\mathcal{L}(X, Y)$ . If  $X = Y$ , we simply  
 126 write  $\mathcal{L}(X)$ . The dual space of  $X$  is denoted by  $X^* = \mathcal{L}(X, \mathbb{R})$ . If  $H$  is a Hilbert  
 127 space, we denote its scalarproduct by  $(\cdot, \cdot)_H$ . For the whole paper, we fix the final  
 128 time  $T > 0$ . To shorten the notation, Bochner-spaces are abbreviated by  $L^p(X) :=$

129  $L^p(0, T; X)$ ,  $W^{1,p}(X) := W^{1,p}(0, T; X)$  ( $p \in [1, \infty]$ ), and  $C(X) := C([0, T]; X)$ . Note  
 130 that functions in  $C(X)$  are continuous on the whole time interval. When  $G \in \mathcal{L}(X; Y)$   
 131 is a linear and continuous operator, we can define an operator in  $\mathcal{L}(L^p(X); L^p(Y))$   
 132 by  $G(u)(t) := G(u(t))$  for all  $u \in L^p(X)$  and for almost all  $t \in [0, T]$ , we denote this  
 133 operator also by  $G$ , that is,  $G \in \mathcal{L}(L^p(X); L^p(Y))$ , and analog for Bochner-Sobolev  
 134 spaces, i.e.,  $G \in \mathcal{L}(W^{1,p}(X); W^{1,p}(Y))$ .

135 Given a coercive operator  $G \in \mathcal{L}(H)$  in a Hilbert space  $H$ , we denote its coercivity  
 136 constant by  $\gamma_G$ , i.e.,  $(Gh, h)_H \geq \gamma_G \|h\|_H^2$  for all  $h \in H$ . With this operator we can  
 137 define a new scalar product, which induces an equivalent norm, by  $H \times H \ni (h_1, h_2) \mapsto$   
 138  $(Gh_1, h_2)_H \in \mathbb{R}$ . We denote the Hilbert space equipped with this scalar product by  
 139  $H_G$ , that is  $(h_1, h_2)_{H_G} = (Gh_1, h_2)_H$  for all  $h_1, h_2 \in H$ .

140 If  $p \in [1, \infty]$ , then we denote its conjugate exponent by  $p'$ , that is  $\frac{1}{p} + \frac{1}{p'} = 1$ .  
 141 Furthermore,  $c, C > 0$  are generic constants.

142 *Matrices.* Given a matrix  $\tau \in \mathbb{R}^{n \times n}$ , we define its deviatoric (i.e., trace-free) part  
 143 as

$$144 \quad \tau^D := \tau - \frac{1}{n} \operatorname{tr}(\tau) I$$

145 and use the same notation for matrix-valued functions. The Frobenius norm is denoted  
 146 by  $|A|_F^2 = \sum_{i,j=1}^n A_{ij}^2$  for  $A \in \mathbb{R}^{n \times n}$  and for the associated scalar product, we write  
 147  $A : B = \sum_{i,j=1}^n A_{ij} B_{ij}$ ,  $A, B \in \mathbb{R}^{n \times n}$ . By  $\mathbb{R}_{\operatorname{sym}}^{n \times n}$ , we denote the space of symmetric  
 148 matrices.

149 *Domain.* The domain  $\Omega \subset \mathbb{R}^n$ ,  $n \in \mathbb{N}$ ,  $n \geq 2$ , is bounded of class  $C^1$ . The  
 150 boundary consists of two disjoint measurable parts  $\Gamma_N$  and  $\Gamma_D$  such that  $\Gamma = \Gamma_N \cup \Gamma_D$ .  
 151 While  $\Gamma_N$  is a relatively open subset,  $\Gamma_D$  is a relatively closed. We moreover suppose  
 152 that  $\Gamma_D$  has a nonempty relative interior. In addition, the set  $\Omega \cup \Gamma_N$  is regular in the  
 153 sense of Gröger, cf. [15]. Throughout the article,  $\nu : \partial\Omega \rightarrow \mathbb{R}^n$  denotes the outward  
 154 unit normal vector.

155 Thanks to the regularity of  $\Omega$ , the harmonic extension  $\mathfrak{E}$  maps  $C^1(\Gamma)$  to  $W^{1,p}(\Omega)$   
 156 for some  $p > n$ . Moreover, the maximum principle implies that

$$157 \quad (2.1) \quad \|\mathfrak{E}\varphi\|_{L^\infty(\Omega)} \leq \|\varphi\|_{L^\infty(\Gamma)} \quad \forall \varphi \in C^1(\Gamma).$$

158 *Remark 2.1.* The  $C^1$ -regularity of  $\Omega$  and its boundary, respectively, is required  
 159 for the trace theorem and the formula of integration by parts for BD-functions in [31,  
 160 Chap. II, Theorem 2.1], which will be used several times throughout the paper. In [12,  
 161 Section 6], it is claimed that this formula integration by parts also holds in Lipschitz  
 162 domains, but no proof is provided. Since the minimal regularity of the boundary is  
 163 not in the focus of this paper and would go beyond the scope of our work, we restrict  
 164 to domains of class  $C^1$ .

165 *Spaces.* Throughout the paper, by  $L^p(\Omega; M)$  we denote Lebesgue spaces with  
 166 values in  $M$ , where  $p \in [1, \infty]$  and  $M$  is a finite dimensional space. To shorten  
 167 notation, we abbreviate

$$168 \quad \mathbb{L}^p(\Omega) := L^p(\Omega; \mathbb{R}^n) \quad \text{and} \quad \mathbb{L}^p(\Omega) := L^p(\Omega; \mathbb{R}_{\operatorname{sym}}^{n \times n}).$$

169 Given  $s \in \mathbb{N}$  and  $p \in [1, \infty]$ , the Sobolev spaces of vector- resp. tensor-valued functions  
 170 are denoted by

$$171 \quad \begin{aligned} \mathbf{W}^{s,p}(\Omega) &:= W^{s,p}(\Omega; \mathbb{R}^n), & \mathbf{H}^s(\Omega) &:= \mathbf{W}^{s,2}(\Omega), \\ \mathbb{W}^{s,p}(\Omega) &:= W^{s,p}(\Omega; \mathbb{R}_{\operatorname{sym}}^{n \times n}), & \mathbb{H}^s(\Omega) &:= \mathbb{W}^{s,2}(\Omega). \end{aligned}$$

172 Furthermore, set

$$173 \quad (2.2) \quad \mathbf{W}_D^{1,p}(\Omega) := \overline{\{\psi|_\Omega : \psi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n), \text{supp}(\psi) \cap \Gamma_D = \emptyset\}}^{\mathbf{W}^{1,p}(\Omega)}$$

174 and define  $\mathbf{H}_D^1(\Omega)$  analogously. The dual of  $\mathbf{H}_D^1(\Omega)$  is denoted by  $\mathbf{H}_D^{-1}(\Omega)$ . The space  
175 of *bounded deformation* is abbreviated by

$$176 \quad \text{BD}(\Omega) := \{u \in \mathbf{L}^1(\Omega) : \frac{1}{2}(\partial_i u_j + \partial_j u_i) \in \mathfrak{M}(\Omega) \forall i, j = 1, \dots, n\},$$

177 where  $\mathfrak{M}(\Omega)$  denotes the space of regular Borel measures on  $\Omega$  and the (partial)  
178 derivatives are of course understood in a distributional sense. Equipped with the  
179 norm

$$180 \quad \|u\|_{\text{BD}(\Omega)} := \|u\|_{\mathbf{L}^1(\Omega)} + \sum_{i,j=1}^n \frac{1}{2} \|\partial_i u_j + \partial_j u_i\|_{\mathfrak{M}(\Omega)},$$

181 it becomes a Banach space.

182 *Coefficients.* The elasticity tensor satisfies  $\mathbb{C} \in \mathcal{L}(\mathbb{R}_{\text{sym}}^{d \times d})$  and is symmetric and  
183 coercive. In addition we set  $\mathbb{A} := \mathbb{C}^{-1}$  and note that  $\mathbb{A}$  is symmetric and coercive,  
184 too. Let us note that  $\mathbb{C}$  could also depend on space, however, to keep the discussion  
185 concise, we restrict ourselves to constant elasticity tensors.

186 *Yield condition.* The set defining the yield condition is denoted by  $K \subset \mathbb{R}_{\text{sym}}^{n \times n}$   
187 and is closed and convex and there exists  $0 < \rho < R$  such that

$$188 \quad (2.3) \quad \overline{B_{\mathbb{R}^{n \times n}}(0; \rho)} \subset K \subset \overline{B_{\mathbb{R}^{n \times n}}(0; R)}.$$

189 Given this set, we define the *set of admissible stresses* as

$$190 \quad \mathcal{K}(\Omega) := \{\tau \in \mathbb{L}^2(\Omega) : \tau^D(x) \in K \text{ f.a.a. } x \in \Omega\}.$$

191

192 *Remark 2.2.* The boundedness of the set  $K$  is not really needed for our analysis.  
193 It is only required for the formula of integration by parts in (3.9), which we only  
194 need to compare our notion of solution to the one in [8]. Nevertheless, we kept the  
195 boundedness assumption on the set  $K$ , since it is fulfilled in all practically relevant  
196 examples such as e.g. the von Mises or the Tresca yield condition.

197 *Operators.* Throughout the paper,  $\nabla^s := \frac{1}{2}(\nabla + \nabla^\top) : \mathbf{W}^{1,p}(\Omega) \rightarrow \mathbb{L}^p(\Omega)$  denotes  
198 the linearized strain. Its restriction to  $\mathbf{W}_D^{1,p}(\Omega)$  is denoted by the same symbol and,  
199 for the adjoint of this restriction, we write  $-\text{div} := (\nabla^s)^* : \mathbb{L}^{p'}(\Omega) \rightarrow \mathbf{W}_D^{1,p}(\Omega)^*$ .

200 Let  $\mathcal{K} \subset \mathbb{L}^2(\Omega)$  be a closed and convex set. We denote the indicator function by

$$201 \quad I_{\mathcal{K}} : \mathbb{L}^2(\Omega) \rightarrow \{0, \infty\}, \quad \tau \mapsto \begin{cases} 0, & \tau \in \mathcal{K}, \\ \infty, & \tau \notin \mathcal{K}. \end{cases}$$

202 By  $\partial I_{\mathcal{K}} : \mathbb{L}^2(\Omega) \rightarrow 2^{\mathbb{L}^2(\Omega)}$  we denote the subdifferential of the indicator function. For  
203  $\lambda > 0$ , the Yosida regularization is given by

$$204 \quad (2.4) \quad I_\lambda : \mathbb{L}^2(\Omega) \rightarrow \mathbb{R}, \quad \tau \mapsto \frac{1}{2\lambda} \|\tau - \pi_{\mathcal{K}}(\tau)\|_{\mathbb{L}^2(\Omega)}^2,$$

205 where  $\pi_{\mathcal{K}}$  is the projection onto  $\mathcal{K}$  in  $\mathbb{L}^2(\Omega)$ , and its Fréchet derivative is

$$206 \quad \partial I_{\lambda}(\tau) = \frac{1}{\lambda}(\tau - \pi_{\mathcal{K}}(\tau)).$$

208 When  $\lambda = 0$  we define  $I_{\lambda} = I_0 := I_{\mathcal{K}}$ . For a sequence  $\{\lambda_n\}_{n \in \mathbb{N}} \subset (0, \infty)$  we abbreviate  
209  $I_n := I_{\lambda_n}$ .

210 *Initial data.* For the initial stress field  $\sigma_0$ , we assume that  $\sigma_0 \in \mathbf{W}^{1,p}(\Omega)$  with  
211 some  $p > n$ . Moreover,  $\sigma_0$  satisfies the equilibrium condition, i.e.,  $\operatorname{div} \sigma_0 = 0$  a.e. in  
212  $\Omega$ , and the yield condition, i.e.,  $\sigma_0 \in \mathcal{K}(\Omega)$ . The initial displacement  $u_0$  is supposed  
213 to be an element of  $\mathbf{H}^2(\Omega)$  and we require  $\operatorname{tr}(\nabla^s u_0 - \mathbb{A}\sigma_0) = 0$  a.e. in  $\Omega$  in order to  
214 obtain a purely deviatoric initial plastic strain.

215 *Remark 2.3.* The high regularity of  $u_0$  is just needed to ensure that the feasible  
216 set of (P) is nonempty. For the mere discussion of the state system, this is not  
217 necessary. The same holds for the assumption  $\sigma_0 \in \mathbf{W}^{1,p}(\Omega)$ , which will be needed to  
218 construct a recovery sequence for the optimal control problem.

219 *Optimization Problem.* The Tikhonov parameter  $\alpha$  is a positive constant and  $\Psi$   
220 is a functional that is bounded from below and satisfies a certain lower semicontinuity  
221 assumption w.r.t. weak convergence in the displacement space, which will be made  
222 precise in section 4 below, see (4.8).

223 **3. State Equation.** We start our investigations with the analysis of the state  
224 system and recall some known results concerning quasi-static perfect plasticity. Al-  
225 ready since the pioneering work of Suquet [30], it is well known that a precise definition  
226 of a solution to the system of perfect plasticity is all but straight forward, since a so-  
227 lution of the system in its “natural” form (below termed strong solution) does in  
228 general not exist due to a lack of regularity of the displacement and the plastic strain,  
229 respectively. We start with the definition of the function spaces already indicating  
230 this lack of regularity:

231 **DEFINITION 3.1** (State spaces).

232 1. Stress space:

$$233 \quad \Sigma(\Omega) := \{\tau \in \mathbb{L}^2(\Omega) : \operatorname{div} \tau \in \mathbf{L}^n(\Omega), \tau^D \in \mathbb{L}^\infty(\Omega)\}$$

234 2. Displacement space:

$$235 \quad \mathcal{U} := \{u \in H^1(\mathbf{L}^{\frac{n}{n-1}}(\Omega)) : \nabla^s \dot{u} \in L_w^2(\mathfrak{M}(\Omega; \mathbb{R}_{\operatorname{sym}}^{n \times n}))\}.$$

236 *Herein,  $L_w^2(\mathfrak{M}(\Omega; \mathbb{R}_{\operatorname{sym}}^{n \times n}))$  is the space of weakly measurable functions with*  
237 *values in  $\mathfrak{M}(\Omega; \mathbb{R}_{\operatorname{sym}}^{n \times n})$ , for which  $t \mapsto \|\mu(t)\|_{\mathfrak{M}}$  is an element of  $L^2(0, T; \mathbb{R})$ .*  
238 *For the definition of weak measurability, we refer to [9, Section 8].*

239 *We say that a sequence  $\{u_n\} \subset \mathcal{U}$  converges weakly in  $\mathcal{U}$  to  $u$  and write*  
240  *$u_n \rightharpoonup u$  in  $\mathcal{U}$ , iff*

$$241 \quad (3.1) \quad u_n \rightharpoonup u \text{ in } H^1(\mathbf{L}^{\frac{n}{n-1}}(\Omega)), \quad \nabla^s \dot{u}_n \rightharpoonup^* \nabla^s \dot{u} \text{ in } L_w^2(\mathfrak{M}(\Omega; \mathbb{R}_{\operatorname{sym}}^{n \times n})).$$

242 *Note that, by [9, Theorem 8.20.3],  $L_w^2(\mathfrak{M}(\Omega; \mathbb{R}_{\operatorname{sym}}^{n \times n})) = L^2(C_0(\Omega; \mathbb{R}_{\operatorname{sym}}^{n \times n}))^*$ ,*  
243 *which gives a meaning to the weak-\* convergence in (3.1).*

244 *Remark 3.2.* Unfortunately,  $\operatorname{BD}(\Omega)$  does not admit the Radon-Nikodým property  
245 and therefore weak measurability does not imply Bochner-measurability.

246 DEFINITION 3.3 (Equilibrium condition). *We define the set of stresses which ful-*  
 247 *fill the equilibrium condition as*

$$248 \quad \mathcal{E}(\Omega) := \ker(\operatorname{div}) = \{\tau \in \mathbb{L}^2(\Omega) : (\tau, \nabla^s \varphi)_{\mathbb{L}^2(\Omega)} = 0 \quad \forall \varphi \in \mathbf{H}_D^1(\Omega)\}.$$

249 Note that  $\sigma \in \mathcal{E}(\Omega) \cap \mathcal{K}(\Omega)$  implies  $\sigma \in \Sigma(\Omega)$ .

251 With the above definitions at hand, we can now define a hierarchy of three dif-  
 252 ferent solutions:

253 DEFINITION 3.4 (Notions of solutions). *Let  $u_D \in H^1(\mathbf{H}^1(\Omega))$  with  $u_D(0) = u_0$*   
 254 *a.e. on  $\Gamma_D$  be given. Then we define the following notions of solutions to the perfect*  
 255 *plasticity system:*

256 1. Reduced solution: *A function  $\sigma \in H^1(\mathbb{L}^2(\Omega))$  is called reduced solution of*  
 257 *the state equation, if, for almost all  $t \in (0, T)$ , the following holds true:*

259 • *Equilibrium and yield condition:*

$$260 \quad (3.2a) \quad \sigma(t) \in \mathcal{E}(\Omega) \cap \mathcal{K}(\Omega),$$

262 • *Reduced flow rule inequality:*

$$263 \quad (3.2b) \quad \int_{\Omega} (\mathbb{A}\dot{\sigma}(t) - \nabla^s \dot{u}_D(t)) : (\tau - \sigma(t)) \, dx \geq 0 \quad \forall \tau \in \mathcal{E}(\Omega) \cap \mathcal{K}(\Omega),$$

265 • *Initial condition:*

$$266 \quad (3.2c) \quad \sigma(0) = \sigma_0.$$

268 2. Weak solution: *A tuple  $(u, \sigma) \in \mathcal{U} \times H^1(\mathbb{L}^2(\Omega))$  is called weak solution of the*  
 269 *state equation, if, for almost all  $t \in (0, T)$ , there holds*

271 • *Equilibrium and yield condition:*

$$272 \quad (3.3a) \quad \sigma(t) \in \mathcal{E}(\Omega) \cap \mathcal{K}(\Omega),$$

274 • *Weak flow rule inequality:*

$$275 \quad (3.3b) \quad \begin{aligned} & \int_{\Omega} \mathbb{A}\dot{\sigma}(t) : (\tau - \sigma(t)) \, dx + \int_{\Omega} \dot{u}(t) \cdot \operatorname{div}(\tau - \sigma(t)) \, dx \\ & \geq \int_{\Omega} \nabla^s \dot{u}_D(t) : (\tau - \sigma(t)) + \dot{u}_D(t) \cdot \operatorname{div}(\tau - \sigma(t)) \, dx \\ & \quad \forall \tau \in \Sigma(\Omega) \cap \mathcal{K}(\Omega), \end{aligned}$$

277 • *Initial condition:*

$$278 \quad (3.3c) \quad u(0) = u_0, \quad \sigma(0) = \sigma_0.$$

280 3. Strong solution: *A tuple  $(u, \sigma) \in H^1(\mathbf{H}^1(\Omega)) \times H^1(\mathbb{L}^2(\Omega))$  is called strong*  
 281 *solution of the state equation, if, for almost all  $t \in (0, T)$ , there holds*

283 • *Equilibrium and yield condition:*

$$284 \quad (3.4a) \quad \sigma(t) \in \mathcal{E}(\Omega) \cap \mathcal{K}(\Omega),$$

- *Strong flow rule inequality:*

$$(3.4b) \quad \int_{\Omega} \mathbb{A}\dot{\sigma}(t) : (\tau - \sigma(t)) \, dx + \int_{\Omega} \nabla^s \dot{u}(t) : (\tau - \sigma(t)) \, dx \geq 0$$

$$\forall \tau \in \mathcal{K}(\Omega),$$

- *Dirichlet boundary condition:*

$$(3.4c) \quad u(t) - u_D(t) \in \mathbf{H}_D^1(\Omega)$$

- *Initial condition:*

$$(3.4d) \quad u(0) = u_0, \quad \sigma(0) = \sigma_0.$$

Some words concerning this definition are in order. First, let us shortly investigate the relationship between the three different solution concepts. By restricting the test functions in (3.3b) to functions in  $\mathcal{E}(\Omega)$ , one immediately observes that every weak solution is also a reduced solution. Moreover, by integration by parts, it is evident that (3.4b) and (3.4c) imply (3.3b). On the other hand, if a weak solution satisfies  $u \in H^1(\mathbf{H}^1(\Omega))$  and the Dirichlet boundary conditions in (3.4c), then integration by parts yields (3.4b), provided that  $\Sigma(\Omega) \cap \mathcal{K}(\Omega)$  is dense in  $\mathcal{K}(\Omega)$ , which is a direct consequence of Lemma A.1 proven in the appendix. Thus, we have the following relations between the three different solution concepts:

COROLLARY 3.5 (Relations between the solution concepts).

1. If  $(u, \sigma)$  is a weak solution, then  $\sigma$  is automatically a reduced solution.
2. A weak solution  $(u, \sigma)$  is a strong solution, if and only if  $u \in H^1(\mathbf{H}^1(\Omega))$  and  $(u - u_D)(t) \in \mathbf{H}_D^1(\Omega)$  for all  $t \in [0, T]$ .

One may further ask why no Dirichlet boundary conditions appear in the definition of a weak solution. In fact, from a mechanical point of view, it is reasonable that no boundary conditions are imposed, since plastic slips may well develop on the Dirichlet part on the boundary, too, in form of tangential jumps of the displacement perpendicular to the outward normal  $\nu$ . This observation is implicitly contained in the above definition as demonstrated in [8, Theorem 6.1]. For convenience of the reader, we shortly sketch the underlying arguments. To this end, suppose that a weak solution is given and let us define the *plastic strain*  $z \in L_w^2(\mathfrak{M}(\Omega \cup \Gamma_D; \mathbb{R}_{\text{sym}}^{n \times n}))$  by

$$(3.5) \quad z|_{\Omega} := \nabla^s(u) - \mathbb{A}\sigma \, dx, \quad z|_{\Gamma_D} := (u - u_D) \odot \nu \mathcal{H}^{n-1},$$

where  $\odot$  refers to the symmetrized dyadic product, i.e.,  $a \odot b = 1/2(a_i b_j + a_j b_i)_{i,j=1}^n$  for  $a, b \in \mathbb{R}^n$ . Note that functions in  $\text{BD}(\Omega)$  admit traces in  $L^1(\partial\Omega; \mathbb{R}^n)$  (see e.g. [31, Chap. II, Thm. 2.1]) so that  $z|_{\Gamma_D}$  is well defined. According to [8, Lemma 5.5], these equations carry over to the time derivatives for almost all  $t \in (0, T)$ , i.e.,

$$(3.6) \quad \dot{z}|_{\Omega} := \nabla^s(\dot{u}) - \mathbb{A}\dot{\sigma} \, dx, \quad \dot{z}|_{\Gamma_D} := (\dot{u} - \dot{u}_D) \odot \nu \mathcal{H}^{n-1}.$$

Let us prove that the trace of  $p$  vanishes. For this purpose, we need the following formula of integration by parts:

LEMMA 3.6 ([31, Chap. II, Thm. 2.1]). *For every  $v \in \text{BD}(\Omega)$  and every  $\varphi \in W^{1,p}(\Omega)$ ,  $p > n$ , there holds*

$$(3.7) \quad \int_{\Omega} \frac{1}{2}(v_i \partial_j \varphi + v_j \partial_i \varphi) \, dx + \int_{\Omega} \varphi \, d(\nabla^s v)_{ij} = \int_{\partial\Omega} \varphi \frac{1}{2}(v_i \nu_j + v_j \nu_i) \, ds$$

for all  $i, j = 1, \dots, n$ .

328 *Remark 3.7.* The result in [31] is only stated for test functions in  $C^1(\bar{\Omega})$ . However,  
 329 the embeddings  $\text{BD}(\Omega) \hookrightarrow \mathbf{L}^{\frac{n}{n-1}}(\Omega)$  and  $W^{1,p}(\Omega) \hookrightarrow C(\bar{\Omega})$ ,  $p > n$ , along with the  
 330 trace theorem for BD-functions and the density of  $C^1(\bar{\Omega})$  in  $W^{1,p}(\Omega)$  imply that the  
 331 integration by parts also holds for test functions in  $W^{1,p}(\Omega)$ .

332 Now, let  $\varphi \in C_c^\infty(\Omega)$  be arbitrary. Then, since  $\mathcal{K}(\Omega)$  just acts on the deviatoric  
 333 part,  $\varphi \delta_{ij} + \sigma_{ij}(t) \in \Sigma(\Omega) \cap \mathcal{K}(\Omega)$  for all  $t \in [0, T]$  and therefore (3.3b) and the above  
 334 formula of integration by parts give

$$335 \quad \sum_i \left( \int_{\Omega} (\mathbb{A}\dot{\sigma})_{ii} \varphi \, dx + \int_{\Omega} \varphi \, d(\nabla^s \dot{u})_{ii} \right) = 0 \quad \forall \varphi \in C_c^\infty(\Omega)$$

336 and therefore  $\text{tr } \dot{z}|_{\Omega} = 0$  f.a.a.  $t \in (0, T)$ . Since  $\text{tr}(\nabla^s(u_0) - \mathbb{A}\sigma_0) = 0$ , [8, Theorem 7.1]  
 337 yields  $\text{tr } z|_{\Omega} = 0$  for all  $t \in [0, T]$ . Similarly, we choose an arbitrary test function  
 338  $\psi \in C^\infty(\Gamma)$  with  $\text{supp}(\psi) \subset \Gamma_D$  and test (3.3b) with  $\mathfrak{E}\psi \delta_{ij} + \sigma_{ij}(t) \in \Sigma(\Omega) \cap \mathcal{K}(\Omega)$ .  
 339 Note that  $\mathfrak{E}\psi \delta_{ij} \in \Sigma(\Omega)$ , since the harmonic extension maps into  $W^{1,p}(\Omega)$  with  $p > n$ .  
 340 Applying then again the formula of integration by parts implies, in view of  $\text{tr } \dot{z}|_{\Omega} = 0$ ,  
 341 that

$$342 \quad (3.8) \quad (\dot{u} - \dot{u}_D) \cdot \nu = 0 \quad \text{a.e. on } \Gamma_D.$$

343 As  $u_0 = u_D(0)$  a.e. on  $\Gamma_D$ , this yields  $(u - u_D) \cdot \nu = 0$  a.e. on  $\Gamma_D$ , giving in turn  
 344  $\text{tr } z|_{\Gamma_D} = 0$  for all  $t \in [0, T]$ . Now that we know that  $z$  is deviatoric, the formula of  
 345 integration by parts from [8, Proposition 2.2] is applicable, which yields

$$346 \quad (3.9) \quad \langle \tau^D, \dot{z}(t) \rangle + \int_{\Omega} \tau : (\mathbb{A}\dot{\sigma}(t) - \nabla^s(\dot{u}_D(t))) \, dx = \int_{\Omega} \text{div } \tau \cdot (\dot{u}(t) - \dot{u}_D(t)) \, dx$$

347 for almost all  $t \in (0, T)$  and all  $\tau \in \Sigma(\Omega)$ . It is to be noted that the duality product  
 348  $\langle \tau^D, \dot{z} \rangle$  has to be treated with care, since, in general,  $\tau^D \notin C(\bar{\Omega}; \mathbb{R}_{\text{sym}}^{n \times n})$ , but  $\dot{z}$  is  
 349 only a measure. For a detailed and rigorous discussion of this issue, we refer to [8,  
 350 Section 2.3]. Inserting (3.9) in the flow rule inequality (3.3b) then results in

$$351 \quad (3.10) \quad \langle \tau^D - \sigma^D(t), \dot{z}(t) \rangle \geq 0 \quad \forall \tau \in \Sigma(\Omega) \cap \mathcal{K}(\Omega),$$

352 which is just the maximum plastic work inequality illustrating that  $z$  as defined in  
 353 (3.5) is indeed the correct object for the plastic strain. As a byproduct, we obtain the  
 354 second equation in (3.5) as boundary condition on  $\Gamma_D$  indicating that the Dirichlet  
 355 boundary condition in (3.4c) as part of the definition of a strong solution is in general  
 356 too restrictive as already mentioned above. Accordingly, a strong solution does in  
 357 general not exist, while we have the following result for a weak solution:

358 **PROPOSITION 3.8** (Existence of weak solutions, [30, Résultat 2]). *For all  $u_D \in$*   
 359  *$H^1(\mathbf{H}^1(\Omega))$ , there exists a weak solution in the sense of Definition 3.4.*

360 *Proof.* Using the Yosida regularization, Suquet showed in [30] the existence of a  
 361 functions  $\sigma \in H^1(\mathbb{L}^2(\Omega))$  and  $v \in L_w^2(\text{BD}(\Omega))$  so that, for almost all  $t \in (0, T)$ ,

$$362 \quad (3.11) \quad \begin{aligned} & -\text{div } \sigma(t) \in \mathcal{E}(\Omega) \cap \mathcal{K}(\Omega), \\ & \int_{\Omega} \mathbb{A}\dot{\sigma}(t) : (\tau - \sigma(t)) \, dx + \int_{\Omega} v(t) \cdot \text{div}(\tau - \sigma(t)) \, dx \\ & \quad \geq \langle \dot{u}_D(t), (\tau - \sigma(t))\nu \rangle_{H^{1/2}(\Gamma_D), H^{-1/2}(\Gamma_D)} \quad \forall \tau \in \Sigma(\Omega) \cap \mathcal{K}(\Omega), \\ & \quad \sigma(0) = \sigma_0. \end{aligned}$$

363 Due to the continuous embedding  $\text{BD}(\Omega) \hookrightarrow \mathbf{L}^{\frac{n}{n-1}}(\Omega)$  (see e.g. [31, Chap. II, Theo-  
 364 rem 2.2]) and the Radon-Nikodym property of  $\mathbf{L}^{\frac{n}{n-1}}(\Omega)$ , we have that  $v \in L^2(\mathbf{L}^{\frac{n}{n-1}}(\Omega))$ .  
 365 Therefore,

$$366 \quad u(t) := u_0 + \int_0^t v(r) \, dr$$

367 is an element of  $H^1(\mathbf{L}^{\frac{n}{n-1}}(\Omega))$  and satisfies the initial condition in (3.3c). Inserting  
 368 this in (3.11) and integrating the right hand side by parts gives the desired flow rule  
 369 inequality (3.3b). The claimed regularity of  $u$  directly follows from the regularity of  
 370  $v = \dot{u}$ .  $\square$

371 *Remark 3.9* (Other equivalent notions of solutions). Beside the reformulation of  
 372 the flow rule in terms of the maximum plastic work inequality (3.10), there are other  
 373 solutions concepts, which are equivalent to the definition of a weak solution, such  
 374 as the notion of a *quasi-static evolution*, which in essence corresponds to a global  
 375 energetic solution in the sense of [23]. For an overview over the various notions of  
 376 solutions and a rigorous proof of their equivalence, we refer to [8, Section 6]. A slightly  
 377 sloppy, but very illustrating derivation of the flow rule out of the quasi-static evolution  
 378 can also be found in [13].

379 Unfortunately, the weak solution is not unique, as the following example shows:

380 **EXAMPLE 3.10** ([30, Section 2.1]). *We choose  $\Omega = (0, 1)$ ,  $\Gamma_D = \partial\Omega = \{0, 1\}$ ,  
 381  $T = 1$ ,  $K = [-1, 1]$ ,  $\mathbb{C} = 1$ ,  $(\sigma_0, u_0) = 0$ , and  $u_D(t, x) := 2tx$ . One easily verifies that  
 382 the stress does only depend on the time with  $\sigma(t) = 2t$  for  $t \in (0, \frac{1}{2})$  and  $\sigma(t) = 1$  for  
 383  $t \in (\frac{1}{2}, 1)$ . For the displacement one obtains  $u(t, x) = 2tx$  for  $(t, x) \in (0, \frac{1}{2}) \times (0, 1)$   
 384 so that it is unique for  $t \in (0, \frac{1}{2})$ . For  $t \in (\frac{1}{2}, 1)$  there are more than one solution, for  
 385 example*

$$386 \quad u(t, x) = 2tx, \quad \text{if } (t, x) \in (\tfrac{1}{2}, 1) \times (0, 1),$$

$$387 \quad u(t, x) = \begin{cases} \frac{2tx}{\beta} + x - \frac{x}{\beta}, & \text{if } (t, x) \in (\tfrac{1}{2}, 1) \times [0, \beta], \\ 2t + x - 1, & \text{if } (t, x) \in (\tfrac{1}{2}, 1) \times [\beta, 1], \end{cases}$$

$$388 \quad u(t, x) = \begin{cases} x, & \text{if } (t, x) \in (\tfrac{1}{2}, 1) \times [0, \beta], \\ \alpha t + x - \frac{\alpha}{2}, & \text{if } (t, x) \in (\tfrac{1}{2}, 1) \times [\beta, 1], \end{cases}$$

389

390 where  $\alpha \in [0, 2]$  and  $\beta \in [0, 1]$  can be freely chosen. Note that the last solution just  
 391 provides the minimal regularity, i.e.,  $\partial_x \dot{u}(t) \in \mathfrak{M}(\Omega)$ .

392 The uniqueness of the stress field observed in the above example is a general  
 393 result:

394 **LEMMA 3.11** (Uniqueness of the stress, [17, Theorem 1], [21, Lemma 3.5]). *As-  
 395 sume that  $\sigma_1, \sigma_2 \in H^1(\mathbb{L}^2(\Omega))$  are two reduced solutions. Then  $\sigma_1 = \sigma_2$ .*

396 *Remark 3.12* (Optimal control vs. optimization). Since the displacement field as  
 397 part of a weak solution is not unique in general, there is no (single-valued) control-  
 398 to-state operator mapping  $u_D$  to  $u$ . Therefore, one might argue that (P) is actually  
 399 no real optimal control problem. Strictly speaking, one should rather regard it as an  
 400 *optimization problem* with the triple  $(u, \sigma, u_D)$  as optimization variables.

401 **4. Existence of Optimal Solutions.** Before we come to the main point of our  
 402 analysis, which concerns the approximation of (P) by means of regularized optimal

403 control problems, let us address the existence of optimal solutions. The proof in prin-  
 404 ciple follows the classical direct method, for which we need the following boundedness  
 405 and continuity results:

406 LEMMA 4.1 ([21, Lemma 3.6]). *Let  $u_D \in H^1(\mathbf{H}^1(\Omega))$  be given and  $\sigma$  be the*  
 407 *associated reduced solution. Then there holds*

$$408 \quad (4.1) \quad \|\dot{\sigma}\|_{L^2(\mathbb{L}^2(\Omega))} \leq \gamma_{\mathbb{A}}^{-1} \|u_D\|_{H^1(\mathbf{H}^1(\Omega))},$$

409 where  $\gamma_{\mathbb{A}}$  is the coercivity constant of  $\mathbb{A}$ . Consequently, there is a constant  $C > 0$  such  
 410 that  $\|\sigma\|_{H^1(\mathbb{L}^2(\Omega))} \leq C(\|\sigma_0\|_{L^2(\Omega)} + \|u_D\|_{H^1(\mathbf{H}^1(\Omega))}$ .

411 LEMMA 4.2 (Continuity of reduced solutions, [21, Proposition 3.10]). *Let  $\{u_{D,n}\} \subset$*   
 412  *$H^1(\mathbf{H}^1(\Omega))$  be a sequence such that*

$$413 \quad (4.2) \quad \begin{aligned} u_{D,n} &\rightharpoonup u_D \text{ in } H^1(\mathbf{H}^1(\Omega)), & u_{D,n} &\rightarrow u_D \text{ in } L^2(\mathbf{H}^1(\Omega)), \\ & & u_{D,n}(T) &\rightarrow u_D(T) \text{ in } \mathbf{H}^1(\Omega) \end{aligned}$$

414 and denote the (unique) reduced solution associated with  $u_{D,n}$  by  $\sigma_n$ . Then  $\sigma_n \rightharpoonup \sigma$   
 415 in  $H^1(\mathbb{L}^2(\Omega))$ , where  $\sigma$  is the reduced solution w.r.t.  $u_D$ .

416 LEMMA 4.3. *There is a constant  $C > 0$ , independent of  $u_D$ , such that every weak*  
 417 *solution w.r.t.  $u_D$  fulfills*

$$418 \quad \left( \int_0^T \|\dot{u}(t)\|_{\text{BD}(\Omega)}^2 dt \right)^{1/2} \leq C \|u_D\|_{H^1(\mathbf{H}^1(\Omega))} (1 + \|u_D\|_{H^1(\mathbf{H}^1(\Omega))}).$$

419 *Proof.* Let  $\varphi \in C_c^\infty(\Omega)$  with  $\|\varphi\|_{L^\infty(\Omega)} \leq 1$  and  $i, j \in \{1, \dots, n\}$  be arbitrary.  
 420 According to (2.3), the test function

$$421 \quad (\tau_\varphi)_{ij} = (\tau_\varphi)_{ji} := -\frac{\varrho}{\sqrt{2}} \varphi, \quad (\tau_\varphi)_{kl} = 0 \quad \forall (k, l) \notin \{(i, j), (j, i)\}$$

422 is admissible for (3.3b). Using  $\text{div } \sigma = 0$ , we deduce

$$423 \quad \int_\Omega \varphi d(\nabla^s \dot{u})_{ij} \leq \frac{\sqrt{2}}{\varrho} \left( \int_\Omega \nabla^s \dot{u}_D : \sigma dx - \int_\Omega \mathbb{A} \dot{\sigma} : (\tau_\varphi - \sigma) dx \right)$$

424 and consequently, since  $\varphi \in C_c^\infty(\Omega)$  with  $\|\varphi\|_{L^\infty(\Omega)} \leq 1$  was arbitrary,

$$425 \quad \begin{aligned} \|\nabla^s \dot{u}\|_{L_w^2(\mathfrak{M}(\Omega; \mathbb{R}_{\text{sym}}^{n \times n}))} &\leq C(\|u_D\|_{H^1(\mathbf{H}^1(\Omega))} \|\sigma\|_{L^\infty(\mathbb{L}^2(\Omega))} \\ &\quad + \|\dot{\sigma}\|_{L^2(\mathbb{L}^2(\Omega))} + \|\dot{\sigma}\|_{L^2(\mathbb{L}^2(\Omega))} \|\sigma\|_{L^\infty(\mathbb{L}^2(\Omega))}) \\ &\leq C \|u_D\|_{H^1(\mathbf{H}^1(\Omega))} (1 + \|u_D\|_{H^1(\mathbf{H}^1(\Omega))}), \end{aligned}$$

426 where we used Lemma 4.1.

427 Since  $\Gamma_D$  is assumed to have a nonempty relative interior, there is a set  $\Lambda \subset \Gamma_D$   
 428 and a constant  $\delta > 0$  such that  $\Lambda$  has positive boundary measure and  $\text{dist}(\Lambda, \partial\Gamma_D) \geq \delta$ .  
 429 By [31, Chap. II, Theorem 2.1],  $\dot{u}(t)$  admits a trace in  $\mathbf{L}^1(\Gamma)$  for almost all  $t \in (0, T)$ .  
 430 In the following, we neglect the variable  $t$  for the sake of readability. The restriction  
 431 of this trace to  $\Lambda$  is denoted by  $\dot{u}|_\Lambda$ . We extend  $\text{sign}(\dot{u}|_\Lambda)$  (where the sign is to be  
 432 understood componentwise) to the whole boundary  $\Gamma$  by zero and apply convolution  
 433 with a smoothing kernel to obtain a sequence of functions  $\{\varphi_n\} \subset C^\infty(\Gamma; \mathbb{R}^n)$  with

434  $\text{supp}(\varphi_n) \subset \Gamma_D$  (thanks to  $\text{dist}(\Lambda, \partial\Gamma_D) \geq \delta$ ) and  $\|\varphi_n\|_{L^\infty(\Gamma; \mathbb{R}^n)} \leq 1$  for all  $n \in \mathbb{N}$ .  
 435 Given these functions, let us define

$$436 \quad (\tau_n)_{ij} = \frac{\varrho}{\sqrt{2}} \mathfrak{E}(\varphi_{n,i} \nu_j + \varphi_{n,j} \nu_i),$$

437 where  $\mathfrak{E}$  denoted the harmonic extension and  $\nu$  is the outward normal. Then, (2.1)  
 438 implies  $\|\tau_n\|_{\mathbb{L}^\infty(\Omega)} \leq \varrho$  and, since in addition  $\tau_n$  vanishes on  $\Gamma_N$  by construction, we  
 439 have  $\tau_n \in \Sigma(\Omega) \cap \mathcal{K}(\Omega)$ . Note that, by the mapping properties of  $\mathfrak{E}$ ,  $\tau_n \in \mathbb{W}^{1,p}(\Omega) \hookrightarrow$   
 440  $\Sigma(\Omega)$ . If we insert this as test function in (3.3b) and apply again the integration by  
 441 parts from Lemma 3.6, then  $\text{div } \sigma = 0$  and (3.8) imply

$$442 \quad \int_{\Gamma_D} \varphi_n \cdot \dot{u} \, ds \leq \frac{\sqrt{2}}{\varrho} \left( \int_{\Omega} \tau_n : d\nabla^s(\dot{u}) - \int_{\Omega} \nabla^s \dot{u}_D : \sigma \, dx + \int_{\Omega} \mathbb{A} \dot{\sigma} : (\tau_n - \sigma) \, dx \right).$$

443 Now, since  $\varphi_n \rightarrow \text{sign}(\dot{u})$  a.e. in  $\Lambda$ ,  $\varphi_n \rightarrow 0$  a.e. in  $\Gamma_D \setminus \Lambda$  and  $|\varphi_n \cdot \dot{u}| \leq |\dot{u}|$  a.e. on  $\Gamma_D$ ,  
 444 Lebesgue's dominated convergence theorem along with our previous estimate gives

$$445 \quad \|\dot{u}\|_{L^2(\mathbf{L}^1(\Lambda))} \leq C \|u_D\|_{H^1(\mathbf{H}^1(\Omega))} (1 + \|u_D\|_{H^1(\mathbf{H}^1(\Omega))}).$$

446 Thanks to [31, Chap. II, Proposition 2.4], this completes the proof.  $\square$

447 *Remark 4.4.* A priori estimates for quasistatic evolutions (which is an equivalent  
 448 notion of solution as mentioned above) are already proven in [8, Thm. 5.2] in a slightly  
 449 different setting.

450 **LEMMA 4.5.** *Let  $\{u_n\} \subset \mathcal{U}$  be a sequence such that, for all  $n \in \mathbb{N}$ ,*

$$451 \quad (4.3) \quad u_n(0) = u_0 \quad \text{and} \quad \int_0^T \|\dot{u}_n(t)\|_{\text{BD}(\Omega)}^2 \, dt \leq C$$

452 *with a constant  $C > 0$ . Then there exists a subsequence converging weakly in  $\mathcal{U}$  as*  
 453 *defined in (3.1).*

454 *Proof.* Owing to (4.3),  $\{\nabla^s \dot{u}_n\}$  is bounded in  $L_w^2(\mathfrak{M}(\Omega; \mathbb{R}_{\text{sym}}^{n \times n}))$ , which, accord-  
 455 ing to [9, Theorem 8.20.3], is the dual of  $L^2(C_0(\Omega; \mathbb{R}_{\text{sym}}^{n \times n}))^*$ . Thus, there exists a  
 456 subsequence such that

$$457 \quad (4.4) \quad \nabla^s \dot{u}_{n_k} \rightharpoonup^* w \quad \text{in } L_w^2(\mathfrak{M}(\Omega; \mathbb{R}_{\text{sym}}^{n \times n})).$$

458 Due to  $\text{BD}(\Omega) \hookrightarrow \mathbf{L}^{\frac{n}{n-1}}(\Omega)$ ,  $\{\dot{u}_{n_k}\}$  is bounded in  $L^2(\mathbf{L}^{\frac{n}{n-1}}(\Omega))$  and, since all  $u_n$  share  
 459 the same initial value,  $\{u_{n_k}\}$  is bounded in  $H^1(\mathbf{L}^{\frac{n}{n-1}}(\Omega))$  so that, by reflexivity, there  
 460 is another subsequence (denoted w.l.o.g. by the same symbol) such that

$$461 \quad (4.5) \quad u_{n_k} \rightharpoonup u \quad \text{in } H^1(\mathbf{L}^{\frac{n}{n-1}}(\Omega)).$$

462 Now, for every  $\tau \in C_c^\infty(\Omega; \mathbb{R}_{\text{sym}}^{n \times n})$  and every  $\varphi \in C_c^\infty(0, T)$ , (4.4) and (4.5) imply

$$\begin{aligned} \int_0^T \langle w(t), \tau \rangle \varphi(t) \, dt &= \lim_{k \rightarrow \infty} \int_0^T \langle \nabla^s \dot{u}_{n_k}(t), \tau \rangle \varphi(t) \, dt \\ &= \lim_{k \rightarrow \infty} \int_0^T \int_{\Omega} \dot{u}_{n_k}(t) \cdot \text{div } \tau \, dx \, \varphi(t) \, dt \\ &= \int_0^T \int_{\Omega} \dot{u}(t) \cdot \text{div } \tau \, dx \, \varphi(t) \, dt \end{aligned}$$

464 and hence  $w(t) = \nabla^s \dot{u}(t)$  a.e. in  $(0, T)$ .  $\square$

465 PROPOSITION 4.6 (Continuity properties of weak solutions). *Let  $\{u_{D,n}\}_{n \in \mathbb{N}} \subset$*   
 466  *$H^1(\mathbf{H}^1(\Omega))$  be a sequence fulfilling (4.2). Then, there is a subsequence of weak solu-*  
 467 *tions  $\{u_{n_k}, \sigma_{n_k}\}_{k \in \mathbb{N}}$  associated with  $\{u_{D,n_k}\}$  such that*

$$468 \quad \sigma_{n_k} \rightharpoonup \sigma \text{ in } H^1(\mathbb{L}^2(\Omega)), \quad u_{n_k} \rightharpoonup u \text{ in } \mathcal{U},$$

470 *and the weak limit  $(u, \sigma)$  is a weak solution associated with the limit  $u_D$ .*

471 *Proof.* Since we already know that the stress component of every weak solution  
 472 is also a reduced one and the latter is unique by Lemma 3.11, the convergence of the  
 473 stresses follows from Lemma 4.2 (even for the whole sequence).

474 Owing to Lemma 4.3,  $\{\dot{u}_n\}$  fulfills the boundedness assumption in (4.3) so that,  
 475 by Lemma 4.5, there is a subsequence  $\{u_{n_k}\}$  converging weakly in  $\mathcal{U}$  to some limit  
 476  $u \in \mathcal{U}$ . Due to  $H^1(\mathbf{L}^1(\Omega)) \hookrightarrow C(\mathbf{L}^1(\Omega))$ , the weak limit  $u$  also satisfies the initial  
 477 condition.

478 It remains to prove that  $(u, \sigma)$  fulfills the flow rule inequality (3.3b). To this end,  
 479 choose an arbitrary  $\tau \in L^2(\mathbb{L}^2(\Omega))$  with  $\tau(t) \in \Sigma(\Omega) \cap \mathcal{K}(\Omega)$  for almost all  $t \in [0, T]$ .  
 480 Then, the flow rule inequality for  $(u_{n_k}, \sigma_{n_k})$  along with  $\operatorname{div} \sigma_{n_k} = 0$  and the (weak)  
 481 convergences of  $u_{D,n_k}$ ,  $u_{n_k}$ , and  $\sigma_{n_k}$  yields

$$482 \quad \begin{aligned} & \liminf_{k \rightarrow \infty} (\mathbb{A} \dot{\sigma}_{n_k}, \sigma_{n_k})_{L^2(\mathbb{L}^2(\Omega))} \\ & \leq \lim_{k \rightarrow \infty} \left[ (\mathbb{A} \dot{\sigma}_{n_k} - \nabla^s \dot{u}_{D,n_k}, \tau)_{L^2(\mathbb{L}^2(\Omega))} \right. \\ & \quad \left. + \int_0^T \int_{\Omega} (\dot{u}_{n_k} - \dot{u}_{D,n_k}) \operatorname{div} \tau \, dx dt - (\nabla^s \dot{u}_{D,n_k}, \sigma_{n_k})_{L^2(\mathbb{L}^2(\Omega))} \right] \\ & = (\mathbb{A} \dot{\sigma} - \nabla^s \dot{u}_D, \tau)_{L^2(\mathbb{L}^2(\Omega))} + \int_0^T \int_{\Omega} (\dot{u} - \dot{u}_D) \operatorname{div} \tau \, dx dt - (\nabla^s \dot{u}_D, \sigma)_{L^2(\mathbb{L}^2(\Omega))} \end{aligned}$$

483 where we used Lemma 3.9 in our companion paper [21] for the convergence of the last  
 484 term. On the other hand, the weak lower semicontinuity of  $\|\cdot\|_{\mathbb{L}^2(\Omega)_{\mathbb{A}}}$  together with  
 485  $H^1(\mathbb{L}^2(\Omega)) \hookrightarrow C(\mathbb{L}^2(\Omega))$  gives

$$486 \quad \begin{aligned} & \liminf_{k \rightarrow \infty} (\mathbb{A} \dot{\sigma}_{n_k}, \sigma_{n_k})_{L^2(\mathbb{L}^2(\Omega))} \\ & = \frac{1}{2} \liminf_{k \rightarrow \infty} \|\sigma_{n_k}(T)\|_{\mathbb{L}^2(\Omega)_{\mathbb{A}}}^2 - \frac{1}{2} \|\sigma_0\|_{\mathbb{L}^2(\Omega)_{\mathbb{A}}}^2 \\ & \geq \frac{1}{2} \|\sigma(T)\|_{\mathbb{L}^2(\Omega)_{\mathbb{A}}}^2 - \frac{1}{2} \|\sigma_0\|_{\mathbb{L}^2(\Omega)_{\mathbb{A}}}^2 = (\mathbb{A} \dot{\sigma}, \sigma)_{L^2(\mathbb{L}^2(\Omega))}. \end{aligned}$$

487 Together with (4.6) and  $\operatorname{div} \sigma = 0$ , this implies the flow rule inequality for the weak  
 488 limit.  $\square$

489 Given these boundedness and continuity results, we can now establish the existence  
 490 of at least one optimal solution. Before we do so, let us recall our optimization  
 491 problem and state it in a rigorous manner:

$$492 \quad (\text{P}) \quad \begin{cases} \min & J(u, u_D) := \Psi(u) + \frac{\alpha}{2} \|u_D\|_{H^1(\mathbf{H}^2(\Omega))}^2 \\ \text{s.t.} & u_D \in H^1(\mathbf{H}^2(\Omega)), \quad (u, \sigma) \in \mathcal{U} \times H^1(\mathbb{L}^2(\Omega)), \\ & (u, \sigma) \text{ is a weak solution w.r.t. } u_D, \quad \text{and } u_D(0) - u_0 \in \mathbf{H}_D^1(\Omega), \end{cases}$$

493 where  $\Psi : \mathcal{U} \rightarrow \mathbb{R}$  is bounded from below and lower semicontinuous w.r.t. weak conver-  
 494 gence in  $\mathcal{U}$  as defined in (3.1), i.e.,

$$495 \quad (4.8) \quad u_n \rightharpoonup u \text{ in } \mathcal{U} \implies \liminf_{n \rightarrow \infty} \Psi(u_n) \geq \Psi(u).$$

496 An example for such a functional  $\Psi$  will be given in [section 6](#) below.

497 **THEOREM 4.7** (Existence of optimal solutions). *There exists a globally optimal*  
 498 *solution of (P).*

499 *Proof.* Based on our above findings, the assertion immediately follows from the  
 500 standard direct method of calculus of variations. Nevertheless, let us shortly sketch the  
 501 arguments. First, we observe that the triple  $(u, \sigma, u_D) \equiv (u_0, \sigma_0, u_0)$  (constant in time)  
 502 satisfies the constraints in (P) so that the feasible set is nonempty. (At this point,  
 503 we need the additional regularity  $u_0 \in \mathbf{H}^2(\Omega)$ .) Let  $(u_n, \sigma_n, u_{D,n})$  be a minimizing  
 504 sequence. Then either  $(u_0, u_0)$  is already optimal or  $J(u_n, u_{D,n}) \leq J(u_0, u_0) < \infty$  for  
 505  $n \in \mathbb{N}$  sufficiently large. Thus, since  $\Psi$  is bounded from below,  $\{u_{D,n}\}$  is bounded  
 506 in  $H^1(\mathbf{H}^2(\Omega))$ . Via continuous and compact embedding, there is thus a subsequence  
 507 satisfying (4.2). Clearly, the associated limit satisfies the conditions on the initial  
 508 value in (P). Moreover, according to [Proposition 4.6](#), a subsequence of weak solutions  
 509 converges weakly in  $\mathcal{U} \times H^1(\mathbb{L}^2(\Omega))$  to a weak solution. Thus the weak limit is feasible  
 510 and the weak lower semicontinuity of norms and of  $\Psi$  implies its optimality.  $\square$

511 *Remark 4.8* (More general objectives). The proof of existence readily transfers  
 512 to slightly more general objectives than the one in (P). For instance, one could  
 513 add a term of the form  $\Phi(\sigma)$  with a function  $\Phi : H^1(\mathbb{L}^2(\Omega)) \rightarrow \mathbb{R}$ , which weakly  
 514 lower semicontinuous and bounded from below. Since objectives of this form have  
 515 already been discussed in the companion paper, we restrict ourselves to objectives  
 516 just depending on  $u$  in order to keep the discussion concise. Moreover, one could use  
 517 other Tikhonov terms different from the  $H^1(\mathbf{H}^2(\Omega))$ -norm to ensure the convergence  
 518 properties in (4.2) required for [Proposition 4.6](#). For example, thanks to the Aubin-  
 519 Lions lemma, a Tikhonov term of the form

$$520 \quad \frac{\alpha}{2} \left( \|u_D\|_{H^1(\mathbf{H}^1(\Omega))}^2 + \|u_D\|_{L^2(X)}^2 \right)$$

521 with any Banach space  $X$  embedding compactly in  $\mathbf{H}^1(\Omega)$  (such as e.g.  $\mathbf{H}^2(\Omega)$ ) is  
 522 sufficient to guarantee (4.2) for (a subsequence of) a minimizing sequence. However,  
 523 in order to shorten presentation, we just consider the  $H^1(\mathbf{H}^2(\Omega))$ -norm.

524 **5. Yosida Regularization and Reverse Approximation.** As already men-  
 525 tioned above, the ultimate goal of our analysis is to establish conditions that guarantee  
 526 that optimal solutions to the optimization problem (P) governed by perfect plasticity  
 527 can be approximated via Yosida regularization. The most crucial point in this  
 528 respect is the so-called *reverse approximation*, which essentially means to construct  
 529 a *recovery sequence* for a given perfect plastic solution. This is a rather challenging  
 530 task, as [Example 3.10](#) illustrates: one easily verifies that every sequence of regular-  
 531 ized solutions tends to the linear solution  $u(t, x) = 2tx$  for regularization parameter  
 532 tending to zero, although there are infinitely many other solutions. There is thus no  
 533 hope that every perfect plastic solution can be approximated via Yosida regulariza-  
 534 tion! However, when it comes to optimization, there is not only the state (i.e., the  
 535 solution of the perfect plasticity system), but also the *control* variables, which can be  
 536 used to construct a recovery sequence. Unfortunately, the Dirichlet data  $u_D$ , which  
 537 serve as control variables in our case, are not sufficient for this purpose. Instead we  
 538 need a set of control variables that is rich enough to generate a sufficiently large set  
 539 of regularized solutions. For this purpose, we introduce an *additional control variable*  
 540 *in form of distributed loads* and end up with the following regularized version of the

541 state equation:

$$542 \quad (5.1a) \quad -\operatorname{div} \sigma_\lambda(t) = \ell(t) \quad \text{in } \mathbf{H}_D^{-1}(\Omega),$$

$$543 \quad (5.1b) \quad \sigma_\lambda(t) = \mathbb{C}(\nabla^s u_\lambda(t) - z_\lambda(t)) \quad \text{in } \mathbb{L}^2(\Omega),$$

$$544 \quad (5.1c) \quad \dot{z}_\lambda(t) = \partial I_\lambda(\sigma_\lambda(t)) \quad \text{in } \mathbb{L}^2(\Omega),$$

$$545 \quad (5.1d) \quad u_\lambda(t) - u_D(t) \in \mathbf{H}_D^1(\Omega),$$

$$546 \quad (5.1e) \quad (u_\lambda, \sigma_\lambda)(0) = (u_0, \sigma_0) \quad \text{in } \mathbf{H}^1(\Omega) \times \mathbb{L}^2(\Omega).$$

548 where  $\lambda > 0$  is the regularization parameter,  $I_\lambda$  is the Yosida regularization of the  
 549 indicator functional, see (2.4), and  $\ell \in H^1(\mathbf{H}_D^{-1}(\Omega))$  is the mentioned load. Existence  
 550 and uniqueness of a solution to the regularized state equation (5.1) follows from  
 551 Banach's fixed point theorem and can be proven by a reduction of the system to  
 552 an equation in the variable  $z$  only, cf. e.g. [21, Proposition 3.15]. This gives rise to  
 553 the following

554 LEMMA 5.1 (Existence of solutions to the regularized state system, [21, Corol-  
 555 lary 3.16]). *For every  $\lambda > 0$ ,  $\ell \in H^1(\mathbf{H}_D^{-1}(\Omega))$ , and  $u_D \in H^1(\mathbf{H}^1(\Omega))$  with  $\ell(0) = 0$   
 556 and  $u_D(0)|_{\Gamma_D} = u_0|_{\Gamma_D}$ , there exists a unique solution  $(u_\lambda, \sigma_\lambda, z_\lambda) \in H^1(\mathbf{H}^1(\Omega)) \times$   
 557  $H^1(\mathbb{L}^2(\Omega)) \times H^1(\mathbb{L}^2(\Omega))$  of (5.1).*

558 *The associated solution operator is globally Lipschitz continuous with a Lipschitz*  
 559 *constant proportional to  $\lambda^{-1}$ .*

560 The proof of existence is a direct consequence of the Lipschitz continuity of  $\partial I_\lambda$   
 561 and Banach's contraction principle. In [21], the external loads are set to zero, but  
 562 it is straightforward to incorporate them into the existence theory. The Lipschitz  
 563 continuity of the solution mapping directly follows from the Lipschitz estimate for the  
 564 Yosida approximation, see e.g. [3, Proposition 55.2(b)].

565 Before we address the approximation properties of this regularization approach  
 566 and its convergence behavior for  $\lambda$  tending to zero in section 6 below, see Proposi-  
 567 tion 6.2, we first lay the foundations for the construction of a recovery sequence in  
 568 the upcoming three lemmas. Unfortunately, as already indicated in the introduction,  
 569 the passage to the limit in the regularized state equation in Proposition 6.2 below  
 570 requires a rather high regularity of the stress field, and the recovery sequence has  
 571 to fulfill this regularity, too, as it is a constraint in the regularized optimal control  
 572 problem  $(\mathbf{P}_\lambda)$ . The key issue for our reverse approximation argument is therefore to  
 573 improve the regularity of the stress field provided a displacement field with higher  
 574 regularity is given. To this end, we first need an auxiliary result on the derivative  
 575 of the Yosida regularization. Since the set of admissible stresses admits a pointwise  
 576 representation by the set  $K$ , the Fréchet-derivative of the Yosida regularization does  
 577 the same, i.e., given an arbitrary  $\tau \in \mathbb{L}^2(\Omega)$ , it holds

$$578 \quad (5.2) \quad \partial I_\lambda(\tau)(x) = \frac{1}{\lambda} [\tau(x) - \pi_K(\tau(x))] \quad \text{f.a.a. } x \in \Omega,$$

579 where  $\pi_K : \mathbb{R}_{\text{sym}}^{n \times n} \rightarrow \mathbb{R}_{\text{sym}}^{n \times n}$  is the projection on  $K$ . This pointwise representation  
 580 allows to derive the following

581 LEMMA 5.2. *Let  $\lambda > 0$ ,  $p > 2$ , and  $\tau \in \mathbb{W}^{1,p}(\Omega)$  be arbitrary. Then  $\partial I_\lambda(\tau) \in$   
 582  $\mathbb{W}^{1,p}(\Omega)$  and there holds*

$$583 \quad (5.3) \quad \|\partial I_\lambda(\tau)\|_{\mathbb{W}^{1,p}(\Omega)} \leq \frac{1}{\lambda} \|\tau\|_{\mathbb{W}^{1,p}(\Omega)}$$

584

585 *and*

$$586 \quad (5.4) \quad (\partial_i(\partial I_\lambda(\tau)) : \partial_i \tau)(x) \geq 0 \quad \text{a.e. in } \Omega, \quad \forall i = 1, \dots, n.$$

588 *Proof.* As a projection,  $\pi_k : \mathbb{R}_{\text{sym}}^{n \times n} \rightarrow \mathbb{R}_{\text{sym}}^{n \times n}$  is globally Lipschitz continuous.  
 589 Thus, the chain rule for Sobolev functions (see e.g. [36, Thm 2.1.11]) implies that  
 590  $\partial I_\lambda(\tau) \in \mathbb{W}^{1,p}(\Omega)$  with

$$591 \quad (5.5) \quad \frac{\partial}{\partial x_m} [\partial I_\lambda(\tau)]_{ij} = \frac{1}{\lambda} \left( \frac{\partial \tau_{ij}}{\partial x_m} - \sum_{kl} \frac{\partial}{\partial \tau_{kl}} [\pi_K(\tau)]_{ij} \frac{\partial \tau_{kl}}{\partial x_m} \right).$$

592 Since the Lipschitz constant of the projection equals one, its directional derivative  
 593 clearly satisfies  $|\pi'_K(A; B)|_F \leq |B|_F$  for all  $A, B \in \mathbb{R}_{\text{sym}}^{n \times n}$  and, consequently,

$$594 \quad \partial_m(\partial I_\lambda(\tau)) : \partial_m \tau = \frac{1}{\lambda} (|\partial_m \tau|_F^2 - \pi'_K(\tau; \partial_m \tau) : \partial_m \tau) \geq 0,$$

595 which is (5.4). It is moreover easily seen that  $\text{Id} - \pi_K : \mathbb{R}_{\text{sym}}^{n \times n} \rightarrow \mathbb{R}_{\text{sym}}^{n \times n}$  is globally  
 596 Lipschitz with Lipschitz constant 1, too. Thus, for every  $A, B \in \mathbb{R}_{\text{sym}}^{n \times n}$ , there holds  
 597  $|(\text{Id} - \pi_K)'(A; B)|_F \leq |B|_F$ . Since  $(\text{Id} - \pi_K)(0) = 0$ , the Lipschitz continuity moreover  
 598 entails  $|(\text{Id} - \pi_K)(A)|_F \leq |A|_F$  for all  $A \in \mathbb{R}_{\text{sym}}^{n \times n}$ . In view of (5.5), this yields (5.3).  $\square$

599 The next lemma addresses the crucial regularity result for the stress field  $\sigma_\lambda$  as  
 600 solution of

$$601 \quad (5.6) \quad w - \mathbb{A} \dot{\sigma}_\lambda = \partial I_\lambda(\sigma_\lambda), \quad \sigma_\lambda(0) = \sigma_0.$$

602 In the proof of our main result in [Theorem 6.3](#), an optimal strain rate will play the  
 603 role of  $w$  and the following regularity result will be essential for the construction of a  
 604 recovery sequence associated with that strain rate. The required regularity of  $w$  will  
 605 carry over to this optimal strain rate and represents the most restrictive assumption  
 606 of our reverse approximation approach.

607 **LEMMA 5.3** (Higher regularity of the stress field). *Let  $\lambda > 0$  be arbitrary and*  
 608  *$w \in L^2(\mathbb{L}^2(\Omega)) \cap L^1(\mathbb{W}^{1,p}(\Omega))$  with  $p \geq 2$  be given. Then (5.6) admits a unique*  
 609 *solution  $\sigma_\lambda \in H^1(\mathbb{L}^2(\Omega)) \cap L^\infty(\mathbb{W}^{1,p}(\Omega))$  and there holds*

$$610 \quad (5.7) \quad \|\sigma_\lambda\|_{L^\infty(\mathbb{W}^{1,p}(\Omega))} \leq C_p \left( \|w\|_{L^1(\mathbb{W}^{1,p}(\Omega))} + \|\sigma_0\|_{\mathbb{W}^{1,p}(\Omega)}^p \right)$$

611 *with  $C_p := p \|\mathbb{A}\|^{p/2-1}$ .*

612 *Proof. Step 1. Existence of solutions in  $H^1(\mathbb{L}^2(\Omega))$ :* First we note that (5.6) is  
 613 just an ODE in  $\mathbb{L}^2(\Omega)$  and  $\partial I_\lambda$  is globally Lipschitz in  $\mathbb{L}^2(\Omega)$ . Thus, the existence  
 614 and uniqueness of solutions in  $H^1(\mathbb{L}^2(\Omega))$  follows from the generalized Picard-Lindelöf  
 615 theorem in Banach spaces. However, a pointwise projection is in general not Lipschitz  
 616 continuous in Sobolev spaces. Therefore, we cannot apply this simple argument to  
 617 show that the solution is an element of  $W^{1,1}(\mathbb{W}^{1,p}(\Omega))$ .

618 *Step 2. Higher regularity in case of smooth data:* To prove this, we apply a time  
 619 discretization scheme, namely the explicit Euler method. At first we consider the case  
 620  $w \in C(\mathbb{W}^{1,p}(\Omega))$ . For  $N \in \mathbb{N}$  and  $n \in \{0, \dots, N\}$ , we set  $d_t^N := \frac{T}{N}$  and  $t_n^N := nd_t^N$  such  
 621 that  $0 = t_0^N < t_1^N < \dots < t_N^N = T$ . Now define  $\sigma_0^N := \sigma_0 \in \mathbb{W}^{1,p}(\Omega)$  and

$$622 \quad \sigma_n^N := \sigma_{n-1}^N + d_t^N \mathbb{C}(w(t_{n-1}^N) - \partial I_\lambda(\sigma_{n-1}^N)) \in \mathbb{W}^{1,p}(\Omega) \quad (\text{by Lemma 5.2})$$

623 such that

$$624 \quad (5.8) \quad \mathbb{A} \frac{\sigma_n^N - \sigma_{n-1}^N}{d_t^N} + \partial I_\lambda(\sigma_{n-1}^N) = w(t_{n-1}^N)$$

625 for all  $N \in \mathbb{N}$  and  $n \in \{1, \dots, N\}$ . We define the piecewise linear approximation  
626  $\sigma^N \in W^{1,\infty}(\mathbb{W}^{1,p}(\Omega))$  by

$$627 \quad \sigma^N(t) := \sigma_{n-1}^N + \frac{t - t_{n-1}^N}{d_t^N} (\sigma_n^N - \sigma_{n-1}^N)$$

628 and the piecewise constant approximation  $\tilde{\sigma}^N \in L^\infty(\mathbb{W}^{1,p}(\Omega))$  by  $\tilde{\sigma}^N(t) := \sigma_{n-1}^N$  for  
629  $t \in [t_{n-1}^N, t_n^N)$ . Using (5.3), we deduce from (5.8) that

$$630 \quad \|\sigma_n^N\|_{\mathbb{W}^{1,p}(\Omega)} \leq \|\sigma_0\|_{\mathbb{W}^{1,p}(\Omega)} + d_t^N C \left( \sum_{i=0}^{n-1} \|\sigma_i^N\|_{\mathbb{W}^{1,p}(\Omega)} \right) + C \|w\|_{C(\mathbb{W}^{1,p}(\Omega))},$$

631 which, together with the discrete Gronwall lemma (cf. [16, Lemma 5.1 and the follow-  
632 ing remark]), shows that  $\sigma^N$  is bounded in  $L^\infty(\mathbb{W}^{1,p}(\Omega))$  by a constant independent  
633 of  $d_t^N$ . Thus, again owing to (5.8) and (5.3),  $\dot{\sigma}^N(t) = \frac{\sigma_n^N - \sigma_{n-1}^N}{d_t^N}$ ,  $t \in (t_{n-1}^N, t_n^N)$  is  
634 also bounded in  $L^\infty(\mathbb{W}^{1,p}(\Omega))$ . Therefore,  $\sigma^N$  is bounded in  $H^1(\mathbb{L}^2(\Omega))$  and conse-  
635 quently, there is a weakly converging subsequence, for simplicity also denoted by  $\sigma^N$ ,  
636 such that  $\sigma^N \rightharpoonup \sigma$  in  $H^1(\mathbb{L}^2(\Omega))$  and  $\sigma^N \rightharpoonup^* \sigma$  in  $L^\infty(\mathbb{W}^{1,p}(\Omega))$  as  $N \rightarrow \infty$ . Note  
637 that, due to the reflexivity of  $\mathbb{W}^{1,p}(\Omega)$ ,  $L^\infty(\mathbb{W}^{1,p}(\Omega))$  can be identified with the dual  
638 of  $L^1(\mathbb{W}^{1,p'}(\Omega))$  so there is a weakly-\* converging subsequence. It remains to show  
639 that  $\sigma$  solves (5.6). Since  $\sigma^N$  is bounded in  $W^{1,\infty}(\mathbb{W}^{1,p}(\Omega))$  as seen above, we have  
640 by compact embeddings that  $\sigma^N \rightarrow \sigma$  in  $C(\mathbb{L}^2(\Omega))$ . Thus, we find for the piecewise  
641 constant interpolation that, for every  $t \in [t_{n-1}^N, t_n^N)$ ,

$$642 \quad \|\tilde{\sigma}^N(t) - \sigma(t)\|_{\mathbb{L}^2(\Omega)} \leq \|\sigma^N(t_{n-1}^N) - \sigma(t)\|_{\mathbb{L}^2(\Omega)} \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

643 Therefore, (5.8) and the Lipschitz continuity of  $\partial I_\lambda$  in  $\mathbb{L}^2(\Omega)$  give

$$\begin{aligned} 644 \quad & \|\mathbb{A} \dot{\sigma}^N + \partial I_\lambda(\sigma^N) - w\|_{L^2(\mathbb{L}^2(\Omega))} \\ 645 \quad & \leq \|\mathbb{A} \dot{\sigma}^N + \partial I_\lambda(\tilde{\sigma}^N) - \tilde{w}^N\|_{L^2(\mathbb{L}^2(\Omega))} + \|\partial I_\lambda(\sigma^N) - \partial I_\lambda(\tilde{\sigma}^N)\|_{L^2(\mathbb{L}^2(\Omega))} + \|\tilde{w}^N - w\|_{L^2(\mathbb{L}^2(\Omega))} \\ 646 \quad & \leq \frac{1}{\lambda} \|\sigma^N - \tilde{\sigma}^N\|_{L^2(\mathbb{L}^2(\Omega))} + \|\tilde{w}^N - w\|_{L^2(\mathbb{L}^2(\Omega))} \rightarrow 0 \quad \text{as } N \rightarrow \infty, \end{aligned}$$

648 where  $\tilde{w}^N$  denotes the piecewise constant interpolation of  $w$ , which converges strongly  
649 in  $C(\mathbb{L}^2(\Omega))$  to  $w$  thanks to the assumed regularity of  $w$ . Therefore, by the weak lower  
650 semicontinuity of the  $L^2(\mathbb{L}^2(\Omega))$ -norm, we see that the limit satisfies (5.6).

651 *Step 3. Higher regularity for nonsmooth data:* Let now  $w \in L^2(\mathbb{L}^2(\Omega)) \cap L^1(\mathbb{W}^{1,p}(\Omega))$   
652 be arbitrary and take a sequence  $\{w_n\} \subset C(\mathbb{W}^{1,p}(\Omega))$  such that  $w_n \rightarrow w$  in  $L^1(\mathbb{W}^{1,p}(\Omega))$ .  
653 Let  $\sigma_\lambda \in H^1(\mathbb{L}^2(\Omega))$  be the solution of (5.6) and denote by  $\sigma_{\lambda,n} \in H^1(\mathbb{L}^2(\Omega)) \cap$   
654  $L^\infty(\mathbb{W}^{1,p}(\Omega))$  the solution of

$$655 \quad (5.9) \quad w_n - \mathbb{A} \dot{\sigma}_{\lambda,n} = \partial I_\lambda(\sigma_{\lambda,n}), \quad \sigma_{\lambda,n}(0) = \sigma_0.$$

657 Since  $\partial I_\lambda : \mathbb{L}^2(\Omega) \rightarrow \mathbb{L}^2(\Omega)$  is monotone, one obtains  $\sigma_{\lambda,n} \rightarrow \sigma_\lambda$  in  $H^1(\mathbb{L}^2(\Omega))$  by  
658 standard arguments. Moreover, (5.9) holds almost everywhere in time and space and

659 so that, f.a.a.  $t \in [0, T]$ ,

$$660 \quad \partial_j w_n(t) - \mathbb{A} \partial_j \dot{\sigma}_{\lambda,n}(t) = \partial_j \partial I_\lambda(\sigma_{\lambda,n})(t) \quad \text{a.e. in } \Omega.$$

661 follows. Testing this equation with  $((\mathbb{A} \partial_j \sigma_{\lambda,n} : \partial_j \sigma_{\lambda,n})^{p/2-1} \partial_j \sigma_{\lambda,n})(t) \in \mathbb{W}^{1,p'}(\Omega)$  and  
662 using (5.4) leads to

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} (\mathbb{A} \partial_j \sigma_{\lambda,n} : \partial_j \sigma_{\lambda,n})^{p/2} dx \\ & \leq p \int_{\Omega} (\mathbb{A} \partial_j \sigma_{\lambda,n} : \partial_j \sigma_{\lambda,n})^{p/2-1} (\mathbb{A} \partial_j \sigma_{\lambda,n} : \partial_j \dot{\sigma}_{\lambda,n} + \partial_j \partial I_\lambda(\sigma_{\lambda,n}) : \partial_j \sigma_{\lambda,n}) dx \\ 663 \quad (5.10) \quad & = p \int_{\Omega} (\mathbb{A} \partial_j \sigma_{\lambda,n} : \partial_j \sigma_{\lambda,n})^{p/2-1} \partial_j w_n : \partial \sigma_n dx \\ & \leq C_p \int_{\Omega} |\partial_j w|_F |\partial_j \sigma_{\lambda,n}|_F^{p-1} dx \leq C_p \|w\|_{\mathbb{W}^{1,p}(\Omega)} \|\sigma_{\lambda,n}\|_{\mathbb{W}^{1,p}(\Omega)}^{p-1}, \end{aligned}$$

664 with  $C_p$  as defined in the statement of the lemma. Integrating this inequality in time  
665 and taking the coercivity of  $\mathbb{A}$  into account gives

$$666 \quad \|\sigma_{\lambda,n}\|_{L^\infty(\mathbb{W}^{1,p}(\Omega))} \leq C_p \left( \|w_n\|_{L^1(\mathbb{W}^{1,p}(\Omega))} + \|\sigma_0\|_{\mathbb{W}^{1,p}(\Omega)}^p \right).$$

667 Therefore,  $\sigma_{\lambda,n}$  is bounded in  $L^\infty(\mathbb{W}^{1,p}(\Omega))$  and we can select a weakly-\* converging  
668 subsequence. The uniqueness of the weak limit then gives  $\sigma \in L^\infty(\mathbb{W}^{1,p}(\Omega))$  as  
669 claimed. The estimate in (5.7) finally follows from the above inequality and the lower  
670 semicontinuity of the norm w.r.t. weak-\* convergence.  $\square$

671 *Remark 5.4.* We observe that (5.3), (5.6), and the proven regularity of  $\sigma_\lambda$  even  
672 imply that  $\sigma_\lambda \in W^{1,1}(\mathbb{W}^{1,p}(\Omega))$ . However, we do not obtain an estimate independent  
673 of  $\lambda$  in this norm (in contrast to (5.7)) and therefore, this additional regularity is not  
674 useful for us.

675 **LEMMA 5.5** ([20, Section 3]). *Let  $w \in L^2(\mathbb{L}^2(\Omega))$  be given and  $\lambda \searrow 0$ . Then*  
676  *$\sigma_\lambda \rightarrow \sigma$  in  $H^1(\mathbb{L}^2(\Omega))$ , where  $\sigma$  is the solution of*

$$677 \quad (5.11) \quad w - \mathbb{A} \dot{\sigma} \in \partial I_{\mathcal{K}(\Omega)}(\sigma), \quad \sigma(0) = \sigma_0.$$

678 *Moreover, there holds*

$$679 \quad (5.12) \quad \|\sigma_\lambda - \sigma\|_{\mathbb{C}(\mathbb{L}^2(\Omega))}^2 \leq \lambda \frac{\|\mathbb{C}\|^2}{\gamma_{\mathbb{C}}} \|w - \mathbb{A} \dot{\sigma}\|_{L^2(\mathbb{L}^2(\Omega))}^2,$$

680 *where  $\gamma_{\mathbb{C}} > 0$  is the coercivity constant of  $\mathbb{C}$ .*

681 *Proof.* The assertion is proven in [20], but, for convenience of the reader, we  
682 shortly sketch the arguments. First, observe that  $\sigma_\lambda \in H^1(\mathbb{L}^2(\Omega))$  and  $\sigma \in H^1(\mathbb{L}^2(\Omega))$   
683 solve (5.6) and (5.11), respectively, if and only if  $z_\lambda := W - \mathbb{A} \sigma_\lambda$  and  $z := W - \mathbb{A} \sigma$   
684 with  $W(t) := \int_0^t w(s) ds$  solve

$$685 \quad (5.13) \quad \dot{z}_\lambda = \partial I_\lambda(\mathbb{C}W - \mathbb{C}z_\lambda), \quad z_\lambda(0) = z_0 := -\mathbb{A} \sigma_0$$

687 and

$$688 \quad (5.14) \quad \dot{z} \in \partial I_{\mathcal{K}(\Omega)}(\mathbb{C}W - \mathbb{C}z), \quad z(0) = z_0,$$

690 respectively. These equations are exactly of the form studied in [20, Section 3] with  
 691 the setting  $A := \partial I_{\mathcal{K}(\Omega)}$ ,  $Q = R := \mathbb{C}$ , and  $\ell := W$ . The existence of  $\sigma$  in  $H^1(\mathbb{L}^2(\Omega))$   
 692 then follows from [20, Theorem 3.3], while the convergence  $\sigma_\lambda \rightarrow \sigma$  in  $H^1(\mathbb{L}^2(\Omega))$  as  
 693 well as the estimate

$$694 \quad \|\mathbb{A}(\sigma_\lambda - \sigma)\|_{\mathbb{C}(\mathbb{L}^2(\Omega))}^2 \leq \frac{\lambda}{\gamma_{\mathbb{C}}} \|w - \mathbb{A}\dot{\sigma}\|_{L^2(\mathbb{L}^2(\Omega))}^2$$

695 are consequences of [20, Proposition 3.5]. (Note that  $D(A) = \mathcal{K}(\Omega)$  is closed and  
 696  $A^0 \equiv 0$  in this case, hence, the assumptions in [20, Section 2] are fulfilled). The  
 697 inequality in (5.12) now follows easily using  $\|\sigma_\lambda - \sigma\|_{\mathbb{L}^2(\Omega)} = \|\mathbb{C}\mathbb{A}(\sigma_\lambda - \sigma)\|_{\mathbb{L}^2(\Omega)} \leq$   
 698  $\|\mathbb{C}\| \|\mathbb{A}(\sigma_\lambda - \sigma)\|_{\mathbb{L}^2(\Omega)}$ .  $\square$

699 *Remark 5.6.* As a consequence of (5.7), the solution of (5.11) is an element of  
 700  $L^\infty(\mathbb{W}^{1,p}(\Omega))$ , provided that  $w \in L^1(\mathbb{W}^{1,p}(\Omega))$ . However, we do not need this regu-  
 701 larity result for the upcoming analysis.

702 As already mentioned, in the proof of our final convergence result in Theorem 6.3,  
 703  $\nabla^s \bar{u}$  will play the role of the function  $w$ , where  $\bar{u}$  is an optimal solution of (P). This  
 704 already indicates our most restrictive assumption, namely the existence of an optimal  
 705 solution providing the high regularity required for  $w$ . We will come back to this point  
 706 in Remark 6.4.

707 **6. Convergence of Minimizers.** We are now in the position to state the regu-  
 708 larized optimal control problems. Beside the additional control variable  $\ell$  required for  
 709 the reverse approximation, they differ from (P) in an additional inequality constraint  
 710 on the stress field, which is needed to improve the regularity of the stress in order to  
 711 pass to the limit in the regularized state equation, see the proof of Proposition 6.2  
 712 below. This additional regularity of the stresses is unfortunately not enough to pass  
 713 to the limit in the state system. We additionally need to bound the displacement  
 714 in  $\mathcal{U}$ , since this is not guaranteed a priori by the regularized state system itself, un-  
 715 less the loads fulfill a safe load condition. This however cannot be ensured for the  
 716 loads arising in the construction of the recovery sequence in the proof of our main  
 717 Theorem 6.3 (at least, we were not able to verify it). Therefore, we directly enforce  
 718 this boundedness by a special choice of the objective functional as a tracking type  
 719 objective of the following form:

$$720 \quad (6.1) \quad \Psi(u) := \int_0^T \|\nabla^s \dot{u}(t) - \mu(t)\|_{\mathfrak{M}(\Omega; \mathbb{R}_{\text{sym}}^{n \times n})}^2 + \|\dot{u}(t) - v(t)\|_{\mathbf{L}^1(\Omega)}^2 dt$$

721 with a given desired strain rate  $\mu \in L^2(\mathbb{L}^1(\Omega))$  and a desired displacement rate  
 722  $v \in L^2(\mathbf{L}^1(\Omega))$ . Note that this objective trivially fulfills the lower semicontinuity  
 723 assumption in (4.8). One could even allow for less regular desired strain rates (in the  
 724 space of measures), but for convenience, we restrict to functions in  $L^2(\mathbb{L}^1(\Omega))$ . The

725 regularized counterpart of  $(\mathbf{P})$  now reads as follows:

$$726 \quad (\mathbf{P}_\lambda) \quad \left\{ \begin{array}{l} \min \quad J_\lambda(u, u_D, \ell) := \|\nabla^s \dot{u} - \mu\|_{L^2(\mathbb{L}^1(\Omega))}^2 + \|\dot{u} - v\|_{L^2(\mathbb{L}^1(\Omega))}^2 \\ \quad \quad \quad + \frac{\alpha}{2} \|u_D\|_{H^1(\mathbf{H}^2(\Omega))}^2 + \lambda^{-\theta} \|\ell\|_{L^2(\mathbf{H}_D^{-1}(\Omega))}^2 + \|\dot{\ell}\|_{L^2(\mathbf{H}_D^{-1}(\Omega))}^2 \\ \text{s.t.} \quad u_D \in H^1(\mathbf{H}^2(\Omega)), \quad \ell \in H^1(\mathbf{H}_D^{-1}(\Omega)), \\ \quad \quad u_D(0) - u_0 \in \mathbf{H}_D^1(\Omega), \quad \ell(0) = 0, \\ \quad \quad (u, \sigma, z) \in \mathcal{U} \cap L^2(\mathbf{H}^1(\Omega)) \times L^2(\mathbb{L}^2(\Omega)) \times H^1(\mathbb{L}^2(\Omega)), \\ \quad \quad (u, \sigma, z) \text{ is the solution of (5.1) w.r.t. } u_D \text{ and } \ell, \\ \quad \quad \|\dot{\sigma}\|_{L^2(\mathbb{L}^2(\Omega))} + \|\sigma\|_{L^s(\mathbb{W}^{1,p}(\Omega))} \leq R \end{array} \right.$$

727 with  $0 < \theta < 1$  and

$$728 \quad (6.2) \quad p > n \quad \text{and} \quad s > \max \left\{ 1, \frac{2np}{np + 2(p-n)} \right\}$$

729 and  $R \geq \|\sigma_0\|_{\mathbb{W}^{1,p}(\Omega)}$  to be specified later, see (6.6) below. With the exponents in  
730 (6.2), [19, Lemma 4.2(i)] is applicable and tells us that  $H^1(\mathbb{L}^2(\Omega)) \cap L^s(\mathbb{W}^{1,p}(\Omega))$   
731 embeds compactly in  $L^2(C(\bar{\Omega}; \mathbb{R}_{\text{sym}}^{n \times n}))$ , which will be useful at several places in the  
732 upcoming proofs. The term in the objective associated with  $\theta$  will be used to force  
733 the additional loads to zero in the limit.

734 **PROPOSITION 6.1.** *For every  $\lambda > 0$ , there exists a globally optimal solution of*  
735  $(\mathbf{P}_\lambda)$ .

736 *Proof.* The proof is almost standard, except for a lack of compactness with re-  
737 gard to the control space. Let  $(u_n, \sigma_n, z_n, u_{D,n}, \ell_n)$  be a minimizing sequence. As in  
738 the proof of [Theorem 4.7](#),  $(u, \sigma, z, u_D, \ell) \equiv (u_0, \sigma_0, \nabla^s u_0 - \mathbb{A}\sigma_0, u_0, 0)$  is feasible for  
739  $(\mathbf{P}_\lambda)$ . Thus,  $\{u_{D,n}, \ell_n\}$  is bounded in  $H^1(\mathbf{H}^2(\Omega)) \times H^1(\mathbf{H}_D^{-1}(\Omega))$ . Hence the Lipschitz  
740 continuity of the solution operator associated with (5.1) implies that  $\{(u_n, \sigma_n, z_n)\}$   
741 is bounded in  $H^1(\mathbf{H}^1(\Omega) \times \mathbb{L}^2(\Omega) \times \mathbb{L}^2(\Omega))$ . Therefore, there exist weakly conver-  
742 gent subsequences and we can pass to the limit in (5.1) except for the nonlinearity in  
743  $\partial I_\lambda$ . However, the additional constraint on the stress implies that  $\sigma_n$  also converges  
744 weakly in  $H^1(\mathbb{L}^2(\Omega)) \cap L^s(\mathbb{W}^{1,p}(\Omega))$ , which is compactly embedded in  $L^2(C(\bar{\Omega}; \mathbb{R}_{\text{sym}}^{n \times n}))$   
745 as mentioned above. Thus  $\{\sigma_n\}$  converges strongly in  $L^2(C(\bar{\Omega}; \mathbb{R}_{\text{sym}}^{n \times n}))$ , which allows  
746 to pass to the limit in  $\partial I_\lambda(\sigma_n)$  so that the weak limit solves (5.1). Moreover, the  
747 inequality constraint on  $\sigma_n$  is clearly weakly closed so that the weak limit is indeed  
748 feasible for  $(\mathbf{P}_\lambda)$ . Since the objective is convex and continuous and thus weakly lower  
749 semicontinuous, the weak limit is also optimal.  $\square$

750 **PROPOSITION 6.2** (Convergence of the Yosida regularization with varying loads).

751 *Let  $\{\lambda_n\}_{n \in \mathbb{N}}$  be a sequence converging to zero. Suppose moreover that two sequences*  
752  $\{\ell_n\} \subset H^1(\mathbf{H}_D^{-1}(\Omega))$  *and*  $\{u_{D,n}\} \subset H^1(\mathbf{H}^1(\Omega))$  *are given and denote the solution of*  
753  $(5.1)$  *associated with*  $\lambda_n$ ,  $\ell_n$ , *and*  $u_{D,n}$  *by*  $(u_n, \sigma_n, z_n)$ . *Furthermore, we assume that*  
754  $\{u_{D,n}\}$  *satisfies the convergence properties in (4.2), i.e.,*

$$755 \quad (6.3) \quad \begin{aligned} u_{D,n} &\rightharpoonup u_D \quad \text{in } H^1(\mathbf{H}^1(\Omega)), \quad u_{D,n} \rightarrow u_D \quad \text{in } L^2(\mathbf{H}^1(\Omega)), \\ &u_{D,n}(T) \rightarrow u_D(T) \quad \text{in } \mathbf{H}^1(\Omega), \end{aligned}$$

756 and that

$$757 \quad (6.4) \quad \ell_n \rightharpoonup 0 \quad \text{in } L^2(\mathbf{H}_D^{-1}(\Omega)), \quad u_n \rightharpoonup u \quad \text{in } \mathcal{U},$$

$$758 \quad (6.5) \quad \sigma_n \rightharpoonup \sigma \quad \text{in } H^1(\mathbb{L}^2(\Omega)), \quad \sigma_n \rightarrow \sigma \quad \text{in } L^2(C(\bar{\Omega}; \mathbb{R}_{\text{sym}}^{n \times n})).$$

760 Then  $(u, \sigma)$  is a weak solution associated with  $u_D$ .

761 *Proof.* The arguments are similar to the proof of [Proposition 4.6](#). First, since  
 762  $\sigma_n \rightharpoonup \sigma$  in  $H^1(\mathbb{L}^2(\Omega))$ ,  $\ell_n \rightharpoonup 0$  in  $L^2(\mathbf{H}_D^{-1}(\Omega))$ , and  $-\operatorname{div} \sigma_n = \ell_n$  for all  $n \in \mathbb{N}$ , it  
 763 follows that  $\sigma(t) \in \mathcal{E}(\Omega)$  f.a.a.  $t \in (0, T)$ . Moreover, from [Lemma 3.20](#) and [3.21](#) in our  
 764 compaion paper [\[21\]](#), we deduce that  $\sigma(t) \in \mathcal{K}(\Omega)$  a.e. in  $(0, T)$ , cf. also the first part  
 765 of the proof of [\[21, Theorem 3.22\]](#). Moreover, due to  $H^1(\mathbf{L}^{\frac{n}{n-1}}(\Omega)) \hookrightarrow C(\mathbf{L}^{\frac{n}{n-1}}(\Omega))$   
 766 and  $H^1(\mathbb{L}^2(\Omega)) \hookrightarrow C(\mathbb{L}^2(\Omega))$ , the weak limit satisfies the initial conditions.

767 To show the flow rule inequality, let  $\tau \in L^2(\mathbb{L}^2(\Omega))$  with  $\tau(t) \in \Sigma(\Omega) \cap \mathcal{K}(\Omega)$  f.a.a.  
 768  $t \in (0, T)$  be arbitrary. Then, [\(5.1b\)](#) and [\(5.1c\)](#) along with  $I_n(a) = 0$  for  $a \in \mathcal{K}(\Omega)$   
 769 and  $I_n \geq 0$ , imply

$$\begin{aligned} 0 &= \int_0^T I_n(\tau(t)) dt \\ &\geq (\nabla^s \dot{u}_n - \mathbb{A} \dot{\sigma}_n, \tau - \sigma_n)_{L^2(\mathbb{L}^2(\Omega))} \\ 770 &= (\nabla^s \dot{u}_{D,n} - \mathbb{A} \dot{\sigma}_n, \tau - \sigma_n)_{L^2(\mathbb{L}^2(\Omega))} - \int_0^T \int_{\Omega} (\dot{u}_n - \dot{u}_{D,n}) \operatorname{div} \tau \, dx dt \\ &\quad + (\nabla^s \dot{u}_{D,n} - \nabla^s \dot{u}_n, \sigma_n)_{L^2(\mathbb{L}^2(\Omega))}. \end{aligned}$$

771 Now one can pass to the limit with the first two terms on the right hand side exactly as  
 772 described at the end of the proof of [Proposition 4.6](#), see [\(4.6\)](#) and [\(4.7\)](#). Concerning  
 773 the last term, we argue as follows: Since  $\operatorname{div} \sigma = 0$  and  $u_n$  satisfies the Dirichlet  
 774 boundary condition, i.e.,  $\dot{u}_n = \dot{u}_{D,n}$  on  $\Gamma_D$ , we obtain

$$\begin{aligned} &| (\nabla^s \dot{u}_{D,n} - \nabla^s \dot{u}_n, \sigma_n)_{L^2(\mathbb{L}^2(\Omega))} | \\ 775 &= | (\nabla^s \dot{u}_{D,n} - \nabla^s \dot{u}_n, \sigma_n - \sigma)_{L^2(\mathbb{L}^2(\Omega))} | \\ &\leq \|\nabla^s \dot{u}_{D,n} - \nabla^s \dot{u}_n\|_{L^2(\mathbb{L}^1(\Omega))} \|\sigma_n - \sigma\|_{L^2(C(\bar{\Omega}; \mathbb{R}_{\operatorname{sym}}^{n \times n}))} \rightarrow 0, \end{aligned}$$

776 thanks to the boundedness of  $u_n$  in  $\mathcal{U}$  and the convergence of  $\sigma_n$ .  $\square$

777 The last step of the above proof illustrates, where the high regularity of the stress  
 778 field enforced by the additional inequality constraint in  $(\mathbf{P}_\lambda)$  comes into play: we need  
 779 the strong convergence of the stress in  $L^2(C(\bar{\Omega}; \mathbb{R}_{\operatorname{sym}}^{n \times n}))$  in order to pass to the limit  
 780 in the flow rule inequality. Unfortunately, the recovery sequence needs to be feasible  
 781 for  $(\mathbf{P}_\lambda)$  and thus has to fulfill this inequality constraint, too. Using our results from  
 782 [section 5](#), this can be guaranteed, provided that there is at least one optimal solution,  
 783 whose strain rate admits higher regularity. This is the most severe restriction for our  
 784 main result:

785 **THEOREM 6.3** (Approximation of global minimizers). *Let the objective in  $(\mathbf{P})$  be*  
 786 *of the form [\(6.1\)](#). Assume moreover that there exists a global minimizer  $(\bar{u}, \bar{\sigma}, \bar{u}_D)$  of*  
 787  *$(\mathbf{P})$  such that  $\nabla^s \dot{\bar{u}} \in L^2(\mathbb{L}^2(\Omega)) \cap L^1(\mathbb{W}^{1,p}(\Omega))$  and  $\bar{u} - \bar{u}_D \in \mathbf{H}_D^1(\Omega)$  for all  $t \in (0, T)$ .*  
 788 *Suppose in addition that  $R$  in  $(\mathbf{P}_\lambda)$  is chosen so large that*

$$789 \quad (6.6) \quad R \geq \frac{1}{\gamma_{\mathbb{A}}} \|\bar{u}_D\|_{H^1(\mathbf{H}^1(\Omega))} + p \|\mathbb{A}\|^{p/2-1} \left( \|\nabla^s \dot{\bar{u}}\|_{L^1(\mathbb{W}^{1,p}(\Omega))} + \|\sigma_0\|_{\mathbb{W}^{1,p}(\Omega)}^p \right).$$

790 Furthermore, let  $\{\bar{u}_\lambda, \bar{\sigma}_\lambda, \bar{z}_\lambda, \bar{u}_{D,\lambda}, \bar{\ell}_\lambda\}_{\lambda>0}$  be a sequence of global minimizers of  $(\mathbf{P}_\lambda)$   
 791 for  $\lambda \searrow 0$ .

792 Then there exists an accumulation point of  $\{\bar{u}_\lambda, \bar{\sigma}_\lambda, \bar{u}_{D,\lambda}\}_{\lambda>0}$  w.r.t. weak conver-  
 793 gence in  $\mathcal{U} \times H^1(\mathbb{L}^2(\Omega)) \cap L^s(\mathbb{W}^{1,p}(\Omega)) \times H^1(\mathbf{H}^2(\Omega))$ . Moreover, every such accumu-  
 794 lation point is a global minimizer of  $(\mathbf{P})$ .

795 Furthermore, if  $(\tilde{u}, \tilde{\sigma}, \tilde{u}_D)$  is such an accumulation point and  $\{\bar{u}_\lambda, \bar{\sigma}_\lambda, \bar{u}_{D,\lambda}\}_{\lambda>0}$   
 796 the associated sequence converging weakly to it, then

$$797 \quad (6.7) \quad \bar{u} \rightarrow \tilde{u} \quad \text{in } H^1(L^1(\Omega; \mathbb{R}^n)), \quad \bar{u}_{D,\lambda} \rightarrow \tilde{u}_D \quad \text{in } H^1(\mathbf{H}^2(\Omega)),$$

$$798 \quad (6.8) \quad \bar{\sigma}_\lambda \rightarrow \tilde{\sigma} \quad \text{in } L^2(C(\bar{\Omega}; \mathbb{R}_{\text{sym}}^{n \times n})), \quad \bar{\ell}_\lambda \rightarrow 0 \quad \text{in } H^1(\mathbf{H}_D^{-1}(\Omega)).$$

800 *Proof. Step 1. Existence of an accumulation point.* Since  $\{\bar{u}_\lambda, \bar{\sigma}_\lambda, \bar{u}_{D,\lambda}, \bar{\ell}_\lambda\}_{\lambda>0}$   
 801 is a global solution of  $(\mathbf{P}_\lambda)$  and the constant tuple  $(u, \sigma, z, u_D, \ell) \equiv (u_0, \sigma_0, \nabla^s u_0 -$   
 802  $\mathbb{A}\sigma_0, u_0, 0)$  is feasible for  $(\mathbf{P}_\lambda)$ , we obtain

$$803 \quad (6.9) \quad J_\lambda(\bar{u}_\lambda, \bar{u}_{D,\lambda}, \bar{\ell}_\lambda) \leq J_\lambda(u_0, u_0, 0) = \frac{\alpha}{2} \|u_0\|_{L^2(\mathbf{H}^2(\Omega))}^2 =: C < \infty.$$

804 Since all  $\bar{u}_\lambda$  share the same initial value and due to the special structure of the objec-  
 805 tive in (6.1), this implies that  $\bar{u}_\lambda$  satisfies the boundedness assumption in (4.3) such  
 806 that Lemma 4.5 yields the existence of a subsequence converging weakly in  $\mathcal{U}$ . More-  
 807 over, the inequality constraint on the stress and the  $H^1(H^2)$ -norm in the objective  
 808 immediately yield the boundedness of  $\bar{\sigma}_\lambda$  and  $\bar{u}_{D,\lambda}$  in their respective spaces, and the  
 809 reflexivity of the latter imply the existence of a weakly convergent subsequence.

810 *Step 2. Feasibility of an accumulation point.* Let us now assume that a given sub-  
 811 sequence of  $\{\bar{u}_\lambda, \bar{\sigma}_\lambda, \bar{u}_{D,\lambda}\}_{\lambda>0}$ , denoted by the same symbol for simplicity, converges  
 812 weakly to  $(\tilde{u}, \tilde{\sigma}, \tilde{u}_D)$  in  $\mathcal{U} \times H^1(\mathbb{L}^2(\Omega)) \cap L^s(\mathbb{W}^{1,p}(\Omega)) \times H^1(\mathbf{H}^2(\Omega))$ . By the com-  
 813 pact embedding of  $H^1(\mathbf{H}^2(\Omega))$  in  $C(\mathbf{H}^1(\Omega))$ , this ensures the convergence properties  
 814 required in (6.3) and in addition  $\tilde{u}_D(0) - u_0 \in \mathbf{H}_D^1(\Omega)$ . Moreover, the assumptions  
 815 on  $p$  and  $s$  in (6.2) guarantee that  $H^1(\mathbb{L}^2(\Omega)) \cap L^s(\mathbb{W}^{1,p}(\Omega))$  embeds compactly in  
 816  $L^2(C(\bar{\Omega}; \mathbb{R}_{\text{sym}}^{n \times n}))$ , as already mentioned above, so that (6.5) is valid. Furthermore,  
 817 considering again (6.9), we see that  $\lambda^{-\theta} \|\bar{\ell}_\lambda\|_{L^2(\mathbf{H}_D^{-1}(\Omega))}$  is bounded, hence,  $\bar{\ell}_\lambda \rightarrow \ell = 0$   
 818 in  $L^2(\mathbf{H}_D^{-1}(\Omega))$  (even with strong convergence). Altogether, we observe that the con-  
 819 vergence properties in (6.3)–(6.5) are fulfilled such that Proposition 6.2 yields that the  
 820 weak accumulation point  $(\tilde{u}, \tilde{\sigma})$  is a weak solution associated with  $\tilde{u}_D$  and therefore  
 821 feasible for the original optimization problem  $(\mathbf{P})$ .

822 *Step 3. Construction of a recovery sequence.* First, observe that, since  $\bar{u}$  is  
 823 assumed to be in  $H^1(\mathbf{H}^1(\Omega))$  and to satisfy the Dirichlet boundary conditions, Corol-  
 824 lary 3.5 gives that  $(\bar{\sigma}, \bar{u})$  is a strong solution associated with  $\bar{u}_D$ .

825 The recovery sequence for  $(\bar{u}, \bar{\sigma}, \bar{u}_D)$  is constructed based on our findings in sec-  
 826 tion 5. To be more precise, we apply Lemma 5.3 and Lemma 5.5 with  $w = \nabla^s \bar{u}$ .  
 827 According to these lemmas,  $\sigma_\lambda \in H^1(\mathbb{L}^2(\Omega))$  defined as unique solution of

$$828 \quad \nabla^s \dot{\bar{u}} - \mathbb{A}\dot{\sigma}_\lambda = \partial I_\lambda(\sigma_\lambda), \quad \sigma_\lambda(0) = \sigma_0,$$

829 satisfies the bound in (5.7) and converges strongly in  $H^1(\mathbb{L}^2(\Omega))$  to  $\sigma$ , which is the  
 830 solution to

$$831 \quad \nabla^s \dot{\bar{u}} - \mathbb{A}\dot{\sigma} \in \partial I_{\mathcal{K}(\Omega)}(\sigma), \quad \sigma(0) = \sigma_0.$$

832 This equation is just the strong form of the flow rule in (3.4b). The monotonicity  
 833 of  $\partial I_{\mathcal{K}(\Omega)}$  immediately gives that (3.4b) is uniquely solvable. Therefore, the limit  $\sigma$   
 834 coincides with  $\bar{\sigma}$ , i.e., the stress associated with  $\bar{u}_D$ . If we now define

$$835 \quad z_\lambda := \nabla^s \bar{u} - \mathbb{A}\sigma_\lambda \in H^1(\mathbb{L}^2(\Omega)) \quad \text{and} \quad \ell_\lambda := -\text{div } \sigma_\lambda \in H^1(\mathbf{H}_D^{-1}(\Omega)),$$

836 then we observe that  $(\bar{u}, \sigma_\lambda, z_\lambda)$  is the solution of the regularized plasticity system in  
 837 (5.1) w.r.t.  $\bar{u}_D$  and  $\ell_\lambda$ . In addition, we have  $\ell_\lambda(0) = -\operatorname{div} \sigma_0 = 0$  and  $\bar{u}_D(0) - u_0 =$   
 838  $\bar{u}_D(0) - \bar{u}(0) \in \mathbf{H}_D^1(\Omega)$ . Therefore, since  $\sigma_\lambda$  satisfies the bounds in (5.7) and (4.1) (by  
 839 Lemma 4.1),  $(\bar{u}, \sigma_\lambda, z_\lambda, \bar{u}_D, \ell_\lambda)$  satisfies all constraints in  $(\mathbf{P}_\lambda)$ .

840 Next we show the convergence of the objective functional. As  $\bar{\sigma}$  fulfills the equi-  
 841 librium condition, i.e.,  $\bar{\sigma} \in \mathcal{E}(\Omega)$ , the convergence of  $\sigma_\lambda$  by Lemma 5.5 implies

$$842 \quad \ell_\lambda = -\operatorname{div} \sigma_\lambda \rightarrow -\operatorname{div} \bar{\sigma} = 0 \quad \text{in } H^1(\mathbf{H}_D^{-1}(\Omega)).$$

843 Furthermore, (5.12) gives

$$\begin{aligned} \lambda^{-\theta} \|\ell_\lambda\|_{L^2(\mathbf{H}_D^{-1}(\Omega))}^2 &= \lambda^{-\theta} \|\operatorname{div} \sigma_\lambda - \operatorname{div} \bar{\sigma}\|_{L^2(\mathbf{H}_D^{-1}(\Omega))}^2 \\ 844 \quad &\leq C \lambda^{-\theta} \|\sigma_\lambda - \bar{\sigma}\|_{L^2(\mathbb{L}^2(\Omega))}^2 \\ &\leq C \lambda^{1-\theta} \|\nabla^s \bar{u} - \mathbb{A} \bar{\sigma}\|_{L^2(\mathbb{L}^2(\Omega))}^2 \rightarrow 0 \quad \text{as } \lambda \searrow 0. \end{aligned}$$

845 To summarize, we found that  $(\bar{u}, \sigma_\lambda, z_\lambda, \bar{u}_D, \ell_\lambda)$  is feasible for  $(\mathbf{P}_\lambda)$  and fulfills

$$846 \quad (6.10) \quad J_\lambda(\bar{u}, \bar{u}_D, \ell_\lambda) \rightarrow J(\bar{u}, \bar{u}_D).$$

847 *Step 4. Strong convergence and global minimizer.* The feasibility and the conver-  
 848 gence of the recovery sequence and the optimality of  $(\bar{u}_\lambda, \bar{u}_{D,\lambda}, \bar{\ell}_\lambda)$  give

$$\begin{aligned} J(\bar{u}, \bar{u}_D) &\leq \liminf_{\lambda \searrow 0} J(\bar{u}_\lambda, \bar{u}_{D,\lambda}) \\ 849 \quad (6.11) \quad &\leq \limsup_{\lambda \searrow 0} J(\bar{u}_\lambda, \bar{u}_{D,\lambda}) \\ &\leq \limsup_{\lambda \searrow 0} J_\lambda(\bar{u}_\lambda, \bar{u}_{D,\lambda}, \bar{\ell}_\lambda) \leq \limsup_{\lambda \searrow 0} J_\lambda(\bar{u}, \bar{u}_D, \ell_\lambda) = J(\bar{u}, \bar{u}_D), \end{aligned}$$

850 which, together with the feasibility of  $(\bar{u}, \bar{\sigma}, \bar{u}_D)$  for  $(\mathbf{P})$  shown in step 2, implies that  
 851  $(\bar{u}, \bar{\sigma}, \bar{u}_D)$  is a global minimizer of  $(\mathbf{P})$ .

852 To show the strong convergence in (6.7) and (6.8), we first observe that (6.11)  
 853 yields  $J(\bar{u}_\lambda, \bar{u}_{D,\lambda}) \rightarrow J(\bar{u}, \bar{u}_D)$ , from which we deduce the convergence of the norms  
 854  $\|\dot{\bar{u}}_\lambda\|_{L^2(\mathbf{L}^1(\Omega))}$  and  $\|\bar{u}_{D,\lambda}\|_{H^1(\mathbf{H}^2(\Omega))}$  to  $\|\dot{\bar{u}}\|_{L^2(\mathbf{L}^1(\Omega))}$  and  $\|\bar{u}_D\|_{H^1(\mathbf{H}^2(\Omega))}$ , respectively.  
 855 Since both norms are Kadec norms and we already have weak convergence in the  
 856 respective spaces, this implies (6.7). Similarly, (6.11) yields  $\|\bar{\ell}_\lambda\|_{H^1(\mathbf{H}_D^{-1}(\Omega))} \rightarrow 0$ .  
 857 Finally, the strong convergence of the stresses follows from the compact embedding  
 858 of  $H^1(\mathbb{L}^2(\Omega)) \cap L^s(\mathbb{W}^{1,p}(\Omega))$  in  $L^2(C(\bar{\Omega}; \mathbb{R}_{\text{sym}}^{n \times n}))$ , already used above.  $\square$

859 Some comments concerning our approximation result are in order:

860 *Remark 6.4* (Crucial regularity assumption). The assumption of existence of a  
 861 global minimizer  $(\bar{u}, \bar{\sigma}, \bar{u}_D)$  with the properties listed in Theorem 6.3 is admittedly  
 862 very restrictive. Notice in particular that the regularity assumptions on  $\bar{u}$  imply that  
 863  $(\bar{u}, \bar{\sigma})$  is a *strong solution* w.r.t.  $\bar{u}_D$ , whose existence can in general not be guaranteed.  
 864 The regularity assumption however seems to be indispensable, as the above proof  
 865 demonstrates: In order to pass to the limit in the flow rule inequality to show the  
 866 feasibility of an accumulation point in step 2 of the proof, we need the additional  
 867 regularity of the stress ensured by the inequality constraint in  $(\mathbf{P}_\lambda)$ . The generic  
 868 regularity of the stress, which is  $H^1(\mathbb{L}^2(\Omega))$  (see Lemma 4.1), is by far not sufficient  
 869 for this passage to the limit. It therefore appears to be unavoidable to enforce the  
 870 required regularity by additional inequality constraints in  $(\mathbf{P}_\lambda)$ . The elements of the

871 recovery sequence however have to be feasible for  $(\mathbf{P}_\lambda)$  and thus have to fulfill this  
 872 inequality constraint, too. As the generic regularity of the stress is  $H^1(\mathbb{L}^2(\Omega))$ , it is  
 873 not possible to guarantee this constraint to be fulfilled without further hypotheses  
 874 on the recovery sequence and its limit, respectively. At the end, this leads to the  
 875 regularity assumption on  $\nabla^s \dot{\bar{u}}$ .

876 We however emphasize that we do not require the existence of a strong solution  
 877 with the addition regularity of the strain rate for every Dirichlet displacement  $u_D \in$   
 878  $H^1(\mathbf{H}^2(\Omega))$  (which would really be unrealistic), but only for one optimal  $\bar{u}_D$ . (Of  
 879 course, there might be many optimal solutions, since  $(\mathbf{P})$  is a non-convex problem).  
 880 Whether an optimal solution fulfilling these regularity assumptions exists or not,  
 881 clearly depends on the data, especially on the smoothness of the desired strain rate  
 882  $\mu$  in (6.1).

883 *Remark 6.5* (Extensions and modifications of the approximation result).

- 884 (i) One essential drawback of the approximation result is that the bound  $R$  given  
 885 in (6.6) depends on the unknown solution  $(\bar{u}, \bar{u}_D)$  and is therefore in general  
 886 unknown, too. One could replace the inequality constraints on the stress  
 887 involving this bound in  $(\mathbf{P}_\lambda)$  by an additional tracking term in the objective of  
 888 the form  $\|\sigma - \sigma_d\|_{H^1(\mathbb{L}^2(\Omega))}^2 + \|\sigma - \sigma_d\|_X$  with a given desired stress distribution  
 889  $\sigma_d$  and a reflexive Banach space  $X$  with the following properties: On the one  
 890 hand,  $H^1(\mathbb{L}^2(\Omega)) \cap X$  should compactly embed in  $L^2(C(\bar{\Omega}; \mathbb{R}_{\text{sym}}^{n \times n}))$ . On the  
 891 other hand,  $H^1(\mathbb{L}^2(\Omega)) \cap L^\infty(\mathbb{W}^{1,p}(\Omega))$  should compactly be embedded in  $X$ .  
 892 Provided these embeddings hold, the steps 2 and 3 of the previous proof can  
 893 easily be adapted. At this point, one benefits from the strong convergence of  
 894 the recovery sequence in  $H^1(\mathbb{L}^2(\Omega))$  by Lemma 5.5.
- 895 (ii) The above analysis is restricted to objectives of the type (6.1) or other types  
 896 of objectives ensuring the boundedness of  $\{\bar{u}_\lambda\}$  in  $\mathcal{U}$ . This bound cannot  
 897 be deduced from the regularized plasticity system in (5.1) unless the loads  
 898 fulfill a *safe load condition*, see [30]. One could thus allow for more general  
 899 objectives, if a safe load condition would be included in the set of constraints  
 900 in  $(\mathbf{P}_\lambda)$ . We were however not able to find a safe load condition that is  
 901 satisfied by the loads associated with the recovery sequence. This is due to  
 902 several reasons, among these a lack of regularity of the recovery sequence.  
 903 This issue is subject to future research.
- 904 (iii) By contrast, it is well possible to consider objectives, which give the bound-  
 905 edness of the displacement in more regular spaces such as  $H^1(\mathbf{H}^1(\Omega))$ . In  
 906 this case, the inequality constraints on the stress in  $(\mathbf{P}_\lambda)$  can be weakened or  
 907 even be completely left out, since the higher regularity of the displacement  
 908 enables the passage to the limit at the end of the proof of Lemma 5.5. Such  
 909 a setting is treated in [35].
- 910 (iv) We have chosen the space  $H^1(\mathbf{H}^2(\Omega))$  as the control space for the Dirichlet  
 911 displacement in order to guarantee the compact embeddings in step 2 of the  
 912 above proof and in the proof of Theorem 4.7. Of course, one might want to  
 913 avoid the  $\mathbf{H}^2(\Omega)$ -norm in the objective, which could be achieved by an addi-  
 914 tional (pseudo-)force-to-Dirichlet-map, for example by solving an additional  
 915 linear elasticity system. This strategy was employed in [21, Subsection 6.1].

916 *Remark 6.6* (Numerical treatment of  $(\mathbf{P}_\lambda)$ ). Although they are still nonsmooth  
 917 optimization problems, the regularized problems in  $(\mathbf{P}_\lambda)$  offer ample possibilities for  
 918 a numerical treatment. A popular strategy is to further regularize the problem by  
 919 smoothing the Yosida approximation  $\partial I_\lambda$ . This has been used for the numerical

920 computations in the companion paper [21]. Moreover, the non-smooth objective in  
 921  $(P_\lambda)$  calls for an additional regularization of the  $L^1$ -norms for instance in terms of  
 922 a Huber-regularization. In this way, one ends up with a smooth optimal control  
 923 problem, which can be treated by the classical adjoint approach. Our convergence  
 924 result in [Theorem 6.3](#) implies that, under the certainly restrictive assumptions of this  
 925 theorem, there is an optimal solution of the original optimization problem governed  
 926 by the perfect plasticity system that can be approximated by this procedure.

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 929 tion of the recovery sequence.

### 930 **Appendix A. Auxiliary results.**

931 **LEMMA A.1.** *Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain and  $X$  be a Banach space and*  
 932  *$A \subset X$  be a convex and closed set with  $0 \in A^\circ$ . Set  $\mathcal{A}(\Omega) := \{v \in L^2(\Omega; X) : v \in$*   
 933  *$A$  a.e. in  $\Omega\}$ . Then  $C_c^\infty(\Omega; X) \cap \mathcal{A}(\Omega)$  is dense in  $\mathcal{A}(\Omega)$ .*

934 *Proof.* Let  $v \in \mathcal{A}(\Omega)$  and  $\varepsilon \in (0, 1)$  be arbitrary. By assumption there exists  
 935  $\delta > 0$  such that  $\overline{B_X(0, \delta)} \subset A$ . We set  $\bar{v} := (1 - \varepsilon)v$  and select a sequence  $\{v_n\}_{n \in \mathbb{N}} \subset$   
 936  $C_c^\infty(\Omega; X)$  such that

$$937 \quad (\text{A.1}) \quad \|v_n - \bar{v}\|_{L^2(\Omega; X)}^2 \leq \frac{\delta^2 \varepsilon^3}{4n} \quad \forall n \in \mathbb{N}.$$

938 We moreover define

$$939 \quad S_n^c := \{x \in \Omega : v_n(x) \in X \setminus A^\circ\}, \quad S_n^o := \{x \in \Omega : v_n(x) \in X \setminus (1 - \frac{\varepsilon}{2})A\}.$$

941 Hence,  $S_n^c \subset S_n^o$  and, by continuity and compact support of  $v_n$ ,  $S_n^c$  is compact, while  
 942  $S_n^o$  is open. Thus, for every  $n \in \mathbb{N}$ , there is a function  $\varphi_n \in C^\infty(\mathbb{R}^n; [0, 1])$  with  
 943  $\varphi_n \equiv 1$  in  $\mathbb{R}^n \setminus S_n^o$  and  $\varphi_n \equiv 0$  in  $S_n^c$ . Furthermore, if  $\|v_n(x) - \bar{v}(x)\|_X \leq \frac{\varepsilon}{2} \delta$ , then  
 944 the convexity of  $A$  and  $\overline{B_X(0, \delta)} \subset A$  imply

$$945 \quad \frac{v_n(x)}{1 - \frac{\varepsilon}{2}} = \frac{1 - \varepsilon}{1 - \frac{\varepsilon}{2}} v(x) + \left(1 - \frac{1 - \varepsilon}{1 - \frac{\varepsilon}{2}}\right) \frac{2}{\varepsilon} (v_n(x) - \bar{v}(x)) \in A$$

946 Therefore, we obtain by contraposition that

$$948 \quad \|v_n - \bar{v}\|_{L^2(\Omega; X)}^2 \geq \int_{S_n^o} \|v_n - \bar{v}\|_X^2 dx \geq \frac{\varepsilon^2}{4} \delta^2 |S_n^o|$$

950 so that (A.1) yields  $|S_n^c| \leq |S_n^o| \leq \varepsilon/n$ . Thus, due to Lebesgue's dominated conver-  
 951 gence theorem, there exists  $N = N(\varepsilon) \in \mathbb{N}$  such that

$$952 \quad (\text{A.2}) \quad \|\bar{v}\|_{L^2(S_N^o; X)} \leq \|v\|_{L^2(S_N^o; X)} \leq \varepsilon.$$

953 Now we define  $v_s := \varphi_N v_N$ . Then, by construction  $v_s \in \mathcal{A}(\Omega) \cap C_c^\infty(\Omega; X)$  and, in  
 954 addition, (A.1) and (A.2) imply

$$\begin{aligned} 955 \quad \|v - v_s\|_{L^2(\Omega; X)} &\leq \|v - \bar{v}\|_{L^2(\Omega; X)} + \|\bar{v} - v_N\|_{L^2(\Omega; X)} + \|v_N - v_s\|_{L^2(\Omega; X)} \\ 956 &\leq \varepsilon \|v\|_{L^2(\Omega; X)} + \|\bar{v} - v_N\|_{L^2(\Omega; X)} + \|v_N\|_{L^2(S_N^o; X)} \\ 957 &\leq \varepsilon \|v\|_{L^2(\Omega; X)} + 2 \|\bar{v} - v_N\|_{L^2(\Omega; X)} + \|\bar{v}\|_{L^2(S_N^o; X)} \\ 958 &\leq \varepsilon \left( \|v\|_{L^2(\Omega; X)} + \frac{\delta \sqrt{\varepsilon}}{\sqrt{N}} + 1 \right). \\ 959 \end{aligned}$$

960 Since  $\varepsilon$  was arbitrary, this finishes the proof.  $\square$

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