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Full Stability for Variational Nash Equilibriums of Parametric Optimal Control Problems of PDEs*

Nguyen Thanh Qui† and Daniel Wachsmuth‡

February 20, 2020

Abstract. This paper investigates full stability properties for *variational Nash equilibriums* of a system of parametric nonconvex optimal control problems governed by semilinear elliptic partial differential equations. We first obtain some new results on the existence of variational Nash equilibriums for the system of original/parametric nonconvex optimal control problems. Then we establish explicit characterizations of the Lipschitzian and Hölderian full stability of variational Nash equilibriums under perturbations. These results deduce the equivalence between variational Nash equilibriums and local Nash equilibriums in the classical sense.

Key words. Lipschitzian and Hölderian full stability, variational inequality, optimal control, partial differential equation, coderivative, subdifferential.

AMS subject classification. 35J61, 49J52, 49J53, 49K20, 49K30.

1 Introduction

It is well-known that the noncooperative game interactions of m players, where every player finds to optimize individually under the effects of each other's choices, is a valuable model for competitive circumstances in economics and operations management. This model is one of the most importance in the theory of equilibrium problems, where solutions of the model will be called (Nash) equilibriums. If each optimization problem in the equilibrium model is convex, the equilibriums are understood in the classical sense. Otherwise, these equilibriums will be defined via the variational sense, i.e., they satisfy a first-order necessary optimality condition in the form of a generalized equation (a variational system/inequality). Concerning with the equilibrium model, standard questions related to existence, uniqueness and stability of equilibriums are interesting and they need to be addressed. Recently, Rockafellar [39] has studied the strongly stable local optimality and the strong metric regularity of variational Nash equilibriums for an abstract game-like framework of multi-agent optimization via tools of variational analysis.

In this paper, we will consider an equilibrium model for multi-agent optimization, where each agent is an optimal control problem governed by a semilinear elliptic partial differential

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equation with the cost functional being nonconvex. By applying techniques of the optimal control theory and results in the perturbation theory of maximal monotone and m-accretive operators we obtain some new results on the existence of variational Nash equilibriums for the original/parametric (nonconvex) equilibrium problem. In addition, we establish criteria of Lipschitzian and Hölderian full stability for the equilibrium problem under (basic and tilt) perturbations by means of the Mordukhovich's generalized differentiation. The full stability of the parametric equilibrium problem deduces the local uniqueness of a variational Nash equilibrium in question and ensures the equivalence between variational Nash equilibriums and local Nash equilibriums in the classical sense.

Recent results for generalized Nash equilibrium problems associated to convex optimal control problems governed by partial differential equations can be found in [11, 16, 17, 19]. In comparison with these works, the methods and techniques we apply to deal with the nonconvex Nash equilibrium problems in this paper are different and new (even applying for the convex setting).

The concept of full Lipschitzian stability for local minimizers was introduced and studied by Levy, Poliquin, and Rockafellar [21] in the setting of finite dimensions. Then this property together with the full Hölderian stability were investigated in the infinite dimensional setting by Mordukhovich and Nghia in the work [25], where the notion of full Hölderian stability was also introduced in [25]. Many researchers have been interested in the full stability for local minimizers; see, e.g., [21, 25, 29, 30, 32, 33, 37]. The notions of full stability are defined via basic parameters and tilt ones, where the tilt parameters are related to the concept of tilt stability introduced by Poliquin and Rockafellar in the paper [36]; see, e.g., [12, 13, 14, 22, 31] for more results on the tilt stability. Note that the concepts of Lipschitzian and Hölderian full stability were defined for local minimizers of parametric optimization problems. Extensions of these concepts for parametric variational systems (PVSs), Mordukhovich and Nghia [26] have studied local strong maximal monotonicity of set-valued operators in Hilbert spaces with applications to full Lipschitzian and Hölderian stability for solutions of PVSs, where both notions of Lipschitzian and Hölderian full stability for solutions of PVSs were also introduced in [26]. In addition, the authors in [26] established characterizations of these notions for PVSs. Note that the notions of full stability for solutions of PVSs are generalized the ones for local minimizers of optimization problems, and these notions are equivalent in some special cases; see the comments in [26] for more details. To the best of our knowledge, there were only a few applications of the results of [26]; see, e.g., [28]. Recently, the results of [26] were applied to some models that can be regarded as general PVSs in [28]. In particular, necessary conditions and sufficient conditions of full stability for solutions to general parametric variational inequalities (PVIs) were established in [28]. Note that the authors of [28] considered the PVIs over fixed constraint sets.

In order to investigate stability for variational Nash equilibriums of our equilibrium model when the optimal control problems in the equilibrium model undergone full perturbations, we have recognized that the results on the Lipschitzian and Hölderian full stability for solutions of PVSs given in [26] are suitable and effective in applications for our parametric equilibrium problem. For this reason, we will apply the results of [26] and the techniques of the optimal control theory to establish characterizations of the Lipschitzian and Hölderian full stability of variational Nash equilibriums. A crucial role for our stability results is that a quadratic form defined via the second-order directional derivatives of the cost functionals is a Legendre form. The latter fact will be also proved in this paper.

The rest of the paper is organized as follows. Preliminaries given in Section 2 consist of the definitions for classical/local/variational Nash equilibriums, standard assumptions and auxiliary results for optimal control problems, and some material from variational analysis. Section 3 is devoted to prove results on existence of variational/classical Nash equilibriums associated to many finite optimal control problems governed by a semilinear elliptic partial differential equation. The results provided in this section are new and they are among our main results in this paper. In Section 4, we investigate the full stability of variational Nash equilibriums to the parametric equilibrium problem. We first prove that the quadratic form defined via the second-order directional derivatives of the cost functionals is a Legendre form. Then we establish necessary conditions and sufficient conditions (resp., explicit characterizations) of the Lipschitzian and Hölderian full stability for variational Nash equilibriums with respect to general nonempty bounded closed convex admissible control sets (resp., admissible control sets of box constraint type). We will also prove for the case of admissible control sets of box constraint type that variational Nash equilibriums and local Nash equilibriums are equivalent under the full stability condition of variational Nash equilibriums. Finally, some concluding remarks will be given in the last section.

2 Preliminaries

2.1 Variational Nash Equilibrium

For each $k \in \{1, \ldots, m\}$, consider the optimal control problem

$$\begin{cases} \text{Minimize} \quad J_k(u_1, \ldots, u_m) = \displaystyle\int_\Omega L_k(x, y_u(x))dx + \frac{1}{2}\int_\Omega \zeta_k(x) u_k(x)^2 dx \\ \text{subject to} \quad u_k \in \mathcal{U}_{ad}^k \subset L^2(\Omega), \end{cases} \tag{2.1}$$

where $\zeta_k \in L^\infty(\Omega)$ satisfies $\zeta_k(x) \geq \zeta_{0k} > 0$ for a.a. $x \in \Omega \subset I\!R^N$, and y_u is the weak solution associated to the control $u = \sum_{i=1}^m B_i u_i$ of the Dirichlet problem

$$\begin{cases} \mathcal{A}y + f(x, y) = \displaystyle\sum_{i=1}^m B_i u_i & \text{in } \Omega \\ y = 0 & \text{on } \Gamma, \end{cases} \tag{2.2}$$

where $B_i \in L^\infty(\Omega)$ for all $i \in \{1, \ldots, m\}$ and $\mathcal{A}$ denotes the second-order differential elliptic operator of the form

$$\mathcal{A}y(x) = -\sum_{i=1}^N \sum_{j=1}^N \partial_{x_j}\big(a_{ij}(x)\partial_{x_i} y(x)\big), \tag{2.3}$$

and the admissible control set $\mathcal{U}_{ad}^k$ in (2.1) is nonempty, convex, closed and bounded in $L^2(\Omega)$.

Example 2.1 Let us provide a specific example for the admissible control sets $\mathcal{U}_{ad}^k$ in (2.1), for $k \in \{1, \ldots, m\}$, that are very frequently appearing in the applications as follows

$$\begin{cases} \mathcal{U}_{ad}^k = \big\{v \in L^2(\Omega) \big| \, \alpha_k(x) \leq v(x) \leq \beta_k(x) \text{ for a.a. } x \in \Omega\big\} \\ \text{where } \alpha_k, \beta_k \in L^\infty(\Omega) \text{ with } \alpha_k(x) < \beta_k(x) \text{ for a.a. } x \in \Omega. \end{cases} \tag{2.4}$$

An element $\bar{u}_k \in \mathcal{U}_{ad}^k$ is said to be a *solution/global minimum* of the control problem (2.1) if $J_k(\bar{u}_k) \leq J_k(u_k)$ holds for all $u_k \in \mathcal{U}_{ad}^k$. We will say that $\bar{u}_k$ is a *local solution/local minimum* of the problem (2.1) if there exists a closed ball $\bar{B}_\varepsilon(\bar{u}_k) \subset L^2(\Omega)$ with the center $\bar{u}_k$ and the radius $\varepsilon > 0$ such that $J_k(\bar{u}_k) \leq J_k(u_k)$ holds for all $u_k \in \mathcal{U}_{ad}^k \cap \bar{B}_\varepsilon(\bar{u}_k)$. The local solution $\bar{u}_k$ is called *strict* if we have $J_k(\bar{u}_k) < J_k(u_k)$ for all $u_k \in \mathcal{U}_{ad}^k \cap \bar{B}_\varepsilon(\bar{u}_k)$ with $u_k \neq \bar{u}_k$.

Definition 2.2 A *Nash equilibrium* (in the classical sense) associated to the optimal control problems of the form (2.1) is the following combination

$$(\bar{u}_1, \ldots, \bar{u}_m) \in \mathcal{U}_{ad}^1 \times \cdots \times \mathcal{U}_{ad}^m \tag{2.5}$$

such that

$$\bar{u}_k \in \underset{u_k \in \mathcal{U}_{ad}^k}{\operatorname{argmin}} J_k(u_k, \bar{u}_{-k}), \ \forall k = 1, \ldots, m, \tag{2.6}$$

where $\bar{u}_k$ is a decision associated to the k-th player and $\bar{u}_{-k}$ stands for those decisions of all other players. We say that $(\bar{u}_1, \ldots, \bar{u}_m)$ is a *local Nash equilibrium* if for each $k \in \{1, \ldots, m\}$ there exists a closed ball $\bar{B}_\varepsilon(\bar{u}_k) \subset L^2(\Omega)$ with the center $\bar{u}_k$ and the radius $\varepsilon > 0$ such that

$$\bar{u}_k \in \underset{u_k \in \mathcal{U}_{ad}^k \cap \bar{B}_\varepsilon(\bar{u}_k)}{\operatorname{argmin}} J_k(u_k, \bar{u}_{-k}), \ \forall k = 1, \ldots, m. \tag{2.7}$$

Definition 2.3 A *variational Nash equilibrium* associated to the optimal control problems of the form (2.1) is the following combination

$$(\bar{u}_1, \ldots, \bar{u}_m) \in \mathcal{U}_{ad}^1 \times \cdots \times \mathcal{U}_{ad}^m \tag{2.8}$$

such that

$$0 \in \nabla_{u_k} J_k(\bar{u}_k, \bar{u}_{-k}) + N(\bar{u}_k; \mathcal{U}_{ad}^k), \ \forall k = 1, \ldots, m, \tag{2.9}$$

where $N(u; \Theta)$ is the normal cone to the convex set Θ at u in the sense of convex analysis. In other words, a variational Nash equilibrium $(\bar{u}_1, \ldots, \bar{u}_m) \in \mathcal{U}_{ad}^1 \times \cdots \times \mathcal{U}_{ad}^m$ is a solution of the system of necessary optimality conditions (2.9) associated to (2.6).

For each $k \in \{1, \ldots, m\}$, we define the basic parametric cost functional

$$\mathcal{J}_k(u_k, u_{-k}, e_Y, e_k) = J_k(u_k + e_Y, u_{-k}) + (e_{k,J}, y_{u+e_Y})_{L^2(\Omega)} \tag{2.10}$$

and consider the corresponding fully perturbed problem of the control problem (2.1)–(2.3) as follows

$$\begin{cases} \text{Minimize} \quad \mathcal{J}_k(u_k, u_{-k}, e_Y, e_k) - (u_k^*, u_k)_{L^2(\Omega)} \\ \text{subject to} \quad u_k \in \mathcal{U}_{ad}^k(e_k) \subset L^2(\Omega), \end{cases} \tag{2.11}$$

where y_{u+e_Y} is the weak solution associated to the control $u = \sum_{i=1}^m B_i u_i$ of the perturbed Dirichlet problem

$$\begin{cases} \mathcal{A}y + f(x, y) = \displaystyle\sum_{i=1}^m B_i u_i + e_Y & \text{in } \Omega \\ y = 0 & \text{on } \Gamma. \end{cases} \tag{2.12}$$

Here, $e_Y \in E_Y$ and $e_k = (e_{k,J}, e_{k,\bullet}) \in E_{k,J} \times E_{k,\bullet} = E_k$, where $E_Y \subset L^2(\Omega)$, $E_{k,J} \subset L^2(\Omega)$ and $E_{k,\bullet}$ are metric parametric spaces.

Example 2.4 When the admissible control sets $\mathcal{U}_{ad}^k$ are given by (2.4), we can consider the corresponding parametric admissible control sets $\mathcal{U}_{ad}^k(e_k)$ of the perturbed problem (2.11) by

$$\mathcal{U}_{ad}^k(e_k) = \big\{ v \in L^2(\Omega) \big| \, \alpha_k(x) + e_{k,\alpha}(x) \le v(x) \le \beta_k(x) + e_{k,\beta}(x) \text{ for a.a. } x \in \Omega \big\}, \tag{2.13}$$

where $e_k = (e_{k,J}, e_{k,\bullet}) = (e_{k,J}, e_{k,\alpha}, e_{k,\beta})$ with $e_{k,\bullet} = (e_{k,\alpha}, e_{k,\beta}) \in L^\infty(\Omega) \times L^\infty(\Omega)$. For this case, the metric parametric space is given by

$$E_Y \times E_k = E_Y \times E_{k,J} \times E_{k,\bullet} \subset L^2(\Omega) \times L^2(\Omega) \times L^\infty(\Omega) \times L^\infty(\Omega) \tag{2.14}$$

with the induced metric defined via the norm

$$\begin{aligned}\|(e_Y, e_k)\| &= \|(e_Y, e_{k,J}, e_{k,\alpha}, e_{k,\beta})\| \\ &= \|e_Y\|_{L^2(\Omega)} + \|e_{k,J}\|_{L^2(\Omega)} + \|e_{k,\alpha}\|_{L^\infty(\Omega)} + \|e_{k,\beta}\|_{L^\infty(\Omega)}\end{aligned} \tag{2.15}$$

of the product space $L^2(\Omega) \times L^2(\Omega) \times L^\infty(\Omega) \times L^\infty(\Omega)$.

Let us define $\mathbf{L}^2(\Omega) = L^2(\Omega)^m = L^2(\Omega) \times \cdots \times L^2(\Omega)$ and $\mathbf{E} = E_Y \times E_1 \times \cdots \times E_m$ the product spaces endowed with the sum norms therein. In what follows we use the notations for $\mathbf{u} \in \mathbf{L}^2(\Omega)$, $\mathbf{u}^* \in \mathbf{L}^2(\Omega)$, $\mathbf{e} \in \mathbf{E}$ by setting

$$\begin{aligned}\mathbf{u} &= (u_1, \ldots, u_m) \in \mathbf{L}^2(\Omega), \\ \mathbf{u}^* &= (u_1^*, \ldots, u_m^*) \in \mathbf{L}^2(\Omega), \\ \mathbf{e} &= (e_Y, e_1, \ldots, e_m) \in \mathbf{E},\end{aligned} \tag{2.16}$$

and denote

$$\mathcal{U}_{ad}(\mathbf{e}) = \mathcal{U}_{ad}^1(e_1) \times \cdots \times \mathcal{U}_{ad}^m(e_m) \subset \mathbf{L}^2(\Omega), \tag{2.17}$$

where $\mathcal{U}_{ad}^k(e_k)$ is given by (2.11) for every $k = 1, \ldots, m$.

Definition 2.5 Given $\bar{\mathbf{u}}^* = (\bar{u}_1^*, \ldots, \bar{u}_m^*) \in \mathbf{L}^2(\Omega)$ and $\bar{\mathbf{e}} = (\bar{e}_Y, \bar{e}_1, \ldots, \bar{e}_m) \in \mathbf{E}$, associated to the parametric control problem (2.11), the following combination

$$(\bar{u}_1, \ldots, \bar{u}_m) \in \mathcal{U}_{ad}^1(\bar{e}_1) \times \cdots \times \mathcal{U}_{ad}^m(\bar{e}_m) \tag{2.18}$$

such that

$$\bar{u}_k^* \in \nabla_{u_k} \mathcal{J}_k(\bar{u}_k, \bar{u}_{-k}, \bar{e}_Y, \bar{e}_k) + N(\bar{u}_k; \mathcal{U}_{ad}^k(\bar{e}_k)),\ \forall k = 1, \ldots, m, \tag{2.19}$$

is called a *variational Nash equilibrium with respect to the parameters* $(\bar{\mathbf{u}}^*, \bar{\mathbf{e}})$.

2.2 Assumptions and auxiliary results

Let us give some standard assumptions in optimal control and provide some auxiliary results related to the optimal control problems (2.1)–(2.3).

We assume that $\Omega \subset I\!R^N$ with $N \in \{1, 2, 3\}$. In addition, for every $k \in \{1, \ldots, m\}$, the functions $L_k, f : \Omega \times I\!R \to I\!R$ are Carathéodory functions of the class $\mathcal{C}^2$ with respect to the second variable. We now consider the following assumptions:

(A1) The function f satisfies

$$f(\cdot, 0) \in L^2(\Omega) \quad \text{and} \quad \frac{\partial f}{\partial y}(x, y) \geq 0 \quad \text{for a.a. } x \in \Omega,$$

and for all $M > 0$ there exists a constant $C_{f,M} > 0$ such that

$$\left|\frac{\partial f}{\partial y}(x, y)\right| + \left|\frac{\partial^2 f}{\partial y^2}(x, y)\right| \leq C_{f,M},$$

and

$$\left|\frac{\partial^2 f}{\partial y^2}(x, y_2) - \frac{\partial^2 f}{\partial y^2}(x, y_1)\right| \leq C_{f,M}|y_2 - y_1|,$$

for a.a. $x \in \Omega$ and $|y|, |y_1|, |y_2| \leq M$.

(A2) For every $k \in \{1, \ldots, m\}$, the function $L_k(\cdot, 0) \in L^1(\Omega)$, and for all $M > 0$ there are a constant $C_{L_k,M} > 0$ and a function $\psi_{k,M} \in L^2(\Omega)$ such that

$$\left|\frac{\partial L_k}{\partial y}(x, y)\right| \leq \psi_{k,M}(x), \quad \left|\frac{\partial^2 L_k}{\partial y^2}(x, y)\right| \leq C_{L_k,M},$$

and

$$\left|\frac{\partial^2 L_k}{\partial y^2}(x, y_2) - \frac{\partial^2 L_k}{\partial y^2}(x, y_1)\right| \le C_{L_k,M}|y_2 - y_1|,$$

for a.a. $x \in \Omega$ and $|y|, |y_1|, |y_2| \le M$.

(A3) The set Ω is an open and bounded domain in $I\!R^N$ with Lipschitz boundary Γ, the coefficients $a_{ij} \in L^\infty(\Omega)$ of the second-order elliptic differential operator $\mathcal{A}$ defined by (2.3) satisfy the condition

$$\lambda_{\mathcal{A}}\|\xi\|^2_{I\!R^N} \le \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j, \ \forall \xi \in I\!R^N, \text{ for a.a. } x \in \Omega,$$

for some constant $\lambda_{\mathcal{A}} > 0$.

For the sake of convenience in order to investigate properties of weak solutions to the state equation (2.2) which depend on controls given on the right-hand side of (2.2), we consider the following auxiliary state equation

$$\begin{cases} \mathcal{A}y + f(x, y) = u & \text{in } \Omega \\ y = 0 & \text{on } \Gamma. \end{cases} \tag{2.20}$$

Then properties of weak solutions to the state equation (2.2) will be deduced from properties of weak solutions to the equation (2.20).

Theorem 2.6 *Assume that the assumptions* **(A1)**–**(A3)** *hold. Then, for each* $u \in L^2(\Omega)$, *the state equation* (2.20) *admits a unique weak solution* $y_u \in H_0^1(\Omega) \cap C(\bar{\Omega})$. *In addition, for every* $k \in \{1, \ldots, m\}$, *there exists a constant* $M_k > 0$ *such that*

$$\|y_u\|_{H_0^1(\Omega)} + \|y_u\|_{C(\bar{\Omega})} \le M_k, \ \forall u \in \mathcal{U}^k_{ad}. \tag{2.21}$$

Furthermore, if $u^n \rightharpoonup u$ *weakly in* $L^2(\Omega)$, *then* $y_{u^n} \to y_u$ *strongly in* $H_0^1(\Omega) \cap C(\bar{\Omega})$.

Proof. By applying [5, Theorem 2.1], we obtain the assertions of the theorem. □

Theorem 2.7 *Assume that the assumptions* **(A1)**–**(A3)** *hold. Then, associated to* (2.20), *the control-to-state operator* $G : L^2(\Omega) \to H_0^1(\Omega) \cap C(\bar{\Omega})$ *defined by* $G(u) = y_u$ *is of class* $\mathcal{C}^2$. *Moreover, for every* $u, v \in L^2(\Omega)$, $z_{u,v} = G'(u)v$ *is the unique weak solution of*

$$\begin{cases} \mathcal{A}z + \dfrac{\partial f}{\partial y}(x, y)z = v & \text{in } \Omega \\ z = 0 & \text{on } \Gamma. \end{cases} \tag{2.22}$$

Finally, for every $v_1, v_2 \in L^2(\Omega)$, $z_{v_1 v_2} = G''(u)(v_1, v_2)$ *is the unique weak solution of*

$$\begin{cases} \mathcal{A}z + \dfrac{\partial f}{\partial y}(x, y)z + \dfrac{\partial^2 f}{\partial y^2}(x, y)z_{u,v_1}z_{u,v_2} = 0 & \text{in } \Omega \\ z = 0 & \text{on } \Gamma, \end{cases} \tag{2.23}$$

where $y = G(u)$ *and* $z_{u,v_i} = G'(u)v_i$ *for* $i = 1, 2$.

Proof. The assertions of the theorem are deduced from [5, Theorem 2.4]. □

More details related to weak solutions of the state equations (2.20) as well as (2.2) can be found in [41, Chapter 4]. By Theorem 2.7, we denote the space containing weak solutions of (2.20) by $Y := H_0^1(\Omega) \cap C(\bar{\Omega})$ endowed with the norm

$$\|y\|_Y = \|y\|_{H_0^1(\Omega)} + \|y\|_{L^\infty(\Omega)}.$$

The forthcoming theorem provide us with formulas for computing the first and second-order directional derivatives of the cost functional of the control problem (2.1).

Theorem 2.8 *Assume that the assumptions* **(A1)–(A3)** *hold. For every* $k \in \{1, \dots, m\}$, *the cost functional* $J_k : \mathbf{L}^2(\Omega) \to I\!R$ *is of class* $\mathcal{C}^2$. *Moreover, for every* $(u_1, \dots, u_m) \in \mathbf{L}^2(\Omega)$ *and* $h, h_1, \dots, h_m \in L^2(\Omega)$, *the first and second derivatives of* $J_k(\cdot)$ *are given by*

$$\nabla_{u_k} J_k(u_k, u_{-k})h = \int_\Omega (\zeta_k u_k + \varphi_{k,u}) h dx, \tag{2.24}$$

and

$$\begin{aligned}\nabla^2_{u_k u_j} J_k(u_1, \dots, u_m)(h_k, h_j) = &\int_\Omega \left(\frac{\partial^2 L_k}{\partial y^2}(x, y_u) - \varphi_{k,u} \frac{\partial^2 f}{\partial y^2}(x, y_u) \right) z_{u,h_k} z_{u,h_j} dx \\ &+ \int_\Omega \chi_{\{k\}}(j) \zeta_k h_k h_j dx,\end{aligned} \tag{2.25}$$

where $u = \sum_{i=1}^m B_i u_i$, $y_u = G(u)$, $z_{u,h_i} = G'(u)h_i$ *for* $i \in \{k, j\}$, *and* $\varphi_{k,u} \in W^{2,2}(\Omega)$ *is the adjoint state of* y_u *defined as the unique weak solution of*

$$\begin{cases} \mathcal{A}^* \varphi + \dfrac{\partial f}{\partial y}(x, y_u)\varphi = \dfrac{\partial L_k}{\partial y}(x, y_u) & \text{in } \Omega \\ \varphi = 0 & \text{on } \Gamma \end{cases}$$

with $\mathcal{A}^*$ *being the adjoint operator of* $\mathcal{A}$.

Proof. It follows from [5, Theorem 2.6 and Remark 2.8]; see also [41, Chapter 4]. □

Theorem 2.9 *Assume that the assumptions* **(A1)–(A3)** *hold. For every* $k \in \{1, \dots, m\}$, *the optimal control problem* (2.1) *has at least one solution.*

Proof. Arguing similarly to the proof of [41, Theorem 4.15], we obtain the assertion of the theorem; see also [5, Theorem 2.2] for the related result. □

2.3 Material from variational analysis

This subsection recalls some concepts and facts of variational analysis taken from [23]; see also [40]. Unless otherwise stated, every reference norm in a product normed space is the sum norm. Let us denote the open ball of center $u \in X$ and radius $\rho > 0$ in a Banach space X by $B_\rho(u)$, and $\bar{B}_\rho(u)$ is the corresponding closed ball. Let $F : X \rightrightarrows W$ be a multifunction between Banach spaces. The set $\operatorname{gph} F := \{(u, v) \in X \times W \,|\, v \in F(u)\}$ is the graph of F. We say that F is locally closed around $\bar{\omega} := (\bar{u}, \bar{v}) \in \operatorname{gph} F$ if the graph of F is locally closed around $\bar{\omega}$, i.e., there is a closed ball $\bar{B}_\rho(\bar{\omega})$ such that $\bar{B}_\rho(\bar{\omega}) \cap \operatorname{gph} F$ is closed in $X \times W$. The *sequential Painlevé-Kuratowski outer/upper limit* of $\Phi : X \rightrightarrows X^*$ as $u \to \bar{u}$ is defined by

$$\begin{aligned}\operatorname*{Limsup}_{u \to \bar{u}} \Phi(u) = \Big\{ u^* \in X^* \Big| &\text{ there exist } u_n \to \bar{u} \text{ and } u_n^* \xrightarrow{w^*} u^* \text{ with} \\ &u_n^* \in \Phi(u_n) \text{ for every } n \in I\!N := \{1, 2, \dots\} \Big\}.\end{aligned} \tag{2.26}$$

Let $\phi : X \to \overline{I\!R}$ be a proper extended-real-valued function on an *Asplund* space X [1]; see [23, 24, 34] for more applications of Asplund spaces. Assume that ϕ is lower semicontinuous around $\bar{u}$, where $\bar{u} \in \operatorname{dom} \phi := \{u \in X \,|\, \phi(u) < \infty\}$. The *regular subdifferential* of ϕ at $\bar{u} \in \operatorname{dom} \phi$ is

$$\widehat{\partial}\phi(\bar{u}) = \Big\{ u^* \in X^* \Big| \liminf_{u \to \bar{u}} \frac{\phi(u) - \phi(\bar{u}) - \langle u^*, u - \bar{u} \rangle}{\|u - \bar{u}\|} \geq 0 \Big\}. \tag{2.27}$$

This concept is also known as the *subdifferential in the sense of viscosity solutions*; see [8, 9]. The *limiting subdifferential* (known also as *Mordukhovich subdifferential*) of ϕ at $\bar{u}$ is defined via the Painlevé-Kuratowski sequential outer limit (2.26) by

$$\partial\phi(\bar{u}) = \mathop{\mathrm{Lim\,sup}}_{u \xrightarrow{\phi} \bar{u}} \widehat{\partial}\phi(u), \tag{2.28}$$

where the notation $u \xrightarrow{\phi} \bar{u}$ means that $u \to \bar{u}$ with $\phi(u) \to \phi(\bar{u})$.

For a subset $\Theta \subset X$ locally closed around $\bar{u} \in \Theta$, the *regular and limiting normal cones* to Θ at $\bar{u} \in \Theta$ are respectively defined by

$$\widehat{N}(\bar{u};\Theta) = \widehat{\partial}\delta(\bar{u};\Theta) \quad \text{and} \quad N(\bar{u};\Theta) = \partial\delta(\bar{u};\Theta), \tag{2.29}$$

where $\delta(\cdot;\Theta)$ is the indicator function of Θ defined by $\delta(u;\Theta) = 0$ for $u \in \Theta$ and $\delta(u;\Theta) = \infty$ otherwise. The *regular* and *Mordukhovich coderivatives* of the multifunction $F : X \rightrightarrows W$ at the point $(\bar{u},\bar{v}) \in \operatorname{gph} F$ are respectively the multifunction $\widehat{D}^*F(\bar{u},\bar{v}) : W^* \rightrightarrows X^*$ defined by

$$\widehat{D}^*F(\bar{u},\bar{v})(v^*) = \big\{u^* \in X^* \big| \, (u^*, -v^*) \in \widehat{N}\big((\bar{u},\bar{v}); \operatorname{gph} F\big)\big\}, \ \forall v^* \in W^*,$$

and the multifunction $D^*F(\bar{u},\bar{v}) : W^* \rightrightarrows X^*$ given by

$$D^*F(\bar{u},\bar{v})(v^*) = \big\{u^* \in X^* \big| \, (u^*, -v^*) \in N\big((\bar{u},\bar{v}); \operatorname{gph} F\big)\big\}, \ \forall v^* \in W^*.$$

Given any $\bar{u}^* \in \partial\phi(\bar{u})$, following [25, 27] the *combined second-order subdifferential* of ϕ at $\bar{u}$ relative to $\bar{u}^*$ is the multifunction $\breve{\partial}^2\phi(\bar{u},\bar{u}^*) : X^{**} \rightrightarrows X^*$ with the values

$$\breve{\partial}^2\phi(\bar{u},\bar{u}^*)(u) = (\widehat{D}^*\partial\phi)(\bar{u},\bar{u}^*)(u), \ \forall u \in X^{**}. \tag{2.30}$$

Note that for $\phi \in \mathcal{C}^2$ around $\bar{u}$ with $\bar{u}^* = \nabla\phi(\bar{u})$ we have $\breve{\partial}^2\phi(\bar{u},\bar{u}^*)(u) = \{\nabla^2\phi(\bar{u})u\}$ for all $u \in X^{**}$ via the symmetric Hessian operator $\nabla^2\phi(\bar{u})$.

We say that the multifunction $F : X \rightrightarrows W$ is *locally Lipschitz-like*, or F has the *Aubin property* [10], around a point $(\bar{u},\bar{v}) \in \operatorname{gph} F$ if there exist $\ell > 0$ and neighborhoods U of $\bar{u}$, V of $\bar{v}$ such that

$$F(u_1) \cap V \subset F(u_2) + \ell\|u_1 - u_2\|\bar{B}_W, \ \forall u_1, u_2 \in U,$$

where $\bar{B}_W$ denotes the closed unit ball in W. Characterization of this property via the mixed Mordukhovich coderivative of F can be found in [23, Theorem 4.10].

3 Existence of variational Nash equilibriums

In this section, under the standard assumptions **(A1)**–**(A3)** we will prove the existence of variational Nash equilibriums to the system (2.9) associated to the optimal control problems given in (2.1). This result will lead to the existence of variational Nash equilibriums to the parametric system (2.19) associated to the optimal control problems given in (2.11). These variational Nash equilibriums are also local Nash equilibriums in the classical sense when the optimal control problems in question are all convex.

Theorem 3.1 *Assume that the assumptions* **(A1)**–**(A3)** *hold. There exists a variational Nash equilibrium*

$$\bar{\mathbf{u}} = (\bar{u}_1, \ldots, \bar{u}_m) \in \mathcal{U}_{ad}^1 \times \cdots \times \mathcal{U}_{ad}^m \tag{3.1}$$

to the system

$$0 \in \nabla_{u_k} J_k(\bar{u}_k, \bar{u}_{-k}) + N(\bar{u}_k; \mathcal{U}_{ad}^k), \ \forall k = 1, \ldots, m. \tag{3.2}$$

Proof. Let us put $\mathbf{X} = \mathbf{L}^2(\Omega)$ and put $\zeta = (\zeta_1, \ldots, \zeta_m)$. For every $\mathbf{u} = (u_1, \ldots, u_m) \in \mathbf{X}$, we denote $\zeta \odot \mathbf{u} = (\zeta_1 u_1, \ldots, \zeta_m u_m)$ and $\varphi_{\mathbf{u}} = (\varphi_{1,u}, \ldots, \varphi_{m,u})$, where $u = \sum_{i=1}^m B_i u_i$ and $\varphi_{k,u}$ is the adjoint state of y_u with respect to the k-th control problem in (2.1). We now define the map

$$\begin{aligned} T : \mathbf{X} &\rightrightarrows \mathbf{X} \\ \mathbf{u} &\mapsto \zeta \odot \mathbf{u} + N(\mathbf{u}; \mathcal{U}_{ad}). \end{aligned} \tag{3.3}$$

Denote $D(T) = \{\mathbf{u} \in \mathbf{X} | \, T(\mathbf{u}) \neq \emptyset\}$ and choose the set $\mathcal{G} \equiv \mathbf{X}$. Then, $D(T) \equiv \mathcal{U}_{ad}$ and the operator C defined by

$$\begin{aligned} C : \operatorname{cl} \mathcal{G} \equiv \mathbf{X} &\to \mathbf{X} \\ \mathbf{u} &\mapsto \varphi_{\mathbf{u}} \end{aligned} \tag{3.4}$$

is compact. Since C is a compact operator, C is completely continuous (i.e., C maps weakly convergent sequences into strongly convergent sequences). In addition, we observe that T is a maximal monotone operator. Hence, T is accretive and $R(T + \lambda I) = \mathbf{X}$ for every $\lambda > 0$ by [35, Corollary 3.7], where I denotes the identity operator on $\mathbf{X}$ which can be regarded as a duality mapping and $R(T + \lambda I)$ stands for the range of $T + \lambda I$. Thus, T is m-accretive; see definitions of accretive/m-accretive properties in [20]. Since $\mathcal{G} \equiv \mathbf{X}$, the conditions

$$D(T) \cap \mathcal{G} \neq \emptyset \quad \text{and} \quad \big(I - (T + C)\big)\big(D(T) \cap \partial \mathcal{G}\big) \subset \operatorname{cl} \mathcal{G} \tag{3.5}$$

are trivial. Of course, $\mathbf{X}$ is a uniformly convex space. Therefore, summarizing the above and applying [20, Theorem 1] we deduce that $0 \in R(T + C)$, i.e., there exists $\bar{\mathbf{u}} \in \mathbf{X}$ such that

$$0 \in T(\bar{\mathbf{u}}) + C(\bar{\mathbf{u}}), \tag{3.6}$$

or, equivalently, as follows

$$0 \in \zeta \odot \bar{\mathbf{u}} + \varphi_{\bar{\mathbf{u}}} + N(\bar{\mathbf{u}}; \mathcal{U}_{ad}), \tag{3.7}$$

which yields (3.2) due to $\nabla_{u_k} J_k(\bar{u}_k, \bar{u}_{-k}) = \zeta_k \bar{u}_k + \varphi_{k,\bar{u}}$ for every $k = 1, \ldots, m$. □

Based on Theorem 3.1, the forthcoming theorem provides us with an existence result of local Nash equilibriums in the classical sense to the equilibrium problems associated to the optimal control problems (2.1) for the convex setting.

Theorem 3.2 *Assume that the assumptions* **(A1)**–**(A3)** *hold and that the optimal control problems* (2.1) *are all convex optimization problems. There exists a classical Nash equilibrium*

$$\bar{\mathbf{u}} = (\bar{u}_1, \ldots, \bar{u}_m) \in \mathcal{U}_{ad}^1 \times \cdots \times \mathcal{U}_{ad}^m \tag{3.8}$$

to the equilibrium problem associated to the optimal control problems (2.1).

Proof. According to Theorem 3.1, there exists a variational Nash equilibrium

$$\bar{\mathbf{u}} = (\bar{u}_1, \ldots, \bar{u}_m) \in \mathcal{U}_{ad}^1 \times \cdots \times \mathcal{U}_{ad}^m$$

satisfying the system (3.2). Since the control problems (2.1) are all convex, the system of conditions (3.2) implies that

$$\bar{u}_k \in \underset{u_k \in \mathcal{U}_{ad}^k}{\operatorname{argmin}} J_k(u_k, \bar{u}_{-k}), \ \forall k = 1, \ldots, m, \tag{3.9}$$

This means that $\bar{\mathbf{u}}$ is a classical Nash equilibrium associated to the problems (2.1). □

By applying the techniques given in the proof of Theorem 3.1, we can deduce the existence of variational Nash equilibriums to the parametric system (2.19).

Theorem 3.3 *Let $\bar{\mathbf{u}}^* = (\bar{u}_1^*, \ldots, \bar{u}_m^*) \in \mathbf{L}^2(\Omega)$ and let $\bar{\mathbf{e}} = (\bar{e}_Y, \bar{e}_1, \ldots, \bar{e}_m) \in \mathbf{E}$ be such that*

$$\mathcal{U}_{ad}(\bar{\mathbf{e}}) = \mathcal{U}_{ad}^1(\bar{e}_1) \times \cdots \times \mathcal{U}_{ad}^m(\bar{e}_m) \neq \emptyset. \tag{3.10}$$

Assume that the assumptions **(A1)**–**(A3)** *hold. There exists a variational Nash equilibrium*

$$\bar{\mathbf{u}} = (\bar{u}_1, \ldots, \bar{u}_m) \in \mathcal{U}_{ad}^1(\bar{e}_1) \times \cdots \times \mathcal{U}_{ad}^m(\bar{e}_m) \tag{3.11}$$

to the parametric system

$$\bar{u}_k^* \in \nabla_{u_k} \mathcal{J}_k(\bar{u}_k, \bar{u}_{-k}, \bar{e}_Y, \bar{e}_k) + N(\bar{u}_k; \mathcal{U}_{ad}^k(\bar{e}_k)), \ \forall k = 1, \ldots, m, \tag{3.12}$$

with respect to the parameters $(\bar{\mathbf{u}}^*, \bar{\mathbf{e}})$.

Proof. Let $\mathbf{X} = \mathbf{L}^2(\Omega)$ and let $\zeta = (\zeta_1, \ldots, \zeta_m)$. For every $\mathbf{u} = (u_1, \ldots, u_m) \in \mathbf{X}$, we denote

$$\varphi_{\mathbf{u},\bar{\mathbf{e}}} = (\varphi_{1,u+\bar{e}_Y,\bar{e}_1}, \ldots, \varphi_{m,u+\bar{e}_Y,\bar{e}_m}), \tag{3.13}$$

where $u + \bar{e}_Y = \sum_{i=1}^m B_i u_i + \bar{e}_Y$, the state $y_{u+\bar{e}_Y}$ is the weak solution of (2.12), and $\varphi_{k,u+\bar{e}_Y,\bar{e}_k}$ is the adjoint state of $y_{u+\bar{e}_Y}$ with respect to the k-th parametric control problem in (2.11) that is the weak solution of the following equation

$$\begin{cases} \mathcal{A}^*\varphi + \dfrac{\partial f}{\partial y}(x, y_{u+\bar{e}_Y})\varphi = \dfrac{\partial L_k}{\partial y}(x, y_{u+\bar{e}_Y}) + \bar{e}_{k,J} & \text{in } \Omega \\ \varphi = 0 & \text{on } \Gamma. \end{cases} \tag{3.14}$$

Then, we have $\nabla_{u_k} \mathcal{J}_k(u_k, u_{-k}, \bar{e}_Y, \bar{e}_k)$ can be represented as follows

$$\nabla_{u_k} \mathcal{J}_k(u_k, u_{-k}, \bar{e}_Y, \bar{e}_k) = \zeta_k u_k + \varphi_{k,u+\bar{e}_Y,\bar{e}_k}, \ \forall k = 1, \ldots, m. \tag{3.15}$$

We now define the map

$$\begin{aligned} T : \mathbf{X} &\rightrightarrows \mathbf{X} \\ \mathbf{u} &\mapsto \zeta \odot \mathbf{u} + N(\mathbf{u}; \mathcal{U}_{ad}(\bar{\mathbf{e}})). \end{aligned} \tag{3.16}$$

Denote $D(T) = \{\mathbf{u} \in \mathbf{X} | \, T(\mathbf{u}) \neq \emptyset\}$ and choose the set $\mathcal{G} \equiv \mathbf{X}$. Then, $D(T) \equiv \mathcal{U}_{ad}(\bar{\mathbf{e}})$ and the operator C defined by

$$\begin{aligned} C : \operatorname{cl}\mathcal{G} \equiv \mathbf{X} &\to \mathbf{X} \\ \mathbf{u} &\mapsto \varphi_{\mathbf{u},\bar{\mathbf{e}}} - \bar{\mathbf{u}}^* \end{aligned} \tag{3.17}$$

is a compact operator. Arguing similarly to the proof of Theorem 3.1 we obtain the assertion of the theorem. □

Remark 3.4 Similar to Theorem 3.2, if the parametric optimal control problems considered in Theorem 3.3 are all convex, then variational Nash equilibriums to the parametric system (3.12) are also local Nash equilibriums to the equilibrium problem associated to the optimal control problems (2.11) with respect to the parameters $(\bar{\mathbf{u}}^*, \bar{\mathbf{e}})$.

4 Full stability of variational Nash equilibriums

In this section, we will investigate the Lipschitzian and Hölderian full stability for variational Nash equilibriums to the system (2.9) via the parametric system (2.19). We will apply the results on full stability for parametric variational systems provided in [26] to establish explicit characterizations of Lipschitzian and Hölderian full stability for parametric variational Nash equilibriums to the system (2.19). Related to the solution stability for the optimal control problem (2.1)–(2.3) via the perturbed problem (2.11)–(2.13) with $m = 1$ and $\zeta_1 = 0$, we refer the reader to [38]. In the setting of [38], stability for bang-bang optimal controls was investigated by means of tools of the optimal control theory.

To apply the stability results of [26], we need to reformulate the parametric system (2.19) to a corresponding parametric variational inequality (4.2) below. Let us define the operator $F : \mathbf{L}^2(\Omega) \times \mathbf{E} \to \mathbf{L}^2(\Omega)$ by setting

$$F(\mathbf{u}, \mathbf{e}) = \big(\nabla_{u_1}\mathcal{J}_1(\mathbf{u}, e_Y, e_1), \ldots, \nabla_{u_m}\mathcal{J}_m(\mathbf{u}, e_Y, e_m)\big),\ \forall(\mathbf{u}, \mathbf{e}) \in \mathbf{L}^2(\Omega) \times \mathbf{E}. \tag{4.1}$$

Then, for each $\mathbf{e} \in \mathbf{E}$, we have $F'_{\mathbf{u}}(\mathbf{u}, \mathbf{e})(\cdot) : \mathbf{L}^2(\Omega) \to \mathbf{L}^2(\Omega)$ (or equivalently, we can rewrite as follows $\langle F'_{\mathbf{u}}(\mathbf{u}, \mathbf{e})(\cdot), \cdot\rangle : \mathbf{L}^2(\Omega) \times \mathbf{L}^2(\Omega) \to I\!R$) with

$$F'_{\mathbf{u}}(\mathbf{u}, \mathbf{e})\mathbf{v} \in \mathbf{L}^2(\Omega) \quad \text{and} \quad \langle F'_{\mathbf{u}}(\mathbf{u}, \mathbf{e})\mathbf{v}, \mathbf{h}\rangle = \sum_{k=1}^{m}\sum_{j=1}^{m} \nabla^2_{u_k u_j}\mathcal{J}_k(u_1, \ldots, u_m, e_Y, e_k)v_k h_j.$$

Using the notations given above, we consider the parametric variational inequality associated to the system (2.19) as follows

$$\mathbf{u}^* \in F(\mathbf{u}, \mathbf{e}) + N(\mathbf{u}; \mathcal{U}_{ad}(\mathbf{e})). \tag{4.2}$$

The solution map $\mathcal{S} : \mathbf{L}^2(\Omega) \times \mathbf{E} \rightrightarrows \mathbf{L}^2(\Omega)$ of the PVI (4.2) is defined by

$$\begin{aligned} \mathcal{S}(\mathbf{u}^*, \mathbf{e}) &= \{\mathbf{u} \in \mathcal{U}_{ad}(\mathbf{e}) |\ \mathbf{u}^* \in F(\mathbf{u}, \mathbf{e}) + N(\mathbf{u}; \mathcal{U}_{ad}(\mathbf{e}))\} \\ &= \{\mathbf{u} \in \mathcal{U}_{ad}(\mathbf{e}) |\ \mathbf{u}^* \in F(\mathbf{u}, \mathbf{e}) + \mathcal{N}(\mathbf{u}, \mathbf{e})\}, \end{aligned} \tag{4.3}$$

where the normal cone mapping $\mathcal{N} : \mathbf{L}^2(\Omega) \times \mathbf{E} \rightrightarrows \mathbf{L}^2(\Omega)$ is defined by setting

$$\mathcal{N}(\mathbf{u}, \mathbf{e}) = N(\mathbf{u}; \mathcal{U}_{ad}(\mathbf{e})),\ \forall(\mathbf{u}, \mathbf{e}) \in \mathbf{L}^2(\Omega) \times \mathbf{E}. \tag{4.4}$$

Following [26], we now recall the concepts of Lipschitzian and Hölderian full stability for the PVI (4.2).

Definition 4.1 Let $\bar{\mathbf{u}} = (\bar{u}_1, \ldots, \bar{u}_m) \in \mathcal{S}(\bar{\mathbf{u}}^*, \bar{\mathbf{e}})$ from (4.3) with $(\bar{\mathbf{u}}^*, \bar{\mathbf{e}}) \in \mathbf{L}^2(\Omega) \times \mathbf{E}$.

- We say that $\bar{\mathbf{u}}$ is a *Lipschitzian fully stable solution to* the PVI (4.2) corresponding to the pair $(\bar{\mathbf{u}}^*, \bar{\mathbf{e}})$ if the solution map (4.3) admits a single-valued localization ϑ relative to some neighborhood $\mathcal{V}_{\bar{\mathbf{u}}^*} \times \mathcal{V}_{\bar{\mathbf{e}}} \times \mathcal{V}_{\bar{\mathbf{u}}}$ such that for any $(\mathbf{u}_1^*, \mathbf{e}_1), (\mathbf{u}_2^*, \mathbf{e}_2) \in \mathcal{V}_{\bar{\mathbf{u}}^*} \times \mathcal{V}_{\bar{\mathbf{e}}}$ we have

$$\big\|(\mathbf{u}_1^* - \mathbf{u}_2^*) - 2\kappa\big(\vartheta(\mathbf{u}_1^*, \mathbf{e}_1) - \vartheta(\mathbf{u}_2^*, \mathbf{e}_2)\big)\big\|_{\mathbf{L}^2(\Omega)} \le \|\mathbf{u}_1^* - \mathbf{u}_2^*\|_{\mathbf{L}^2(\Omega)} + \ell\|\mathbf{e}_1 - \mathbf{e}_2\|_{\mathbf{E}} \tag{4.5}$$

with some constants $\kappa > 0$ and $\ell > 0$.

- We say that $\bar{\mathbf{u}}$ is a *Hölderian fully stable solution to* the PVI (4.2) corresponding to the pair $(\bar{\mathbf{u}}^*, \bar{\mathbf{e}})$ if the solution map (4.3) admits a single-valued localization ϑ relative to some neighborhood $\mathcal{V}_{\bar{\mathbf{u}}^*} \times \mathcal{V}_{\bar{\mathbf{e}}} \times \mathcal{V}_{\bar{\mathbf{u}}}$ such that for any $(\mathbf{u}_1^*, \mathbf{e}_1), (\mathbf{u}_2^*, \mathbf{e}_2) \in \mathcal{V}_{\bar{\mathbf{u}}^*} \times \mathcal{V}_{\bar{\mathbf{e}}}$ we have

$$\big\|(\mathbf{u}_1^* - \mathbf{u}_2^*) - 2\kappa\big(\vartheta(\mathbf{u}_1^*, \mathbf{e}_1) - \vartheta(\mathbf{u}_2^*, \mathbf{e}_2)\big)\big\|_{\mathbf{L}^2(\Omega)} \le \|\mathbf{u}_1^* - \mathbf{u}_2^*\|_{\mathbf{L}^2(\Omega)} + \ell\|\mathbf{e}_1 - \mathbf{e}_2\|_{\mathbf{E}}^{1/2} \tag{4.6}$$

with some constants $\kappa > 0$ and $\ell > 0$.

Note that the concepts of full stability for parametric variational systems provided in [26] are really extensions of the ones for local minimizers of parametric optimization problems given in [21] and [25], and they are very effective for studying solution stability for parametric variational inequalities.

By applying the results of the previous section we deduce that the PVI (4.2) has solutions under our standard assumptions. Beside the assumptions **(A1)**–**(A3)** mentioned above, the following condition with respect to the reference parameter $\bar{\mathbf{e}} \in \mathbf{E}$ is also a crucial role in our investigation:

(A4) There exists $\varrho > 0$ such that $\mathcal{U}_{ad}(\mathbf{e}) \neq \emptyset$ for all $\mathbf{e} \in \bar{B}_\varrho(\bar{\mathbf{e}}) \subset \mathbf{E}$.

We see that when $\mathcal{U}_{ad}$ and $\mathcal{U}_{ad}(\mathbf{e})$ are defined via $\mathcal{U}_{ad}^k$ and $\mathcal{U}_{ad}^k(e_k)$ for $k = 1, \ldots, m$ given in (2.4) and (2.13) respectively, we can use the following condition as a sufficient condition to ensure that the assumption **(A4)** holds:

$$\begin{cases} \exists \sigma > 0, \forall k = 1, \ldots, m, \\ \alpha_k(x) + \bar{e}_{k,\alpha}(x) + \sigma \leq \beta_k(x) + \bar{e}_{k,\beta}(x) \text{ for a.a. } x \in \Omega. \end{cases} \tag{4.7}$$

Related to the condition (4.7) we refer the reader to [37] for more details in applications to the full stability for local minimizers of parametric optimal control problems.

4.1 General case for $\mathcal{U}_{ad}(\mathbf{e})$

In this subsection, we will establish conditions for the Lipschitzian and Hölderian full stability for variational Nash equilibriums (solutions) to the PVI (4.2) via the data with respect to the general convex, closed and bounded admissible control set $\mathcal{U}_{ad}(\mathbf{e}) \subset \mathbf{L}^2(\Omega)$ for $\mathbf{e} \in \mathbf{E}$.

Theorem 4.2 *Let $\bar{\mathbf{u}} \in \mathcal{S}(\bar{\mathbf{u}}^*, \bar{\mathbf{e}})$ from (4.3) with $(\bar{\mathbf{u}}^*, \bar{\mathbf{e}}) \in \mathbf{L}^2(\Omega) \times \mathbf{E}$ and $\widehat{\mathbf{u}}^* := \bar{\mathbf{u}}^* - F(\bar{\mathbf{u}}, \bar{\mathbf{e}})$. Assume that the assumptions* **(A1)**–**(A4)** *hold. Then, $\bar{\mathbf{u}}$ is Lipschitzian fully stable solution to the PVI (4.2) if and only if the both conditions below hold:*

(i) *There exist $\eta > 0$, $\kappa > 0$ such that for $(\mathbf{u}, \mathbf{e}, \mathbf{u}^*) \in \operatorname{gph} \mathcal{N} \cap B_\eta(\bar{\mathbf{u}}, \bar{\mathbf{e}}, \widehat{\mathbf{u}}^*)$ and $\mathbf{u}^{**} \in \mathbf{L}^2(\Omega)$ we have*

$$\langle F'_{\mathbf{u}}(\bar{\mathbf{u}}, \bar{\mathbf{e}})\mathbf{u}^{**}, \mathbf{u}^{**} \rangle + \langle \mathbf{v}, \mathbf{u}^{**} \rangle \geq \kappa \|\mathbf{u}^{**}\|_{\mathbf{L}^2(\Omega)}, \ \forall \mathbf{v} \in \widehat{D}^* \mathcal{N}_{\mathbf{e}}(\mathbf{u}, \mathbf{u}^*)(\mathbf{u}^{**}), \tag{4.8}$$

where $\mathcal{N}_{\mathbf{e}}(\cdot) = \mathcal{N}(\cdot, \mathbf{e})$ from (4.4).

(ii) *The graphical mapping $\mathbf{e} \mapsto \operatorname{gph} \mathcal{N}(\cdot, \mathbf{e})$ is locally Lipschitz-like around $(\bar{\mathbf{e}}, \bar{\mathbf{u}}, \widehat{\mathbf{u}}^*)$.*

Proof. Using the assumptions **(A1)**–**(A3)** and applying Theorem 2.8 we deduce that $F(\cdot, \cdot)$ given in (4.1) is differentiable with respect to $\mathbf{u}$ around $(\bar{\mathbf{u}}, \bar{\mathbf{e}})$ uniformly in $\mathbf{e}$ and the partial derivative $F'_{\mathbf{u}}(\cdot, \cdot)$ is continuous at $(\bar{\mathbf{u}}, \bar{\mathbf{e}})$. Moreover, $F(\cdot, \cdot)$ is also Lipschitz continuous with respect to $\mathbf{e}$ uniformly in $\mathbf{u}$ around $(\bar{\mathbf{u}}, \bar{\mathbf{e}})$. In addition, by arguing similarly to the proof of [37, Theorem 4.1] we can verify that the parametric indicator $\delta(\cdot\,; \mathcal{U}_{ad}(\cdot))$ is parametrically continuously prox-regular at $(\bar{\mathbf{u}}, \bar{\mathbf{e}})$ for $\widehat{\mathbf{u}}^*$ and the basic constraint qualification (BCQ) holds at $(\bar{\mathbf{u}}, \bar{\mathbf{e}})$ under the assumption **(A4)**; see the parametric continuous prox-regularity and the BCQ in [30]. Summarizing the above, we infer that all the assumptions stated in [26, Theorem 4.7] are satisfied. Therefore, applying [26, Theorem 4.7] we obtain the assertion of the theorem. □

Theorem 4.3 *Let $\bar{\mathbf{u}} \in \mathcal{S}(\bar{\mathbf{u}}^*, \bar{\mathbf{e}})$ from (4.3) with $(\bar{\mathbf{u}}^*, \bar{\mathbf{e}}) \in \mathbf{L}^2(\Omega) \times \mathbf{E}$ and $\widehat{\mathbf{u}}^* := \bar{\mathbf{u}}^* - F(\bar{\mathbf{u}}, \bar{\mathbf{e}})$. Assume that the assumptions* **(A1)**–**(A4)** *hold. Let us consider the following two statements:*

(i) *$\bar{\mathbf{u}}$ is Hölderian fully stable solution to the PVI (4.2) corresponding to the parameter pair $(\bar{\mathbf{u}}^*, \bar{\mathbf{e}})$ with the moduli $\kappa > 0$ and $\ell > 0$ taken from (4.6).*

(ii) *There exist some* $\eta > 0$, $\kappa_0 > 0$ *such that for* $(\mathbf{u}, \mathbf{e}, \mathbf{u}^*) \in \operatorname{gph} \mathcal{N} \cap B_\eta(\bar{\mathbf{u}}, \bar{\mathbf{e}}, \widehat{\mathbf{u}}^*)$ *we have*

$$\langle F'_{\mathbf{u}}(\bar{\mathbf{u}}, \bar{\mathbf{e}})\mathbf{u}^{**}, \mathbf{u}^{**}\rangle + \langle \mathbf{v}, \mathbf{u}^{**}\rangle \geq \kappa_0 \|\mathbf{u}^{**}\|_{\mathbf{L}^2(\Omega)}, \ \forall \mathbf{v} \in \widehat{D}^*\mathcal{N}_{\mathbf{e}}(\mathbf{u}, \mathbf{u}^*)(\mathbf{u}^{**}). \tag{4.9}$$

Then, (i) *implies* (ii) *with constant* κ_0 *that can be chosen smaller than but arbitrarily closed to* κ*. Conversely, the validity of* (ii) *ensures that* (i) *holds, where* κ *can be chosen smaller than but arbitrarily closed to* κ_0*.*

Proof. Arguing similarly to the proof of Theorem 4.2, we can check that all the assumptions of [26, Theorem 4.3] hold. Therefore, applying [26, Theorem 4.3] we obtain the assertions of the theorem. □

Following [15], we say that a closed and convex subset Θ of a Banach space X is *polyhedric* at $\bar{u} \in \Theta$ for $\widehat{u}^* \in N(\bar{u}; \Theta)$ if we have the representation

$$T(\bar{u}; \Theta) \cap \{\widehat{u}^*\}^\perp = \operatorname{cl}\big(\operatorname{cone}(\Theta - \bar{u}) \cap \{\widehat{u}^*\}^\perp\big), \tag{4.10}$$

where

$$\operatorname{cone}(\Theta - \bar{u}) = \bigcup_{t>0} \frac{\Theta - \bar{u}}{t} \tag{4.11}$$

is the radial cone, and

$$T(\bar{u}; \Theta) = \operatorname{cl}(\operatorname{cone}(\Theta - \bar{u})) \tag{4.12}$$

is the tangent cone to Θ at $\bar{u}$. The set Θ is said to be *polyhedric* if Θ is polyhedric at every $u \in \Theta$ for any $u^* \in N(u; \Theta)$. The polyhedricity property of a set is first introduced in [15] and then applied extensively in optimal control; see, e.g., [2, 3, 18] and the references therein.

Theorem 4.4 (See [25, Theorem 6.2]) *Let* $(\bar{\mathbf{u}}, \bar{\mathbf{e}}) \in \mathcal{U}_{ad}(\bar{\mathbf{e}}) \times \mathbf{E}$ *and let* $\widehat{\mathbf{u}}^* \in N(\bar{\mathbf{u}}; \mathcal{U}_{ad}(\bar{\mathbf{e}}))$*. Then, we have*

$$\operatorname{dom} \widehat{D}^*\mathcal{N}_{\bar{\mathbf{e}}}(\bar{\mathbf{u}}, \widehat{\mathbf{u}}^*) \subset -T(\bar{\mathbf{u}}; \mathcal{U}_{ad}(\bar{\mathbf{e}})) \cap \{\widehat{\mathbf{u}}^*\}^\perp. \tag{4.13}$$

If, in addition, $\mathcal{U}_{ad}(\bar{\mathbf{e}})$ *is polyhedric at* $\bar{\mathbf{u}} \in \mathcal{U}_{ad}(\bar{\mathbf{e}})$ *for* $\widehat{\mathbf{u}}^*$*, then the equality*

$$\widehat{D}^*\mathcal{N}_{\bar{\mathbf{e}}}(\bar{\mathbf{u}}, \widehat{\mathbf{u}}^*)(\mathbf{u}) = \big(T(\bar{\mathbf{u}}; \mathcal{U}_{ad}(\bar{\mathbf{e}})) \cap \{\widehat{\mathbf{u}}^*\}^\perp\big)^* \tag{4.14}$$

holds for all $\mathbf{u} \in -T(\bar{\mathbf{u}}; \mathcal{U}_{ad}(\bar{\mathbf{e}})) \cap \{\widehat{\mathbf{u}}^*\}^\perp$.

For each $(\mathbf{u}, \mathbf{e}, \mathbf{u}^*) \in \mathbf{L}^2(\Omega) \times \mathbf{E} \times \mathbf{L}^2(\Omega)$ with $\mathbf{u}^* \in \mathcal{N}(\mathbf{u}, \mathbf{e})$, we define the critical cone

$$\mathcal{C}(\mathbf{u}, \mathbf{e}, \mathbf{u}^*) = T(\mathbf{u}; \mathcal{U}_{ad}(\mathbf{e})) \cap \{\mathbf{u}^*\}^\perp. \tag{4.15}$$

Note that $\mathcal{U}_{ad}(\mathbf{e}) = \mathcal{U}^1_{ad}(e_1) \times \cdots \times \mathcal{U}^m_{ad}(e_m) \subset \mathbf{L}^2(\Omega)$ with $\mathcal{U}^k_{ad}(e_k) \subset L^2(\Omega)$ being convex and $u^*_k \in N(u_k; \mathcal{U}^k_{ad}(e_k))$ for every $k \in \{1, \ldots, m\}$. Therefore, from (4.15) we obtain

$$\mathcal{C}(\mathbf{u}, \mathbf{e}, \mathbf{u}^*) = \prod_{k=1}^m \mathcal{C}(u_k, e_k, u^*_k) = \prod_{k=1}^m T(u_k; \mathcal{U}^k_{ad}(e_k)) \cap \{u^*_k\}^\perp, \tag{4.16}$$

where $\mathcal{C}(u_k, e_k, u^*_k) := T(u_k; \mathcal{U}^k_{ad}(e_k)) \cap \{u^*_k\}^\perp$ for every $k \in \{1, \ldots, m\}$. Let us now define the sequential outer limits of the critical cones $\mathcal{C}(\mathbf{u}, \mathbf{e}, \mathbf{u}^*)$ respectively in the weak* topology of $\mathbf{L}^2(\Omega)$ by

$$\mathcal{C}_{w^*}(\bar{\mathbf{u}}, \bar{\mathbf{e}}, \widehat{\mathbf{u}}^*) = \operatorname*{Limsup}_{(\mathbf{u},\mathbf{e},\mathbf{u}^*) \overset{\operatorname{gph}\mathcal{N}}{\longrightarrow} (\bar{\mathbf{u}},\bar{\mathbf{e}},\widehat{\mathbf{u}}^*)} \mathcal{C}(\mathbf{u}, \mathbf{e}, \mathbf{u}^*), \tag{4.17}$$

and in the strong topology of $\mathbf{L}^2(\Omega)$ as follows

$$\mathcal{C}_s(\bar{\mathbf{u}},\bar{\mathbf{e}},\widehat{\mathbf{u}}^*)=\Big\{\mathbf{v}\in\mathbf{L}^2(\Omega)\Big|\,\exists(\mathbf{u}_n,\mathbf{e}_n,\mathbf{u}_n^*)\overset{\text{gph}\mathcal{N}}{\longrightarrow}(\bar{\mathbf{u}},\bar{\mathbf{e}},\widehat{\mathbf{u}}^*),\mathbf{v}_n\in\mathcal{C}(\mathbf{u}_n,\mathbf{e}_n,\mathbf{u}_n^*),\mathbf{v}_n\to\mathbf{v}\Big\}. \tag{4.18}$$

Then, using (4.16) we deduce from (4.17) and (4.18) that

$$\mathcal{C}_{w^*}(\bar{\mathbf{u}},\bar{\mathbf{e}},\widehat{\mathbf{u}}^*)=\prod_{k=1}^{m}\mathcal{C}_{w^*}(\bar{u}_k,\bar{e}_k,\widehat{u}_k^*)\quad\text{and}\quad\mathcal{C}_s(\bar{\mathbf{u}},\bar{\mathbf{e}},\widehat{\mathbf{u}}^*)=\prod_{k=1}^{m}\mathcal{C}_s(\bar{u}_k,\bar{e}_k,\widehat{u}_k^*), \tag{4.19}$$

where

$$\mathcal{C}_{w^*}(\bar{u}_k,\bar{e}_k,\widehat{u}_k^*)=\operatorname*{Limsup}_{(u,e,u^*)\overset{\text{gph}\mathcal{N}_k}{\longrightarrow}(\bar{u}_k,\bar{e}_k,\bar{u}_k^*)}\mathcal{C}(u,e,u^*) \tag{4.20}$$

and

$$\mathcal{C}_s(\bar{u}_k,\bar{e}_k,\widehat{u}_k^*)=\Big\{v\in L^2(\Omega)\Big|\,\exists(u_{kn},e_{kn},u_{kn}^*)\overset{\text{gph}\mathcal{N}_k}{\longrightarrow}(\bar{u}_k,\bar{e}_k,\widehat{u}_k^*),v_{kn}\in\mathcal{C}(u_{kn},e_{kn},u_{kn}^*),v_{kn}\to v\Big\} \tag{4.21}$$

with $\mathcal{N}_k(u,e)=N(u;\mathcal{U}_{ad}^k(e))$ for all $(u,e)\in L^2(\Omega)\times E_k$.

Definition 4.5 A quadratic form $Q:H\to I\!R$ on a Hilbert space H is said to be a *Legendre form* if Q is sequentially weakly lower semicontinuous and that if h_n converges weakly to h in H and $Q(h_n)\to Q(h)$ then h_n converges strongly to h in H.

Theorem 4.6 *Let* $\bar{\mathbf{u}}=(\bar{u}_1,\ldots,\bar{u}_m)\in\mathcal{S}(\bar{\mathbf{u}}^*,\bar{\mathbf{e}})$ *from* (4.3) *and* $\widehat{\mathbf{u}}^*:=\bar{\mathbf{u}}^*-F(\bar{\mathbf{u}},\bar{\mathbf{e}})$. *Assume that the assumptions* **(A1)–(A4)** *hold. Let us define the quadratic form* $\mathcal{Q}:\mathbf{L}^2(\Omega)\to I\!R$ *by*

$$\mathcal{Q}(\mathbf{h}):=\langle F'_{\mathbf{u}}(\bar{\mathbf{u}},\bar{\mathbf{e}})\mathbf{h},\mathbf{h}\rangle,\ \forall\mathbf{h}\in\mathbf{L}^2(\Omega). \tag{4.22}$$

Then, the following assertions are valid:

(i) *If the quadratic form* $\mathcal{Q}(\cdot)$ *is a Legendre form on* $\mathbf{L}^2(\Omega)$ *and*

$$\langle F'_{\mathbf{u}}(\bar{\mathbf{u}},\bar{\mathbf{e}})\mathbf{v},\mathbf{v}\rangle>0,\ \forall\mathbf{v}\in\mathcal{C}_{w^*}(\bar{\mathbf{u}},\bar{\mathbf{e}},\widehat{\mathbf{u}}^*)\ \text{with}\ \mathbf{v}\neq0, \tag{4.23}$$

then $\bar{\mathbf{u}}$ *is a fully stable solution to the PVI* (4.2).

(ii) *If* $\bar{\mathbf{u}}$ *is a fully stable solution to the PVI* (4.2) *and that* $\mathcal{U}_{ad}(\mathbf{e})$ *is polyhedric around* $\bar{\mathbf{e}}$, *then we have the positive definiteness condition*

$$\langle F'_{\mathbf{u}}(\bar{\mathbf{u}},\bar{\mathbf{e}})\mathbf{v},\mathbf{v}\rangle>0,\ \forall\mathbf{v}\in\mathcal{C}_s(\bar{\mathbf{u}},\bar{\mathbf{e}},\widehat{\mathbf{u}}^*)\ \text{with}\ \mathbf{v}\neq0. \tag{4.24}$$

Proof. To prove the assertion (i) we suppose to the contrary that $\bar{\mathbf{u}}$ is not a fully stable solution to the PVI (4.2). By Theorem 4.2, one can find some sequences $(\mathbf{u}_n,\mathbf{e}_n,\mathbf{u}_n^*)\to(\bar{\mathbf{u}},\bar{\mathbf{e}},\widehat{\mathbf{u}}^*)$ with $(\mathbf{u}_n,\mathbf{e}_n,\mathbf{u}_n^*)\in\text{gph}\,\mathcal{N}$ and $(\mathbf{u}_n^{**},\mathbf{v}_n)\in\mathbf{L}^2(\Omega)\times\mathbf{L}^2(\Omega)$ with $\mathbf{v}_n\in\widehat{D}^*\mathcal{N}_{\mathbf{e}_n}(\mathbf{u}_n,\mathbf{u}_n^*)(\mathbf{u}_n^{**})$ such that

$$\langle F'_{\mathbf{u}}(\bar{\mathbf{u}},\bar{\mathbf{e}})\mathbf{u}_n^{**},\mathbf{u}_n^{**}\rangle+\langle\mathbf{v}_n,\mathbf{u}_n^{**}\rangle<\frac{1}{n}\|\mathbf{u}_n^{**}\|^2_{\mathbf{L}^2(\Omega)},\ \forall n\in I\!N. \tag{4.25}$$

Since $\mathcal{N}_{\mathbf{e}_n}:\mathbf{L}^2(\Omega)\times\mathbf{E}\rightrightarrows\mathbf{L}^2(\Omega)$ is maximal monotone, by [7, Lemma 3.3] we get $\langle\mathbf{v}_n,\mathbf{u}_n^{**}\rangle\geq0$. According to Theorem 4.4, we have $\mathbf{u}_n^{**}\in-\mathcal{C}(\mathbf{u}_n,\mathbf{e}_n,\mathbf{u}_n^*)$, which yields $\mathbf{u}_n^{**}\neq0$. Combining this and (4.25) with $\langle\mathbf{v}_n,\mathbf{u}_n^{**}\rangle\geq0$, we have

$$\langle F'_{\mathbf{u}}(\bar{\mathbf{u}},\bar{\mathbf{e}})\mathbf{u}_n^{**},\mathbf{u}_n^{**}\rangle<\frac{1}{n}\|\mathbf{u}_n^{**}\|^2_{\mathbf{L}^2(\Omega)}\quad\text{with}\quad\mathbf{u}_n^{**}\in-\mathcal{C}(\mathbf{u}_n,\mathbf{e}_n,\mathbf{u}_n^*). \tag{4.26}$$

By setting $\widetilde{\mathbf{u}}_n^{**} := \mathbf{u}_n^{**}\|\mathbf{u}_n^{**}\|^{-1}_{\mathbf{L}^2(\Omega)}$, we deduce from (4.26) that

$$\mathcal{Q}(\widetilde{\mathbf{u}}_n^{**}) = \langle F'_{\mathbf{u}}(\bar{\mathbf{u}},\bar{\mathbf{e}})\widetilde{\mathbf{u}}_n^{**},\widetilde{\mathbf{u}}_n^{**}\rangle < \frac{1}{n} \quad \text{with} \quad \widetilde{\mathbf{u}}_n^{**} \in -\mathcal{C}(\mathbf{u}_n,\mathbf{e}_n,\mathbf{u}_n^*). \tag{4.27}$$

We may assume that $\widetilde{\mathbf{u}}_n^{**}$ weakly converges to some $\widetilde{\mathbf{u}}^{**}$. It follows from (4.17) that

$$\widetilde{\mathbf{u}}^{**} \in -\mathcal{C}_{w^*}(\bar{\mathbf{u}},\bar{\mathbf{e}},\widehat{\mathbf{u}}^*) \quad \text{or equivalently as} \quad -\widetilde{\mathbf{u}}^{**} \in \mathcal{C}_{w^*}(\bar{\mathbf{u}},\bar{\mathbf{e}},\widehat{\mathbf{u}}^*). \tag{4.28}$$

From the weak lower semicontinuity of $\mathcal{Q}$ and from (4.22), (4.23), (4.27) and (4.28) it follows that

$$0 \le \mathcal{Q}(-\widetilde{\mathbf{u}}^{**}) = \mathcal{Q}(\widetilde{\mathbf{u}}^{**}) \le \liminf_{n\to\infty} \mathcal{Q}(\widetilde{\mathbf{u}}_n^{**}) \le 0, \tag{4.29}$$

which yields $\mathcal{Q}(-\widetilde{\mathbf{u}}^{**}) = 0$ with $-\widetilde{\mathbf{u}}^{**} \in \mathcal{C}_{w^*}(\bar{\mathbf{u}},\bar{e},\widehat{\mathbf{u}}^*)$. Since $\mathcal{Q}(\cdot)$ is a Legendre form on $\mathbf{L}^2(\Omega)$, we obtain $\widetilde{\mathbf{u}}_n^{**} \to \widetilde{\mathbf{u}}^{**}$. This implies that $\|\widetilde{\mathbf{u}}^{**}\|_{\mathbf{L}^2(\Omega)} = 1$, and thus $-\mathbf{u}^{**} \neq 0$. We have arrived at a contradiction.

We now prove the assertion (ii). By Theorem 4.2, there exist $\kappa > 0$, $\eta > 0$ such that for any $(\mathbf{u},\mathbf{e},\mathbf{u}^*) \in \operatorname{gph}\mathcal{N} \cap B_\eta(\bar{\mathbf{u}},\bar{\mathbf{e}},\widehat{\mathbf{u}}^*)$ and $\mathbf{u}^{**} \in \mathbf{L}^2(\Omega)$ we have

$$\langle F'_{\mathbf{u}}(\bar{\mathbf{u}},\bar{\mathbf{e}})\mathbf{u}^{**},\mathbf{u}^{**}\rangle + \langle \mathbf{v},\mathbf{u}^{**}\rangle \ge \kappa\|\mathbf{u}^{**}\|_{\mathbf{L}^2(\Omega)}, \ \forall \mathbf{v} \in \widehat{D}^*\mathcal{N}_{\mathbf{e}}(\mathbf{u},\mathbf{u}^*)(\mathbf{u}^{**}). \tag{4.30}$$

Since $\mathcal{U}_{ad}(\mathbf{e})$ is polyhedric around $\bar{\mathbf{e}}$, by Theorem 4.4 we deduce that

$$\mathbf{u}^{**} \in -\mathcal{C}(\mathbf{u},\mathbf{e},\mathbf{u}^*) \quad \text{and} \quad \widehat{D}^*\mathcal{N}_{\mathbf{e}}(\mathbf{u},\mathbf{u}^*)(\mathbf{u}^{**}) = \mathcal{C}(\mathbf{u},\mathbf{e},\mathbf{u}^*)^*,$$

which yields $0 \in \widehat{D}^*\mathcal{N}_{\mathbf{e}}(\mathbf{u},\mathbf{u}^*)(\mathbf{u}^{**})$. Consequently, for all $(\mathbf{u},\mathbf{e},\mathbf{u}^*) \in \operatorname{gph}\mathcal{N} \cap B_\eta(\bar{\mathbf{u}},\bar{\mathbf{e}},\widehat{\mathbf{u}}^*)$, we have

$$\langle F'_{\mathbf{u}}(\bar{\mathbf{u}},\bar{\mathbf{e}})\mathbf{u}^{**},\mathbf{u}^{**}\rangle \ge \kappa\|\mathbf{u}^{**}\|_{\mathbf{L}^2(\Omega)}, \ \forall \mathbf{u}^{**} \in -\mathcal{C}(\mathbf{u},\mathbf{e},\mathbf{u}^*). \tag{4.31}$$

Using the strong convergence in (4.18) and passing (4.31) to the limit when $\eta \downarrow 0$ we obtain the positive definiteness condition (4.24). □

The following lemma shows that the quadratic $\mathcal{Q}(\cdot)$ defined by (4.22) is a Legendre form on the space $\mathbf{L}^2(\Omega)$. This is one of important results that will help us to establish explicit characterizations of full stability for parametric variational Nash equilibriums.

Lemma 4.7 *Assume that the assumptions* **(A1)–(A4)** *hold. Let* $(\bar{\mathbf{u}},\bar{\mathbf{e}}) \in \mathbf{L}^2(\Omega) \times \mathbf{E}$ *and define the quadratic form* $\mathcal{Q} : \mathbf{L}^2(\Omega) \to I\!R$ *by*

$$\mathcal{Q}(\mathbf{h}) = \langle F'_{\mathbf{u}}(\bar{\mathbf{u}},\bar{\mathbf{e}})\mathbf{h},\mathbf{h}\rangle, \ \forall \mathbf{h} \in \mathbf{L}^2(\Omega). \tag{4.32}$$

Then, $\mathcal{Q}(\cdot)$ *is a Legendre form on* $\mathbf{L}^2(\Omega)$.

Proof. For $\mathbf{h} = (h_1,\ldots,h_m) \in \mathbf{L}^2(\Omega)$, we have

$$\mathcal{Q}(\mathbf{h}) = \langle F'_{\mathbf{u}}(\bar{\mathbf{u}},\bar{\mathbf{e}})\mathbf{h},\mathbf{h}\rangle = \sum_{k=1}^m\sum_{j=1}^m \nabla^2_{u_ku_j}\mathcal{J}_k(\bar{u}_1,\ldots,\bar{u}_m,\bar{e}_Y,\bar{e}_k)h_kh_j. \tag{4.33}$$

For $k,j \in \{1,\ldots,m\}$, we define $Q_{kj} : L^2(\Omega) \times L^2(\Omega) \to I\!R$ by

$$Q_{kj}(v,h) = \nabla^2_{u_ku_j}\mathcal{J}_k(\bar{u}_1,\ldots,\bar{u}_m,\bar{e}_Y,\bar{e}_k)(v,h), \ \forall (v,h) \in L^2(\Omega) \times L^2(\Omega). \tag{4.34}$$

Then, from (4.33) we have

$$\mathcal{Q}(\mathbf{h}) = \sum_{k=1}^m\sum_{j=1}^m Q_{kj}(h_k,h_j). \tag{4.35}$$

We have

$$\begin{aligned}
Q_{kj}(h_k,h_j) &= \nabla^2_{u_k u_j}\mathcal{J}_k(\bar u_1,\dots,\bar u_m,\bar e_Y,\bar e_k)(h_k,h_j)\\
&= \int_\Omega \left(\frac{\partial^2 L_k}{\partial y^2}(x,y_{\bar u+\bar e_Y}) - \varphi_{k,\bar u+\bar e_Y}\frac{\partial^2 f}{\partial y^2}(x,y_{\bar u+\bar e_Y})\right) z_{\bar u+\bar e_Y,h_k} z_{\bar u+\bar e_Y,h_j}dx\\
&\quad + \int_\Omega \bar e_{k,J} G''(\bar u+\bar e_Y)(h_k,h_j)dx + \int_\Omega \chi_{\{k\}}(j)\zeta_k h_k h_j dx\\
&= Q^1_{kj}(h_k,h_j) + Q^2_{kj}(h_k,h_j),
\end{aligned}$$

where

$$\begin{aligned}
Q^1_{kj}(h_k,h_j) &= \int_\Omega \left(\frac{\partial^2 L_k}{\partial y^2}(x,y_{\bar u+\bar e_Y}) - \varphi_{k,\bar u+\bar e_Y}\frac{\partial^2 f}{\partial y^2}(x,y_{\bar u+\bar e_Y})\right) z_{\bar u+\bar e_Y,h_k} z_{\bar u+\bar e_Y,h_j}dx\\
&\quad + \int_\Omega \bar e_{k,J} G''(\bar u+\bar e_Y)(h_k,h_j)dx
\end{aligned}$$

and

$$Q^2_{kj}(h_k,h_j) = \int_\Omega \chi_{\{k\}}(j)\zeta_k h_k h_j dx.$$

From (4.35) we have

$$\mathcal{Q}(\mathbf{h}) = \sum_{k=1}^m\sum_{j=1}^m Q^1_{kj}(h_k,h_j) + \sum_{k=1}^m Q^2_{kk}(h_k,h_k) = \mathcal{Q}_1(\mathbf{h}) + \mathcal{Q}_2(\mathbf{h}), \tag{4.36}$$

where

$$\mathcal{Q}_1(\mathbf{h}) = \sum_{k=1}^m\sum_{j=1}^m Q^1_{kj}(h_k,h_j) \quad \text{and} \quad \mathcal{Q}_2(\mathbf{h}) = \sum_{k=1}^m Q^2_{kk}(h_k,h_k).$$

Since the operator $G'(\bar u+\bar e_Y): h\mapsto z_{\bar u+\bar e_Y,h}$ from $L^2(\Omega)$ into $L^2(\Omega)$ is compact, $Q^1_{kj}(h_k,h_j)$ is weakly continuous on $L^2(\Omega)\times L^2(\Omega)$. It follows that $\mathcal{Q}_1(\mathbf{h})$ is weakly continuous on $\mathbf{L}^2(\Omega)$. In addition, the quadratic form

$$\mathcal{Q}_2(\mathbf{h}) = \sum_{k=1}^m Q^2_{kk}(h_k,h_k) = \sum_{k=1}^m \int_\Omega \zeta_k h_k^2 dx \tag{4.37}$$

is a Legendre form on $\mathbf{L}^2(\Omega)$.

Suppose that $\mathbf{h}_n \rightharpoonup \mathbf{h}$ in $\mathbf{L}^2(\Omega)$ and that $\mathcal{Q}(\mathbf{h}_n)\to\mathcal{Q}(\mathbf{h})$, where $\mathbf{h}_n = (h_{1n},\dots,h_{mn})$ and $\mathbf{h}=(h_1,\dots,h_m)$. Then, we have

$$h_{kn} \rightharpoonup h_k,\ \forall k\in\{1,\dots,m\}. \tag{4.38}$$

It follows that

$$\lim_{n\to\infty} Q^1_{kj}(h_{kn},h_{jn}) = Q^1_{kj}(h_k,h_j),\ \forall k,j\in\{1,\dots,m\}. \tag{4.39}$$

Consequently, we have

$$\lim_{n\to\infty}\mathcal{Q}_1(\mathbf{h}_n) = \lim_{n\to\infty}\sum_{k=1}^m\sum_{j=1}^m Q^1_{kj}(h_{kn},h_{jn}) = \sum_{k=1}^m\sum_{j=1}^m Q^1_{kj}(h_k,h_j) = \mathcal{Q}_1(\mathbf{h}). \tag{4.40}$$

Combining this with (4.36) yields

$$\lim_{n\to\infty}\mathcal{Q}_2(\mathbf{h}_n) = \lim_{n\to\infty}\Big(\mathcal{Q}(\mathbf{h}_n) - \mathcal{Q}_1(\mathbf{h}_n)\Big) = \mathcal{Q}(\mathbf{h}) - \mathcal{Q}_1(\mathbf{h}) = \mathcal{Q}_2(\mathbf{h}). \tag{4.41}$$

Since $\mathcal{Q}_2(\cdot)$ is a Legendre form on $\mathbf{L}^2(\Omega)$, we deduce that $\mathbf{h}_n\to\mathbf{h}$ as $n\to\infty$. We have shown that the quadratic form $\mathcal{Q}(\cdot)$ defined by (4.32) is a Legendre form on $\mathbf{L}^2(\Omega)$. □

Lemma 4.8 (Mazur's lemma) *Let X be a Banach space and let $\{u_n\}_{n\in\mathbb{N}}\subset X$ be such that u_n converges weakly to some $\bar u$ in X. Then, there exist a function $\sigma:\mathbb{N}\to\mathbb{N}$ and a sequence $\{\varrho(n)_r|\, r=n,\dots,\sigma(n)\}$ satisfying $\varrho(n)_r\ge 0$ and $\sum_{r=n}^{\sigma(n)}\varrho(n)_r=1$ such that for the sequence $\{v_n\}_{n\in\mathbb{N}}$ defined by $v_n=\sum_{r=n}^{\sigma(n)}\varrho(n)_r u_r$ we have v_n converges strongly to $\bar u$ in X.*

Theorem 4.9 *Let $(\bar{\mathbf{u}},\bar{\mathbf{e}},\bar{\mathbf{u}}^*)\in\mathcal{U}_{ad}(\bar{\mathbf{e}})\times\mathbf{E}\times\mathbf{L}^2(\Omega)$ such that $\widehat{\mathbf{u}}^*:=\bar{\mathbf{u}}^*-F(\bar{\mathbf{u}},\bar{\mathbf{e}})\in\mathcal{N}(\bar{\mathbf{u}},\bar{\mathbf{e}})$. Assume that the assumptions* **(A1)–(A4)** *hold. Then, we have*

$$\mathcal{C}_{w^*}(\bar{\mathbf{u}},\bar{\mathbf{e}},\widehat{\mathbf{u}}^*)\subset\prod_{k=1}^m\Big(\mathrm{cl}\big[T(\bar u_k;\mathcal{U}_{ad}^k(\bar e_k))-T(\bar u_k;\mathcal{U}_{ad}^k(\bar e_k))\big]\cap\{\widehat u_k^*\}^\perp\Big). \tag{4.42}$$

If, in addition, $\mathcal{U}_{ad}(\mathbf{e})$ is polyhedric around $\bar{\mathbf{e}}$, then we have the following lower estimate

$$\prod_{k=1}^m\Big(\mathcal{C}(\bar u_k,\bar e_k,\widehat u_k^*)-\mathcal{C}(\bar u_k,\bar e_k,\widehat u_k^*)\Big)\subset\mathcal{C}_s(\bar{\mathbf{u}},\bar{\mathbf{e}},\widehat{\mathbf{u}}^*). \tag{4.43}$$

Proof. To prove (4.42) we take any $\mathbf{v}\in\mathcal{C}_{w^*}(\bar{\mathbf{u}},\bar{\mathbf{e}},\widehat{\mathbf{u}}^*)$. By (4.17), there exist some sequences $(\mathbf{u}_n,\mathbf{e}_n,\mathbf{u}_n^*)\to(\bar{\mathbf{u}},\bar{\mathbf{e}},\widehat{\mathbf{u}}^*)$ with $(\mathbf{u}_n,\mathbf{e}_n,\mathbf{u}_n^*)\in\mathrm{gph}\,\mathcal{N}$ and $\mathbf{v}_n\overset{w}{\rightharpoonup}\mathbf{v}$ with $\mathbf{v}_n\in\mathcal{C}(\mathbf{u}_n,\mathbf{e}_n,\mathbf{u}_n^*)$. Note that we have

$$\mathbf{v}_n\in\mathcal{C}(\mathbf{u}_n,\mathbf{e}_n,\mathbf{u}_n^*)=T(\mathbf{u}_n;\mathcal{U}_{ad}(\mathbf{e}_n))\cap\{\mathbf{u}_n^*\}^\perp.$$

Hence, for each n, there exist sequences $\mathbf{w}_{nr}\to\mathbf{u}_n$ with $\mathbf{w}_{nr}\in\mathcal{U}_{ad}(\mathbf{e}_n)$ and $t_{nr}\downarrow 0$ satisfying $t_{nr}^{-1}(\mathbf{w}_{nr}-\mathbf{u}_n)\to\mathbf{v}_n$. We define $(\mathbf{d}_n,\varrho_n)\in\mathcal{U}_{ad}(\mathbf{e}_n)\times(0,+\infty)$ by

$$(\mathbf{d}_n,\varrho_n)\in\{(\mathbf{w}_{nr},t_{nr})|\, r\in\mathbb{N}\}$$

such that $\mathbf{d}_n\to\bar{\mathbf{u}}$, $\varrho_n\downarrow 0$, and $\varrho_n^{-1}(\mathbf{d}_n-\mathbf{u}_n)-\mathbf{v}_n\to 0$ when $n\to\infty$. It follows that

$$\begin{aligned}\mathbf{v}=\lim_{n\to\infty}^{w}\mathbf{v}_n=\lim_{n\to\infty}^{w}\frac{\mathbf{d}_n-\mathbf{u}_n}{\varrho_n}&=\lim_{n\to\infty}^{w}\frac{(\mathbf{d}_n-\bar{\mathbf{u}})-(\mathbf{u}_n-\bar{\mathbf{u}})}{\varrho_n}\\&\in\mathrm{cl}^w\big[T(\bar{\mathbf{u}};\mathcal{U}_{ad}(\bar{\mathbf{e}}))-T(\bar{\mathbf{u}};\mathcal{U}_{ad}(\bar{\mathbf{e}}))\big],\end{aligned}$$

where "$\lim_{n\to\infty}^w\mathbf{v}_n$" stands for the weak limit of the sequence $\mathbf{v}_n$ in $\mathbf{L}^2(\Omega)$. Since $\langle\mathbf{u}_n^*,\mathbf{v}_n\rangle=0$, by passing to the limit we get $\langle\widehat{\mathbf{u}}^*,\mathbf{v}\rangle=0$. Therefore, we obtain

$$\mathbf{v}\in\mathrm{cl}^w\big[T(\bar{\mathbf{u}};\mathcal{U}_{ad}(\bar{\mathbf{e}}))-T(\bar{\mathbf{u}};\mathcal{U}_{ad}(\bar{\mathbf{e}}))\big]\cap\{\widehat{\mathbf{u}}^*\}^\perp.$$

Since $T(\bar{\mathbf{u}};\mathcal{U}_{ad}(\bar{\mathbf{e}}))-T(\bar{\mathbf{u}};\mathcal{U}_{ad}(\bar{\mathbf{e}}))$ is convex, by Lemma 4.8 (Mazur's lemma) we deduce that

$$\mathrm{cl}^w\big[T(\bar{\mathbf{u}};\mathcal{U}_{ad}(\bar{\mathbf{e}}))-T(\bar{\mathbf{u}};\mathcal{U}_{ad}(\bar{\mathbf{e}}))\big]=\mathrm{cl}\big[T(\bar{\mathbf{u}};\mathcal{U}_{ad}(\bar{\mathbf{e}}))-T(\bar{\mathbf{u}};\mathcal{U}_{ad}(\bar{\mathbf{e}}))\big].$$

This implies that $\mathbf{v}\in\mathrm{cl}\big[T(\bar{\mathbf{u}};\mathcal{U}_{ad}(\bar{\mathbf{e}}))-T(\bar{\mathbf{u}};\mathcal{U}_{ad}(\bar{\mathbf{e}}))\big]$, which yields

$$\mathcal{C}_{w^*}(\bar{\mathbf{u}},\bar{\mathbf{e}},\widehat{\mathbf{u}}^*)\subset\mathrm{cl}\big[T(\bar{\mathbf{u}};\mathcal{U}_{ad}(\bar{\mathbf{e}}))-T(\bar{\mathbf{u}};\mathcal{U}_{ad}(\bar{\mathbf{e}}))\big]\cap\{\widehat{\mathbf{u}}^*\}^\perp. \tag{4.44}$$

In addition, we can verify that

$$\begin{aligned}&\mathrm{cl}\big[T(\bar{\mathbf{u}};\mathcal{U}_{ad}(\bar{\mathbf{e}}))-T(\bar{\mathbf{u}};\mathcal{U}_{ad}(\bar{\mathbf{e}}))\big]\cap\{\widehat{\mathbf{u}}^*\}^\perp\\&=\prod_{k=1}^m\Big(\mathrm{cl}\big[T(\bar u_k;\mathcal{U}_{ad}^k(\bar e_k))-T(\bar u_k;\mathcal{U}_{ad}^k(\bar e_k))\big]\cap\{\widehat u_k^*\}^\perp\Big).\end{aligned} \tag{4.45}$$

Combining (4.44) and (4.45) we obtain (4.42).

To prove (4.43) we take any $\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2 \in \mathcal{C}(\bar{\mathbf{u}}, \bar{\mathbf{e}}, \widehat{\mathbf{u}}^*) - \mathcal{C}(\bar{\mathbf{u}}, \bar{\mathbf{e}}, \widehat{\mathbf{u}}^*)$, where we can verify that

$$\mathcal{C}(\bar{\mathbf{u}}, \bar{\mathbf{e}}, \widehat{\mathbf{u}}^*) - \mathcal{C}(\bar{\mathbf{u}}, \bar{\mathbf{e}}, \widehat{\mathbf{u}}^*) = \prod_{k=1}^{m} \Big(\mathcal{C}(\bar{u}_k, \bar{e}_k, \widehat{u}_k^*) - \mathcal{C}(\bar{u}_k, \bar{e}_k, \widehat{u}_k^*) \Big).$$

By the polyhedricity of $\mathcal{U}_{ad}(\bar{\mathbf{e}})$ and by (4.10) we can find sequences $\mathbf{v}_{1n} \to \mathbf{v}_1$, $\mathbf{v}_{2n} \to \mathbf{v}_2$ and $t_{1n} \downarrow 0$, $t_{2n} \downarrow 0$ such that $\bar{\mathbf{u}} + t_{1n}\mathbf{v}_{1n} \in \mathcal{U}_{ad}(\bar{\mathbf{e}})$, $\bar{\mathbf{u}} + t_{2n}\mathbf{v}_{2n} \in \mathcal{U}_{ad}(\bar{\mathbf{e}})$ and $\mathbf{v}_{1n}, \mathbf{v}_{2n} \in \{\widehat{\mathbf{u}}^*\}^{\perp}$. We define $t_n := \min\{t_{1n}, t_{2n}\}$ and deduce from the convexity of $\mathcal{U}_{ad}(\bar{\mathbf{e}})$ that

$$\begin{cases} \mathbf{w}_n := \bar{\mathbf{u}} + t_n\mathbf{v}_{1n} = (1 - t_n t_{1n}^{-1})\bar{\mathbf{u}} + t_n t_{1n}^{-1}(\bar{\mathbf{u}} + t_{1n}\mathbf{v}_{1n}) \in \mathcal{U}_{ad}(\bar{\mathbf{e}}) \\ \mathbf{u}_n := \bar{\mathbf{u}} + t_n\mathbf{v}_{2n} = (1 - t_n t_{2n}^{-1})\bar{\mathbf{u}} + t_n t_{2n}^{-1}(\bar{\mathbf{u}} + t_{2n}\mathbf{v}_{2n}) \in \mathcal{U}_{ad}(\bar{\mathbf{e}}). \end{cases}$$

Therefore, by choosing $\mathbf{e}_n = \bar{\mathbf{e}}$, $\mathbf{u}_n^* = \widehat{\mathbf{u}}^*$, and $\mathbf{v}_n = \mathbf{v}_{1n} - \mathbf{v}_{2n}$, we have $(\mathbf{u}_n, \mathbf{e}_n, \mathbf{u}_n^*) \to (\bar{\mathbf{u}}, \bar{\mathbf{e}}, \widehat{\mathbf{u}}^*)$ with $(\mathbf{u}_n, \mathbf{e}_n, \mathbf{u}_n^*) \in \operatorname{gph} \mathcal{N}$ and $\mathbf{v}_n \to \mathbf{v}$ with

$$\mathbf{v}_n = \mathbf{v}_{1n} - \mathbf{v}_{2n} = \frac{\mathbf{w}_n - \mathbf{u}_n}{t_n} \in \operatorname{cone}\big(\mathcal{U}_{ad}(\mathbf{e}_n) - \mathbf{u}_n\big) \cap \{\mathbf{u}_n^*\}^{\perp} \subset \mathcal{C}(\mathbf{u}_n, \mathbf{e}_n, \mathbf{u}_n^*).$$

This yields $\mathbf{v} \in \mathcal{C}_s(\bar{\mathbf{u}}, \bar{\mathbf{e}}, \widehat{\mathbf{u}}^*)$ due to (4.18). □

Theorem 4.10 *For the setting of Theorem 4.6, the following assertions are valid:*

(i) *If the positive definiteness condition*

$$\langle F'_{\mathbf{u}}(\bar{\mathbf{u}}, \bar{\mathbf{e}})\mathbf{v}, \mathbf{v} \rangle = \sum_{k=1}^{m} \sum_{j=1}^{m} \nabla^2_{u_k u_j} \mathcal{J}_k(\bar{u}_1, \ldots, \bar{u}_m, \bar{e}_Y, \bar{e}_k) v_k v_j > 0 \tag{4.46}$$

holds for all $\mathbf{v} = (v_1, \ldots, v_m) \in \mathbf{L}^2(\Omega)$ *satisfying*

$$0 \neq \mathbf{v} \in \prod_{k=1}^{m} \Big(\operatorname{cl}\big[T(\bar{u}_k; \mathcal{U}_{ad}^k(\bar{e}_k)) - T(\bar{u}_k; \mathcal{U}_{ad}^k(\bar{e}_k))\big] \cap \{\widehat{u}_k^*\}^{\perp} \Big), \tag{4.47}$$

then $\bar{\mathbf{u}}$ *is a fully stable solution to the PVI* (4.2).

(ii) *If* $\bar{\mathbf{u}}$ *is a fully stable solution to the PVI* (4.2) *and that* $\mathcal{U}_{ad}(\mathbf{e})$ *is polyhedric around* $\bar{\mathbf{e}}$, *then the condition* (4.46) *holds for all* $\mathbf{v} = (v_1, \ldots, v_m) \in \mathbf{L}^2(\Omega)$ *satisfying*

$$0 \neq \mathbf{v} \in \prod_{k=1}^{m} \Big(\mathcal{C}(\bar{u}_k, \bar{e}_k, \widehat{u}_k^*) - \mathcal{C}(\bar{u}_k, \bar{e}_k, \widehat{u}_k^*) \Big), \tag{4.48}$$

where $\mathcal{C}(u_k, e_k, u_k^*) = T(u_k; \mathcal{U}_{ad}^k(e_k)) \cap \{u_k^*\}^{\perp}$ *for every* $k = 1, \ldots, m$.

Proof. It follows from Theorems 4.6, 4.9 and Lemma 4.7. □

4.2 Case for $\mathcal{U}_{ad}(\mathbf{e})$ of box constraint type

It is worthy mentioning that the structure of the admissible control set $\mathcal{U}_{ad}(\mathbf{e})$ defined in (2.17) via (2.13) is standard and it has many nice properties. Therefore, it is very frequently appearing in the optimal control theory and applications. Our stability results established in the previous subsection can be refined for the specific case of the admissible control set. In this subsection, we will establish explicit characterizations of full stability for variational Nash equilibriums to the system (2.9) via the parametric system (2.19) with respect to $\mathcal{U}_{ad}(\mathbf{e})$ given by (2.17) and (2.13).

Theorem 4.11 *Let* $(\bar{\mathbf{u}}, \bar{\mathbf{e}}, \bar{\mathbf{u}}^*) \in \mathcal{U}_{ad}(\bar{\mathbf{e}}) \times \mathbf{E} \times \mathbf{L}^2(\Omega)$ *such that* $\widehat{\mathbf{u}}^* := \bar{\mathbf{u}}^* - F(\bar{\mathbf{u}}, \bar{\mathbf{e}}) \in \mathcal{N}(\bar{\mathbf{u}}, \bar{\mathbf{e}})$, *where* $\mathcal{U}_{ad}(\bar{\mathbf{e}})$ *is defined by* (2.17) *and* (2.13). *Assume that the assumptions* **(A1)–(A4)** *hold. Then, we have*

$$\mathcal{C}_{w^*}(\bar{\mathbf{u}}, \bar{\mathbf{e}}, \widehat{\mathbf{u}}^*) = \mathcal{C}_s(\bar{\mathbf{u}}, \bar{\mathbf{e}}, \widehat{\mathbf{u}}^*) = \prod_{k=1}^{m} \big\{v \in L^2(\Omega) \big| \, v(x)\widehat{u}_k^*(x) = 0 \textit{ for a.a. } x \in \Omega\big\}. \tag{4.49}$$

Proof. By our assumptions, for every $k \in \{1, \ldots, m\}$, the admissible control set $\mathcal{U}_{ad}^k(e_k)$ in (2.13) is convex and polyhedric for $e_k \in E_k$; see, e.g., [37, Remark 3.5]. Arguing similarly to the proof of [37, Lemma 4.5], we deduce for that

$$\mathcal{C}_{w^*}(\bar{u}_k, \bar{e}_k, \widehat{u}_k^*) = \mathcal{C}_s(\bar{u}_k, \bar{e}_k, \widehat{u}_k^*) = \big\{v \in L^2(\Omega) \big| \, v(x)\widehat{u}_k^*(x) = 0 \text{ for a.a. } x \in \Omega\big\}.$$

Combining this with (4.19) we obtain (4.49). □

The forthcoming theorem provides us with an explicit characterization of full stability for solutions (variational Nash equilibriums) to the PVI (4.2).

Theorem 4.12 *Assume that all the assumptions of Theorem 4.6 hold, where* $\mathcal{U}_{ad}(\mathbf{e})$ *is given by* (2.17) *and* (2.13) *for* $\mathbf{e} \in \mathbf{E}$. *Then* $\bar{\mathbf{u}}$ *is a fully stable solution to the PVI* (4.2) *if and only if the positive definiteness condition* (4.46) *holds for all* $\mathbf{v} = (v_1, \ldots, v_m) \in \mathbf{L}^2(\Omega)$ *satisfying*

$$0 \neq \mathbf{v} \in \prod_{k=1}^{m} \big\{v \in L^2(\Omega) \big| \, v(x)\widehat{u}_k^*(x) = 0 \textit{ for a.a. } x \in \Omega\big\}. \tag{4.50}$$

Proof. It follows from Theorems 4.6, 4.11 and Lemma 4.7. □

It is interesting to know that there is a relationship between variational Nash equilibriums that are fully stable under perturbations and local Nash equilibriums in the classical sense. From Theorem 4.12 we obtain the following result on the aforementioned relationship.

Theorem 4.13 *Let* $\bar{\mathbf{u}} \in \mathcal{S}(\bar{\mathbf{u}}^*, \bar{\mathbf{e}})$ *and put* $\widehat{\mathbf{u}}^* := \bar{\mathbf{u}}^* - F(\bar{\mathbf{u}}, \bar{\mathbf{e}})$, *where* $\mathcal{U}_{ad}(\cdot)$ *is defined by* (2.17) *and* (2.13). *Assume that the assumptions* **(A1)–(A4)** *hold. If* $\bar{\mathbf{u}}$ *is a fully stable solution to the PVI* (4.2), *then* $\bar{\mathbf{u}}$ *is a local Nash equilibrium associated to the parametric optimal control problems*

$$\text{Minimize} \quad \mathcal{J}_k(u_k, u_{-k}, e_Y, e_k) - (u_k^*, u_k)_{L^2(\Omega)} \quad \text{subject to} \quad u_k \in \mathcal{U}_{ad}^k(e_k) \tag{4.51}$$

with respect to $(\bar{\mathbf{u}}^*, \bar{\mathbf{e}})$, *where the functional* $\mathcal{J}_k(u_k, u_{-k}, e_Y, e_k)$ *is defined in* (2.10).

Proof. Applying Theorem 4.12, we infer that the positive definiteness condition (4.46) holds for all $\mathbf{v} = (v_1, \ldots, v_m) \in \mathbf{L}^2(\Omega)$ satisfying (4.50). This implies that for every $k \in \{1, \ldots, m\}$ the condition

$$\nabla^2_{u_k u_k} \mathcal{J}_k(\bar{u}_k, \bar{u}_{-k}, \bar{e}_Y, \bar{e}_k) v_k v_k > 0 \tag{4.52}$$

holds for all $v_k \in L^2(\Omega)$ with $v_k \neq 0$ and $v_k(x)\widehat{u}_k^*(x) = 0$ for a.a. $x \in \Omega$. Combining this with [37, Theorem 4.8] we deduce that $\bar{u}_k$ is a (Lipschitzian and Hölderian) fully stable local minimizer of the parametric control problem (4.51) with respect to $(\bar{u}_k^*, \bar{e}_Y, \bar{e}_{k,J}, \bar{e}_{k,\alpha}, \bar{e}_{k,\beta})$ for every $k = 1, \ldots, m$; see definitions of Lipschitzian and Hölderian fully stable local minimizers in [25, 37]. This implies that $\bar{\mathbf{u}}$ is a local Nash equilibrium associated to the parametric control problem (4.51) with respect to $(\bar{\mathbf{u}}^*, \bar{\mathbf{e}})$. □

Remark 4.14 Theorems 4.12 and 4.13 can be viewed as generalizations of [4, Theorem 2.2] (see also the results in [6]) to the perturbed cases.

5 Concluding remarks

In this paper, we have provided some new results on the existence of variational/classical Nash equilibriums to the equilibrium problem associated to the nonconvex/convex optimal control problems governed by semilinear elliptic partial differential equations. In addition, we have established a necessary condition and a sufficient condition (resp., an explicit characterization) of full stability for variational Nash equilibrium to the parametric equilibrium problem under full perturbations for general nonempty bounded closed convex component admissible control sets (resp., for component admissible control sets of box constraint type). Furthermore, for the case where the component admissible control sets of box constraint type, we have also proved that variational Nash equilibriums and local Nash equilibriums to the parametric equilibrium problem are equivalent provided that the variational Nash equilibriums are fully stable.

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