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VANISHING VISCOSITY SOLUTIONS TO A RATE-INDEPENDENT TWO-FIELD GRADIENT DAMAGE MODEL

LIVIA BETZ[§]

Abstract. A rate-independent damage model which features two damage variables coupled through a penalty term in the stored energy is considered. Since the energy functional is nonconvex, solutions may be discontinuous in time. This calls for suitable notions of (weak) solutions which allow for jumps. We resort to a vanishing viscosity approach based on an $L^2(\Omega)-$arclength parametrization, where solutions arise as a limit of the graphs of the viscous solutions in the extended state space. This enables us to prove that vanishing viscosity solutions exist and belong to the class of parametrized solutions. We show that the latter can be characterized in various different ways. These alternative formulations highlight the influence of the viscous effects in the jump points, while in the continuous points, the evolution displays a rate-indepedent behaviour. As it turns out, each jump point is related to an ordinary differential equation in Banach space during which the physical time is constant.

Key words. Rate-independent systems, damage evolution, vanishing viscosity, parametrized solutions, penalization

1. Introduction. In this paper, we investigate a rate-independent damage model involving *two damage variables* which are connected through a penalty term in the energy functional. Its examination is of particular interest, since models of this type are frequently employed in computational mechanics, due to the numerical benefits offered by the additional damage variable, cf. e.g. [2, 3, 16, 26–29]. From a theoretical point of view, the use of two-field damage models has been justified in [18]. Therein, the limit problem for penalization tending to infinity was identified with a classical single-field damage model [5, 6] in the rate-dependent case.

The model analyzed in this contribution is inspired by the one presented in [2], where two damage variables are introduced, to which the authors refer as 'local' and 'nonlocal' (for the background of these notions, see [17, Rem. 2.3]). It describes the evolution of damage in an elastic body occupying the domain $\Omega \subset \mathbb{R}^N$, $N = 2, 3$, which has a part of its boundary clamped. During the process, a time dependent volume and boundary load, denoted by ℓ, is applied upon Ω. This induces a certain displacement $\boldsymbol{u} : [0, T] \times \Omega \to \mathbb{R}^N$, as well as local and nonlocal damage. The latter one is denoted by $\varphi : [0, T] \times \Omega \to \mathbb{R}$, while the local damage is called $d : [0, T] \times \Omega \to \mathbb{R}$. Its values measure the degree of the material rigidity loss. Therefore, $d(t, x) = 0$ means that the body is completely sound, while $d(t, x) \to \infty$ means that the body is so damaged that there is no more opponence from its side. Indeed, in the classical literature [5, 6], the damage variable is set to 1, if the material is fully sound, and 0, if it is completely damaged. However, the paper [18] demonstrates that one can transfer our setting into the ones presented in [5, 6].

The considered rate-independent two-field damage problem reads

$$\left.\begin{aligned} (\boldsymbol{u}(t), \varphi(t)) &\in \operatorname*{arg\,min}_{(\boldsymbol{u},\varphi)\in V\times H^1(\Omega)} \mathcal{E}(t, \boldsymbol{u}, \varphi, d(t)), \\ -\partial_d \mathcal{E}(t, \boldsymbol{u}(t), \varphi(t), d(t)) &\in \partial\mathcal{R}(d'(t)), \quad d(0) = d_0 \text{ a.e. in } \Omega \end{aligned}\right\} \tag{P}$$

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for almost all $t \in (0,T)$. For more details regarding its motivation, we refer to [17, Section 2] and [2]. The stored energy $\mathcal{E} : [0,T] \times V \times H^1(\Omega) \times L^2(\Omega) \to \mathbb{R}$ is given by

$$\mathcal{E}(t, \boldsymbol{u}, \varphi, d) := \frac{1}{2} \int_\Omega g(\varphi) \mathbb{C}\varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{u}) \, dx - \langle \ell(t), \boldsymbol{u} \rangle_V + \frac{\alpha}{2} \|\nabla \varphi\|_2^2 + \frac{\beta}{2} \|\varphi - d\|_2^2, \tag{1.1}$$

where $\varepsilon = 1/2(\nabla + \nabla^\top)$ is the linearized strain, $\mathbb{C}$ the elasticity tensor, and ℓ the load applied to the body. The function g describes the influence of the damage on the elastic behavior of the body. Furthermore, $\alpha > 0$ denotes the gradient regularization and $\beta > 0$ stands for the penalization parameter. The problem (P) features the dissipation functional $\mathcal{R} : L^2(\Omega) \to [0, \infty]$ defined as

$$\mathcal{R}(v) := \begin{cases} r \int_\Omega v \, dx & \text{if } v \geq 0 \text{ a.e. in } \Omega, \\ \infty & \text{otherwise,} \end{cases} \tag{1.2}$$

where $r > 0$ stands for the threshold value which triggers the damage evolution. The positively homogeneous functional $\mathcal{R}$ accounts for the rate-independent nature of the model, i.e., the values of the damage do not depend on the rate at which the external load ℓ changes in time. As a consequence, one ignores inertial and viscosity effects. Moreover, as a result of the differential inclusion in (P), we deal with an irreversible damage process, i.e., healing of the body is not allowed.

Since the reduced energy associated to the minimization problem in (P) is nonconvex w.r.t. the local damage variable, one cannot expect (P) to admit smooth solutions in time. This calls for suitable notions of weak solutions which allow for jumps, such as global energetic solutions. This concept was introduced for rate-independent systems in [23,24] and involves a global stability criterion as well as an energy balance. However, this type of solutions fails to describe accurately the behaviour of the system at discontinuous points, as they tend to jump earlier than expected, see [10, Ex. 6.3] and [20, Ex. 7.1].

In response to this, a vanishing viscosity approach was proposed in [4]. The idea is that rate-independence should be considered as limit of viscous systems with viscosity parameter δ tending to 0. This gives rise to the concept of vanishing viscosity solutions and parametrized solutions. We refer here to [1,4,10–12,14,15,20,21,25] and the references therein. Roughly speaking, the vanishing viscosity solutions are approximable solutions, that is, they are limits of sequences of solutions to viscous systems for $\delta \searrow 0$. The set of parametrized solutions is larger, and generalizes the outcome of the vanishing viscosity approach, see e.g. [19, Sec. 4.4] and Theorem 5.9. The main advantage thereof is that the behaviour at jumps can be better understood, as it follows a path which is reminiscent of the viscous approximation. By contrast to global energetic solutions, they jump as late as possible. The existence of parametrized solutions can be shown by proceeding as follows. One starts by reparametrizing the solution d_δ associated to the viscous version of the rate-independent model, in our case (P_δ), by going over to the extended state space $[0,T] \times L^2(\Omega)$. To each $\delta > 0$, one associates a parametrized trajectory $\{(\hat{t}_\delta(s), \hat{d}_\delta(s)) : s \in [0, S_\delta]\} \subset [0,T] \times L^2(\Omega)$. This features a new time variable s which is defined by means of the $L^2(\Omega)-$arclength parametrization of the graph of d_δ. The vanishing viscosity limit of the sequence $\{(\hat{t}_\delta, \hat{d}_\delta)\}_\delta$ is then given by a pair $(\hat{t}, \hat{d})$ which is smooth w.r.t. s (vanishing viscosity solution). This is expected to fit in the setting of parametrized solutions, meaning that its evolution encompasses both dry friction effects and, when the system jumps, the influence of rate-dependent dissipation. Its non-smooth (non-parametrized) counterpart is related to the notion of BV solutions, which are defined on the original time interval $[0, T]$, see [13,20–22]. These can be turned into parametrized solutions by filling in the jumps and vice versa, cf. e.g. [22, Prop. 4.7].

The present contribution is concerned with the existence of parametrized solutions to (P) as well as with the investigation of their properties. To the best of our knowledge, a *rate-independent two-field* damage model has not yet been examined with regard to solvability, although such models are often used in numerical simulations [2, 3, 16, 26–29]. However, the viscous counterpart of (P) has already been studied in the papers [17, 18]. Therein, the unique solvability of (P_δ) and the behaviour of the viscous solutions for penalization parameter tending to infinity have been thoroughly discussed. To achieve our main goal, we will perform a vanishing viscosity analysis as described above. The starting point therefor is a uniform bound w.r.t. δ for the $L^2(\Omega)-$arclength of the viscous curve, which is encoded by S_δ. This shall be provided by employing a time-discretization technique along with a refined estimate from [17] and arguments inspired by [11]. Then, the passage to the limit $\delta \searrow 0$ will be carefully carried out by relying on two crucial findings from the previous contributions regarding our viscous model. In [17] it was established that the entire problem (P_δ) can be reduced to an ordinary differential equation (ODE) in Banach space that is uniquely solved by the local damage, while in [18] a characterization of (P_δ) via an energy-dissipation balance was provided. These will enable us to describe the vanishing viscosity solutions to (P) in terms of a (parametrized) energy inequality, i.e., (4.11), which turns out to admit various equivalent formulations.

The most noteworthy is the complementarity system in (5.3), which, at jump points, reveals an ODE that strongly resembles the viscous one. Our vanishing viscosity limit thus manages to capture the viscous transition paths at jump points, while, in the continuous points, it leads to rate-independent evolution. In fact, each jump point is in correspondence to an ODE during which the external time is constant. See Remarks 5.11 and 5.12 for more details. Moreover, (5.3) has the particularity that it features the precise value of the viscous contribution to the total dissipation that remains in the vanishing viscosity limit, i.e., $\|\xi(s)\|_2$. By relying on convex analysis tools, we provide an additional description of the energy inequality (4.11), namely the differential inclusion (5.20). Here, the term which accounts for viscosity effects in the limit has the same value as in (5.3). In the end, the afore mentioned equivalent formulations turn out to be just different characterizations of the notion of parametrized solutions as presented in [4,25], see also [19, Eq. (90)] and Definition 5.2. Thus, they all bear the same mechanical interpretation and offer an accurate depiction of the behaviour of (P) at jumps.

Furthermore, we prove bounds from below for the $W^{1,1}(0, S)-$norm of the vanishing viscosity solutions (Proposition 4.2). Here we employ the energy-dissipation balance and a uniform bound for the viscous nonlocal damage from [18] along with arguments inspired by [25]. As therein, one can then introduce a new time interval $[0, \bar{S}]$ which is connected via a bijection to $[0, S]$ such that (P) admits parametrized solutions that satisfy a normalization condition on $[0, \bar{S}]$ (Corollary 5.10). Nevertheless, it seems that the present contribution might give some new insight concerning existence of parametrized solutions to single-field damage models as introduced in [5,6], cf. (5.44).

The paper is organized as follows. Section 2 collects the notations and standing assumptions, as well as known results regarding the viscous counterpart of (P) from

[17, 18]. In Section 3 we derive some preliminary estimates which will serve as basis for the upcoming analysis. Section 4 is devoted to the vanishing viscosity analysis. By following the classical approach (see e.g. [4]), we prove that vanishing viscosity solutions exist and fulfill a (parametrized) energy inequality (Theorem 4.1). Here we shall also provide the afore mentioned $W^{1,1}(0,S)-$bounds. Section 5.1 is dedicated to the various equivalent formulations of the energy estimate (4.11) such as the complementarity system in Theorem 5.3 and the differential characterization in Proposition 5.7. Section 5.2 provides an additional characterization via the notion of parametrized solutions (Theorem 5.9). The existence of solutions to (P) in the sense of Definition 5.2 is then a consequence of the fact that each vanishing viscosity solution is a parametrized solution. The non-degeneracy thereof is discussed as well. We also give remarks on their mechanical interpretation with an emphasis on the behaviour at jump points. The paper ends with some thoughts on future research. For the reader's convenience, we include the computation of a Fenchel-conjugate in Appendix A.

2. Notation, standing assumptions, and known results. Throughout the paper, C, C_1, C_2 and C_3 denote generic positive constants. If X and Y are two linear normed spaces, the space of linear and bounded operators from X to Y is called $\mathcal{L}(X, Y)$. The dual of a linear normed space X will be denoted by X^*. For the dual pairing between X and X^* we write $\langle ., . \rangle_X$. The notation $X \hookrightarrow\hookrightarrow Y$ means that X is compactly embedded in Y. In all what follows, $T > 0$ is fixed and $\Omega \subset \mathbb{R}^N$ is a bounded domain, where $N \in \{2, 3\}$ stands for the spatial dimension. By $\| \cdot \|_p$ we denote the $L^p(\Omega)-$norm for $p \in [1, \infty]$ and by $(\cdot, \cdot)_2$ the $L^2(\Omega)-$scalar product. If a Lebesgue integrable function $f : \Omega \to \mathbb{R}$ satisfies $f(x) \geq 0$ a.e. in Ω, we simply write $f \geq 0$. By bold-face case letters we denote vector valued variables and vector valued spaces. For convenience, we sometimes write $\max(\cdot)$ instead of $\max\{\cdot; 0\}$.

Let us now introduce our standing assumptions. We emphasize that these are tacitly assumed in all what follows without mentioning them every time.

Assumption 2.1. *The domain $\Omega \subset \mathbb{R}^N$, $N \in \{2, 3\}$, is bounded with Lipschitz boundary Γ. The boundary consists of two disjoint measurable parts Γ_N and Γ_D such that $\Gamma = \Gamma_N \cup \Gamma_D$. While Γ_N is a relatively open subset, Γ_D is a relatively closed subset of Γ with positive measure.*

In addition, the set $\Omega \cup \Gamma_N$ is regular in the sense of Gröger, cf. [7]. That is, for every point $x \in \Gamma$, there exists an open neighborhood $\mathcal{U}_x \subset \mathbb{R}^N$ of x and a bi-Lipschitz map (a Lipschitz continuous and bijective map with Lipschitz continuous inverse) $\Psi_x : \mathcal{U}_x \to \mathbb{R}^N$ such that $\Psi_x(x) = 0 \in \mathbb{R}^N$ and $\Psi_x\big(\mathcal{U}_x \cap (\Omega \cup \Gamma_N)\big)$ equals one of the following sets:

$$
\begin{aligned}
E_1 &:= \left\{ y \in \mathbb{R}^N : |y| < 1,\ y_N < 0 \right\}, \\
E_2 &:= \left\{ y \in \mathbb{R}^N : |y| < 1,\ y_N \leq 0 \right\}, \\
E_3 &:= \{ y \in E_2 : y_N < 0 \text{ or } y_1 > 0 \}.
\end{aligned}
$$

A detailed characterization of Gröger-regular sets in two and three spatial dimensions is given in [8].

Assumption 2.2. *The function $g : \mathbb{R} \to [\epsilon, 1]$ satisfies $g \in C^2(\mathbb{R})$ and $g', g'' \in L^\infty(\mathbb{R})$ with some $\epsilon > 0$. With a little abuse of notation the Nemystkii-operators associated* *with g and g', considered with different domains and ranges, will be denoted by the same symbol.*

The condition $g(\cdot) \geq \epsilon > 0$ is essential for the coercivity of the bilinear form associated with the balance of momentum equation (2.7a) below, and thus, for the solvability of the viscous version of (P), see [17] for details. Hence, as in the most mathematical literature, we deal with a *partial damage model*.

ASSUMPTION 2.3. *The fourth-order tensor $\mathbb{C} \in L^\infty(\Omega; \mathcal{L}(\mathbb{R}^{N\times N}_{\text{sym}}))$ is symmetric and uniformly coercive, i.e., there is a constant $\gamma_{\mathbb{C}} > 0$ such that*

$$\mathbb{C}(x)\boldsymbol{\sigma} : \boldsymbol{\sigma} \geq \gamma_{\mathbb{C}} |\boldsymbol{\sigma}|^2 \quad \forall \boldsymbol{\sigma} \in \mathbb{R}^{N\times N}_{\text{sym}} \text{ and a.e. in } \Omega, \tag{2.1}$$

where $|\cdot|$ denotes the Frobenius norm on $\mathbb{R}^{N\times N}$ and the symbol " $:$ " stands for the scalar product inducing this norm.

DEFINITION 2.4. *For $p \in [1, \infty]$ we define the following subspace of $\boldsymbol{W}^{1,p}(\Omega)$:*

$$\boldsymbol{W}^{1,p}_D(\Omega) := \{\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega) : \boldsymbol{v}|_{\Gamma_D} = 0\}.$$

The dual space of $\boldsymbol{W}^{1,p'}_D(\Omega)$ is denoted by $\boldsymbol{W}^{-1,p}_D(\Omega)$, where p' is the conjugate exponent of p. If $p = 2$, we abbreviate $V := \boldsymbol{W}^{1,2}_D(\Omega)$.

ASSUMPTION 2.5. *For the applied volume and boundary load we require*

$$\ell \in C^1([0,T]; \boldsymbol{W}^{-1,p}_D(\Omega)),$$

where $p > N$ is specified below, see Assumption 2.7.1. Moreover, the initial damage is supposed to satisfy $d_0 \in L^2(\Omega)$.

Our last assumption concerns the balance of momentum associated with the energy functional in (1.1). For its precise statement we need the following

DEFINITION 2.6. *For given $\varphi \in L^1(\Omega)$ we define the linear form $A_\varphi : V \to V^*$ as*

$$\langle A_\varphi \boldsymbol{u}, \boldsymbol{v}\rangle_V := \int_\Omega g(\varphi)\mathbb{C}\varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{v})\, dx.$$

The operator A_φ considered with different domains and ranges will be denoted by the same symbol for the sake of convenience.

ASSUMPTION 2.7. *In the rest of the paper, we require:*

1. *There exists $p > N$ such that, for all $\overline{p} \in [2, p]$ and all $\varphi \in L^1(\Omega)$, the operator $A_\varphi : \boldsymbol{W}^{1,\overline{p}}_D(\Omega) \to \boldsymbol{W}^{-1,\overline{p}}_D(\Omega)$ is continuously invertible. Moreover, there exists a constant $c > 0$, independent of φ and $\overline{p}$, such that*

$$\|A_\varphi^{-1}\|_{\mathcal{L}(\boldsymbol{W}^{-1,\overline{p}}_D(\Omega), \boldsymbol{W}^{1,\overline{p}}_D(\Omega))} \leq c$$

   holds for all $\overline{p} \in [2, p]$ and all $\varphi \in L^1(\Omega)$.
2. *The penalization parameter β is fixed, sufficiently large, depending only on the given data, see [17, Eq. (3.33)].*

REMARK 2.8. *(i) If $N = 2$, Assumption 2.7.1 is automatically fulfilled, see [17, Lemma 3.3]. The situation changes, if one turns to $N = 3$. In this case this assumption can be guaranteed by imposing additional conditions on the data, in particular on the ellipticity and boundedness constants associated with $\mathbb{C}$ and g, see [17, Remark 3.11] for more details. However, as explained in [17, Remark 3.12], one could* *alternatively modify the energy functional in* (1.1) *by replacing* $\|\nabla\varphi\|_2^2$ *with its* $H^{3/2}$*-seminorm. This would allow us to drop Assumption 2.7.1 in the three dimensional case, too. We have chosen not to work with the* $H^{3/2}$*-seminorm, as the associated bilinear form is difficult to realize in numerical computations.*

(ii) For convenience of the reader, we mention here that in [17, Eq. (3.33)] one requires $\beta > \alpha + C\,C_k$*, where the constant* $C > 0$ *depends on the supremum norm of* ℓ *and* ℓ'*, the Lipschitz constants of* g *and* g'*,* $\|\mathbb{C}\|_{L^\infty(\Omega;\mathcal{L}(\mathbb{R}^{N\times N}_{\rm sym}))}$ *and embedding constants, while the fixed value of* $C_k > 0$ *depends on* k *and* p*, see the proof of [17, Lemma 3.18].*

Next we present some known results from [17, 18], which will be employed in the present paper. Let us point out that the model considered in these contributions differs from (P) only regarding the dissipation functional. Therein, a viscous regularization is applied, which consists of adding an L^2-viscosity term in the dissipation. This approach is frequently used in the context of damage modelling and leads to a rate-dependent process, since the dissipation loses its positive homogeneity. Hence, the damage model from [17, 18] reads as follows

$$\left.\begin{aligned} (\boldsymbol{u}(t),\varphi(t)) \in \operatorname*{arg\,min}_{(\boldsymbol{u},\varphi)\in V\times H^1(\Omega)} \mathcal{E}(t,\boldsymbol{u},\varphi,d(t)),\\ -\partial_d\mathcal{E}(t,\boldsymbol{u}(t),\varphi(t),d(t)) \in \partial\mathcal{R}_\delta(d'(t)), \quad d(0)=d_0 \text{ a.e. in } \Omega \end{aligned}\right\} \qquad (P_\delta)$$

for almost all $t \in (0,T)$. The viscous dissipation functional $\mathcal{R}_\delta : L^2(\Omega) \to [0,\infty]$ appearing in (P_δ) is given by

$$\mathcal{R}_\delta(v) := \begin{cases} r\int_\Omega v\,dx + \frac{\delta}{2}\|v\|_2^2, & \text{if } v \geq 0 \text{ a.e. in } \Omega,\\ \infty, & \text{otherwise,} \end{cases} \tag{2.2}$$

where $\delta > 0$ is the fixed viscosity parameter.

We begin our presentation of the findings from [17, 18] by recalling the main result on the energy minimization:

Lemma 2.9 (Energy minimizer, [17, Prop. 3.14, Thm. 3.20]). *For every* $(t,d) \in [0,T]\times L^2(\Omega)$*, the optimization problem*

$$\min_{(\boldsymbol{u},\varphi)\in V\times H^1(\Omega)} \mathcal{E}(t,\boldsymbol{u},\varphi,d)$$

admits a unique minimizer $(\boldsymbol{u},\varphi) \in \boldsymbol{W}^{1,p}_D(\Omega)\times H^1(\Omega)$ *characterized by* $\boldsymbol{u} = \mathcal{U}(t,\varphi)$ *and* $\varphi = \Phi(t,d)$*, where the operator* $\mathcal{U} : [0,T]\times H^1(\Omega) \to \boldsymbol{W}^{1,p}_D(\Omega)$ *is defined as* $\mathcal{U}(t,\varphi) := A_\varphi^{-1}\ell(t)$*, while* $\Phi : [0,T]\times L^2(\Omega) \to H^1(\Omega)$*,* $\Phi : (t,d) \mapsto \varphi$*, is the solution operator of*

$$-\alpha\triangle\varphi + \beta\varphi + \frac{1}{2}g'(\varphi)\mathbb{C}\varepsilon(\mathcal{U}(t,\varphi)) : \varepsilon(\mathcal{U}(t,\varphi)) = \beta d.$$

Thanks to Assumptions 2.5 and 2.7.1 there exists a constant $c > 0$, independent of t and φ, such that

$$\|\mathcal{U}(t,\varphi)\|_{\boldsymbol{W}^{1,p}_D(\Omega)} \leq c \quad \forall\,(t,\varphi) \in [0,T]\times H^1(\Omega). \tag{2.3}$$

Lemma 2.10 (Continuity of $\mathcal{U}$, [17, Lemma 3.9]). *Let* $\{t_n,\varphi_n\} \subset [0,T]\times H^1(\Omega)$ *and* $(t,\varphi) \in [0,T]\times H^1(\Omega)$ *be given such that* $(t_n,\varphi_n) \to (t,\varphi)$ *in* $\mathbb{R}\times L^1(\Omega)$*. Then it holds* $\mathcal{U}(t_n,\varphi_n) \to \mathcal{U}(t,\varphi)$ *in* $\boldsymbol{W}^{1,s}_D(\Omega)$ *as* $n\to\infty$ *for every* $s \in [2,p)$*.*

Since Φ maps to $H^1(\Omega)$, the last inequality in [17, Eq. (3.34)] can be refined such that [17, Eq. (3.35)] is replaced by

$$\|\Phi(t_1, d_1) - \Phi(t_2, d_2)\|_{H^1(\Omega)} \leq L\left(\|d_1 - d_2\|_{H^1(\Omega)^*} + |t_1 - t_2|\right) \;\; \forall\, t_1, t_2 \in [0, T],\, d_1, d_2 \in L^2(\Omega), \tag{2.4}$$

where $L > 0$ is a constant depending only on the given data.

In the light of the embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$, we get the following estimate, which turns out to be crucial in the proof of Proposition 3.1 below:

Lemma 2.11. *There exists a constant $L > 0$ so that*

$$\|\Phi(t_1, d_1) - \Phi(t_2, d_2)\|_6 \leq L\left(|t_1 - t_2| + \|d_1 - d_2\|_{6/5}\right) \;\; \forall\, t_1, t_2 \in [0, T],\, d_1, d_2 \in L^2(\Omega).$$

Moreover, by relying again on (2.4) this time in combination with the compact embedding $L^2(\Omega) \hookrightarrow\hookrightarrow H^1(\Omega)^*$, we obtain

Lemma 2.12 (Weak-to-strong continuity of Φ). *Let $\{t_n, d_n\} \subset [0, T] \times L^2(\Omega)$ and $(t, d) \in [0, T] \times L^2(\Omega)$ be given such that $(t_n, d_n) \rightharpoonup (t, d)$ in $\mathbb{R} \times L^2(\Omega)$ as $n \to \infty$. Then, it holds $\Phi(t_n, d_n) \to \Phi(t, d)$ in $H^1(\Omega)$ as $n \to \infty$.*

Finally we turn our attention to the differential inclusion in (P_δ). First note that the functional $\mathcal{E}$ is partially Fréchet differentiable w.r.t. d on $[0, T] \times V \times H^1(\Omega) \times L^2(\Omega)$, and its partial derivative is given by $\partial_d \mathcal{E}(t, \boldsymbol{u}, \varphi, d) = \beta(d - \varphi)$. Therefore, in view Lemma 2.9, (P_δ) reduces to the following evolutionary equation

$$-\beta\big(d(t) - \Phi(t, d(t))\big) \in \partial\mathcal{R}_\delta(d'(t)) \quad \text{a.e. in } (0, T), \quad d(0) = d_0. \tag{2.5}$$

The evolution (2.5) is equivalent to the following non-smooth operator differential equation:

$$d'(t) = \frac{1}{\delta}\max\{-\beta\big(d(t) - \Phi(t, d(t))\big) - r, 0\} \quad \text{a.e. in } (0, T), \quad d(0) = d_0. \tag{2.6}$$

cf. [17, Lemma 3.22]. This handy reformulation of (P_δ) and (2.5), respectively, is a main advantage of the penalty-type regularization of partial damage models and it will turn out to be very useful in the upcoming sections.

Theorem 2.13 (Existence and uniqueness for the viscous penalized damage model, [17, Thm. 5.13]). *There exists a unique solution $(\boldsymbol{u}, \varphi, d)$ of the problem (P_δ), satisfying $\boldsymbol{u} \in C^1([0, T]; V)$, $\varphi \in C^1([0, T]; H^1(\Omega))$, $d \in C^{1,1}([0, T]; L^2(\Omega))$, which is characterized by the following system of differential equations:*

$$-\operatorname{div} g(\varphi(t))\mathbb{C}\varepsilon(\boldsymbol{u}(t)) = \ell(t) \qquad \text{in } \boldsymbol{W}_D^{-1,p}(\Omega) \tag{2.7a}$$

$$-\alpha\Delta\varphi(t) + \beta\,\varphi(t) + \frac{1}{2}\,g'(\varphi(t))\mathbb{C}\,\varepsilon(\boldsymbol{u}(t)) : \varepsilon(\boldsymbol{u}(t)) = \beta d(t) \quad \text{in } H^1(\Omega)^* \tag{2.7b}$$

$$d'(t) - \frac{1}{\delta}\max\{-\beta(d(t) - \varphi(t)) - r, 0\} = 0, \qquad d(0) = d_0 \tag{2.7c}$$

for every $t \in [0, T]$.

Note that the equations (2.7a) and (2.7b) are just equivalent to $\boldsymbol{u}(t) = \mathcal{U}(t, \varphi(t))$ and $\varphi(t) = \Phi(t, d(t))$, respectively.

Next we state some known results from [18].

Proposition 2.14 (Uniform bound for the viscous damage variable, [18, Prop. 6.8]). *Assume that there exists a constant $M > 0$ such that $g(x) \geq g(M)$ for all $x \geq M$.*

Then, the nonlocal damage associated with the penalized viscous model (P_δ) *and* (2.7)*, respectively, fulfils* $\varphi(t,x) \leq M$ *a.e. in* $(0,T) \times \Omega$.

Definition 2.15 (The reduced energy, [18, Def. 3.1, Eq. (3.1)]). *The reduced energy functional* $\mathcal{I} : [0,T] \times L^2(\Omega) \to \mathbb{R}$ *is given by* $\mathcal{I}(t,d) := \mathcal{E}(t, \mathcal{U}(t, \Phi(t,d)), \Phi(t,d), d)$*, i.e.,*

$$\mathcal{I}(t,d) = -\frac{1}{2}\langle \ell(t), \mathcal{U}(t, \Phi(t,d))\rangle_V + \frac{\alpha}{2}\|\nabla \Phi(t,d)\|_2^2 + \frac{\beta}{2}\|\Phi(t,d) - d\|_2^2. \tag{2.8}$$

Lemma 2.16 (Differentiability properties of $\mathcal{I}$, [18, Lemma 3.2]). *It holds* $\mathcal{I} \in C^1([0,T] \times L^2(\Omega))$ *and, at all* $(t,d) \in [0,T] \times L^2(\Omega)$*, we have*

$$\partial_t \mathcal{I}(t,d) = -\langle \ell'(t), \mathcal{U}(t, \Phi(t,d))\rangle_V, \quad \partial_d \mathcal{I}(t,d) = \beta(d - \Phi(t,d)). \tag{2.9}$$

We end this section by recalling another equivalent formulation of the evolutions in (2.5) and (2.6), respectively, namely the so-called energy identity. This was introduced in [18] for the purpose of passing to the limit $\beta \to \infty$. Again, it will turn out to be essential, this time in the context of vanishing viscosity $\delta \searrow 0$.

Proposition 2.17 (Energy-dissipation balance, [18, Prop. 3.5, Lemma 3.4]). *The unique solution* $d \in C^1([0,T]; L^2(\Omega))$ *of the viscous penalized damage evolution* (P_δ) *fulfills for all* $0 \leq s \leq t \leq T$ *the energy identity*

$$\begin{aligned} \int_s^t \mathcal{R}_\delta(d'(\tau))\, d\tau + \int_s^t \mathcal{R}_\delta^*\big(-\partial_d \mathcal{I}(\tau, d(\tau))\big)\, d\tau + \mathcal{I}(t, d(t)) \\ = \mathcal{I}(s, d(s)) + \int_s^t \partial_t \mathcal{I}(\tau, d(\tau))\, d\tau, \end{aligned} \tag{2.10}$$

i.e.,

$$\int_s^t r\|d'(\tau)\|_1 + \delta\|d'(\tau)\|_2^2\, d\tau + \mathcal{I}(t, d(t)) = \mathcal{I}(s, d(s)) + \int_s^t \partial_t \mathcal{I}(\tau, d(\tau))\, d\tau. \tag{2.11}$$

3. Preliminary estimates. Before we begin with our vanishing viscosity analysis we need to prove that the $L^2(\Omega)$-arclength of the viscous curve (see (4.1) below) is uniformly bounded w.r.t. the viscosity parameter [11, 25]. This is ensured by the following

Proposition 3.1 (Uniform L^1-estimate for the time derivative of the viscous local damage). *There exist constants* $C, \hat{C} > 0$*, independent of* δ*, such that*

$$\int_0^T \|d'(t)\|_2 \, dt \leq C + C \int_0^T \|d'(t)\|_1 \, dt \leq \hat{C}, \tag{3.1}$$

where d *is the unique solution of the viscous damage model* (P_δ).

Proof. Note that the second inequality in (3.1) follows directly from (2.11) (where we set $s := 0$ and $t := T$), combined with (2.8), (2.3), and (2.9). Regarding the first estimate in (3.1), we proceed as in [11, Prop. 4.3], i.e., we first show the time-discrete version of the desired inequality, i.e., (3.4) below, where $C > 0$ is independent of δ and τ (the fineness of the time-discretization). Letting $\tau \searrow 0$ therein will then give (3.1), in view of [18, Prop. 6.2]. To this end, we first recall the time-discretization scheme from [18, Sec. 6]. Let a number of time-steps $n \in \mathbb{N}^+$ be given, set $\tau := T/n$, and denote by $\{t_k^\tau = k\tau\}_{k=0,...,n}$ the corresponding partition of the time interval $[0, T]$. Beginning with $d_0^\tau := d_0$, we define the approximation of the local damage at time point t_{k+1}^τ, $k \in \{0, ..., n-1\}$, as the unique solution of the fixed-point equation

$$d_{k+1}^\tau = d_k^\tau + \frac{\tau}{\delta} \max\big\{ -\beta(d_{k+1}^\tau - \Phi(t_{k+1}^\tau, d_{k+1}^\tau)) - r, 0\big\}. \tag{P_k^δ}$$

In [18, Sec. 6] it was shown that (P_k^δ) admits indeed a unique solution d_{k+1}^τ (for $\tau > 0$ sufficiently small). We introduce the notations

$$\bar{t}_\tau(t) := t_{k+1}^\tau, \quad \bar{d}_\tau(t) := d_{k+1}^\tau \text{ for } t \in (t_k^\tau, t_{k+1}^\tau],\ k \in \{0, ..., n-1\},$$

and $\bar{t}_\tau(0) := 0$, $\bar{d}_\tau(0) := d_0$, as well as the piecewise linear interpolation $d_\tau : [0, T] \to L^2(\Omega)$, which is given by

$$d_\tau(t) := d_k^\tau + \frac{t - t_k^\tau}{\tau}(d_{k+1}^\tau - d_k^\tau) \text{ for } t \in [t_k^\tau, t_{k+1}^\tau],\ k \in \{0, ..., n-1\}.$$

Notice that d_τ is differentiable on $[0, T] \setminus \{t_0, t_1, ..., t_n\}$ with

$$d_\tau'(t) = \frac{d_{k+1}^\tau - d_k^\tau}{\tau} \quad \text{for } t \in (t_k^\tau, t_{k+1}^\tau),\ k \in \{0, ..., n-1\}. \tag{3.2}$$

Thus, (P_k^δ) can be rewritten for all $t \in [0, T] \setminus \{t_0, t_1, ..., t_n\}$ as

$$d_\tau'(t) = \frac{1}{\delta} \max\{-\beta(\bar{d}_\tau(t) - \bar{\varphi}_\tau(t)) - r, 0\},$$

which is equivalent to

$$-\beta\big(\bar{d}_\tau(t) - \bar{\varphi}_\tau(t)\big) - \delta d_\tau'(t) \in \partial\mathcal{R}(d_\tau'(t)), \tag{3.3}$$

where we abbreviate $\bar{\varphi}_\tau := \Phi(\bar{t}_\tau(\cdot), \bar{d}_\tau(\cdot))$. See proof of [17, Lem. 3.22]. With the above time-discretization at hand, we can now focus on showing

$$\int_0^T \|d_\tau'(t)\|_2 \, dt \le C + C \int_0^T \|d_\tau'(t)\|_1 \, dt, \tag{3.4}$$

where $C > 0$ is independent of δ and τ. We employ arguments from the proofs of [11, Prop. 4.1 and 4.3]. Let $k \in \{1, ..., n-1\}$ be arbitrary, but fixed and fix $t \in (t_k^\tau, t_{k+1}^\tau)$ and $s \in (t_{k-1}^\tau, t_k^\tau)$. From (3.3) we have

$$\begin{aligned}
(-\beta\big(\bar{d}_\tau(t) - \bar{\varphi}_\tau(t)\big) - \delta d_\tau'(t), d_\tau'(t))_2 &= \mathcal{R}(d_\tau'(t)), \\
(-\beta\big(\bar{d}_\tau(s) - \bar{\varphi}_\tau(s)\big) - \delta d_\tau'(s), d_\tau'(t))_2 &\le \mathcal{R}(d_\tau'(t)),
\end{aligned} \tag{3.5}$$

by the positive homogeneity of $\mathcal{R}$, see also [17, Eq. (3.41)]. By subtracting the identity from the inequality in (3.5) we get

$$\begin{aligned}
&\delta\|d_\tau'(t)\|_2(\|d_\tau'(t)\|_2 - \|d_\tau'(s)\|_2) \le \delta(\|d_\tau'(t)\|_2^2 - (d_\tau'(s), d_\tau'(t))_2) \\
&\le -\beta(\bar{d}_\tau(t) - \bar{d}_\tau(s), d_\tau'(t))_2 + \beta\big(\bar{\varphi}_\tau(t) - \bar{\varphi}_\tau(s), d_\tau'(t))_2 \\
&\le -\beta\tau\|d_\tau'(t)\|_2^2 + \beta L(\tau + \|\bar{d}_\tau(t) - \bar{d}_\tau(s)\|_{6/5})\|d_\tau'(t)\|_{6/5} \\
&\le -\beta\tau\|d_\tau'(t)\|_2^2 + \beta L\tau(c\|d_\tau'(t)\|_2 + \|d_\tau'(t)\|_{6/5}^2),
\end{aligned} \tag{3.6}$$

where L is the Lipschitz constant of Φ, which is independent of δ and τ. Note that for the last two inequalities we used (3.2), the Lipschitz continuity of Φ, cf. Lemma 2.11, as well as $\bar{t}_\tau(t) - \bar{t}_\tau(s) = \tau$ and the embedding $L^2(\Omega) \hookrightarrow L^{6/5}(\Omega)$. Since $1 < 6/5 < 2$, we can estimate, similarly to [17, Eq. (3.31),(3.32)], as follows

$$\|d'_\tau(t)\|_{6/5}^2 \leq \frac{1}{2L}\|d'_\tau(t)\|_2^2 + c(L)\|d'_\tau(t)\|_1^2 \leq \frac{1}{2L}\|d'_\tau(t)\|_2^2 + c\|d'_\tau(t)\|_1\|d'_\tau(t)\|_2, \quad (3.7)$$

in view of Lyapunov's and generalized Young's inequality. Note that $c > 0$ depends only on L and Ω, and is thus, independent of δ and τ. Using (3.7) in (3.6) now results in

$$\begin{aligned}
&\delta\|d'_\tau(t)\|_2(\|d'_\tau(t)\|_2 - \|d'_\tau(s)\|_2) \\
&\qquad \leq -\beta\tau\|d'_\tau(t)\|_2^2 + \beta L\tau\big(c\|d'_\tau(t)\|_2 + \frac{1}{2L}\|d'_\tau(t)\|_2^2 + c\|d'_\tau(t)\|_1\|d'_\tau(t)\|_2\big) \\
&\qquad \leq C_1\tau\|d'_\tau(t)\|_2 - \frac{\beta}{2}\tau\|d'_\tau(t)\|_2^2 + C_2\tau\|d'_\tau(t)\|_1\|d'_\tau(t)\|_2,
\end{aligned} \quad (3.8)$$

whence

$$\delta(\|d'_\tau(t)\|_2 - \|d'_\tau(s)\|_2) + C_3\tau\|d'_\tau(t)\|_2 \leq C_1\tau + C_2\tau\|d'_\tau(t)\|_1 \quad (3.9)$$

follows. Further, as a consequence of (3.2), we have for all $j = 0, ..., n-1$ that $d'_\tau(\rho) = d'_\tau\left(\frac{t^\tau_j + t^\tau_{j+1}}{2}\right)\ \forall\, \rho \in (t^\tau_j, t^\tau_{j+1})$, by means of which we rewrite (3.9). Recall that $t \in (t^\tau_k, t^\tau_{k+1})$ and $s \in (t^\tau_{k-1}, t^\tau_k)$, where $k \in \{1, ..., n-1\}$ was arbitrary, but fixed. Summing up the inequalities (3.9) for $k = 1, ..., n-1$ implies

$$C_3\tau\sum_{k=1}^{n-1}\Big\|d'_\tau\Big(\frac{t^\tau_k + t^\tau_{k+1}}{2}\Big)\Big\|_2 \leq \delta\Big\|d'_\tau\Big(\frac{\tau}{2}\Big)\Big\|_2 + C_1T + C_2\tau\sum_{k=1}^{n-1}\Big\|d'_\tau\Big(\frac{t^\tau_k + t^\tau_{k+1}}{2}\Big)\Big\|_1. \quad (3.10)$$

Let us now estimate the term $\delta\|d'_\tau(\tau/2)\|_2$. By the identity in (3.5) we have

$$\begin{aligned}
\delta\|d'_\tau(\tau/2)\|_2^2 = &-\mathcal{R}(d'_\tau(\tau/2)) - \beta(\bar{d}_\tau(\tau/2) - \bar{d}_\tau(0), d'_\tau(\tau/2))_2 \\
&+ \beta(\bar{\varphi}_\tau(\tau/2) - \bar{\varphi}_\tau(0), d'_\tau(\tau/2))_2 - \beta(\bar{d}_\tau(0) - \bar{\varphi}_\tau(0), d'_\tau(\tau/2))_2.
\end{aligned}$$

Arguing as above gives in turn

$$\delta\|d'_\tau(\tau/2)\|_2 + C_3\tau\|d'_\tau(\tau/2)\|_2 \leq C_1\tau + C_2c\tau\|d'_\tau(\tau/2)\|_2 + \beta\|d_0 - \Phi(0, d_0)\|_2.$$

By choosing τ small enough e.g. such that $C_2c\tau \leq \delta/2$ we deduce $\delta\|d'_\tau(\tau/2)\|_2 \leq C$, where $C > 0$ is a constant independent of δ and τ. This together with (3.10) yields (3.4). The first estimate in (3.1) can now be deduced from [18, Prop. 6.2], which says that $d_\tau \to d$ in $W^{1,\infty}(0, T; L^2(\Omega))$ as $\tau \searrow 0$. Since the second estimate is (3.1) is satisfied as well, the proof is now complete. □

The next result will be used in a later discussion regarding the so-called *non-degenerate parametrized solutions*, see Section 5 below. For this, we need the following

Assumption 3.2. *There exists a constant $M > 0$ such that $g(x) \geq g(M)$ for all $x \geq M$. In addition, we suppose that, at the beginning of the process, the body is completely sound, i.e., $d_0 \equiv 0$.*

Remark 3.3. *Recall that the function $g : \mathbb{R} \to [\epsilon, 1]$ measures the material rigidity of the body, where, according to Assumption 2.2, $\epsilon > 0$ (which turns the model into* *a partial damage model). Thus, there is always a minimum rigidity remaining, and Assumption 3.2 is for instance fulfilled, if $g(x) \equiv \epsilon$ for all $x \geq M$, which means that this minimum rigidity is already achieved with finite values of the (nonlocal) damage variable.*

LEMMA 3.4. *Let Assumption 3.2 be satisfied. Then, there exists a constant $C > 0$, independent of δ, such that*

$$\int_{T_1}^{T_2} \|d'(t)\|_2 \, dt \leq C(T_2 - T_1) + C\Big(\int_{T_1}^{T_2} \|d'(t)\|_1 \, dt\Big)^{1/2} \quad \forall\, 0 \leq T_1 \leq T_2 \leq T, \tag{3.11}$$

where d is the unique solution of the viscous damage model (P_δ).

Proof. We assume that $T_1 < T_2$, otherwise (3.11) follows immediately. By arguing exactly as in the proof of Proposition 3.1 we deduce the counterpart of (3.1) on some arbitrary (fixed) interval $[T_1, T_2]$. This reads

$$\frac{2\delta}{\beta}(\|d'(T_2)\|_2 - \|d'(T_1)\|_2) + \int_{T_1}^{T_2} \|d'(t)\|_2 \, dt \leq c\,(T_2 - T_1) + c\int_{T_1}^{T_2} \|d'(t)\|_1 \, dt, \tag{3.12}$$

where $c > 0$ is a constant independent of δ. Note that, to arrive at (3.12), we divided by $\beta/2$ in (3.8) and relied again on (3.9) as well as the convergence $d_\tau \to d$ in $W^{1,\infty}(0, T; L^2(\Omega))$ for $\tau \searrow 0$.

Now we want to estimate the first term on the left hand side in (3.12). Similarly to (3.5), we obtain from (2.5) combined with the sum rule for convex subdifferentials and the positive homogeneity of $\mathcal{R}$:

$$(-\beta\big(d(t) - \varphi(t)\big) - \delta d'(t), d'(t))_2 = \mathcal{R}(d'(t)) \quad \forall\, t \in [0, T], \tag{3.13}$$

where $\varphi := \Phi(\cdot, d(\cdot))$ stands for the nonlocal damage associated to d. On the other hand, Proposition 2.14 together with $\dot{d} \geq 0$ (see (2.7c)) implies

$$(-\beta\big(d(t) - \varphi(t)\big), d'(t))_2 \leq -\frac{\beta}{2}\frac{d}{dt}\|d(t)\|_2^2 + \beta M\|d'(t)\|_1 \quad \forall\, t \in [0, T]. \tag{3.14}$$

Inserting this in (3.13) and integrating over (T_1, T_2) leads to the following

$$\delta\int_{T_1}^{T_2} \|d'(t)\|_2^2 \, dt + \frac{\beta}{2}(\|d(T_2)\|_2^2 - \|d(T_1)\|_2^2) \leq (\beta M - r)\int_{T_1}^{T_2} \|d'(t)\|_1 \, dt. \tag{3.15}$$

In view of $d_0 \equiv 0$ and (2.7c), we have $0 \leq d(T_1) \leq d(T_2)$, from which we infer

$$\|d(T_2) - d(T_1)\|_2^2 \leq \|d(T_2)\|_2^2 - \|d(T_1)\|_2^2. \tag{3.16}$$

Further, the ODE (2.7c) along with the Lipschitz-continuity of max and Φ, cf. Lemma 2.11, gives in turn

$$\delta\|d'(T_2) - d'(T_1)\|_2 \leq \beta C_1(\|d(T_2) - d(T_1)\|_2 + (T_2 - T_1)), \tag{3.17}$$

where $C_1 > 0$ is a constant independent of δ. Now, building squares on both sides in (3.17) and employing (3.16) in the resulting inequality yields

$$\delta^2\|d'(T_2) - d'(T_1)\|_2^2 \leq \beta^2 C_2(\|d(T_2)\|_2^2 - \|d(T_1)\|_2^2 + (T_2 - T_1)^2). \tag{3.18}$$

We combine (3.18) and (3.15) to obtain

$$\frac{\delta^2}{\beta^2}\|d'(T_2)-d'(T_1)\|_2^2 \le C_3\Big(\int_{T_1}^{T_2}\|d'(t)\|_1\,dt + (T_2-T_1)^2\Big),$$

which results in

$$\begin{aligned}\frac{2\delta}{\beta}(\|d'(T_1)\|_2 - \|d'(T_2)\|_2) &\le \frac{2\delta}{\beta}\|d'(T_2)-d'(T_1)\|_2 \\ &\le \widetilde{c}\Big(\int_{T_1}^{T_2}\|d'(t)\|_1\,dt\Big)^{1/2} + \widetilde{c}(T_2-T_1),\end{aligned} \tag{3.19}$$

with $\widetilde{c}>0$ independent of δ. In addition, in view of the second inequality in (3.1), we deduce that there exists $\hat{c}>0$, independent of δ, so that

$$\int_{T_1}^{T_2}\|d'(t)\|_1\,dt \le \hat{c}\Big(\int_{T_1}^{T_2}\|d'(t)\|_1\,dt\Big)^{1/2}. \tag{3.20}$$

Finally, the desired estimate follows from (3.19), (3.12) and (3.20). ▯

4. Vanishing viscosity analysis. In the sequel, we consider $\delta>0$ arbitrary, but fixed. Throughout this section, d_δ denotes the unique solution to the penalized viscous problem (P_δ). Following the approach in [4] (see also [20, 21]), we define the $L^2(\Omega)$-arclength parametrization of the graph $\mathrm{Graph}(d_\delta) := \{(t,d_\delta(t)) : t\in[0,T]\}\subset [0,T]\times L^2(\Omega)$ as

$$s_\delta(t) := t + \int_0^t \|d_\delta'(\tau)\|_2\,d\tau \quad \forall\, t\in[0,T]. \tag{4.1}$$

We observe that s_δ defines a bijection from $[0,T]$ to $[0,S_\delta]$, where $S_\delta := s_\delta(T)$. This allows us to define the functions $\hat{t}_\delta : [0,S_\delta]\to[0,T]$ and $\hat{d}_\delta : [0,S_\delta]\to L^2(\Omega)$

$$\hat{t}_\delta(s) := s_\delta^{-1}(s), \quad \hat{d}_\delta(s) := d_\delta(\hat{t}_\delta(s)). \tag{4.2}$$

A straight forward computation shows that $\hat{t}_\delta\in C^{1,1}([0,S_\delta];\mathbb{R})$, since d_δ belongs to $C^{1,1}([0,T];L^2(\Omega))$ (cf. Theorem 2.13), with

$$\hat{t}_\delta'(s) = 1/(1+\|{d_\delta}'(\hat{t}_\delta(s))\|_2) > 0 \quad \text{for all } s\in[0,S_\delta]. \tag{4.3}$$

By the chain rule, we have $\hat{d}_\delta\in C^{1,1}([0,S_\delta];L^2(\Omega))$ with

$$\hat{d}_\delta'(s) = {d_\delta}'(\hat{t}_\delta(s))\hat{t}_\delta'(s) \ge 0 \quad \text{for all } s\in[0,S_\delta], \tag{4.4}$$

since ${d_\delta}'(t)\ge 0$ a.e. in $(0,T)$, see (2.7c). As a consequence of (4.3) and (4.4), the pair $(\hat{t}_\delta,\hat{d}_\delta)$ fulfils the normalization condition

$$\hat{t}_\delta'(s) + \|\hat{d}_\delta'(s)\|_2 = 1 \quad \text{for all } s\in[0,S_\delta]. \tag{4.5}$$

The main goal in this section is to study the behaviour of the parametrized trajectories $\{(\hat{t}_\delta(s),\hat{d}_\delta(s)) : s\in[0,S_\delta]\}$ for $\delta\searrow 0$. We start by observing that $\{S_\delta\}_\delta$ admits a convergent subsequence (denoted by the same symbol), in view of Proposition 3.1. Hence, there exists $S(\ge T$, in view of (4.1)), so that $S_\delta\to S$ as $\delta\searrow 0$. With no loss of generality we consider the parametrized trajectories to be defined on the fixed interval $[0, S]$.

With the above parametrization at hand, the energy identity (2.11) leads to

$$\int_0^S r\|\hat{d}_\delta'(s)\|_1 + \frac{\delta}{\hat{t}_\delta'(s)}\|\hat{d}_\delta'(s)\|_2^2 \, ds + \mathcal{I}(\hat{t}_\delta(S), \hat{d}_\delta(S)) \\ = \mathcal{I}(0, \hat{d}_\delta(0)) + \int_0^S \partial_t \mathcal{I}(\hat{t}_\delta(s), \hat{d}_\delta(s))\hat{t}_\delta'(s) ds. \tag{4.6}$$

The energy-dissipation balance (4.6) will be exploited when investigating the limiting behaviour of the sequence $\{(\hat{t}_\delta, \hat{d}_\delta)\}_\delta$ for $\delta \searrow 0$ (see Theorem 4.1 below). Moreover, we shall make use of the ODE (2.6) which characterizes the local damage d_δ. Note that this can be rewritten as

$$\frac{\hat{d}_\delta'(s)}{\hat{t}_\delta'(s)} = \frac{1}{\delta}\max\{-\beta\big(\hat{d}_\delta(s) - \Phi(\hat{t}_\delta(s), \hat{d}_\delta(s))\big) - r, 0\} \quad \text{for every } s \in [0, S], \quad \hat{d}_\delta(0) = d_0, \tag{4.7}$$

in view of (4.2) and (4.4). The equation (4.7) plays an essential role in the proof of Theorem 4.1 below, as it allows us to use (rather standard) weak lower semicontinuity arguments when passing to the limit in the second term in (4.6). Moreover, from (4.7) and (4.5) we deduce that

$$\|\hat{d}_\delta'(s)\|_2 = \frac{\|\xi_\delta(s)\|_2}{\|\xi_\delta(s)\|_2 + \delta} \quad \text{for every } s \in [0, S], \quad \hat{d}_\delta(0) = d_0, \tag{4.8}$$

where we abbreviate $\xi_\delta(s) := \max(-\beta\big(\hat{d}_\delta(s) - \Phi(\hat{t}_\delta(s), \hat{d}_\delta(s))\big) - r)$, and by employing (4.5) again, we get $\hat{t}_\delta'(s) = \delta/(\|\xi_\delta(s)\|_2 + \delta)$ for every $s \in [0, S]$. Inserted in (4.7), this gives in turn that (4.7) can be equivalently written as

$$(\|\xi_\delta(s)\|_2 + \delta)\hat{d}_\delta'(s) = \xi_\delta(s) \quad \text{for every } s \in [0, S], \quad \hat{d}_\delta(0) = d_0. \tag{4.9}$$

Note that (4.9) is an ODE in implicit form in terms of $\hat{d}_\delta$. As we will later see, the equation (4.9) preserves its structure in the vanishing viscosity limit (Theorem 5.3 below).

We are now in the position to prove the existence of so-called vanishing viscosity solutions to (P), i.e., solutions which are obtained by letting $\delta \searrow 0$ in the viscous model described in terms of $(\hat{t}_\delta, \hat{d}_\delta)$. As it will turn out, these belong to the set of *parametrized solutions* [19], see Section 5 below.

Theorem 4.1 (Vanishing viscosity limit of (4.6)/(4.7)). *For every sequence $\delta_n \searrow 0$, there exists a (not relabeled subsequence) $\{(\hat{t}_{\delta_n}, \hat{d}_{\delta_n})\}_n \subset C^{1,1}([0, S]; [0, T] \times L^2(\Omega))$ and a pair $(\hat{t}, \hat{d}) \in C^{0,1}([0, S]; [0, T] \times L^2(\Omega))$ such that*

$$(\hat{t}_{\delta_n}, \hat{d}_{\delta_n}) \rightharpoonup (\hat{t}, \hat{d}) \quad \text{in } W^{1,2}(0, S; \mathbb{R} \times L^2(\Omega)) \quad \text{as } n \to \infty, \tag{4.10}$$

where $(\hat{t}_\delta, \hat{d}_\delta)$ is the parametrized trajectory defined in (4.2). Moreover, $(\hat{t}, \hat{d})$ satisfies

$$\int_0^S r\|\hat{d}'(s)\|_1 + \|\max\big(-\beta\big(\hat{d}(s) - \Phi(\hat{t}(s), \hat{d}(s))\big) - r\big)\|_2 \, ds \\ + \mathcal{I}(\hat{t}(S), \hat{d}(S)) \leq \mathcal{I}(0, \hat{d}(0)) + \int_0^S \partial_t \mathcal{I}(\hat{t}(s), \hat{d}(s))\hat{t}'(s) ds, \tag{4.11}$$

$\hat{d}(0) = d_0$, *and*

$$\begin{aligned}
&\hat{t}(0) = 0,\ \hat{t}(S) = T,\\
&\hat{t}'(s) \geq 0,\ \hat{d}'(s) \geq 0 \quad \textit{a.e. in } (0, S),\\
&\hat{t}'(s) + \|\hat{d}'(s)\|_2 \leq 1 \quad \textit{a.e. in } (0, S).
\end{aligned} \tag{4.12}$$

Proof. Let $\delta_n \searrow 0$ be an arbitrary sequence. From (4.5) we deduce that there exists a (not relabeled subsequence) $\{(\hat{t}_{\delta_n}, \hat{d}_{\delta_n})\} \subset C^{1,1}([0, S]; [0, T] \times L^2(\Omega))$ and a pair $(\hat{t}, \hat{d}) \in W^{1,\infty}(0, S; [0, T] \times L^2(\Omega))$ such that $(\hat{t}_{\delta_n}, \hat{d}_{\delta_n}) \rightharpoonup^* (\hat{t}, \hat{d})$ in $W^{1,\infty}(0, S; \mathbb{R} \times L^2(\Omega))$, which immediately implies (4.10). Here we used that $W^{1,\infty}(0, S; \mathbb{R} \times L^2(\Omega))$ is the dual of a separable Banach space and Alaoglu's theorem. The latter one also yields the last inequality in (4.12), in view of (4.5). From (4.10) we obtain

$$\hat{t}_{\delta_n}(s) \to \hat{t}(s) \quad \text{in } \mathbb{R}, \quad \hat{d}_{\delta_n}(s) \rightharpoonup \hat{d}(s) \quad \text{in } L^2(\Omega) \quad \forall\, s \in [0, S], \tag{4.13}$$

whence $\hat{d}(0) = d_0$ and the identities in (4.12) follow. Note that, in the light of Lemma 2.12, (4.13) implies

$$\Phi(\hat{t}_{\delta_n}(s), \hat{d}_{\delta_n}(s)) \to \Phi(\hat{t}(s), \hat{d}(s)) \quad \text{in } H^1(\Omega), \quad \forall\, s \in [0, S], \tag{4.14}$$

which will be crucial later in the proof. Moreover, we have

$$\hat{t}'(s) \geq 0 \text{ a.e. in } (0, S), \tag{4.15}$$

as a result of (4.3), (4.10), and the weak closedness of the set $\{v \in L^2(0, S; \mathbb{R}) : v(s) \geq 0 \text{ a.e. in } (0, S)\}$ in $L^2(0, S; \mathbb{R})$. Analogously, we conclude from (4.10) that $\hat{d}'(s) \geq 0$ a.e. in $(0, S)$, by (4.4) and the weak closedness of the set $\{v \in L^2(0, S; L^2(\Omega)) : v(s) \geq 0 \text{ a.e. in } (0, S)\}$. Therefore, it only remains to prove (4.11). To this end, we pass to the limit $\delta \searrow 0$ in (4.6).

Thanks to (4.7) combined with (4.8), we have the following crucial identity

$$\frac{\delta}{\hat{t}'_\delta(s)} \|\hat{d}'_\delta(s)\|_2^2 = \frac{\|\xi_\delta(s)\|_2^2}{\delta + \|\xi_\delta(s)\|_2} \quad \forall\, s \in [0, S],\ \forall\, \delta > 0, \tag{4.16}$$

where we abbreviate $\xi_\delta(s) := \max(-\beta\big(\hat{d}_\delta(s) - \Phi(\hat{t}_\delta(s), \hat{d}_\delta(s))\big) - r)$ for every $s \in [0, S]$ and $\delta > 0$. On the other hand, it holds

$$\liminf_{\delta \searrow 0} \frac{\|\xi_\delta(s)\|_2^2}{\delta + \|\xi_\delta(s)\|_2} = \liminf_{\delta \searrow 0} (\|\xi_\delta(s)\|_2 - \delta) + \lim_{\delta \searrow 0} \frac{\delta^2}{\delta + \|\xi_\delta(s)\|_2} = \liminf_{\delta \searrow 0} \|\xi_\delta(s)\|_2 \tag{4.17}$$

for all $s \in [0, S]$. Further, we observe that $L^2(\Omega) \ni w \mapsto \|\max(w)\|_2 \in \mathbb{R}$ is convex and continuous, and thus, weakly lower semicontinuous. Note that the convexity is due to the convexity of $\max : \mathbb{R} \to \mathbb{R}$ and of the norm. Relying on (4.16) and (4.17), we arrive at

$$\begin{aligned}
\liminf_{\delta \searrow 0} \frac{\delta}{\hat{t}'_\delta(s)} \|\hat{d}'_\delta(s)\|_2^2 &= \liminf_{\delta \searrow 0} \|\xi_\delta(s)\|_2\\
&\geq \|\max\Big(-\beta\big(\hat{d}(s) - \Phi(\hat{t}(s), \hat{d}(s))\big) - r\Big)\|_2 \quad \forall\, s \in [0, S],
\end{aligned} \tag{4.18}$$

where the last inequality is due to (4.13) and (4.14). Applying Fatou's lemma in (4.18) together with the weak lower semicontinuity of the norm and (4.10) give in turn

$$\begin{aligned}\int_0^S r\|\hat{d}'(s)\|_1 + \|\max\Big(-\beta\big(\hat{d}(s)-\Phi(\hat{t}(s),\hat{d}(s))\big)-r\Big)\|_2\ ds \\ \leq \liminf_{\delta\searrow 0}\int_0^S r\|\hat{d}_\delta'(s)\|_1 + \frac{\delta}{\hat{t}_\delta'(s)}\|\hat{d}_\delta'(s)\|_2^2\ ds.\end{aligned} \tag{4.19}$$

Next we prove that

$$\mathcal{I}(\hat{t}(s),\hat{d}(s)) \leq \liminf_{\delta\searrow 0}\mathcal{I}(\hat{t}_\delta(s),\hat{d}_\delta(s)) \quad \text{ for all } s\in[0,S]. \tag{4.20}$$

By virtue of (4.13), (4.14), and Lemma 2.10, one has

$$\mathcal{U}(\hat{t}_\delta(s),\Phi(\hat{t}_\delta(s),\hat{d}_\delta(s))) \to \mathcal{U}(\hat{t}(s),\Phi(\hat{t}(s),\hat{d}(s))) \quad \text{in } V, \quad \text{ for all } s\in[0,S]. \tag{4.21}$$

The estimate (4.20) now follows from (2.8), where we employ (4.13), the continuity of ℓ (cf. Assumption 2.5), (4.14), (4.21), and the weak lower semicontinuity of squared norms. From (2.9), the continuity of ℓ', (4.13), and (4.21), we also deduce

$$\partial_t\mathcal{I}(\hat{t}_\delta(s),\hat{d}_\delta(s)) \to \partial_t\mathcal{I}(\hat{t}(s),\hat{d}(s)) \quad \text{ for all } s\in[0,S]. \tag{4.22}$$

Due to (2.3) and Assumption 2.5, we can apply Lebesgue's dominated convergence theorem, which tells us that

$$\partial_t\mathcal{I}(\hat{t}_\delta(\cdot),\hat{d}_\delta(\cdot)) \to \partial_t\mathcal{I}(\hat{t}(\cdot),\hat{d}(\cdot)) \quad \text{ in } L^2(0,S;\mathbb{R}).$$

Moreover, the convergence $\hat{t}_\delta' \rightharpoonup \hat{t}'$ in $L^2(0,S;\mathbb{R})$ is true, by (4.10). Hence,

$$\lim_{\delta\searrow 0}\int_0^S \partial_t\mathcal{I}(\hat{t}_\delta(s),\hat{d}_\delta(s))\hat{t}_\delta'(s)ds = \int_0^S \partial_t\mathcal{I}(\hat{t}(s),\hat{d}(s))\hat{t}'(s)ds. \tag{4.23}$$

Since $\hat{d}_\delta(0)=d_0=\hat{d}(0)$ for any $\delta>0$, by (4.2) and (4.13), we can now finally conclude the estimate (4.11) from (4.19), (4.20) and (4.23), by passing to the limit $\delta\searrow 0$ in (4.6). This completes the proof. □

We end this section with a result that will later enable us to show the existence of *non-degenerate parametrized solutions* for (P) (see Definition 5.2 and Corollary 5.10 below).

PROPOSITION 4.2. *Let* $(\hat{t},\hat{d}) \in C^{0,1}([0,S];[0,T]\times L^2(\Omega))$ *be a vanishing viscosity limit of* (4.6) *and of* (4.7)*, respectively. If Assumption 3.2 is satisfied, then there exists a constant* $c>0$*, dependent only on the given data, such that*

$$\int_{S_1}^{S_2} \hat{t}'(s) + \|\hat{d}'(s)\|_2\, ds \geq c(S_2-S_1)^2 \quad \forall\, 0\leq S_1\leq S_2\leq S.$$

Proof. The proof is based on estimates similar to [25, Eq. (2.82)-(2.83)] combined with Lemma 3.4. Let $\{(\hat{t}_\delta,\hat{d}_\delta)\}$ be an approximating sequence such that (4.10) is true. We integrate (4.5) on (S_1, S_2), and by relying on Lemma 3.4 and the scaling invariance of the L^1-norm of the time derivative (cf. (4.4)), we get

$$\begin{aligned}\hat{t}_\delta(S_2) - \hat{t}_\delta(S_1) &= (S_2 - S_1) - \int_{S_1}^{S_2} \|\hat{d}'_\delta(s)\|_2 \, ds \\ &\geq (S_2 - S_1) - C(\hat{t}_\delta(S_2) - \hat{t}_\delta(S_1)) - C\Big(\int_{S_1}^{S_2} \|\hat{d}'_\delta(s)\|_1 \, ds\Big)^{1/2},\end{aligned} \tag{4.24}$$

where $C > 0$ is independent of δ. As a consequence of (4.10) and the positivity of $\hat{d}'_\delta$ and $\hat{d}'$ (see (4.4) and (4.12)), the following convergence is true

$$\int_{S_1}^{S_2} \|\hat{d}'_\delta(s)\|_1 \, ds \to \int_{S_1}^{S_2} \|\hat{d}'(s)\|_1 \, ds \quad \text{as } \delta \searrow 0. \tag{4.25}$$

Owing to (4.13) and (4.25), passing to the limit $\delta \searrow 0$ in (4.24) yields

$$C_1(\hat{t}(S_2) - \hat{t}(S_1)) \geq (S_2 - S_1) - C_1\Big(\int_{S_1}^{S_2} \|\hat{d}'(s)\|_2 \, ds\Big)^{1/2}, \tag{4.26}$$

where we also used the embedding $L^2(\Omega) \hookrightarrow L^1(\Omega)$. From (4.26) we further deduce

$$\begin{aligned}(S_2 - S_1)^2 &\leq C_2\big((\hat{t}(S_2) - \hat{t}(S_1))^2 + \int_{S_1}^{S_2} \|\hat{d}'(s)\|_2 \, ds\big) \\ &\leq C_3\big(\hat{t}(S_2) - \hat{t}(S_1) + \int_{S_1}^{S_2} \|\hat{d}'(s)\|_2 \, ds\big),\end{aligned} \tag{4.27}$$

with $C_3 = \max\{C_2\,(T + \hat{C}), C_2\}$, where $\hat{C}$ is given by Proposition 3.1. The second estimate in (4.27) is due to the third inequality in (4.12) (which yields the Lipschitz-continuity of $\hat{t}$ with constant 1) combined with $0 \leq S_2 - S_1 \leq S \leq T + \hat{C}$ (see (4.1) and (3.1)). The desired assertion (with $c := 1/C_3$) now follows from (4.27). ◻

5. Parametrized solutions. In this section, our main concern is to find and characterize parameterized solutions for the rate-independent damage model (P). This solution concept proves to be a helpful tool in understanding the limit procedure, as it manages to capture the viscous transition paths at jump points, cf. Remark 5.11 below. We begin with a precise definition thereof. First, we introduce

DEFINITION 5.1. *The functional* $\widetilde{\mathcal{R}} : L^2(\Omega) \to [0, \infty]$ *is defined as*

$$\widetilde{\mathcal{R}}(\eta) := \begin{cases} r \int_\Omega \eta \, dx & \text{if } \eta \geq 0 \text{ a.e. in } \Omega \text{ and } \|\eta\|_2 \leq 1, \\ \infty & \text{otherwise,} \end{cases} \tag{5.1}$$

i.e., $\widetilde{\mathcal{R}}(\eta) = \mathcal{R}(\eta)$ *if* $\|\eta\|_2 \leq 1$ *and* $\widetilde{\mathcal{R}}(\eta) = \infty$ *if* $\|\eta\|_2 > 1$.

DEFINITION 5.2 (Parametrized solutions, [19, Def. 4.11, Eq. (90)]). *A pair* $(\hat{t}, \hat{d}) \in C^{0,1}([0, S]; [0, T] \times L^2(\Omega))$ *is a parametrized solution of* (P) *if it satisfies*

$$\begin{gathered}-\partial_d \mathcal{I}(\hat{t}(s), \hat{d}(s)) \in \partial \widetilde{\mathcal{R}}(\hat{d}'(s)) \quad \text{a.e. in } (0, S), \\ \hat{t}(0) = 0, \ \hat{t}(S) = T, \ \hat{t}'(s) \geq 0, \ \hat{t}'(s) + \|\hat{d}'(s)\|_2 \leq 1 \quad \text{a.e. in } (0, S).\end{gathered} \tag{5.2}$$

A parametrized solution $(\hat{t}, \hat{d})$ is called non-degenerate if $\hat{t}'(s) + \|\hat{d}'(s)\|_2 > 0$ a.e. in $(0, S)$.

Starting from Theorem 4.1 we show that (P) admits parametrized solutions by providing two alternative approaches. The first one involves a series of equivalent formulations of (4.11) (Section 5.1 below), which turn out to be different ways of describing the evolution in (5.2) (Theorem 5.9 below). This approach enables us to get a better understanding of the limiting process $\delta \searrow 0$ for the parametrized viscous sequence $\{(\hat{t}_\delta, \hat{d}_\delta)\}$; in particular, we are able to describe the limit behaviour at each jump point by means of an ODE which is satisfied on the associated jump path. Moreover, this is reminiscent of the viscous ODEs (2.6) and (4.9). See Remarks 5.11 and 5.12 below for more details. In addition, we state the precise value of the viscous contribution to the total dissipation that remains in the vanishing viscosity limit, cf. Proposition 5.7 below.

The second approach is rather straight forward and one may go now directly to the proof of Theorem 5.9 below to see that each parametrized solution satisfies (4.11)-(4.12) and vice versa. Hence, each vanishing viscosity limit is a parametrized solution, in view of Theorem 4.1.

As in e.g. [11, 20, 25], we shall also prove that, by reparametrizing, a *non-degenerate* parametrized solution that fulfils a normalization condition can be obtained, see Corollary 5.10 below.

5.1. Equivalent formulations of the (parametrized) energy inequality.

We begin with the following crucial equivalent formulation of (4.11):

Theorem 5.3. *Let $(\hat{t}, \hat{d}) \in C^{0,1}([0, S]; [0, T] \times L^2(\Omega))$ with $\hat{d}'(s) \geq 0$ and $\|\hat{d}'(s)\|_2 \leq 1$ a.e. in $(0, S)$ be given. Then, $(\hat{t}, \hat{d})$ solves (4.11) if and only if it satisfies the complementarity system*

$$\xi(s) = \|\xi(s)\|_2 \hat{d}'(s) \quad \text{a.e. in } \Omega, \ \text{ a.e. in } (0, S), \tag{5.3a}$$

$$0 \leq \hat{d}'(s) \perp \big(-\beta\big(\hat{d}(s) - \Phi(\hat{t}(s), \hat{d}(s))\big) - r\big) \leq 0 \ \text{ a.e. in } \Omega, \ \text{ a.e. where } \|\xi(s)\|_2 = 0, \tag{5.3b}$$

where we abbreviate $\xi(s) := \max\big(-\beta\big(\hat{d}(s) - \Phi(\hat{t}(s), \hat{d}(s))\big) - r\big)$. Therefore, each vanishing viscosity limit of (4.6), and of (4.7), respectively, fulfills (5.3).

Remark 5.4. *Note that the $\max\{\cdot, 0\}$-function is a well-known complementarity function, so that we may refer to the entire system in (5.3) as complementarity system. We observe that after passing to the limit $\delta \searrow 0$, the structure of (4.9) (see also (4.7) and (2.6)) is preserved in those points where ξ does not vanish. We point out that, in the context of parametrized solutions, this has a certain mechanical interpretation, cf. Remarks 5.11 and 5.12 below. In the points where $\|\xi(s)\|_2 = 0$ holds, we do not have any information from (5.3a). However, the complementarity condition in (5.3b) is satisfied there, which, as we will later see in Remark 5.11, bears a mechanical interpretation too.*

Proof of Theorem 5.3. Let $(\hat{t}, \hat{d}) \in C^{0,1}([0, S]; [0, T] \times L^2(\Omega))$ be a fixed solution of (4.11) with $\hat{d}'(s) \geq 0$ and $\|\hat{d}'(s)\|_2 \leq 1$ a.e. in $(0, S)$. Then, we have the series of inequalities

$$\begin{aligned} r\|\hat d'(s)\|_1 + \|\xi(s)\|_2 &\geq r\|\hat d'(s)\|_1 + \|\xi(s)\|_2\|\hat d'(s)\|_2 \\ &\geq r\|\hat d'(s)\|_1 + (\xi(s), \hat d'(s))_2 \\ &\geq r\|\hat d'(s)\|_1 + (-\beta\big(\hat d(s) - \Phi(\hat t(s), \hat d(s))\big) - r, \hat d'(s))_2 \\ &= (-\beta\big(\hat d(s) - \Phi(\hat t(s), \hat d(s))\big), \hat d'(s))_2 \quad \text{a.e. in } (0,S). \end{aligned} \tag{5.4}$$

Further, $\mathcal{I} \in C^1([0,T]\times L^2(\Omega))$, cf. Lemma 2.16, and $(\hat t, \hat d) \in W^{1,\infty}(0,S;[0,T]\times L^2(\Omega))$, which gives rise to the chain rule

$$\frac{d}{ds}\mathcal{I}(\hat t(s), \hat d(s)) = \partial_t \mathcal{I}(\hat t(s), \hat d(s))\hat t'(s) + (\partial_d \mathcal{I}(\hat t(s), \hat d(s)), \hat d'(s))_2 \quad \text{a.e. in } (0,S). \tag{5.5}$$

Note that the mapping $s \mapsto \mathcal{I}(\hat t(s), \hat d(s))$ belongs to $W^{1,\infty}(0,S)$. Integrating (5.5) over $(0,S)$ and inserting the resulting identity in (4.11) gives in turn

$$\begin{aligned} \int_0^S r\|\hat d'(s)\|_1 + \|\xi(s)\|_2 \; ds &\leq -\int_0^S (\partial_d \mathcal{I}(\hat t(s), \hat d(s)), \hat d'(s))_2 \, ds \\ &= \int_0^S (-\beta\big(\hat d(s) - \Phi(\hat t(s), \hat d(s))\big), \hat d'(s))_2 \, ds, \end{aligned} \tag{5.6}$$

where for the last equality we employed (2.9). Now, combining (5.4) with (5.6) yields

$$r\|\hat d'(s)\|_1 + \|\xi(s)\|_2 = (-\beta\big(\hat d(s) - \Phi(\hat t(s), \hat d(s))\big), \hat d'(s))_2 \quad \text{a.e. in } (0,S). \tag{5.7}$$

This means that all inequalities in (5.4) are in fact equalities, i.e.,

$$\|\xi(s)\|_2 = \|\xi(s)\|_2\|\hat d'(s)\|_2 = (\xi(s), \hat d'(s))_2 = (-\beta\big(\hat d(s) - \Phi(\hat t(s), \hat d(s))\big) - r, \hat d'(s))_2 \tag{5.8}$$

a.e. in $(0,S)$. The first identity in (5.8) is equivalent to

$$\|\hat d'(s)\|_2 = 1 \quad \text{a.e. where } \|\xi(s)\|_2 \neq 0, \tag{5.9}$$

while the second equality in (5.8) states that $\xi(s)$ and $\hat d'(s)$ are linearly dependent a.e. in $(0,S)$. The latter implies that there exists a map $s \mapsto \gamma(s) \in \mathbb{R}^+$ so that

$$\xi(s) = \gamma(s)\hat d'(s) \quad \text{in } L^2(\Omega), \quad \text{a.e. where } \|\hat d'(s)\|_2 \neq 0 \text{ and } \|\xi(s)\|_2 \neq 0. \tag{5.10}$$

Note that the positivity of $\gamma(s)$ is due to $\xi(s) \geq 0$ and $\hat d'(s) \geq 0$ a.e. in $(0,S)$. We also notice that if $\hat d'(s) \equiv 0$, then by the first identity in (5.8), $\xi(s) \equiv 0$ follows. Therefore,

$$\|\xi(s)\|_2 = \gamma(s) \quad \text{a.e. where } \|\xi(s)\|_2 \neq 0, \tag{5.11}$$

in view of (5.9) and (5.10). Now, by inserting (5.11) in (5.10) we get (5.3a), where we keep in mind that the set $\{s \in (0,S) : \|\hat d'(s)\|_2 = 0 \text{ and } \|\xi(s)\|_2 \neq 0\}$ has measure zero, see (5.8). Note that (5.3a) holds automatically a.e. where $\|\xi(s)\|_2 = 0$. To show (5.3b), we employ the third equality in (5.8), which gives in turn

$$(\underbrace{-\beta\big(\hat d(s) - \Phi(\hat t(s), \hat d(s))\big) - r}_{\leq 0}, \underbrace{\hat d'(s)}_{\geq 0})_2 = 0 \quad \text{a.e. where } \|\xi(s)\|_2 = 0. \tag{5.12}$$

In view of the assumed nonnegativity of $\hat{d}'$ and the definition of ξ, this is equivalent to (5.3b).

It remains to prove the opposite implication. To this end, we consider $(\hat{t},\hat{d}) \in C^{0,1}([0,S];[0,T]\times L^2(\Omega))$ with $\hat{d}'(s) \geq 0$ and $\|\hat{d}'(s)\|_2 \leq 1$ a.e. in $(0,S)$ such that (5.3) is satisfied. Our aim is to show that $(\hat{t},\hat{d})$ is a solution of (4.11). From (5.3b) we have

$$(-\beta\big(\hat{d}(s)-\Phi(\hat{t}(s),\hat{d}(s))\big)-r,\hat{d}'(s))_2 = 0 = \|\xi(s)\|_2 \quad \text{a.e. where } \|\xi(s)\|_2 = 0. \tag{5.13}$$

Further, with (5.3a) one deduces

$$\hat{d}'(s) = \frac{\xi(s)}{\|\xi(s)\|_2} \quad \text{a.e. where } \|\xi(s)\|_2 \neq 0. \tag{5.14}$$

Since

$$(\min\big(-\beta\big(\hat{d}(s)-\Phi(\hat{t}(s),\hat{d}(s))\big)-r\big),\xi(s))_2 = 0 \quad \text{a.e. in } (0,S),$$

see the definition of ξ, we obtain by (5.14)

$$(\min\big(-\beta\big(\hat{d}(s)-\Phi(\hat{t}(s),\hat{d}(s))\big)-r\big),\hat{d}'(s))_2 = 0 \quad \text{a.e. where } \|\xi(s)\|_2 \neq 0. \tag{5.15}$$

This means that

$$(-\beta\big(\hat{d}(s)-\Phi(\hat{t}(s),\hat{d}(s))\big)-r,\hat{d}'(s))_2 = (\xi(s),\hat{d}'(s))_2 = \|\xi(s)\|_2 \text{ a.e. where } \|\xi(s)\|_2 \neq 0, \tag{5.16}$$

where the last equality is another consequence of (5.14). Altogether, it follows from (5.13) and (5.16) that (5.7) is true (we recall here that $\hat{d}'(s) \geq 0$ a.e. in $(0,S)$). On the other hand, (2.9) and the chain rule (5.5) yield

$$\frac{d}{ds}\mathcal{I}(\hat{t}(s),\hat{d}(s)) - \partial_t\mathcal{I}(\hat{t}(s),\hat{d}(s))\hat{t}'(s) = (\beta\big(\hat{d}(s)-\Phi(\hat{t}(s),\hat{d}(s))\big),\hat{d}'(s))_2 \quad \text{a.e. in } (0,S). \tag{5.17}$$

By integrating (5.17) over $(0,S)$, where we rely on (5.7), we now deduce (4.11) in equality form. Note that the final assertion is due to Theorem 4.1. □

An inspection of the proof of Theorem 5.3 shows that (5.3) as well as (4.11) are equivalent with the crucial identity (5.7). This will turn out to be very useful later on.

Lemma 5.5. *Let $(\hat{t},\hat{d}) \in C^{0,1}([0,S];[0,T]\times L^2(\Omega))$ be given with $\hat{d}'(s) \geq 0$ and $\|\hat{d}'(s)\|_2 \leq 1$ a.e. in $(0,S)$. Then, (4.11) and (5.3), respectively, is equivalent to*

$$r\|\hat{d}'(s)\|_1 + \|\xi(s)\|_2 = (-\beta\big(\hat{d}(s)-\Phi(\hat{t}(s),\hat{d}(s))\big),\hat{d}'(s))_2 \quad \textit{a.e. in } (0,S). \tag{5.18}$$

From Lemma 5.5 we obtain by arguing as at the end of the proof of Theorem 5.3 the following

Corollary 5.6. *Let $(\hat{t},\hat{d}) \in C^{0,1}([0,S];[0,T]\times L^2(\Omega))$ be given with $\hat{d}'(s) \geq 0$ and $\|\hat{d}'(s)\|_2 \leq 1$ a.e. in $(0,S)$. Then, $(\hat{t},\hat{d})$ is a solution of (4.11) if and only if*

$$\int_{S_1}^{S_2} r\|\hat{d}'(s)\|_1 + \|\max\big(-\beta\big(\hat{d}(s)-\Phi(\hat{t}(s),\hat{d}(s))\big)-r\big)\|_2 \, ds \\ + \mathcal{I}(\hat{t}(S_2),\hat{d}(S_2)) = \mathcal{I}(\hat{t}(S_1),\hat{d}(S_1)) + \int_{S_1}^{S_2} \partial_t\mathcal{I}(\hat{t}(s),\hat{d}(s))\hat{t}'(s)ds \tag{5.19}$$

for all $0 \le S_1 \le S_2 \le S$*. Therefore, each vanishing viscosity limit of* (4.6) *and of* (4.7)*, respectively, satisfies the energy identity* (5.19)*.*

Next we give a characterization of (5.3) via a differential inclusion involving $\partial\mathcal{R}$.

Proposition 5.7. *A pair* $(\hat{t}, \hat{d}) \in C^{0,1}([0,S]; [0,T] \times L^2(\Omega))$ *with* $\hat{d}'(s) \ge 0$ *and* $\|\hat{d}'(s)\|_2 \le 1$ *a.e. in* $(0,S)$ *satisfies the system* (5.3) *if and only if*

$$\begin{aligned} -\partial_d \mathcal{I}(\hat{t}(s), \hat{d}(s)) - \|\xi(s)\|_2 \hat{d}'(s) \in \partial\mathcal{R}(\hat{d}'(s)) \quad &\text{in } L^2(\Omega), \text{ a.e. in } (0,S) \\ \|\xi(s)\|_2 (1 - \|\hat{d}'(s)\|_2) = 0 \quad &\text{a.e. in } (0,S), \end{aligned} \tag{5.20}$$

where we abbreviate again $\xi(s) := \max\big(-\beta\big(\hat{d}(s) - \Phi(\hat{t}(s), \hat{d}(s))\big) - r\big)$*.*

Therefore, each vanishing viscosity limit of (4.6) *and of* (4.7)*, respectively, solves* (5.20)*.*

Proof. Our aim is to prove that (5.20) is equivalent to (5.18). Then, the result will follow by Lemma 5.5. We define the subsets $X_0 = \{s \in [0,S] : \|\xi(s)\|_2 = 0\}$ and $X_+ = \{s \in [0,S] : \|\xi(s)\|_2 \neq 0\}$. A well known convex analysis result then yields

$$\begin{gathered} -\partial_d \mathcal{I}(\hat{t}(s), \hat{d}(s)) \in \partial\mathcal{R}(\hat{d}'(s)) \text{ a.e. in } (0,S) \\ \Longleftrightarrow \\ -(\partial_d \mathcal{I}(\hat{t}(s), \hat{d}(s)), \hat{d}'(s))_2 = \mathcal{R}(\hat{d}'(s)) + \mathcal{R}^*(-\partial_d \mathcal{I}(\hat{t}(s), \hat{d}(s))) \text{ a.e. in } (0,S), \end{gathered} \tag{5.21}$$

where we used that $\mathcal{R}$ is convex and proper, cf. (1.2). In the light of (2.9) and the definition of ξ, we have $-\partial_d \mathcal{I}(\hat{t}(s), \hat{d}(s)) \le r$ a.e. in Ω, for all $s \in X_0$. Since $\mathcal{R}$ is positively homogeneous, its Fenchel conjugate $\mathcal{R}^* : L^2(\Omega) \to [0,\infty]$ satisfies for all $\omega \in L^2(\Omega)$

$$\mathcal{R}^*(\omega) = I_{\partial\mathcal{R}(0)}(\omega) = \begin{cases} 0 & \text{if } \omega \le r \text{ a.e. in } \Omega, \\ \infty & \text{otherwise,} \end{cases} \tag{5.22}$$

where $I_{\partial\mathcal{R}(0)}$ is the indicator functional of the set $\partial\mathcal{R}(0)$. Thus, by (5.22) and (5.21), the following equivalence is true

$$\begin{gathered} -\partial_d \mathcal{I}(\hat{t}(s), \hat{d}(s)) \in \partial\mathcal{R}(\hat{d}'(s)) \quad \text{a.e. in } X_0 \\ \Longleftrightarrow \\ \big(-\beta\big(\hat{d}(s) - \Phi(\hat{t}(s), \hat{d}(s))\big) - r, \hat{d}'(s)\big)_2 = 0 = \|\xi(s)\|_2 \quad \text{a.e. in } X_0, \end{gathered} \tag{5.23}$$

where we employed (2.9), the definition of $\mathcal{R}$ combined with the assumed nonnegativity of $\hat{d}'$, as well as the definition of the set X_0.

Now, for every $s \in X_+$, we define the viscous dissipation (with viscosity parameter depending on s)

$$L^2(\Omega) \ni v \mapsto \mathcal{R}_s(v) := \mathcal{R}(v) + \frac{\|\xi(s)\|_2}{2} \|v\|_2^2 \in [0,\infty]$$

and observe that

$$\partial\mathcal{R}_s(v) = \partial\mathcal{R}(v) + \|\xi(s)\|_2 v \quad \forall\, v \in L^2(\Omega), \quad \forall\, s \in X_+, \tag{5.24a}$$

$$\mathcal{R}_s^*(\omega) = \frac{\|\max(\omega - r)\|_2^2}{2\|\xi(s)\|_2} \quad \forall\, w \in L^2(\Omega), \quad \forall\, s \in X_+. \tag{5.24b}$$

The identity (5.24a) is a result of the sum rule for convex subdifferentials and the Fréchet-differentiability of the mapping $L^2(\Omega) \ni v \mapsto \frac{\|\xi(s)\|_2}{2}\|v\|_2^2$ for every $s \in X_+$. The equality (5.24b) is obtained by the sum rule for conjugate functionals [9, Thm. 3.3.4.1], see also the proof of [17, Lem. 3.22]. Thanks to (5.24a) combined with (2.9), (5.24b), and the definitions of $\mathcal{R}_s$ and ξ, one can write the equivalencies

$$
\begin{gathered}
-\partial_d \mathcal{I}(\hat{t}(s), \hat{d}(s)) - \|\xi(s)\|_2 \hat{d}'(s) \in \partial \mathcal{R}(\hat{d}'(s)) \quad \text{a.e. in } X_+ \\
\Longleftrightarrow \\
-\partial_d \mathcal{I}(\hat{t}(s), \hat{d}(s)) \in \partial \mathcal{R}_s(\hat{d}'(s)) \quad \text{a.e. in } X_+ \\
\Longleftrightarrow \\
-(\partial_d \mathcal{I}(\hat{t}(s), \hat{d}(s)), \hat{d}'(s))_2 = \mathcal{R}_s(\hat{d}'(s)) + \mathcal{R}_s^*(-\partial_d \mathcal{I}(\hat{t}(s), \hat{d}(s))) \quad \text{a.e. in } X_+ \\
\Longleftrightarrow \\
(-\beta\big(\hat{d}(s) - \Phi(\hat{t}(s), \hat{d}(s))\big) - r, \hat{d}'(s))_2 = \frac{\|\xi(s)\|_2}{2}\|\hat{d}'(s)\|_2^2 + \frac{\|\xi(s)\|_2}{2} \quad \text{a.e. in } X_+.
\end{gathered}
\tag{5.25}
$$

This leads to

$$
\begin{gathered}
-\partial_d \mathcal{I}(\hat{t}(s), \hat{d}(s)) - \|\xi(s)\|_2 \hat{d}'(s) \in \partial \mathcal{R}(\hat{d}'(s)), \ \|\hat{d}'(s)\|_2 = 1 \quad \text{a.e. in } X_+ \\
\Longleftrightarrow \\
(-\beta\big(\hat{d}(s) - \Phi(\hat{t}(s), \hat{d}(s))\big) - r, \hat{d}'(s))_2 = \|\xi(s)\|_2 \quad \text{a.e. in } X_+,
\end{gathered}
\tag{5.26}
$$

where for the implication '$\Longleftarrow$' we used the fact that (5.18) implies (5.3) (see Lemma 5.5) and (5.14). In view of Lemma 5.5, the desired assertion follows now from (5.23) and (5.26). Note that each vanishing viscosity limit satisfies (5.20), cf. Theorems 4.1 and 5.3. This concludes the proof. ☐

So far, we saw that the following equivalencies are true

$$
\begin{gathered}
\text{the energy inequality (4.11)} \overset{\text{Thm. 5.3}}{\Longleftrightarrow} \text{the complementarity system (5.3)} \\
\overset{\text{Lem. 5.5}}{\Longleftrightarrow} \text{the 'pointwise' identity (5.18)} \overset{\text{Prop. 5.7}}{\Longleftrightarrow} \text{the differential inclusion (5.20)}
\end{gathered}
\tag{5.27}
$$

for a pair $(\hat{t}, \hat{d}) \in C^{0,1}([0, S]; [0, T] \times L^2(\Omega))$ with $\hat{d}'(s) \geq 0$ and $\|\hat{d}'(s)\|_2 \leq 1$ a.e. in $(0, S)$. In Theorem 5.9 below we show that, together with (4.12), each of the formulas mentioned in (5.27) is just another characterization of a parametrized solution, cf. Definition 5.2.

5.2. Existence and characterizations of parametrized solutions. The next lemma will be employed in the alternative proof of the main theorem below.

LEMMA 5.8. *A pair* $(\hat{t}, \hat{d}) \in C^{0,1}([0, S]; [0, T] \times L^2(\Omega))$ *is a parametrized solution of* (P) *if and only if there exists a mapping* $\lambda : [0, S] \to [0, \infty)$ *so that*

$$
\begin{gathered}
-\partial_d \mathcal{I}(\hat{t}(s), \hat{d}(s)) - \lambda(s)\hat{d}'(s) \in \partial \mathcal{R}(\hat{d}'(s)) \quad \textit{a.e. in } (0, S), \\
\lambda(s)(1 - \|\hat{d}'(s)\|_2) = 0 \quad \textit{a.e. in } (0, S), \\
\hat{t}(0) = 0, \ \hat{t}(S) = T, \ \hat{t}'(s) \geq 0, \ \hat{t}'(s) + \|\hat{d}'(s)\|_2 \leq 1 \quad \textit{a.e. in } (0, S).
\end{gathered}
\tag{5.28}
$$

Proof. Let $(\hat{t}, \hat{d}) \in C^{0,1}([0, S]; [0, T] \times L^2(\Omega))$ be arbitrary, but fixed. In view of the sum rule for convex subdifferentials and Definition 5.1, it holds

$$
\partial \widetilde{\mathcal{R}}(\hat{d}'(s)) = \partial \mathcal{R}(\hat{d}'(s)) + \partial F(\hat{d}'(s)) \quad \text{a.e. in } (0, S), \tag{5.29}
$$

where $F : L^2(\Omega) \to [0,\infty]$ is defined as

$$F(v) := \begin{cases} 0 & \text{if } \|v\|_2 \le 1, \\ \infty & \text{otherwise.} \end{cases}$$

We observe that $\partial F(v) = \{0\}$ if $\|v\|_2 < 1$. For $v \in L^2(\Omega)$ with $\|v\|_2 = 1$ we establish

$$\eta \in \partial F(v) \Leftrightarrow \eta \in L^2(\Omega) \text{ satisfies } (\eta, v)_2 = \|\eta\|_2 \Leftrightarrow \eta \in \{\theta v \,|\, \theta \in [0,\infty)\},$$

where we relied on the definition of the subdifferential and some straight forward computations. Provided that $\|\hat{d}'(s)\|_2 \le 1$ a.e. in $(0,S)$, we now infer from the above

$$-\partial_d \mathcal{I}(\hat{t}(s), \hat{d}(s)) \in \partial\widetilde{\mathcal{R}}(\hat{d}'(s)) \quad \text{a.e. in } (0,S)$$
$$\Longleftrightarrow$$
$$\begin{cases} -\partial_d \mathcal{I}(\hat{t}(s), \hat{d}(s)) \in \partial\mathcal{R}(\hat{d}'(s)) & \text{a.e. where } \|\hat{d}'(s)\|_2 < 1 \\ -\partial_d \mathcal{I}(\hat{t}(s), \hat{d}(s)) - \lambda(s)\hat{d}'(s) \in \partial\mathcal{R}(\hat{d}'(s)) & \text{a.e. where } \|\hat{d}'(s)\|_2 = 1, \end{cases}$$

with $\lambda(s) \in \mathbb{R}^+$ a.e. where $\|\hat{d}'(s)\|_2 = 1$. This yields the desired assertion, cf. Definition 5.2. □

We are now in the position to state the main result of this section.

Theorem 5.9 (Parametrized solutions). *A pair $(\hat{t}, \hat{d}) \in C^{0,1}([0,S]; [0,T] \times L^2(\Omega))$ is a parametrized solution of* (P) *if and only if it satisfies* (4.12) *and one of the following equivalent formulations:*

(i) the energy inequality (4.11),
(ii) the complementarity system (5.3),
(iii) the 'pointwise' identity (5.18),
(iv) the differential inclusion (5.20) .

Therefore, each vanishing viscosity limit of (4.6)*, and of* (4.7)*, respectively, is a parametrized solution of* (P).

Proof. We only show the implication '⇐', since the proof of the reverse statement '⇒' is based on the exact same steps (see (I) below and note that $\partial\widetilde{\mathcal{R}}(\hat{d}'(s)) \neq \emptyset$ a.e. in $(0,S)$ implies $\hat{d}'(s) \in \operatorname{dom}\widetilde{\mathcal{R}}$, and thus, $\hat{d}'(s) \ge 0$ a.e. in $(0,S)$). With (5.27) at hand, there are various ways to show '⇐'. We provide two alternative proofs, which have (i) and (iv), respectively, as starting point.

(I) Let $(\hat{t}, \hat{d}) \in C^{0,1}([0,S]; [0,T] \times L^2(\Omega))$ be arbitrary, but fixed such that (4.11)-(4.12) holds true. Our goal is to prove that $(\hat{t}, \hat{d})$ fulfills (5.2). We first observe that $\widetilde{\mathcal{R}}(\hat{d}'(s)) = r\|\hat{d}'(s)\|_1$ a.e. in $(0,S)$, by Definition 5.1 and in view of the inequalities in (4.12). With Lemma A.1 combined with (2.9), we can thus deduce from (4.11) that $(\hat{t}, \hat{d})$ satisfies

$$\begin{aligned} \int_0^S \widetilde{\mathcal{R}}(\hat{d}'(s)) + \widetilde{\mathcal{R}}^*(-\partial_d \mathcal{I}(\hat{t}(s), \hat{d}(s)))\, ds + \mathcal{I}(\hat{t}(S), \hat{d}(S)) \\ \le \mathcal{I}(\hat{t}(0), \hat{d}(0)) + \int_0^S \partial_t \mathcal{I}(\hat{t}(s), \hat{d}(s))\hat{t}'(s) ds. \end{aligned} \tag{5.30}$$

Inserting (5.5) in (5.30) gives in turn

$$\int_0^S \widetilde{\mathcal{R}}(\hat{d}'(s)) + \widetilde{\mathcal{R}}^*(-\partial_d \mathcal{I}(\hat{t}(s), \hat{d}(s)))\, ds \le \int_0^S (-\partial_d \mathcal{I}(\hat{t}(s), \hat{d}(s)), \hat{d}'(s))_2\, ds, \tag{5.31}$$

which combined with the Fenchel-Young inequality

$$(-\partial_d \mathcal{I}(\hat{t}(s), \hat{d}(s)), \hat{d}'(s)) \le \widetilde{\mathcal{R}}(\hat{d}'(s)) + \widetilde{\mathcal{R}}^*(-\partial_d \mathcal{I}(\hat{t}(s), \hat{d}(s))) \quad \text{a.e. in } (0, S) \tag{5.32}$$

leads to

$$(-\partial_d \mathcal{I}(\hat{t}(s), \hat{d}(s)), \hat{d}'(s)) = \widetilde{\mathcal{R}}(\hat{d}'(s)) + \widetilde{\mathcal{R}}^*(-\partial_d \mathcal{I}(\hat{t}(s), \hat{d}(s))) \quad \text{a.e. in } (0, S). \tag{5.33}$$

Since $\widetilde{\mathcal{R}}$ is convex and proper, we can apply a classical result from convex analysis, which states that (5.33) is equivalent to

$$-\partial_d \mathcal{I}(\hat{t}(s), \hat{d}(s)) \in \partial \widetilde{\mathcal{R}}(\hat{d}'(s)) \quad \text{a.e. in } (0, S).$$

As $(\hat{t}, \hat{d})$ fulfills (4.12), by assumption, we now infer that $(\hat{t}, \hat{d})$ solves (5.2).

(II) Alternatively, one can show the statement '$\Leftarrow$' by employing Lemma 5.8, which implies that each solution $(\hat{t}, \hat{d}) \in C^{0,1}([0, S]; [0, T] \times L^2(\Omega))$ of (5.20)-(4.12) satisfies (5.2).

Finally, the second assertion is just an immediate consequence of Theorem 4.1. □

Similarly to [11, 20, 25], we also conclude that each vanishing viscosity solution to (P) (and in fact, each degenerate parametrized solution, see [20, Rem. 2]) can be transformed into a non-degenerate one by rescaling, as described in the following

Corollary 5.10 (Existence of non-degenerate parametrized solutions). *Suppose that Assumption 3.2 is satisfied. Then, there exists $\bar{S} > 0$ such that (P) admits non-degenerate parametrized solutions $(\bar{t}, \bar{d}) \in C^{0,1}([0, \bar{S}]; [0, T] \times L^2(\Omega))$ with*

$$\bar{t}'(\sigma) + \|\bar{d}'(\sigma)\|_2 = 1 \quad \text{a.e. in } (0, \bar{S}). \tag{5.34}$$

Proof. The assertion follows by rescaling as in [25, Eq. (2.84)], see also [20, Rem. 2]. For convenience of the reader, we give a detailed proof, although the main arguments can be found in [20, Rem. 2] and [25, Rem. 2.19]. Let $(\hat{t}, \hat{d}) \in C^{0,1}([0, S]; [0, T] \times L^2(\Omega))$ be a vanishing viscosity limit of (4.6) and of (4.7), respectively; note that its existence is guaranteed by Theorem 4.1. Throughout the proof we keep in mind that, in view of Theorem 5.9, $(\hat{t}, \hat{d})$ is a parametrized solution of (P), that is, it satisfies (5.2). We consider the reparametrization induced by the mapping

$$\bar{\sigma} : [0, S] \to [0, \bar{S}], \qquad \bar{\sigma}(s) := \int_0^s \hat{t}'(\zeta) + \|\hat{d}'(\zeta)\|_2 \, d\zeta,$$

where $\bar{S} := \int_0^S \hat{t}'(\zeta) + \|\hat{d}'(\zeta)\|_2 \, d\zeta > 0$, by Proposition 4.2. Thanks to the latter, $\bar{\sigma}$ is invertible with (Hölder continuous) inverse $\bar{\sigma}^{-1}$. This allows us to define the functions $\bar{t} : [0, \bar{S}] \to [0, T]$ and $\bar{d} : [0, \bar{S}] \to L^2(\Omega)$ as

$$\bar{t}(\sigma) := \hat{t}(\bar{\sigma}^{-1}(\sigma)), \quad \bar{d}(\sigma) := \hat{d}(\bar{\sigma}^{-1}(\sigma)). \tag{5.35}$$

First, we observe that

$$\begin{aligned} \bar{t}(\sigma_2) - \bar{t}(\sigma_1) + \|\bar{d}(\sigma_2) - \bar{d}(\sigma_1)\|_2 &\le \int_{s_1}^{s_2} \hat{t}'(\zeta) + \|\hat{d}'(\zeta)\|_2 \, d\zeta \\ &= \sigma_2 - \sigma_1 \quad \forall\, 0 \le \sigma_1 \le \sigma_2 \le \bar{S}, \end{aligned} \tag{5.36}$$

where we abbreviate $s_1 = \bar\sigma^{-1}(\sigma_1)$ and $s_2 = \bar\sigma^{-1}(\sigma_2)$. From (5.36) one infers that $(\bar t, \bar d) \in C^{0,1}([0,\bar S];[0,T]\times L^2(\Omega))$ with

$$\bar t'(\sigma) + \|\bar d'(\sigma)\|_2 \le 1 \quad \text{a.e. in } (0,\bar S). \tag{5.37}$$

Note that $\bar t$ is monotone increasing, since $\hat t$ and $\bar\sigma$ do so. Moreover, (5.35) implies

$$\hat t'(s) = \bar t'(\bar\sigma(s))\bar\sigma'(s), \qquad \hat d'(s) = \bar d'(\bar\sigma(s))\bar\sigma'(s) \quad \text{a.e. in } (0,S), \tag{5.38}$$

so that we can write

$$\begin{aligned}\int_0^{\bar S} \bar t'(\sigma) + \|\bar d'(\sigma)\|_2 \, d\sigma &= \int_0^S (\bar t'(\bar\sigma(s)) + \|\bar d'(\bar\sigma(s))\|_2)\bar\sigma'(s)\, ds \\ &= \int_0^S (\hat t'(s) + \|\hat d'(s)\|_2)\, ds = \bar S.\end{aligned} \tag{5.39}$$

Now, as a consequence of (5.37) and (5.39), we arrive at the normalization condition (5.34). Moreover, $\bar t(0) = 0$ and $\bar t(\bar S) = T$. Since the mappings $\bar\sigma$, $\hat t$ and $\hat d$ are monotone increasing, we also have

$$\bar t'(\sigma) \ge 0, \quad \bar d'(\sigma) \ge 0 \quad \text{a.e. in } (0,\bar S). \tag{5.40}$$

It remains to prove that $(\bar t, \bar d)$ satisfies the evolution in (5.2) a.e. in $(0,\bar S)$. To this end, we observe that the inclusion $\partial\widetilde{\mathcal R}(\kappa v) \subset \partial\widetilde{\mathcal R}(v)$ is true for all $\kappa \in [0,1]$ and $v \in \operatorname{dom}\widetilde{\mathcal R}$. This can be deduced by the sum rule for convex subdifferentials (see (5.29)) combined with the positive homogeneity of $\mathcal R$ and the computed subdifferential of F in the proof of Lemma 5.8. Since $\bar\sigma'(s) \in [0,1]$, cf. (5.2), and $\bar d'(\bar\sigma(s)) \in \operatorname{dom}\widetilde{\mathcal R}$, see (5.40) and (5.37), we get

$$-\partial_d \mathcal I(\bar t(\bar\sigma(s)), \bar d(\bar\sigma(s))) \in \partial\widetilde{\mathcal R}(\bar d'(\bar\sigma(s))) \qquad \text{a.e. in } (0,S),$$

where we employed the fact that $(\hat t, \hat d)$ is a parametrized solution of (P) and (5.38). The proof is now complete.

□

Remark 5.11 (Mechanical interpretation). *In e.g. [4, 20, 21], the authors draw conclusions on the evolution of non-degenerate parametrized solutions, by relying on a differential characterization of the type* (5.28). *To interpret the behaviour thereof from a mechanical perspective, we shall additionally make use of the complementarity system* (5.3) *combined with* (4.12). *First, let us recall for convenience the equivalencies established in Section 5.1 for a solution of* (5.2)*, cf. Theorem 5.9. In view of* (5.23) *and* (5.26)*, see also the proof of Lemma 5.5, each parametrized solution* $(\hat t, \hat d)$ *of* (P) *satisfies*

$$\left.\begin{aligned} &-\partial_d \mathcal I(\hat t(s), \hat d(s)) \in \partial\mathcal R(\hat d'(s)) \\ &\qquad \textit{or, equivalently,} \\ &0 \le \hat d'(s) \perp \big(-\beta\big(\hat d(s) - \Phi(\hat t(s), \hat d(s))\big) - r\big) \le 0 \ \ \textit{a.e. in } \Omega \end{aligned}\right\} \textit{a.e. where } \|\xi(s)\|_2 = 0 \tag{5.41}$$

and

$$\left.\begin{aligned} &-\partial_d \mathcal I(\hat t(s), \hat d(s)) - \|\xi(s)\|_2 \hat d'(s) \in \partial\mathcal R(\hat d'(s)), \ \ \|\hat d'(s)\|_2 = 1 \\ &\qquad \textit{or, equivalently,} \\ &\xi(s) = \|\xi(s)\|_2 \hat d'(s) \quad \textit{a.e. in } \Omega \end{aligned}\right\} \textit{a.e. where } \|\xi(s)\|_2 \neq 0 \tag{5.42}$$

where we abbreviate again $\xi(s) := \max\big(-\beta\big(\hat{d}(s) - \Phi(\hat{t}(s), \hat{d}(s))\big) - r\big)$.

Thus, we may comment on the evolution described by a non-degenerate parametrized solution of (P) *as follows:*

- *the regime* ($\hat{t}' > 0, \hat{d}' = 0$) *corresponds to sticking;*
- *the case* ($\hat{t}' > 0, \hat{d}' \neq 0$) *leads to rate-independent evolution. Indeed, if* $\hat{t}'(s) > 0$, *we have* $\|\xi(s)\|_2 = 0$ *(this is a consequence of the last inequality in* (5.2)*, which implies* $\|\hat{d}'(s)\|_2 < 1$*, in combination with the identity in* (5.20) *or* (5.3a)*). Hence, the relations in* (5.41) *are fulfilled, where only the rate-independent dissipation appears;*
- *the situation* ($\hat{t}' = 0, \hat{d}' \neq 0$) *means that the system has switched to a viscous regime. This is seen as a jump in the (slow) external time scale, i.e., the function* $\hat{t}$*, which is frozen. Since* $\hat{t}'(s) = 0$*, we may obtain that* $\|\xi(s)\|_2 \neq 0$ *(note that if* $\hat{t}'(s) = 0$*, then* $\|\hat{d}'(s)\|_2 = 1$ *is allowed in* (5.2) *and the viscous-dissipation may be active in* (5.20)*). In this case, the formulas in* (5.42) *hold true. Thus, at jump points, one can see the influence of the rate-dependent dissipation, where the term* $\|\xi(s)\|_2$ *accounts for the viscous contribution that remains in the limit* $\delta \searrow 0$*. In particular, we read from* (5.42) *that on each interval associated to a jump point,* $\hat{d}$ *solves an equation which is reminiscent of our viscous evolution described in terms of the ODE* (2.6) *and of its equivalent formulation* (4.9)*. We refer to Remark 5.12 below, where we address this issue in more detail.*

Note that the scenario ($\hat{t}' = 0, \hat{d}' = 0$) *is excluded by the assumed non-degeneracy, see also Corollary 5.10.*

Remark 5.12 (Jump paths and associated ODEs).

Let $t \in [0, T]$ *be a jump point with corresponding jump path* $[a_t, b_t] \subset [0, S]$*, i.e.,* $\hat{t}(s) = t$ *for all* $s \in [a_t, b_t]$*. Then, there exists some open subset thereof, which is for convenience denoted by* (a_t, b_t) *and assumed to be non-empty in the following, so that* $\|\xi(s)\|_2 \neq 0$ *on* (a_t, b_t)*. Note that the existence of such a subset is due to the continuity of* ξ*; we also recall that* $\hat{t}'(s) = 0$ *may imply* $\|\xi(s)\|_2 \neq 0$*, see the discussion of the case* ($\hat{t}' = 0, \hat{d}' \neq 0$) *in Remark 5.11. Since* $\hat{t}(s) = t$ *for all* $s \in (a_t, b_t)$*, the second equation in* (5.42) *reads*

$$\hat{d}'(s) = \frac{\xi_t(\hat{d}(s))}{\|\xi_t(\hat{d}(s))\|_2} \quad \text{a.e. in } \Omega \quad \forall\, s \in (a_t, b_t), \tag{5.43}$$

where we abbreviate $\xi_t(d) := \max\big(-\beta\big(d - \Phi(t, d)\big) - r\big)$ *for fixed* $d \in L^2(\Omega)$*. Therefore, to each jump point* t*, we can associate an autonomous ordinary differential equation in Banach space which describes the evolution of* $\hat{d}$ *on (an open subset of) the jump path related to* t*. We remark that the term on the right-hand side in* (5.43) *is continuous w.r.t. the variable* s*, as* $\hat{d}$ *and* ξ_t *are continuous functions (cf. Lemma 2.11). Thus,* $\hat{d}$ *belongs to* $C^1((a_t, b_t); L^2(\Omega))$.

From the above observations we conclude that, on each jump path, the parametrized solutions of (P) *inherit the main characteristics of the viscous model, such as the ODE-structure from* (2.6) *(see also* (4.9)*) and the* C^1*-regularity in time.*

We finish this section with some thoughts regarding future research. Note that, at this point, the parametrized solutions of (P) depend on the fixed penalization parameter β (to emphasize this, we use the subscript β in the following). Thus, it may be adequate to investigate their behaviour when the penalization parameter tends to infinity. As in [18], we expect to obtain a single-field gradient damage model in the limit, where the local (parametrized) damage variable, i.e., $\hat{d}_\beta$, coincides with the nonlocal one, i.e., $\hat{\varphi}_\beta := \Phi(\hat{t}_\beta, \hat{d}_\beta)$ for β approaching ∞. A reasonable starting point for the analysis is the equivalent formulation of the evolution in (5.2) as (5.30). Assuming that uniform bounds with respect to β have been established for the involved variables (namely, $\hat{t}_\beta$, $\hat{d}_\beta$, $\hat{\varphi}_\beta$ and S_β), we can pass to the limit $\beta \to \infty$ in (5.30) by arguing as in [18]. In particular, one introduces a new dissipation functional with domain $H^1(\Omega)$ as well as a new energy functional (independent of β) that is defined on $[0,T] \times H^1(\Omega)$, instead of $[0,T] \times L^2(\Omega)$. The idea behind this is that, since the local and nonlocal damage become equal in the limit, we need to adapt the involved functionals to this new scenario, where only one damage variable with $H^1(\Omega)-$space regularity appears.

We expect that, up to a subsequence, the pairs $\{(\hat{t}_\beta, \hat{d}_\beta)\}$ and $\{(\hat{t}_\beta, \hat{\varphi}_\beta)\}$ converge to the same parametrized trajectory $(\hat{t}, \hat{\varphi}) \in C^{0,1}([0,S];[0,T]) \times (C^{0,1}([0,S]; L^2(\Omega)) \cap H^1(0,S;H^1(\Omega)))$ (with $S = \lim_{\beta\to\infty} S_\beta$) satisfying

$$\begin{aligned} 0 \in \partial\bar{\mathcal{R}}(\hat{\varphi}'(s)) + \partial_\varphi\widetilde{\mathcal{I}}(\hat{t}(s), \hat{\varphi}(s)) \quad \text{in } H^1(\Omega)^*, \\ \hat{t}(0) = 0,\ \hat{t}(S) = T,\ \hat{t}'(s) \geq 0,\ \hat{t}'(s) + \|\hat{\varphi}'(s)\|_2 \leq 1 \quad \text{a.e. in } (0,S). \end{aligned} \tag{5.44}$$

Here, $\bar{\mathcal{R}} : H^1(\Omega) \to [0,\infty]$ is given by

$$\bar{\mathcal{R}}(\eta) := \begin{cases} r\int_\Omega \eta\, dx, & \text{if } \eta \geq 0 \text{ a.e. in } \Omega \text{ and } \|\eta\|_2 \leq 1, \\ \infty, & \text{otherwise,} \end{cases} \tag{5.45}$$

while the energy $\widetilde{\mathcal{I}} : [0,T] \times H^1(\Omega) \to \mathbb{R}$ is defined as in [18, Eq. (4.16)]. This not only establishes the viability of the penalty approach in the non-viscous case, but also shows that the resulting model is in accordance with a class of classical one-field damage models introduced in [6]. In two dimensions, we may transfer (5.44) into (a slightly simplified version of) [11, Eq. (5.25)] by proceeding as in [18, Sec. 7].

In the context of proving (5.44), the main challenge consists in finding uniform bounds with respect to β for the involved variables. While for $\{\hat{t}_\beta\}$ and $\{\hat{d}_\beta\}$ such bounds are already given by (4.12), one still has to show that there exists a constant $c > 0$, independent of β, so that $\|\hat{\varphi}_\beta\|_{H^1(0,S;H^1(\Omega))} \leq c$. This can be proven by employing the differential characterization (5.20) as well as the complementarity system (5.3) and by arguing in a similar way as in the viscous case [18]. However, it remains an open question if the sequence of final time points $\{S_\beta\}$ is uniformly bounded with respect to β. Notice that, from (4.1) and (3.1), we only know that $\{S_\beta\}$ admits bounds independent of the viscosity parameter. Nevertheless, the analysis of the penalization limit of $\{(\hat{t}_\beta, \hat{d}_\beta)\}$ goes beyond the scope of this paper and gives rise to future research.

Appendix A. Fenchel-conjugate of $\widetilde{\mathcal{R}}$.

Lemma A.1. *The conjugate functional of $\widetilde{\mathcal{R}}$, cf. Definition 5.1, satisfies*

$$\widetilde{\mathcal{R}}^*(\omega) = \|\max(\omega - r)\|_2 \quad \forall\, \omega \in L^2(\Omega). \tag{A.1}$$

Proof. Let $\omega \in L^2(\Omega)$ be arbitrary, but fixed. In view of the definition of the Fenchel conjugate and of $\widetilde{\mathcal{R}}$, it holds

$$\widetilde{\mathcal{R}}^*(\omega) = \sup_{v\in L^2(\Omega)} ((\omega, v)_2 - \widetilde{\mathcal{R}}(v)) = \sup_{v\in L^2(\Omega), v\geq 0, \|v\|_2\leq 1} (\omega - r, v)_2. \tag{A.2}$$

Our aim is to show

$$\sup_{v\in L^2(\Omega), v\geq 0, \|v\|_2\leq 1} (\omega - r, v)_2 = \sup_{v\in L^2(\Omega), \|v\|_2\leq 1} (\max(\omega - r), v)_2, \tag{A.3}$$

which in view of (A.2) will give the desired assertion. The inequality $\leq$ in (A.3) follows immediately from $(\omega - r, v)_2 \leq (\max(\omega - r), v)_2$ for all $v \in L^2(\Omega), v \geq 0$. To show $\geq$ in (A.3), we define for each $v \in L^2(\Omega)$ with $\|v\|_2 \leq 1$ the function $y_v \in L^2(\Omega)$ as

$$y_v := \chi_{\{x\in\Omega: v(x)\geq 0, \omega(x)\geq r\}} v.$$

Then, $(\max(\omega - r), v)_2 \leq (\omega - r, y_v)_2$ is true, which implies $\geq$ in (A.3), since $y_v \geq 0$ and $\|y_v\|_2 \leq 1$. This completes the proof. ☐

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