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OPTIMAL CONTROL OF A NON-SMOOTH QUASILINEAR ELLIPTIC EQUATION

Christian Clason* Vu Huu Nhu* Arnd Rösch*

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Abstract This work is concerned with an optimal control problem governed by a non-smooth quasilinear elliptic equation with a nonlinear coefficient in the principal part that is locally Lipschitz continuous and directionally but not Gâteaux differentiable. This leads to a control-to-state operator that is directionally but not Gâteaux differentiable as well. Based on a suitable regularization scheme, we derive C- and strong stationarity conditions. Under the additional assumption that the nonlinearity is a PC^1 function with countably many points of nondifferentiability, we show that both conditions are equivalent. Furthermore, under this assumption we derive a relaxed optimality system that is amenable to numerical solution using a semi-smooth Newton method. This is illustrated by numerical examples.

Key words Optimal control, non-smooth optimization, optimality system, quasilinear elliptic equation.

1 INTRODUCTION

This work is concerned with the non-smooth quasilinear elliptic optimal control problem

$$\begin{cases} \min\limits_{u\in L^p(\Omega),\, y\in H_0^1(\Omega)} J(y,u) \\ \quad\text{s.t.} \quad -\operatorname{div}[a(y)\nabla y + b(\nabla y)] = u \quad \text{in } \Omega \end{cases}$$

with a Fréchet differentiable functional $J : H_0^1(\Omega) \times L^p(\Omega) \to \mathbb{R}$ for a domain $\Omega \subset \mathbb{R}^N$, $p > \frac{N}{2}$, a non-smooth function $a : \mathbb{R} \to \mathbb{R}$, and a continuously differentiable vector-valued function $b : \mathbb{R}^N \to \mathbb{R}^N$; see Section 2 for a precise statement. State equations with a similar structure appear for example in models of heat conduction, where a stands for the heat conductivity and depends on the temperature y (see, e.g., [3, 29]); the vector-valued function b is a generalized advection term. The salient point, of course, is the non-differentiability of the conduction coefficient a that allows for different behavior in different temperature regimes with sharp phase transitions but makes the analytic and numerical treatment challenging.

The study of optimal control problems for non-smooth partial differential equations is relatively recent, and previous works have focused on problems for non-smooth semilinear equations,

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see [21] as well as the pioneering work [25, Chap. 2]. To the best of our knowledge, this is the first work to treat non-smooth quasi-linear problems.

The main difficulty in the treatment of such control problems is the derivation of useful optimality conditions. Following [2, 12, 21, 23, 25], we will use a regularization approach to approximate the original problem by corresponding regularized problems that allows obtaining regularized optimality conditions. By passing to the limit, we then derive so-called C-stationarity conditions involving Clarke's generalized gradient of the non-smooth term. However, unlike non-smooth semilinear equations, the main difficulty in our case is that the nonlinearity appears in the higher-order term, which presents an obstacle to directly passing to the limit in the regularized optimality systems. To circumvent this, we will follow a dual approach, where instead of passing to the limit in the regularized adjoint equation, we shall do so in a linearized state equation and apply a duality argument. We also derive strong stationarity conditions involving a sign condition on the adjoint state and show that under additional assumptions on the non-smooth nonlinearity (piecewise differentiability (PC^1) with countably many points of non-differentiability), both stationarity conditions coincide. Furthermore, in this case a relaxed optimality system can be derived that is amenable to numerical solution using a semi-smooth Newton method.

The paper is organized as follows. This introduction ends with some notations used throughout the paper. Section 2 then gives a precise statement of the optimal control problem together with the fundamental assumptions. The following Section 3 is devoted to the directional differentiability and regularizations of the control-to-state operator. These will be used in Section 4 to derive C- and strong stationarity conditions. Section 5 then considers the special case that a is a countably PC^1. Numerical examples illustrating the solution of the relaxed optimality conditions are presented in Section 6.

Notations. By Id, we denote the identity operator in a Banach space X. For a given point $u \in X$ and $\rho > 0$, we denote by $B_X(u, \rho)$ and $\overline{B}_X(u, \rho)$, the open and closed balls, respectively, of radius ρ centered at u. For $u \in X$ and $\xi \in X^*$, the dual space of X, we denote by $\langle \xi, y \rangle$ their duality product. For Banach spaces X and Y, the notation $X \hookrightarrow Y$ means that X is continuously embedded in Y and $X \Subset Y$ means that X is compact embedded in Y.

If K is a measurable subset in $\mathbb{R}^d$, the notation $|K|$ stands for the d-dimensional Lebesgue measure of K. For a function $f : \Omega \to \mathbb{R}$ defined on a domain $\Omega \subset \mathbb{R}^d$ and $t \in \mathbb{R}$, the symbols $\{f \geq t\}$ and $\{f = t\}$ stand for the sets of a.e. $x \in \Omega$ such that $f(x) \geq t$ and $f(x) = t$, respectively. For any set $A \subset \Omega$, the symbol $\mathbb{1}_A$ denotes the indicator function of A, i.e., $\mathbb{1}_A(x) = 1$ if $x \in A$ and $\mathbb{1}_A(x) = 0$ otherwise.

Finally, C stands for a generic positive constant, which may be different at different places of occurrence. We also write, e.g., $C(\tau)$ for a constant depending only on the parameter τ.

2 PROBLEM STATEMENT

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 2$, with C^1-boundary $\partial\Omega$. We consider for $p > \frac{N}{2}$ the optimal control problem

$$\text{(P)} \qquad \begin{cases} \min\limits_{u \in L^p(\Omega),\, y \in H_0^1(\Omega)} \quad J(y, u) \\ \qquad\qquad \text{s.t.} \quad -\operatorname{div}[a(y)\nabla y + b(\nabla y)] = u \quad \text{in } \Omega, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad y = 0 \quad \text{on } \partial\Omega. \end{cases}$$

For the remainder of this paper, we make the following assumptions.

(A1) The function $a : \mathbb{R} \to \mathbb{R}$ is directionally differentiable, i.e., for any $y, h \in \mathbb{R}$ the limit

$$a'(y; h) := \lim_{t \to 0^+} \frac{a(y + th) - a(y)}{t}$$

exists. Moreover, a satisfies

$$a(y) \geq a_0 > 0 \quad \text{for all } y \in \mathbb{R}$$

for some constant a_0. In addition, for each $M > 0$ there exists a constant $C_M > 0$ such that

$$|a(y_1) - a(y_2)| \leq C_M |y_1 - y_2| \quad \text{for all } y_i \in \mathbb{R}, |y_i| \leq M, i = 1, 2.$$

(A2) The function $b : \mathbb{R}^N \to \mathbb{R}^N$ is monotone, globally Lipschitz continuous, and continuously differentiable. Moreover, if $N \geq 3$, there exist constants $C_b > 0$ and $\sigma \in (0, 1)$ such that

$$|b(\xi)| \leq C_b \left(1 + |\xi|^\sigma\right) \quad \text{for all} \quad \xi \in \mathbb{R}^N.$$

(A3) The cost function $J : H_0^1(\Omega) \times L^p(\Omega) \to \mathbb{R}$ is weakly lower semicontinuous and continuously Fréchet differentiable. Furthermore, for any $M > 0$, there exists a function $g_M : [0, \infty) \to \mathbb{R}$ such that $\lim_{t \to +\infty} g_M(t) = +\infty$ and for any $y \in H_0^1(\Omega) \cap C(\overline{\Omega})$ and $u \in L^p(\Omega)$ with $\|y\|_{H_0^1(\Omega)} + \|y\|_{C(\overline{\Omega})} \leq M \|u\|_{L^p(\Omega)}$, it holds that

$$J(y, u) \geq g_M\left(\|u\|_{L^p(\Omega)}\right).$$

Example 2.1. *Assumption (A1) is satisfied, e.g., for the class of functions defined by*

$$a(t) = a_0 + \sum_{i=1}^{k+1} \mathbb{1}_{(t_{i-1}, t_i]}(t) a_i(t) \quad \textit{for all } t \in \mathbb{R},$$

where $a_0 > 0$, $-\infty =: t_0 < t_1 < t_2 < \cdots < t_k < t_{k+1} := \infty$ *with the convention* $(t_k, t_{k+1}] := (t_k, \infty)$, *and* a_i, $1 \leq i \leq k + 1$, *are nonnegative* C^1*-functions on* $\mathbb{R}$ *satisfying*

$$a_i(t_i) = a_{i+1}(t_i) \quad \textit{for all} \quad 1 \leq i \leq k.$$

Note that such functions are continuous and piecewise continuously differentiable (PC^1) with a finite set of points of non-differentiability, see Section 5; explicit examples from this class are, e.g., $a(t) = 1 + |t|$ (cf. Section 5.2) or $a(t) = \max\{1, t\}$.

An example that is not PC^1 is the following. Let

$$f(t) = \begin{cases} t^2 \sin(t^{-1}) & \text{if } t \neq 0, \\ 0 & \text{if } t = 0, \end{cases}$$

and

$$a(t) = 1 + \max\{f(t), 0\}.$$

Then a is Lipschitz continuous (as the pointwise maximum of Lipschitz continuous functions) and directionally differentiable but not PC^1 since f' is not continuous in $t = 0$. (A more pathological example can be found in [14, Ex. 5.3].)

Example 2.2. *Let f be a C^1 function such that for some constants $C_1, C_2 > 0$ and $\sigma \in (0, 1)$,*

$$|f(t)t| \leq C_1(1 + t^\sigma), \quad 0 \leq f(t) + f'(t)t \leq C_2, \quad \text{for all } t \geq 0.$$

Then the vector-valued function $b : \mathbb{R}^N \to \mathbb{R}^N$ given by $b(\xi) := f(|\xi|)\xi$ satisfies Assumption (A2). For example, we can choose

(i) $b(\xi) = (1 + |\xi|)^{r-2}\, \xi$ *with $r \in (1, 2]$ for $N = 2$ and $r \in (1, 2)$ for $N \geq 3$;*

(ii) $b(\xi) = (1 + |\xi|^2)^{-1/2} \xi$.

3 PROPERTIES OF THE CONTROL-TO-STATE OPERATOR

In this section, we derive the necessary results for the state equation

$$\begin{cases} -\operatorname{div}[a(y)\nabla y + b(\nabla y)] = u & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

as well as its regularization that are required in Section 4.

3.1 EXISTENCE, UNIQUENESS, AND REGULARITY OF SOLUTIONS TO THE STATE EQUATION

We first address existence and uniqueness of solutions to (3.1). Here and in the following, we always consider weak solutions. The following proof is based on the technique from [9, Thm. 2.2] with some modifications.

Theorem 3.1. *Let $p^* > N$ be arbitrary. Assume that Assumptions (A1) and (A2) hold. Then, for any $u \in W^{-1,p^*}(\Omega)$, there exists a unique solution $y_u \in H_0^1(\Omega) \cap C(\overline{\Omega})$ to (3.1) satisfying the a priori estimate*

$$\|y_u\|_{H_0^1(\Omega)} + \|y_u\|_{C(\overline{\Omega})} \leq C_\infty \|u\|_{W^{-1,p^*}(\Omega)} \tag{3.2}$$

for some constant $C_\infty > 0$ depending only on a_0, p^*, N, and $|\Omega|$.

Proof. To show existence, fix $M > 0$ and define the truncated coefficient

$$a_M(y) := \begin{cases} a(M) & \text{if } y > M, \\ a(y) & \text{if } |y| \le M, \\ a(-M) & \text{if } y < -M. \end{cases}$$

Fixing $z \in L^2(\Omega)$, we consider the nonlinear but smooth elliptic equation

$$\begin{cases} -\operatorname{div}[a_M(z)\nabla y + b(\nabla y)] = u & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

Define the mapping $F^z_M : H^1_0(\Omega) \to H^{-1}(\Omega)$ given by

$$\langle F^z_M(y), w \rangle := \int_\Omega [a_M(z(x))\nabla y + b(\nabla y) - b(0)] \cdot \nabla w dx, \quad y, w \in H^1_0(\Omega).$$

Due to the continuity and the monotonicity of b and Assumption (A1), F^z_M is coercive, strongly monotone, and hemicontinuous. Since $p^* > N \ge 2$, we have $W^{-1,p^*}(\Omega) \hookrightarrow H^{-1}(\Omega)$ and so $u \in H^{-1}(\Omega)$. Then, [28, Thm. 26.A] implies that there exists a unique $y^z_M \in H^1_0(\Omega)$ which satisfies equation (3.3). The strong monotonicity of F^z_M thus implies that

$$\|\nabla y^z_M\|_{L^2(\Omega)} \le \frac{1}{a_0}\|u\|_{H^{-1}(\Omega)} \le \frac{1}{a_0} C(p^*, N, \Omega)\|u\|_{W^{-1,p^*}(\Omega)}. \tag{3.4}$$

Due to the mean value theorem, we obtain

$$b(\nabla y^z_M(x)) - b(0) = \int_0^1 J_b(t\nabla y^z_M(x))\nabla y^z_M(x) dt \quad \text{for a.e. } x \in \Omega, \tag{3.5}$$

where J_b stands for the Jacobian matrix of b. Since b is globally Lipschitz continuous, there exists a constant $L_b > 0$ such that $|J_b(\xi)| \le L_b$ for all $\xi \in \mathbb{R}^N$. Setting

$$T^z_M(x) := \int_0^1 J_b(t\nabla y^z_M(x)) dt$$

yields that T^z_M is non-negative definite and $|T^z_M(x)| \le L_b$ for a.a. $x \in \Omega$. Due to (3.3), y^z_M satisfies

$$\begin{cases} -\operatorname{div}[\hat{a}^z_M \nabla y] = u & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\hat{a}^z_M(x) := a_M(z(x)) \operatorname{Id} + T^z_M(x)$. It is easy to see for a.e. $x \in \Omega$ that

$$a_0|\xi|^2 \le \hat{a}^z_M(x)\xi \cdot \xi \le (\max\{a_M(t) : |t| \le M\} + L_b)|\xi|^2 \quad \text{for all} \quad \xi \in \mathbb{R}^N.$$

It follows from the fact $u \in W^{-1,p^*}(\Omega)$ and [1, Thm. 3.10] that

$$u = u_0 + \sum_{i=1}^N \partial_{x_i} u_i \tag{3.6}$$

for some functions $u_i \in L^{p^*}(\Omega)$, $i = 0, 1, \ldots, N$. Moreover, one has

$$\|u\|_{W^{-1,p^*}(\Omega)} = \inf\left\{\sum_{i=0}^{N} \|u_i\|_{L^{p^*}(\Omega)} : u_0, \ldots, u_N \text{ satisfy (3.6)}\right\}.$$

Obviously, $u_0 \in L^{q^*}(\Omega)$ with $q^* := \frac{Np^*}{N+p^*} < p^*$. Since $p^* > N$, the Stampacchia theorem [11, Thm. 12.4] implies that there exists a constant $c_\infty := c_\infty(a_0, p^*, N, |\Omega|)$ such that

$$\begin{aligned}\|y_M^z\|_{L^\infty(\Omega)} &\le c_\infty\left(\|u_0\|_{L^{q^*}(\Omega)} + \sum_{i=1}^{N} \|u_i\|_{L^{p^*}(\Omega)}\right)\\ &\le c_\infty \sum_{i=0}^{N} \|u_i\|_{L^{p^*}(\Omega)}.\end{aligned}$$

As the above estimate holds for all families $\{u_i\}_{0\le i\le N} \subset L^{p^*}(\Omega)$ which satisfy (3.6), there holds

$$\|y_M^z\|_{L^\infty(\Omega)} \le c_\infty \|u\|_{W^{-1,p^*}(\Omega)}. \tag{3.7}$$

Besides, the continuity of y_M^z follows as usual; see, for instance, [15, Thm. 8.29]. Combining this with estimates (3.4) and (3.7) yields

$$\|y_M^z\|_{H_0^1(\Omega)} + \|y_M^z\|_{C(\overline{\Omega})} \le c(a_0, p^*, N, |\Omega|)\|u\|_{W^{-1,p^*}(\Omega)}. \tag{3.8}$$

We now prove that y_M^z is a solution to (3.1) for M large enough. To this end, we define the mapping $F_M : L^2(\Omega) \ni z \mapsto y_M^z \in L^2(\Omega)$. Let us take $z_n \to z$ in $L^2(\Omega)$ and set $y_n := F_M(z_n)$, $y := F_M(z)$. We have

$$\begin{cases} -\operatorname{div}[a_M(z_n)\nabla y_n + b(\nabla y_n)] = u & \text{in } \Omega,\\ y_n = 0 & \text{on } \partial\Omega \end{cases} \tag{3.9}$$

and

$$\begin{cases} -\operatorname{div}[a_M(z)\nabla y + b(\nabla y)] = u & \text{in } \Omega,\\ y = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.10}$$

Subtracting these two equations yields that

$$\begin{cases} -\operatorname{div}\left[a_M(z_n)\left(\nabla y_n - \nabla y\right) + b(\nabla y_n) - b(\nabla y)\right] = \operatorname{div}\left[\left(a_M(z_n) - a_M(z)\right)\nabla y\right] & \text{in } \Omega,\\ y_n - y = 0 & \text{on } \partial\Omega. \end{cases}$$

By multiplying the above equation with $y_n - y$, integration over Ω, and then using the monotonicity of b, we have

$$a_0\|\nabla y_n - \nabla y\|_{L^2(\Omega)} \le \|\left(a_M(z_n) - a_M(z)\right)\nabla y\|_{L^2(\Omega)}. \tag{3.11}$$

By virtue of the Lebesgue dominated convergence theorem, the right hand side of (3.11) tends to zero as $n \to \infty$. Consequently, y_n converges to y in $H_0^1(\Omega)$. It follows that F_M is continuous as a function from $L^2(\Omega)$ to $H_0^1(\Omega)$. On the other hand, due to the compact embedding $H_0^1(\Omega) \Subset L^2(\Omega)$, F_M is a compact operator. As a result of (3.8), the range of F_M is therefore contained in a ball in $L^2(\Omega)$. The Schauder fixed-point theorem guarantees the existence of a function $y_M \in L^2(\Omega)$ satisfying $y_M = F_M(y_M)$.

Choosing now $M \geq c(a_0, p, N, |\Omega|)\|u\|_{W^{-1,p^*}(\Omega)}$, it follows from (3.8) that $\|y_M\|_{C(\overline{\Omega})} \leq M$ and so $a_M(y_M(x)) = a(y_M(x))$ for all $x \in \Omega$. Therefore, y_M solves (3.1).

To show the uniqueness of the solution, assume that y_1 and y_2 are two solutions to (3.1) in $H_0^1(\Omega) \cap C(\overline{\Omega})$. Let us define, for any $\varepsilon > 0$, the open sets

$$K_0 := \{x \in \Omega \mid y_2(x) > y_1(x)\} \quad \text{and} \quad K_\varepsilon := \{x \in \Omega \mid y_2(x) > \varepsilon + y_1(x)\}.$$

We set $z_\varepsilon(x) := \min\{\varepsilon, (y_2(x) - y_1(x))^+\}$, where $t^+ := \max(t, 0)$. We then have $z_\varepsilon \in H_0^1(\Omega)$, $|z_\varepsilon| \leq \varepsilon$, $z_\varepsilon = \varepsilon$ on K_ε, and $\nabla z_\varepsilon = \mathbb{1}_{K_0 \setminus K_\varepsilon} \nabla(y_2 - y_1)$.

Multiplying the equations corresponding to y_i by z_ε, integrating over Ω, and using integration by parts, we have

$$\int_\Omega [a(y_i)\nabla y_i + b(\nabla y_i)] \cdot \nabla z_\varepsilon dx = \langle u, z_\varepsilon \rangle, \quad i = 1, 2.$$

Subtracting these equations yields that

$$\int_\Omega a(y_2)|\nabla z_\varepsilon|^2 + (b(\nabla y_2) - b(\nabla y_1)) \cdot \nabla z_\varepsilon dx = \int_\Omega (a(y_1) - a(y_2)) \nabla y_1 \cdot \nabla z_\varepsilon dx.$$

From this, the monotonicity of b, and Assumption (A1), we obtain

$$\begin{aligned}
a_0\|\nabla z_\varepsilon\|_{L^2(\Omega)}^2 &\leq \int_\Omega a(y_2)|\nabla z_\varepsilon|^2 dx + \int_{K_0 \setminus K_\varepsilon} (b(\nabla y_2) - b(\nabla y_1)) \cdot \nabla z_\varepsilon dx \\
&= \int_\Omega a(y_2)|\nabla z_\varepsilon|^2 + (b(\nabla y_2) - b(\nabla y_1)) \cdot \nabla z_\varepsilon dx \\
&= \int_{K_0 \setminus K_\varepsilon} (a(y_1) - a(y_2)) \nabla y_1 \cdot \nabla z_\varepsilon dx \\
&\leq \|a(y_1) - a(y_2)\|_{L^\infty(K_0 \setminus K_\varepsilon)} \|\nabla y_1\|_{L^2(K_0 \setminus K_\varepsilon)} \|\nabla z_\varepsilon\|_{L^2(K_0 \setminus K_\varepsilon)} \\
&\leq C_M \|y_1 - y_2\|_{L^\infty(K_0 \setminus K_\varepsilon)} \|\nabla y_1\|_{L^2(K_0 \setminus K_\varepsilon)} \|\nabla z_\varepsilon\|_{L^2(\Omega)} \\
&\leq C_M \varepsilon \|\nabla y_1\|_{L^2(K_0 \setminus K_\varepsilon)} \|\nabla z_\varepsilon\|_{L^2(\Omega)}.
\end{aligned}$$

Here $M := \max\{|y_i(x)| \mid x \in \overline{\Omega}, i = 1, 2\}$. Combining this with the Poincaré inequality, we have

$$\|z_\varepsilon\|_{L^2(\Omega)} \leq C\varepsilon \|\nabla y_1\|_{L^2(K_0 \setminus K_\varepsilon)}$$

for some constant C. Since $\lim_{\varepsilon \to 0} |K_0 \setminus K_\varepsilon| = 0$ and $z_\varepsilon = \varepsilon$ on K_ε,

$$|K_\varepsilon| = \varepsilon^{-2} \int_{K_\varepsilon} z_\varepsilon^2 dx \leq C^2 \|\nabla y_1\|_{L^2(K_0 \setminus K_\varepsilon)}^2 \longrightarrow 0$$

as $\varepsilon \longrightarrow 0$. This implies that $|K_0| = \lim_{\varepsilon \to 0} |K_\varepsilon| = 0$. Consequently, $y_1 \geq y_2$ a.e. in Ω.

In the same way, we have $y_2 \geq y_1$ a.e. in Ω and hence $y_1 = y_2$. □

From now on, for each $u \in W^{-1,p^*}(\Omega)$, $p^* > N$, we denote by y_u the unique solution to (3.1). The control-to-state operator $W^{-1,p^*}(\Omega) \ni u \mapsto y_u \in H_0^1(\Omega)$ is denoted by S.

The following theorem on the regularity of solutions to equation (3.1) will be crucial in proving the directional differentiability of the control-to-state operator S.

Theorem 3.2. *Assume that Assumptions (A1) and (A2) are valid. Let U be a bounded set in $W^{-1,p^*}(\Omega)$ with $p^* > N$. Then, there exists a constant $s := s(a_0, \Omega, N, p^*, U) > N$ such that the following assertions hold:*

(i) If $u \in U$, then $y_u \in W_0^{1,s}(\Omega)$ and

$$\|y_u\|_{W_0^{1,s}(\Omega)} \le C_1 \tag{3.12}$$

for some constant $C_1 > 0$ depending only on a_0, Ω, N, p^*, and U.

(ii) If $u_n \to u$ in $W^{-1,p^*}(\Omega)$, then $y_{u_n} \to y_u$ in $W_0^{1,r}(\Omega) \cap C(\overline{\Omega})$ for all $1 \le r < s$.

Proof. Ad (i): Let U be a bounded subset in $W^{-1,p^*}(\Omega)$. Due to the a priori estimate (3.2), there is a constant $M := M(a_0, p^*, N, \Omega, U)$ such that

$$\|y_u\|_{C(\overline{\Omega})} \le M \quad \text{for all} \quad u \in U.$$

Fixing $u \in U$ and using the mean value theorem, we have

$$b(\nabla y_u(x)) - b(0) = \left(\int_0^1 J_b(t\nabla y_u(x))dt \right) \cdot \nabla y_u(x) \quad \text{for a.e. } x \in \Omega,$$

where J_b again denotes the Jacobian matrix of b. Setting $T(x) := \int_0^1 J_b(t\nabla y_u(x))dt$, we see that $T(x)$ is non-negative definite and $|T(x)| \le L_b$ for a.a. $x \in \Omega$ with some constant L_b. Then, y_u satisfies

$$\begin{cases} -\operatorname{div}[\hat{a}(x)\nabla y_u] = u & \text{in } \Omega, \\ y_u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.13}$$

with $\hat{a}(x) := a(y_u(x))\,\mathrm{Id} + T(x)$. As above, we see that

$$a_0|\xi|^2 \le \hat{a}(x)\xi \cdot \xi \le \hat{M}|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^N \text{ and for a.e. } x \in \Omega$$

with $\hat{M} := C_M M + |a(0)| + L_b$. The regularity of solutions to (3.13) (see, e.g. [4, Thm. 2.1]) implies that there exists a constant $\delta := \delta(a_0, \hat{M}, \Omega, N, p) > 0$ such that $y_u \in W_0^{1,\tau}(\Omega)$ for any $2 \le \tau < 2 + \delta$. Moreover, it holds that

$$\|y_u\|_{W_0^{1,\tau}(\Omega)} \le c_\tau \|u\|_{W^{-1,p^*}(\Omega)} \tag{3.14}$$

for all $2 \le \tau < 2 + \delta$ and for some constant c_τ depending only on τ. For $N = 2$, assertion (i) then follows from the Sobolev embedding.

It remains to prove assertion (i) for the case where $N \geq 3$. For this, we rewrite equation (3.1) in the form

$$\begin{cases} -\operatorname{div}[a(y_u(x))\nabla y_u] = u + \hat{u} & \text{in } \Omega, \\ y_u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.15}$$

where $\hat{u} := \operatorname{div}[b(\nabla y_u)]$. We now use the Kirchhoff transformation $K(t) := \int_0^t a(\varsigma)d\varsigma$ (see [27, Chap. V]). By setting $\theta(x) := K(y_u(x))$ for $x \in \overline{\Omega}$, (3.15) can be rewritten as follows

$$\begin{cases} -\Delta\theta = u + \hat{u} & \text{in } \Omega, \\ \theta = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.16}$$

We then have

$$\|\theta\|_{L^1(\Omega)} \leq C(\Omega)\|\theta\|_{L^2(\Omega)} \leq C(\Omega)\|\nabla\theta\|_{L^2(\Omega)} \leq C(\Omega)\|u + \hat{u}\|_{H^{-1}(\Omega)}. \tag{3.17}$$

Fixing $\tau \in (2, 2 + \delta)$, we see from (3.14) that $\nabla y_u \in (L^\tau(\Omega))^N$. From this and Assumption (A2), we can conclude that $b(\nabla y_u) \in (L^{\tau_1}(\Omega))^N$, $\tau_1 := \tau/\sigma$ and

$$|b(\nabla y_u(x))|^{\tau_1} \leq C_b^{\tau_1} 2^{\tau_1 - 1}(1 + |\nabla y_u(x)|^\tau) \quad \text{for a.e. } x \in \Omega.$$

Consequently, we have $\hat{u} \in W^{-1,\tau_1}(\Omega)$ and

$$\begin{aligned} \|\hat{u}\|_{W^{-1,\tau_1}(\Omega)} &\leq C\|b(\nabla y_u)\|_{L^{\tau_1}(\Omega)} \\ &\leq C(\Omega,\tau)\left(1 + \|\nabla y_u\|_{L^\tau(\Omega)}^\tau\right)^{\sigma/\tau} \\ &\leq C(\Omega,\tau)\left(1 + \|u\|_{W^{-1,p^*}(\Omega)}^\tau\right)^{\sigma/\tau} \\ &\leq C(\Omega,\tau)\left(1 + \|u\|_{W^{-1,p^*}(\Omega)}^\sigma\right). \end{aligned} \tag{3.18}$$

Here we have used the estimate (3.14) and the inequality $(r_1 + r_2)^d \leq r_1^d + r_2^d$ for $r_1, r_2 \geq 0$, $0 < d < 1$. Consequently, $u + \hat{u} \in W^{-1,q}(\Omega)$ with $q := \min\{p^*, \tau_1\}$. We now apply [22, Thms. 5.5.4' and 5.5.5'] for problem (3.16) to obtain $\theta \in W_0^{1,q}(\Omega)$ and

$$\|\theta\|_{W_0^{1,q}(\Omega)} \leq C(N,q,\Omega)\left(\|u + \hat{u}\|_{W^{-1,q}(\Omega)} + \|\theta\|_{L^1(\Omega)}\right).$$

Combining this with (3.17) yields

$$\begin{aligned} \|\theta\|_{W_0^{1,q}(\Omega)} &\leq C(N,q,\Omega)\left(\|u + \hat{u}\|_{W^{-1,q}(\Omega)} + \|u + \hat{u}\|_{H^{-1}(\Omega)}\right) \\ &\leq C(N,p^*,\tau,\Omega)\left(\|u\|_{W^{-1,p^*}(\Omega)} + \|\hat{u}\|_{W^{-1,\tau_1}(\Omega)}\right), \end{aligned}$$

which together with (3.18) gives

$$\|\theta\|_{W_0^{1,q}(\Omega)} \leq C(N,p^*,\tau,\Omega)\left(1 + \|u\|_{W^{-1,p^*}(\Omega)} + \|u\|_{W^{-1,p^*}(\Omega)}^\sigma\right).$$

Since $y_u = K^{-1}(\theta)$, we obtain $\nabla y_u = \frac{1}{a(K^{-1}(\theta))}\nabla\theta$. We then have

$$\begin{aligned}\|\nabla y_u\|_{L^q(\Omega)} &\leq \frac{1}{a_0}\|\nabla\theta\|_{L^q(\Omega)} \\ &\leq C(a_0, \Omega, N, p^*, U)\left(1 + \|u\|_{W^{-1,p^*}(\Omega)} + \|u\|^{\sigma}_{W^{-1,p^*}(\Omega)}\right).\end{aligned}$$

We now distinguish the following cases.

Case 1: $q > N$. By setting $s := q$, we obtain (3.12).

Case 2: $q \leq N$. In this case, one has $q = \tau_1 < p^*$. By the same arguments as above, we can deduce that $\nabla y_u \in L^{q_2}(\Omega)$ with $q_2 := \min\{\tau_2, p^*\}$ for $\tau_2 := \frac{\tau_1}{\sigma} = \frac{\tau}{\sigma^2}$ and

$$\|\nabla y_u\|_{L^{q_2}(\Omega)} \leq C(a_0, \Omega, N, p^*, U)\left(1 + \|u\|_{W^{-1,p^*}(\Omega)} + \|u\|^{\sigma}_{W^{-1,p^*}(\Omega)} + \|u\|^{2\sigma}_{W^{-1,p^*}(\Omega)}\right).$$

We now choose the smallest integer $k \geq 2$ such that $\tau_k \geq p^*$ with $\tau_k := \frac{\tau}{\sigma^k}$. Proceeding by induction, we obtain $\nabla y_u \in L^{q_k}(\Omega)$ with $q_k := \min\{\tau_k, p^*\} = p^*$ and

$$\|\nabla y_u\|_{L^{p^*}(\Omega)} \leq C(a_0, \Omega, N, p^*, U)\left(\|u\|_{W^{-1,p^*}(\Omega)} + \sum_{i=0}^{k}\|u\|^{i\sigma}_{W^{-1,p^*}(\Omega)}\right).$$

In this case, we set $s := p^*$ and then obtain estimate (3.12).

Ad (ii): Assume that $u_n \to u$ in $W^{-1,p^*}(\Omega)$. We set $y_n := S(u_n)$. From the convergence of $\{u_n\}$ and the a priori estimate (3.12), we deduce that $\{y_n\}$ is bounded in $W^{1,s}_0(\Omega)$. By passing to a subsequence, we can assume that $y_n \rightharpoonup y$ in $W^{1,s}_0(\Omega)$ and $y_n \to y$ in $C(\overline{\Omega})$ as $n \to \infty$ for some $y \in W^{1,s}_0(\Omega)$. Setting $M := \max\{\|y\|_{C(\overline{\Omega})}, \|y_n\|_{C(\overline{\Omega})}\}$, we obtain from Assumption (A1) that

$$\|a(y_n) - a(y)\|_{C(\overline{\Omega})} \leq C_M\|y_n - y\|_{C(\overline{\Omega})} \to 0 \quad \text{as} \quad n \to \infty$$

for some constant $C_M > 0$. For $n \geq 1$, we have

$$\begin{cases} -\operatorname{div}[(a(y_n) - a_0/2)\nabla y_n] + B(y_n) = u_n & \text{in } \Omega, \\ y_n = 0 & \text{on } \partial\Omega \end{cases}$$

with $B(z) := -\operatorname{div}[a_0/2\nabla z + b(\nabla z)]$. The global Lipschitz continuity of b and the boundedness of $\{y_n\}$ in $H^1_0(\Omega)$ ensure the boundedness of $\{B(y_n)\}$ in $H^{-1}(\Omega)$. There exists a subsequence of $\{B(y_n)\}$, denoted in the same way, such that $B(y_n) \rightharpoonup \psi$ in $H^{-1}(\Omega)$. Letting $n \to \infty$ yields

$$\begin{cases} -\operatorname{div}[(a(y) - a_0/2)\nabla y] + \psi = u & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega. \end{cases}$$

We have

$$
\begin{aligned}
&\liminf_{n\to\infty} \langle B(y_n), y_n\rangle \\
&\qquad = \liminf_{n\to\infty} \langle u_n + \operatorname{div}[(a(y_n) - a_0/2)\nabla y_n], y_n\rangle \\
&\qquad = \liminf_{n\to\infty} \langle u_n, y_n\rangle - \limsup_{n\to\infty} \int_\Omega (a(y_n) - a_0/2)|\nabla y_n|^2 dx \\
&\qquad \le \langle u, y\rangle - \liminf_{n\to\infty} \int_\Omega (a(y_n) - a_0/2)|\nabla y_n|^2 dx \\
&\qquad \le \langle u, y\rangle - \liminf_{n\to\infty} \int_\Omega (a(y) - a_0/2)|\nabla y_n|^2 dx + \lim_{n\to\infty} \|a(y_n) - a(y)\|_{C(\overline{\Omega})} \|\nabla y_n\|^2_{L^2(\Omega)} \\
&\qquad \le \langle u, y\rangle - \int_\Omega (a(y) - a_0/2)|\nabla y|^2 dx \\
&\qquad = \langle \psi, y\rangle .
\end{aligned}
$$

Due to Lemma A.1, B is maximally monotone as an operator from $H^1_0(\Omega)$ to $H^{-1}(\Omega)$. From the strong-to-weak closedness of maximally monotone operators (see, e.g., [27, Lemma 5.1, Chap- XI]), we obtain that $\psi = B(y)$.

Consequently, y_n and y satisfy the equation

$$
\begin{cases}
-\operatorname{div}[a(y_n)(\nabla y_n - \nabla y) + b(\nabla y_n) - b(\nabla y)] = u_n - u + \operatorname{div}[(a(y_n) - a(y))\nabla y] & \text{in } \Omega, \\
\qquad\qquad\qquad\qquad y_n - y = 0 & \text{on } \partial\Omega.
\end{cases}
$$

Multiplying the above equation by $(y_n - y)$, integration over Ω, and using Assumptions (A1) and (A2) gives

$$
a_0\|\nabla(y_n - y)\|^2_{L^2(\Omega)} \le \langle u_n - u, y_n - y\rangle - \int_\Omega (a(y_n) - a(y))\nabla y \cdot (\nabla y_n - \nabla y)dx. \tag{3.19}
$$

Since the embedding $W^{-1,p^*}(\Omega) \hookrightarrow H^{-1}(\Omega)$ is continuous, it follows that $u_n \to u$ in $H^{-1}(\Omega)$. Consequently, the right hand side of (3.19) tends to zero. We then have $y_n \to y$ strongly in $H^1_0(\Omega)$. Therefore, $\nabla y_n \to \nabla y$ in measure. The convergence of ∇y_n to ∇y in $(L^r(\Omega))^N$, $1 \le r < s$, follows from [27, Chap. XI, Prop. 3.10]. This and the uniqueness of solutions to (3.1) yield assertion (ii). □

As a direct consequence of Theorem 3.2, we have

Corollary 3.3. *Let $p^* > N$ be arbitrary. Assume that Assumptions (A1) and (A2) are satisfied. Then, the operator $S : W^{-1,p^*}(\Omega) \to H^1_0(\Omega) \cap C(\overline{\Omega})$ is continuous.*

Since $p > N/2$, we can choose a number $\tilde{p} > N$ such that

$$
\frac{1}{p} < \frac{1}{\tilde{p}} + \frac{1}{N}. \tag{3.20}
$$

This implies that the embedding $L^p(\Omega) \Subset W^{-1,\tilde{p}}(\Omega)$ is compact.

Let $\bar{u} \in L^p(\Omega)$ be arbitrary, but fixed and $\bar{\rho} > 0$ be a constant. From now on, we fix

$$U := B_{L^p(\Omega)}(\bar{u}, 2\bar{\rho}) \quad \text{and} \quad \bar{s} \in (N, s) \tag{3.21}$$

with s as given in Theorem 3.2 corresponding to $p^* := \tilde{p}$ and U. Let us define constants s_1 and s_2 such that

$$s_1 \begin{cases} > 2 & \text{if } N = 2, \\ \in \left(\frac{2\bar{s}}{\bar{s}-2}, \frac{2N}{N-2}\right) & \text{if } N \geq 3, \end{cases} \qquad \text{and} \qquad \frac{1}{\bar{s}} + \frac{1}{s_1} + \frac{1}{s_2} = \frac{1}{2}. \tag{3.22}$$

Note that the embedding $H_0^1(\Omega) \Subset L^{s_1}(\Omega)$ is compact. The following property of S is a direct consequence of Theorem 3.2 and the compact embedding $L^p(\Omega) \Subset W^{-1,\tilde{p}}(\Omega)$.

Corollary 3.4. *Assume that Assumptions (A1) and (A2) are satisfied. Then, the operator $S : U \to W_0^{1,\bar{s}}(\Omega)$ is continuous and completely continuous, i.e., $u_n \rightharpoonup u$ implies that $S(u_n) \to S(u)$.*

3.2 DIRECTIONAL DIFFERENTIABILITY OF THE CONTROL-TO-STATE OPERATOR

In order to derive stationarity conditions for problem (P), we require directional differentiability of the control-to-state operator S. We thus consider for given $y \in H_0^1(\Omega)$ the "linearized" equation

$$\begin{cases} -\operatorname{div}\left[(a(y)\operatorname{Id} + J_b(\nabla y))\nabla z + a'(y; z)\nabla y\right] = v & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.23}$$

We begin with a technical lemma regarding the directional derivatives of the nonsmooth nonlinearity a.

Lemma 3.5. *Let $M > 0$ and $y \in C(\overline{\Omega})$ such that $|y(x)| \leq M$ for all $x \in \overline{\Omega}$. Under Assumption (A1), there hold:*

(i) For all $x \in \overline{\Omega}$ and $h \in \mathbb{R}$,

$$\lim_{\substack{t \to 0^+ \\ h' \to h}} \frac{a(y(x) + th') - a(y(x))}{t} = a'(y(x); h);$$

(ii) For all $x \in \overline{\Omega}$, the mapping $\mathbb{R} \ni \eta \mapsto a'(y(x); \eta) \in \mathbb{R}$ is Lipschitz continuous with Lipschitz constant C_{2M} as given in Assumption (A1).

Proof. Let us fix $x \in \overline{\Omega}$ and define the function $a_M : (-2M, 2M) \to \mathbb{R}$ given by $a_M(t) = a(t)$ for all $t \in (-2M, 2M)$. By virtue of Assumption (A1), a_M is directionally differentiable and Lipschitz continuous with Lipschitz constant C_{2M}. Thanks to [6, Prop. 2.49], for all $\eta \in (-2M, 2M)$, a_M is directionally differentiable at η in the Hadamard sense, i.e.,

$$\lim_{\substack{t \to 0^+ \\ h' \to h}} \frac{a_M(\eta + th') - a(\eta)}{t} = a_M'(\eta; h) \quad \text{for all } h \in \mathbb{R},$$

which together with $y(x) \in (-2M, 2M)$ gives assertion (i). Furthermore, [6, Prop. 2.49] implies that $a_M'(y(x); \cdot)$ is Lipschitz continuous with Lipschitz constant C_{2M} on $\mathbb{R}$. From this and the fact that $a'(y(x); \cdot) = a_M'(y(x); \cdot)$, (ii) is thus derived. □

We now show existence and uniqueness of solutions to (3.23).

Theorem 3.6. *Let Assumptions (A1) and (A2) hold and let $\bar{s}$ be given as in (3.21). Assume that $y \in W_0^{1,\bar{s}}(\Omega)$. Then, for each $v \in H^{-1}(\Omega)$, equation (3.23) admits a unique solution $z \in H_0^1(\Omega)$.*

Proof. Here we first prove the uniqueness and then the existence of solutions. The arguments to show the uniqueness are similar to the ones in the proof of Theorem 3.1.

Step 1: Uniqueness of solutions. Let z_1 and z_2 be two solutions to (3.23) in $H_0^1(\Omega)$. We define the measurable sets

$$K_0 := \{x \in \Omega \mid z_2(x) > z_1(x)\}, \quad K_\varepsilon := \{x \in \Omega \mid z_2(x) > \varepsilon + z_1(x)\}, \quad \varepsilon > 0.$$

We set $z_\varepsilon(x) := \min\{\varepsilon, (z_2(x) - z_1(x))^+\}$. Then, $z_\varepsilon \in H_0^1(\Omega)$, $|z_\varepsilon| \le \varepsilon$, $z_\varepsilon = \varepsilon$ on K_ε, and $\nabla z_\varepsilon = \mathbb{1}_{K_0 \setminus K_\varepsilon} \nabla(z_2 - z_1)$. Multiplying the equations corresponding to z_i by z_ε, integrating over Ω, and using integration by parts yields

$$\int_\Omega [(a(y) + J_b(\nabla y))\nabla z_i + a'(y; z_i)\nabla y] \cdot \nabla z_\varepsilon dx = \langle v, z_\varepsilon \rangle, \quad i = 1, 2.$$

Subtracting these equations gives

$$\int_\Omega a(y)|\nabla z_\varepsilon|^2 + J_b(\nabla y)\nabla z_\varepsilon \cdot \nabla z_\varepsilon dx = \int_\Omega (a'(y; z_1) - a'(y; z_2))\nabla y \cdot \nabla z_\varepsilon dx.$$

Setting $M := \max\{|y(x)| \mid x \in \overline{\Omega}\}$, we see from Lemma 3.5 that for a.e. $x \in \Omega$, the mapping $\eta \mapsto a'(y(x); \eta)$ is Lipschitz continuous with Lipschitz constant C_{2M}. From this, the non-negative definiteness of J_b, and Assumption (A1), we have

$$\begin{aligned} a_0\|\nabla z_\varepsilon\|^2_{L^2(\Omega)} &\le \int_{K_0 \setminus K_\varepsilon} C_{2M}|z_1 - z_2||\nabla y \cdot \nabla z_\varepsilon| dx \\ &\le C_{2M}\varepsilon \|\nabla y\|_{L^2(K_0\setminus K_\varepsilon)} \|\nabla z_\varepsilon\|_{L^2(K_0 \setminus K_\varepsilon)}. \end{aligned}$$

Proceeding as in the proof of Theorem 3.1 yields $z_1 = z_2$.

Step 2: Existence of solutions. Assume that $y \in W_0^{1,\bar{s}}(\Omega)$ and $v \in H^{-1}(\Omega)$. We set $M := \|y\|_{C(\overline{\Omega})}$. We fix $\eta \in L^{s_1}(\Omega)$ with s_1 as given in (3.22) and consider the equation

$$\begin{cases} -\operatorname{div}[(a(y)\operatorname{Id} + J_b(\nabla y))\nabla z] = v + \operatorname{div}[a'(y; \eta)\nabla y] & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.24}$$

Setting $\tilde{a}(x) := a(y(x))\operatorname{Id} + J_b(\nabla y(x))$ and using (A1), the non-negative definiteness of the matrix $J_b(\nabla y(x))$, as well as the global Lipschitz continuity of b, yields

$$a_0|\xi|^2 \le \tilde{a}(x)\xi \cdot \xi \le (C_M M + a(0) + L_b)|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^N \text{and a.e. } x \in \Omega.$$

Furthermore, since the mapping $\eta \mapsto a'(y(x); \eta)$ is Lipschitz continuous with Lipschitz constant C_{2M} for a.e. $x \in \Omega$, we have that $|a'(y; \eta)| \le C_{2M}|\eta|$ almost everywhere in Ω and hence that $a'(y; \eta) \in L^{s_1}(\Omega)$. From this and the choice of s_1 (see (3.22)), it follows that the right hand side of equation (3.24) belongs to $H^{-1}(\Omega)$. Equation (3.24) therefore admits a unique solution $z_\eta \in H_0^1(\Omega)$, which satisfies

$$\begin{aligned}\|\nabla z_\eta\|_{L^2(\Omega)} &\leq \frac{1}{a_0}\left(\|v\|_{H^{-1}(\Omega)} + \|a'(y;\eta)\nabla y\|_{L^2(\Omega)}\right)\\ &\leq \frac{1}{a_0}\left(\|v\|_{H^{-1}(\Omega)} + C_{2M}\||\eta|\nabla y\|_{L^2(\Omega)}\right)\\ &\leq \frac{1}{a_0}\left(\|v\|_{H^{-1}(\Omega)} + C_{2M}|\Omega|^{1/s_2}\|\eta\|_{L^{s_1}(\Omega)}\|\nabla y\|_{L^{\bar{s}}(\Omega)}\right).\end{aligned} \tag{3.25}$$

Here we have just used the Hölder inequality and the second relation in (3.22) to obtain the last estimate. Since the embedding $H_0^1(\Omega) \hookrightarrow L^{s_1}(\Omega)$ is continuous, we can define the operator $T : L^{s_1}(\Omega) \ni \eta \mapsto z_\eta \in L^{s_1}(\Omega)$. Let η_1 and η_2 be arbitrary in $L^{s_1}(\Omega)$ and set $z_i := T(\eta_i)$, $i = 1, 2$. By simple calculation, we obtain

$$\begin{aligned}\|\nabla z_1 - \nabla z_2\|_{L^2(\Omega)} &\leq \frac{1}{a_0}\|\left(a'(y;\eta_1) - a'(y;\eta_2)\right)\nabla y\|_{L^2(\Omega)}\\ &\leq \frac{1}{a_0}C_{2M}\||\eta_1 - \eta_2|\nabla y\|_{L^2(\Omega)}\\ &\leq \frac{1}{a_0}C_{2M}|\Omega|^{1/s_2}\|\eta_1 - \eta_2\|_{L^{s_1}(\Omega)}\|\nabla y\|_{L^{\bar{s}}(\Omega)}.\end{aligned}$$

This implies the continuity of T. Furthermore, as a result of (3.25) and the compact embedding $H_0^1(\Omega) \Subset L^{s_1}(\Omega)$, T is compact. We shall apply the Leray–Schauder principle to show that operator T admits at least one fixed point z, which is then a solution to equation (3.23). To this end, we need prove the set

$$K := \{\eta \in L^{s_1}(\Omega) \mid \exists t \in (0,1) : \eta = T(t\eta)\}$$

is bounded.

We now argue by contradiction. Assume that there exist sequences $\{\eta_k\} \subset K$ and $\{t_k\} \subset (0,1)$ such that $\eta_k = T(t_k\eta_k)$ and $\lim_{k\to\infty}\|\eta_k\|_{L^{s_1}(\Omega)} = \infty$. We have

$$\begin{cases} -\operatorname{div}\left[(a(y)\operatorname{Id} + J_b(\nabla y))\nabla\eta_k\right] = v + \operatorname{div}\left[a'(y;t_k\eta_k)\nabla y\right] & \text{in } \Omega,\\ \eta_k = 0 & \text{on } \partial\Omega.\end{cases} \tag{3.26}$$

Let us set $r_k := \frac{1}{\|\eta_k\|_{L^{s_1}(\Omega)}} \to 0$ and $\hat{\eta}_k := r_k\eta_k$. From this, (3.26), and the positive homogeneity of $a'(y;\cdot)$, we arrive at

$$\begin{cases} -\operatorname{div}\left[(a(y)\operatorname{Id} + J_b(\nabla y))\nabla\hat{\eta}_k\right] = r_k v + \operatorname{div}\left[t_k a'(y;\hat{\eta}_k)\nabla y\right] & \text{in } \Omega,\\ \hat{\eta}_k = 0 & \text{on } \partial\Omega.\end{cases} \tag{3.27}$$

By simple computation, we deduce from $\|\hat{\eta}_k\|_{L^{s_1}(\Omega)} = 1$ that

$$
\begin{aligned}
\|\nabla\hat{\eta}_k\|_{L^2(\Omega)} &\leq \frac{1}{a_0}\left(r_k\|v\|_{H^{-1}(\Omega)} + \|t_k a'(y;\hat{\eta}_k)\nabla y\|_{L^2(\Omega)}\right) \\
&\leq \frac{1}{a_0}\left(r_k\|v\|_{H^{-1}(\Omega)} + C_{2M}|\Omega|^{1/s_2}\|\hat{\eta}_k\|_{L^{s_1}(\Omega)}\|\nabla y\|_{L^{\bar{s}}(\Omega)}\right) \\
&= \frac{1}{a_0}\left(r_k\|v\|_{H^{-1}(\Omega)} + C_{2M}|\Omega|^{1/s_2}\|\nabla y\|_{L^{\bar{s}}(\Omega)}\right) \\
&\leq C
\end{aligned}
\tag{3.28}
$$

for all $k \in \mathbb{N}$ and for some constant C independent of $k \in \mathbb{N}$. From this and the compact embedding $H_0^1(\Omega) \Subset L^{s_1}(\Omega)$, we can assume that $\hat{\eta}_k \rightharpoonup \hat{\eta}$ in $H_0^1(\Omega)$ and $\hat{\eta}_k \to \hat{\eta}$ in $L^{s_1}(\Omega)$ for some $\hat{\eta} \in H_0^1(\Omega)$. The Lipschitz continuity of $a'(y(x);\cdot)$, for a.e. $x \in \Omega$, implies that $a'(y;\hat{\eta}_k) \to a'(y;\hat{\eta})$ in $L^{s_1}(\Omega)$. Moreover, we can assume that $t_k \to t_0 \in [0,1]$. Letting $k \to \infty$ in equation (3.27), we see from the above arguments that

$$
\begin{cases}
-\operatorname{div}\left[(a(y)\operatorname{Id} + J_b(\nabla y))\nabla\hat{\eta}\right] = \operatorname{div}\left[t_0 a'(y;\hat{\eta})\nabla y\right] & \text{in } \Omega, \\
\hat{\eta} = 0 & \text{on } \partial\Omega.
\end{cases}
\tag{3.29}
$$

The uniqueness of solutions to (3.29) gives $\hat{\eta} = 0$, which is in contradiction to $\|\hat{\eta}\|_{L^{s_1}(\Omega)} = \lim_{k\to\infty}\|\hat{\eta}_k\|_{L^{s_1}(\Omega)} = 1$. □

We next show boundedness and continuity properties of (3.23).

Theorem 3.7. *Let Assumptions (A1) and (A2) hold and let $\bar{s}$ be as given in (3.21).*

(i) If $\{y_n\}$ is bounded in $W_0^{1,\bar{s}}(\Omega)$ such that $\nabla y_n \to \nabla y$ in $L^2(\Omega)$ for some $y \in W_0^{1,\bar{s}}(\Omega)$, and if $\{v_n\}$ is bounded $H^{-1}(\Omega)$, then there exists a constant C_2 depending only on a_0, Ω, N, $\bar{s}$, $\|y\|_{W_0^{1,\bar{s}}(\Omega)}$, $\sup\{\|y_n\|_{W_0^{1,\bar{s}}(\Omega)}\}$, and $\sup\{\|v_n\|_{H^{-1}(\Omega)}\}$ such that

$$\|z(y_n, v_n)\|_{H_0^1(\Omega)} \leq C_2 \tag{3.30}$$

for all solutions $z(y_n, v_n)$ of (3.23) corresponding to y_n and v_n.

(ii) If $v_n \to v$ in $H^{-1}(\Omega)$, then $z(y, v_n) \to z(y, v)$ in $H_0^1(\Omega)$.

Proof. *Ad (i):* Assume that $\{y_n\}$ is bounded in $W_0^{1,\bar{s}}(\Omega)$ such that $\nabla y_n \to \nabla y$ in $L^2(\Omega)$ for some $y \in W_0^{1,\bar{s}}(\Omega)$, and $\{v_n\}$ is bounded $H^{-1}(\Omega)$. Then there exists a constant $M > 0$ such that

$$\|y\|_{C(\overline{\Omega})}, \|y_n\|_{C(\overline{\Omega})}, \|y\|_{W_0^{1,\bar{s}}(\Omega)}, \|y_n\|_{W_0^{1,\bar{s}}(\Omega)} \leq M \tag{3.31}$$

for all $n \geq 1$. Let $z_n := z(y_n, v_n)$ be the solution to (3.23) corresponding to y_n and v_n. We shall prove the boundedness of $\{z_n\}$ in $H_0^1(\Omega)$ by contradiction. Suppose that there exists a subsequence, again denoted by $\{z_n\}$, such that

$$\lim_{n\to\infty}\|z_n\|_{H_0^1(\Omega)} = \infty. \tag{3.32}$$

Since z_n satisfies equation (3.23) corresponding to $y := y_n$ and $v := v_n$, one has

$$\int_\Omega [a(y_n) + J_b(\nabla y_n)] \nabla z_n \cdot \nabla z_n dx = \langle v_n, z_n \rangle - \int_\Omega a'(y_n; z_n) \nabla y_n \cdot \nabla z_n dx.$$

Combining this with Assumption (A1), the non-negative definiteness of the matrix $J_b(\nabla y_n(x))$, and the Lipschitz continuity of $a'(y_n(x); \cdot)$, the Hölder inequality and the second relation in (3.22) lead to

$$\begin{aligned} \|\nabla z_n\|_{L^2(\Omega)} &\le \frac{1}{a_0} \left(\|v_n\|_{H^{-1}(\Omega)} + \|a'(y_n; z_n)\nabla y_n\|_{L^2(\Omega)} \right) \\ &\le \frac{1}{a_0} \left(\|v_n\|_{H^{-1}(\Omega)} + C_{2M} |\Omega|^{1/s_2} \|z_n\|_{L^{s_1}(\Omega)} \|\nabla y_n\|_{L^{\bar{s}}(\Omega)} \right). \end{aligned} \tag{3.33}$$

From this and (3.32), we obtain $\lim_{n\to\infty} \|z_n\|_{L^{s_1}(\Omega)} = \infty$. Setting $t_n := \frac{1}{\|z_n\|_{L^{s_1}(\Omega)}}$ and $\hat{z}_n := t_n z_n$ yields that

$$t_n \to 0 \quad \text{and} \quad \lim_{n\to\infty} \|\hat{z}_n\|_{L^{s_1}(\Omega)} = 1. \tag{3.34}$$

On the other hand, $\hat{z}_n$ satisfies

$$\begin{cases} -\operatorname{div}\left[(a(y_n)\operatorname{Id} + J_b(\nabla y_n)) \nabla \hat{z}_n \right] = t_n v_n + \operatorname{div}\left[a'(y_n; \hat{z}_n) \nabla y_n \right] & \text{in } \Omega, \\ \hat{z}_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.35}$$

The same argument as in (3.33) gives

$$\|\nabla \hat{z}_n\|_{L^2(\Omega)} \le \frac{1}{a_0} \left(t_n \|v_n\|_{H^{-1}(\Omega)} + C_{2M} |\Omega|^{1/s_2} \|\hat{z}_n\|_{L^{s_1}(\Omega)} \|\nabla y_n\|_{L^{\bar{s}}(\Omega)} \right) \le C$$

for all $n \in \mathbb{N}$ and for some constant C independent of $n \in \mathbb{N}$. Consequently, $\{\hat{z}_n\}$ is bounded in $H_0^1(\Omega)$. We can thus extract a subsequence, denoted in the same way, such that $\hat{z}_n \rightharpoonup \hat{z}$ in $H_0^1(\Omega)$ and $\hat{z}_n \to \hat{z}$ in $L^{s_1}(\Omega)$ for some $\hat{z} \in H_0^1(\Omega)$. We write $a'(y_n; \hat{z}_n) = c_n \hat{z}_n$, where

$$c_n(x) := \begin{cases} \frac{a'(y_n(x); \hat{z}_n(x))}{\hat{z}_n(x)} & \text{if } \hat{z}_n(x) \neq 0, \\ 0 & \text{otherwise} \end{cases} \tag{3.36}$$

for a.e. $x \in \Omega$. We have $|c_n(x)| \le C_{2M}$ because of the Lipschitz continuity of $a'(y_n(x); \cdot)$ for a.e. $x \in \Omega$. Again, by using a subsequence, we can assume that $c_n \hat{z}_n \rightharpoonup c\hat{z}$ in $L^{s_0}(\Omega)$ for any $s_0 \in [1, s_1)$, particularly for s_0 with $s_0^{-1} + \bar{s}^{-1} = 2^{-1}$. Since $\{y_n\}$ is bounded in $W_0^{1,\bar{s}}(\Omega)$, we can assume that $y_n \to y$ in $C(\overline{\Omega})$, as a result of the compact embedding $W^{1,\bar{s}}(\Omega) \Subset C(\overline{\Omega})$. Consequently, $a(y_n) \to a(y)$ in $C(\overline{\Omega})$. Moreover, the Lebesgue dominated convergence theorem together with the fact that $\nabla y_n \to \nabla y$ in measure implies that $J_b(\nabla y_n) \to J_b(\nabla y)$ in $L^m(\Omega)$ for all $m \ge 1$. Letting $n \to \infty$ in equation (3.35), we arrive at

$$\begin{cases} -\operatorname{div}\left[(a(y)\operatorname{Id} + J_b(\nabla y)) \nabla \hat{z} \right] = \operatorname{div}\left[c\hat{z} \nabla y \right] & \text{in } \Omega, \\ \hat{z} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.37}$$

As in the proof of Theorem 3.6, we can show that (3.37) has at most one solution and hence that $\hat{z} = 0$. However, by virtue of the second limit in (3.34), we have $\|\hat{z}\|_{L^{s_1}(\Omega)} = 1$, which yields a contradiction.

We have thus shown that sequence $\{z_n\}$ is bounded in $H^1_0(\Omega)$. From estimate (3.33) and the choice of s_1, s_2, the upper bound of $\{\|z_n\|_{H^1_0(\Omega)}\}$ depends only on M, $\sup\{\|v_n\|_{H^{-1}(\Omega)}\}$, a_0, Ω, and $\bar{s}, s_1, s_2$ and so on N. This shows the a priori estimate (3.30).

Ad (ii): Assume now that $v_n \to v$ in $H^{-1}(\Omega)$. Setting $\tilde{z}_n := z(y, v_n)$, we see from (3.30) and the compact embedding $H^1_0(\Omega) \Subset L^{s_1}(\Omega)$ that

$$\tilde{z}_{n_k} \rightharpoonup \tilde{z} \text{ in } H^1_0(\Omega) \quad \text{and} \quad \tilde{z}_{n_k} \to \tilde{z} \text{ in } L^{s_1}(\Omega) \tag{3.38}$$

for some subsequence $\{n_k\} \subset \mathbb{N}$ and some function $\tilde{z} \in H^1_0(\Omega)$. By letting $k \to \infty$, the uniqueness of solutions to (3.23) guarantees $\tilde{z} = z(y, v)$. On the other hand, $\tilde{z}_{n_k}$ and $\tilde{z}$ satisfy the equation

$$\begin{cases} -\operatorname{div}\left[(a(y)\operatorname{Id} + J_b(\nabla y))\nabla(\tilde{z}_{n_k} - \tilde{z}) + \left(a'(y;\tilde{z}_{n_k}) - a'(y;\tilde{z})\right)\nabla y\right] = v_{n_k} - v & \text{in } \Omega, \\ \tilde{z}_{n_k} - \tilde{z} = 0 & \text{on } \partial\Omega. \end{cases}$$

The same arguments as above yield that

$$\begin{aligned} \|\nabla(\tilde{z}_{n_k} - \tilde{z})\|_{L^2(\Omega)} &\le \frac{1}{a_0}\left(\|v_{n_k} - v\|_{H^{-1}(\Omega)} + \|(a'(y;\tilde{z}_{n_k}) - a'(y;\tilde{z}))\nabla y\|_{L^2(\Omega)}\right) \\ &\le \frac{1}{a_0}\left(\|v_{n_k} - v\|_{H^{-1}(\Omega)} + C_{2M}|\Omega|^{1/s_2}\|\tilde{z}_{n_k} - \tilde{z}\|_{L^{s_1}(\Omega)}\|\nabla y\|_{L^{\bar{s}}(\Omega)}\right), \end{aligned}$$

which together with the second limit in (3.38) gives $\tilde{z}_{n_k} \to \tilde{z}$ in $H^1_0(\Omega)$ as $k \to \infty$. Recall that s_1 and s_2 are defined by (3.22). From this and the uniqueness of solutions to (3.23), we obtain $\tilde{z}_n \to \tilde{z}$ in $H^1_0(\Omega)$ as $n \to \infty$.

□

As a result of the compact embedding $L^p(\Omega) \Subset W^{-1,\tilde{p}}(\Omega)$ with $\tilde{p}$ as given in (3.20), Theorems 3.2 and 3.7, we have the following corollary.

Corollary 3.8. *Let Assumptions (A1) and (A2) hold true. Assume that U is the open ball given by (3.21) and that V is a bounded set in $H^{-1}(\Omega)$. For each $u \in U$ and $v \in V$, let $z_{u,v}$ stand for the solution to (3.23) corresponding to $y := y_u$ and v. Then, there exists a constant $C_3 := C_3(a_0, \Omega, N, p, U, V)$ such that*

$$\|z_{u,v}\|_{H^1_0(\Omega)} \le C_3 \quad \textit{for all} \quad u \in U, v \in V. \tag{3.39}$$

Theorem 3.9. *Assume that Assumptions (A1) and (A2) are valid and that U is the open ball in $L^p(\Omega)$ defined as in (3.21). Then $S : U \to H^1_0(\Omega)$ is directional differentiable. Moreover, for any $u \in U$ and $v \in L^p(\Omega)$, $z := S'(u; v)$ is the unique solution in $H^1_0(\Omega)$ of the equation*

$$\begin{cases} -\operatorname{div}\left[(a(y_u)\operatorname{Id} + J_b(\nabla y_u))\nabla z + a'(y_u; z)\nabla y_u\right] = v & \textit{in } \Omega, \\ z = 0 & \textit{on } \partial\Omega. \end{cases}$$

Proof. For any $u \in U$ and $v \in L^p(\Omega)$, we set $y := S(u)$, $y_\rho := S(u+\rho v)$, and $z_\rho := \frac{y_\rho - y}{\rho}$ for $\rho > 0$. A simple computation shows that

$$(3.40)\qquad \begin{cases} -\operatorname{div}\left[\dfrac{a(y+\rho z_\rho)-a(y)}{\rho}\nabla y_\rho + a(y)\nabla z_\rho + \dfrac{b(\nabla y+\rho\nabla z_\rho)-b(\nabla y)}{\rho}\right] = v & \text{in } \Omega, \\ z_\rho = 0 & \text{on } \partial\Omega. \end{cases}$$

Multiplying the above equation by z_ρ, integrating over Ω, and using integration by parts, we see from Assumption (A1) and the monotonicity of b that

$$a_0\|\nabla z_\rho\|_{L^2(\Omega)} \leq \|v\|_{H^{-1}(\Omega)} + \left\|\frac{a(y+\rho z_\rho)-a(y)}{\rho}\nabla y_\rho\right\|_{L^2(\Omega)}.$$

This together with Assumption (A1) gives

$$a_0\|\nabla z_\rho\|_{L^2(\Omega)} \leq \|v\|_{H^{-1}(\Omega)} + C_M\left\||z_\rho|\nabla y_\rho\right\|_{L^2(\Omega)}$$

with $M := \sup\{\|y\|_{C(\overline{\Omega})}, \|y_\rho\|_{C(\overline{\Omega})} : \rho \in (0,\hat\rho)\}$ for $\hat\rho$ small enough. Hölder's inequality and the embedding $L^p(\Omega) \hookrightarrow H^{-1}(\Omega)$ yield that

$$(3.41)\qquad a_0\|\nabla z_\rho\|_{L^2(\Omega)} \leq C(\Omega,p)\|v\|_{L^p(\Omega)} + C_M|\Omega|^{1/s_2}\left\|z_\rho\right\|_{L^{s_1}(\Omega)}\left\|\nabla y_\rho\right\|_{L^{\bar s}(\Omega)}$$

with s_1, s_2 defined as in (3.22) and some constant $C(\Omega,p)$.

We now show the boundedness of $\{z_\rho\}$ in $H^1_0(\Omega)$ by an indirect proof that is based on arguments similar to the ones in the proof of estimate (3.30). Assume that $\{z_\rho\}$ is not bounded in $H^1_0(\Omega)$. A subsequence $\{\rho_k\}$ then exists such that $\rho_k \to 0^+$ and $\|\nabla z_{\rho_k}\|_{L^2(\Omega)} \to \infty$. By virtue of (3.41), it thus holds that $\|z_{\rho_k}\|_{L^{s_1}(\Omega)} \to \infty$.

Again, setting $t_k := \frac{1}{\|z_{\rho_k}\|_{L^{s_1}(\Omega)}}$, $\sigma_k := \frac{\rho_k}{t_k}$, and $\hat z_k := t_k z_{\rho_k}$ yields that

$$t_k \to 0, \quad y_{\rho_k} = y + \sigma_k \hat z_k, \quad \text{and} \quad \|\hat z_k\|_{L^{s_1}(\Omega)} = 1.$$

Due to Theorem 3.2, $y_{\rho_k} \to y$ in $C(\overline{\Omega})$ and so in $L^{s_1}(\Omega)$. It therefore holds that $\sigma_k \to 0^+$. On the other hand, $\hat z_k$ satisfies

$$(3.42)\qquad \begin{cases} -\operatorname{div}\left[\dfrac{a(y+\sigma_k \hat z_k)-a(y)}{\sigma_k}\nabla y_{\rho_k} + a(y)\nabla \hat z_k + \dfrac{b(\nabla y+\sigma_k\nabla \hat z_k)-b(\nabla y)}{\sigma_k}\right] = t_k v & \text{in } \Omega \\ \hat z_k = 0 & \text{on } \partial\Omega. \end{cases}$$

From this, we obtain the boundedness of $\{\hat z_k\}$ in $H^1_0(\Omega)$. We can then extract a subsequence, again denoted by $\{\hat z_k\}$, such that $\hat z_k \rightharpoonup \hat z$ in $H^1_0(\Omega)$, $\hat z_k \to \hat z$ in $L^{s_1}(\Omega)$, $\hat z_k(x) \to \hat z(x)$, and $|\hat z_k(x)| \leq g(x)$ for all $k \in \mathbb{N}$, for a.e. $x \in \Omega$, and for some $g \in L^{s_1}(\Omega)$. Since $a : \mathbb{R} \to \mathbb{R}$ satisfies Assumption (A1), we have as a result of Lemma 3.5 that

$$\frac{a(y(x)+\sigma_k \hat z_k(x)) - a(y(x))}{\sigma_k} \to a'(y(x);\hat z(x)) \quad \text{for a.e. } x \in \Omega.$$

Furthermore, Assumption (A1) also gives

$$\left|\frac{a(y(x)+\sigma_k \hat{z}_k(x)-a(y(x))}{\sigma_k}\right| \leq C_M|\hat{z}_k(x)| \leq C_M g(x) \quad \text{for a.e. } x \in \Omega.$$

The Lebesgue dominated convergence theorem thus implies that

$$\frac{a(y+\sigma_k \hat{z}_k)-a(y)}{\sigma_k} \to a'(y;\hat{z}) \quad \text{in } L^{s_1}(\Omega). \tag{3.43}$$

Rewriting

$$\begin{aligned}\frac{b(\nabla y+\sigma_k \nabla \hat{z}_k)-b(\nabla y)}{\sigma_k} &= \frac{b(\nabla y_{\rho_k})-b(\nabla y)}{\sigma_k} \\ &= J_b(\nabla y+\theta_k(\nabla y_{\rho_k}-\nabla y))\nabla \hat{z}_k, \quad (\theta_k := \theta_k(x) \in (0,1)),\end{aligned}$$

using the fact that $y_{\rho_k} \to y$ in $W_0^{1,\bar{s}}(\Omega)$ and the boundedness of J_b, we now apply the Lebesgue dominated convergence theorem for a subsequence, which is denoted in the same way, to obtain

$$\frac{b(\nabla y+\sigma_k \nabla \hat{z}_k)-b(\nabla y)}{\sigma_k} \rightharpoonup J_b(\nabla y)\nabla \hat{z} \quad \text{in } L^{s_3}(\Omega) \tag{3.44}$$

for any $s_3 \in [1,2)$. Letting $k \to \infty$ in equation (3.42) and using limits (3.43) and (3.44) yields that

$$\begin{cases} -\operatorname{div}\left[(a(y)\operatorname{Id}+J_b(\nabla y))\nabla \hat{z}+a'(y;\hat{z})\nabla y\right]=0 & \text{in } \Omega, \\ \hat{z}=0 & \text{on } \partial\Omega. \end{cases}$$

The uniqueness of solutions implies that $\hat{z}=0$, a contradiction of the fact that $\|\hat{z}\|_{L^{s_1}(\Omega)} = \lim_{k\to\infty}\|\hat{z}_k\|_{L^{s_1}(\Omega)}=1$.

Having proved the boundedness of $\{z_\rho\}$ in $H_0^1(\Omega)$, we can assume that $z_\rho \rightharpoonup z$ in $H_0^1(\Omega)$ and $z_\rho \to z$ in $L^{s_1}(\Omega)$. From this and standard arguments as above, we obtain the desired conclusion. □

3.3 REGULARIZATION OF THE CONTROL-TO-STATE OPERATOR

To derive C-stationarity conditions, we apply the adapted penalization method of Barbu [2]. We consider a regularization of the state equation via a classical mollification of the non-smooth nonlinearity. Let ψ be a non-negative function in $C_0^\infty(\mathbb{R})$ such that $\operatorname{supp}(\psi) \subset [-1,1]$, $\int_{\mathbb{R}} \psi(\tau)d\tau = 1$ and define the family $\{a_\varepsilon\}_{\varepsilon>0}$ of functions

$$a_\varepsilon := \frac{1}{\varepsilon} a * \left(\psi \circ (\varepsilon^{-1}\operatorname{Id})\right), \tag{3.45}$$

where $f * g$ stands for the convolution of f and g. Then, $a_\varepsilon \in C^\infty(\mathbb{R})$ by a standard result. In addition, a simple calculation shows that

$$a_\varepsilon(\tau) \geq a_0 \quad \forall \tau \in \mathbb{R}. \tag{3.46}$$

Moreover, for any $M > 0$,

$$|a_\varepsilon(\tau) - a(\tau)| \le C_{M+1}\varepsilon \qquad \text{for all} \quad \tau \in \mathbb{R}, |\tau| \le M, \varepsilon \in (0,1) \tag{3.47}$$

and

$$|a_\varepsilon(\tau_1) - a_\varepsilon(\tau_2)| \le C_{M+1}|\tau_1 - \tau_2| \qquad \text{for all} \quad \tau_i \in \mathbb{R}, |\tau_i| \le M, i = 1,2, \varepsilon \in (0,1) \tag{3.48}$$

with C_{M+1} given in Assumption (A1). We now consider the regularized equation

$$\begin{cases} -\operatorname{div}[a_\varepsilon(y)\nabla y + b(\nabla y)] = u & \text{in } \Omega, \\ \qquad\qquad\qquad\qquad\qquad y = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.49}$$

Since a_ε satisfies Assumption (A1), Theorem 3.1 yields that equation (3.49) admits for each $u \in L^p(\Omega)$ with $p > N/2$ a unique solution $y \in H_0^1(\Omega) \cap C(\overline{\Omega})$.

In the sequel, for each $\varepsilon > 0$, we denote by $S_\varepsilon : L^p(\Omega) \to H_0^1(\Omega)$ the solution operator of (3.49), which by Theorem 3.1 satisfies the a priori estimate

$$\|S_\varepsilon(u)\|_{H_0^1(\Omega)} + \|S_\varepsilon(u)\|_{C(\overline{\Omega})} \le C_\infty \|u\|_{L^p(\Omega)} \quad \text{for all } u \in L^p(\Omega) \tag{3.50}$$

with the constant C_∞ defined as in Theorem 3.1 corresponding to $p^* := \tilde{p}$, where $\tilde{p}$ is defined as in (3.20). The following regularity of solutions is a direct consequence of Theorem 3.2.

Corollary 3.10. *Assume that Assumptions (A1) and (A2) hold true. Let U and $\bar{s}$ be defined as in (3.21). Then, the following assertions are valid.*

(i) If $u \in U$, then $S_\varepsilon(u) \in W_0^{1,\bar{s}}(\Omega)$ and

$$\|S_\varepsilon(u)\|_{W_0^{1,\bar{s}}(\Omega)} \le C_4 \tag{3.51}$$

for some constant C_4 depending only on $a_0, \Omega, N, p,$ and U.

(ii) If $u_n \rightharpoonup u$ in $L^p(\Omega)$ with $u_n, u \in U$, then $S_\varepsilon(u_n) \to S_\varepsilon(u)$ in $W_0^{1,\bar{s}}(\Omega) \cap C(\overline{\Omega})$.

We now study the differentiability of S_ε. For $y \in W_0^{1,\bar{s}}(\Omega)$, we consider the equation

$$\begin{cases} -\operatorname{div}\left[(a_\varepsilon(y)\operatorname{Id} + J_b(\nabla y))\nabla z + a_\varepsilon'(y) z \nabla y\right] = v & \text{in } \Omega, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad z = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.52}$$

The well-posedness of equation (3.52) is proven analogously to Theorems 3.6 and 3.7.

Theorem 3.11. *Let Assumptions (A1) and (A2) hold and let $\bar{s}$ be defined as in (3.21). Assume that $y \in W_0^{1,\bar{s}}(\Omega)$. Then, for each $v \in H^{-1}(\Omega)$, equation (3.52) admits a unique solution $z \in H_0^1(\Omega)$.*

Moreover, if $\{y_n\}$ is bounded in $W_0^{1,\bar{s}}(\Omega)$ such that $\nabla y_n \to \nabla y$ in $L^2(\Omega)$ for some $y \in W_0^{1,\bar{s}}(\Omega)$, and $\{v_n\}$ is bounded $H^{-1}(\Omega)$, then there exists a constant C_5 independent of ε and $n \in \mathbb{N}$ such that

$$\|z_n\|_{H_0^1(\Omega)} \le C_5 \tag{3.53}$$

for all solutions z_n of (3.52) corresponding to y_n and v_n.

Also, similarly to Theorem 3.9 and Corollary 3.8, we obtain the Gâteaux differentiability of S_ε.

Theorem 3.12. *Let Assumptions (A1) and (A2) hold. Assume that U is given as in (3.21). Then, $S_\varepsilon : U \to H_0^1(\Omega)$ is Gâteaux differentiable. Moreover, for any $u \in U$ and $v \in L^p(\Omega)$, the Gâteaux derivative $z := S'_\varepsilon(u)v$ is the unique solution in $H_0^1(\Omega)$ of the equation*

$$\begin{cases} -\operatorname{div}\left[(a_\varepsilon(y)\operatorname{Id} + J_b(\nabla y))\nabla z + a'_\varepsilon(y)z\nabla y\right] = v & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.54}$$

where $y := S_\varepsilon(u)$.

For later use in Section 4.1 (see Theorem 4.5) we also need the uniform boundedness of solutions to (3.54). For this purpose, we define the operator $T_{y,\varepsilon} : H_0^1(\Omega) \to H^{-1}(\Omega)$ by

$$T_{y,\varepsilon}w := -\operatorname{div}\left[(a_\varepsilon(y)\operatorname{Id} + J_b(\nabla y))\nabla w + a'_\varepsilon(y)w\nabla y\right].$$

Proposition 3.13. *Let $\bar{s}$ be defined as in (3.21). For any $y \in W_0^{1,\bar{s}}(\Omega)$ and $\varepsilon > 0$, the operator $T_{y,\varepsilon} : H_0^1(\Omega) \to H^{-1}(\Omega)$ is an isomorphism. Moreover, if $\{y_\varepsilon\}$ is bounded in $W_0^{1,\bar{s}}(\Omega)$ such that $\nabla y_\varepsilon \to \nabla y$ in $L^2(\Omega)$ for some $y \in W_0^{1,\bar{s}}(\Omega)$, then*

$$\sup\left\{\|T^{-1}_{y_\varepsilon,\varepsilon}\|_{\mathcal{L}(H^{-1}(\Omega),H_0^1(\Omega))} \mid \varepsilon \in (0,1)\right\} < \infty. \tag{3.55}$$

Proof. From the definition, we immediately deduce that $T_{y,\varepsilon}$ is continuous. Moreover, Theorem 3.11 yields that $T_{y,\varepsilon}$ is bijective, while the estimate (3.55) follows from (3.53).

It remains to prove that $T^{-1}_{y,\varepsilon}$ is continuous. Let $v_n \to v$ in $H^{-1}(\Omega)$ and set $z_n := T^{-1}_{y,\varepsilon}v_n$, $z := T^{-1}_{y,\varepsilon}v$. It is easy to see that z_n and z satisfy the equation

$$\begin{cases} -\operatorname{div}\left[(a_\varepsilon(y)\operatorname{Id} + J_b(\nabla y))\nabla(z_n - z) + a'_\varepsilon(y)(z_n - z)\nabla y\right] = v_n - v & \text{in } \Omega, \\ z_n - z = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.56}$$

which, together with (3.48), implies that

$$\|\nabla(z_n - z)\|_{L^2(\Omega)} \leq \frac{1}{a_0}\left(\|v_n - v\|_{H^{-1}(\Omega)} + C_{M+1}|\Omega|^{1/s_2}\|z_n - z\|_{L^{s_1}(\Omega)}\|\nabla y\|_{L^{\bar{s}}(\Omega)}\right), \tag{3.57}$$

where $M := \|y\|_{C(\overline{\Omega})}$ and s_1, s_2 are defined as in (3.22). The same argument as in Theorem 3.7 then implies that $\|\nabla(z_n - z)\|_{L^2(\Omega)}$ is bounded. We can thus extract a subsequence, also denoted by $\{z_n - z\}$, such that $z_n - z \rightharpoonup \tilde{z}$ in $H_0^1(\Omega)$ and $z_n - z \to \tilde{z}$ in $L^{s_1}(\Omega)$ for some $\tilde{z} \in H_0^1(\Omega)$. Letting $n \to \infty$ in (3.56) yields

$$\begin{cases} -\operatorname{div}\left[(a_\varepsilon(y)\operatorname{Id} + J_b(\nabla y))\nabla \tilde{z} + a'_\varepsilon(y)\tilde{z}\nabla y\right] = 0 & \text{in } \Omega, \\ \tilde{z} = 0 & \text{on } \partial\Omega, \end{cases}$$

which together with the uniqueness of solutions indicates that $\tilde{z} = 0$. By virtue of this and the limit $z_n - z \to \tilde{z}$ in $L^{s_1}(\Omega)$, the estimate (3.57) shows that $z_n \to z$ in $H_0^1(\Omega)$. Consequently, $T^{-1}_{y,\varepsilon}$ is continuous. □

Since $T_{y,\varepsilon}$ is isomorphic, so is its adjoint $T^*_{y,\varepsilon}$, which immediately yields well-posedness of the regularized adjoint equation

$$\begin{cases} -\operatorname{div}\left[\left(a_\varepsilon(y)\operatorname{Id} + J_b(\nabla y)^T\right)\nabla w\right] + a'_\varepsilon(y)\nabla y\cdot\nabla w = v & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.58}$$

where A^T stands for the transpose of matrix A.

Corollary 3.14. *Let $\bar{s}$ be defined as in (3.21). Under Assumptions (A1) and (A2), for any $y \in W_0^{1,\bar{s}}(\Omega)$, $v \in H^{-1}(\Omega)$, and $\varepsilon > 0$, the equation (3.58) admits a unique solution $w \in H_0^1(\Omega)$.*

Finally, we address convergence of S_ε to S as $\varepsilon \to 0$.

Proposition 3.15. *If $u_\varepsilon \rightharpoonup u$ in $L^p(\Omega)$ with $u_\varepsilon, u \in U$, then $S_\varepsilon(u_\varepsilon) \to S(u)$ in $H_0^1(\Omega)\cap C(\overline{\Omega})$.*

Proof. Set $y_\varepsilon := S_\varepsilon(u_\varepsilon)$. Then by Corollary 3.10, a constant $M > 0$ exists such that

$$\|y_\varepsilon\|_{C(\overline{\Omega})} + \|y_\varepsilon\|_{W_0^{1,\bar{s}}(\Omega)} \leq M \quad \text{for all} \quad \varepsilon > 0 \tag{3.59}$$

with $\bar{s}$ given in (3.21). On the other hand, we have

$$\begin{cases} -\operatorname{div}\left[a_\varepsilon(y_\varepsilon)\nabla y_\varepsilon + b(\nabla y_\varepsilon)\right] = u_\varepsilon & \text{in } \Omega, \\ y_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

Rewriting this equation as

$$\begin{cases} -\operatorname{div}\left[a(y_\varepsilon)\nabla y_\varepsilon + b(\nabla y_\varepsilon)\right] = u_\varepsilon + v_\varepsilon & \text{in } \Omega, \\ y_\varepsilon = 0 & \text{on } \partial\Omega \end{cases}$$

with $v_\varepsilon := \operatorname{div}\left[(a_\varepsilon(y_\varepsilon) - a(y_\varepsilon))\nabla y_\varepsilon\right]$, we then have $y_\varepsilon = S(u_\varepsilon + v_\varepsilon)$. In addition, we deduce from (3.47) that

$$\begin{aligned} \|v_\varepsilon\|_{W^{-1,\bar{s}}(\Omega)} &= \sup\left\{\int_\Omega (a_\varepsilon(y_\varepsilon) - a(y_\varepsilon))\nabla y_\varepsilon\cdot\nabla\varphi dx \mid \|\varphi\|_{W_0^{1,\bar{s}'}(\Omega)} \leq 1, \bar{s}' = \frac{\bar{s}}{\bar{s}-1}\right\} \\ &\leq C_{M+1}\varepsilon\|\nabla y_\varepsilon\|_{L^{\bar{s}}(\Omega)} \\ &\leq C_{M+1}M\varepsilon. \end{aligned} \tag{3.60}$$

It follows that $v_\varepsilon \to 0$ in $W^{-1,\bar{s}}(\Omega)$ as $\varepsilon \to 0^+$. Since $L^p(\Omega) \Subset W^{-1,\tilde{p}}(\Omega)$ is compact, we have $u_\varepsilon \to u$ in $W^{-1,\tilde{p}}(\Omega)$ and so $u_\varepsilon + v_\varepsilon \to u$ in $W^{-1,s_0}(\Omega)$ with $s_0 := \min\{\tilde{p}, \bar{s}\} > N$. Corollary 3.3 then implies that $y_\varepsilon = S(u_\varepsilon + v_\varepsilon) \to S(u) = y$ in $H_0^1(\Omega)\cap C(\overline{\Omega})$, as claimed. ◻

4 EXISTENCE AND OPTIMALITY CONDITIONS

We now turn to the optimal control problem (P), which we recall is given as

$$\begin{cases} \min\limits_{u\in L^p(\Omega),\, y\in H_0^1(\Omega)} J(y,u) \\ \quad \text{s.t.} \quad -\operatorname{div}[a(y)\nabla y + b(\nabla y)] = u \quad \text{in } \Omega \\ \qquad\qquad\qquad\qquad\qquad\qquad y = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{P}$$

It will frequently be useful to rewrite problem (P) using the control-to-state operator S in the reduced form

$$\min_{u \in L^p(\Omega)} j(u) := J(S(u), u). \tag{4.1}$$

We first address the existence of minimizers.

Proposition 4.1. *Under Assumptions (A1) to (A3), there exists a minimizer $(\bar{y}, \bar{u}) \in H^1_0(\Omega) \times L^p(\Omega)$ of (P).*

Proof. Applying Theorem 3.1 to the case where $p^* := \tilde{p}$ with $\tilde{p}$ defined as in (3.20) yields that

$$\|S(u)\|_{H^1_0(\Omega)} + \|S(u)\|_{C(\overline{\Omega})} \leq M\|u\|_{L^p(\Omega)}$$

for all $u \in L^p(\Omega)$ and for some constant M independent of u. By Assumption (A3), there exists a function $g_M : [0, \infty) \to \mathbb{R}$ such that $\lim_{t\to\infty} g_M(t) = +\infty$ and $j(u) \geq g_M(\|u\|_{L^p(\Omega)})$ for all $u \in L^p(\Omega)$. This implies that j is coercive. Together with the weak lower semicontinuity of j, the existence of a minimizer follows by Tonelli's direct method. □

The remainder of this section is devoted to deriving optimality conditions for (P). The weakest conditions are the primal stationarity conditions, which are also obtained by standard arguments.

Proposition 4.2 (primal stationarity). *Assume that Assumptions (A1) to (A3) are satisfied. Then any local minimizer $(\bar{y}, \bar{u}) \in H^1_0(\Omega) \times L^p(\Omega)$ of (P) satisfies*

$$\partial_y J(\bar{y}, \bar{u})S'(\bar{u}; h) + \partial_u J(\bar{y}, \bar{u})h \geq 0 \quad \textit{for all} \quad h \in L^p(\Omega). \tag{4.2}$$

Proof. By virtue of Theorem 3.9, the continuous differentiability of the cost functional J, and [17, Lem. 3.9], the reduced cost functional $j : U \to \mathbb{R}$ is directionally differentiable with its directional derivative $\partial_y J(\bar{y}, \bar{u})S'(\bar{u}; h) + \partial_u J(\bar{y}, \bar{u})h$. The desired result then follows from the local optimality of $\bar{u}$ and a standard argument. □

In the following subsections, we will derive stronger, dual, optimality conditions that involve Lagrange multipliers for the non-smooth quasilinear equation.

4.1 C-stationary conditions

We start with C-stationarity conditions, which can be obtained by regularizing problem (P) and passing to the limit.

Let $(\bar{y}, \bar{u}) \in H^1_0(\Omega) \times L^p(\Omega)$ be a local minimizer of (P) and set $G(u) := \frac{1}{p}\|u\|^p_{L^p(\Omega)}$. We then consider the regularized problem

$$(\mathrm{P}_\varepsilon) \qquad \begin{cases} \min\limits_{u \in L^p(\Omega),\, y \in H^1_0(\Omega)} & J_\varepsilon(y, u) := J(y, u) + G(u - \bar{u}) \\ \quad \text{s.t.} & -\operatorname{div}[a_\varepsilon(y)\nabla y + b(\nabla y)] = u \quad \text{in } \Omega, \\ & \qquad\qquad\qquad y = 0 \quad \text{on } \partial\Omega, \end{cases}$$

with a_ε defined in (3.45). The reduced cost functional of (P_ε) is given by

$$j_\varepsilon(u) := J_\varepsilon(S_\varepsilon(u), u).$$

In addition, we set

$$U_0 := \{u \in L^p(\Omega) \mid \|u - \bar{u}\|_{L^p(\Omega)} \leq \bar{\rho}\}, \tag{4.3}$$

Note that $U_0 \subset U$ with U given in (3.21).

We first show that any minimizer of (P) can be approximated by minimizers of (P_ε).

Proposition 4.3. *Assume that Assumptions (A1) to (A3) are fulfilled. Let $(\bar{y}, \bar{u}) \in H_0^1(\Omega) \times L^p(\Omega)$ be a local minimizer of problem (P). Then there exists a sequence $\{(y_\varepsilon, u_\varepsilon)\}$ of local minimizers of problems (P_ε) such that $u_\varepsilon \in U_0$, $\{y_\varepsilon\}$ is bounded in $W_0^{1,\bar{s}}(\Omega)$ with $\bar{s}$ given in (3.21), and*

$$u_\varepsilon \to \bar{u} \quad \text{in } L^p(\Omega) \qquad \text{as } \varepsilon \to 0^+, \tag{4.4}$$

$$y_\varepsilon \to \bar{y} \quad \text{in } H_0^1(\Omega) \cap C(\overline{\Omega}) \qquad \text{as } \varepsilon \to 0^+. \tag{4.5}$$

Proof. We proceed similarly as in [21]. Since $(\bar{y}, \bar{u})$ is a local optimal solution to problem (P), there exists $\rho \in (0, \bar{\rho})$ such that

$$j(\bar{u}) \leq j(u) \qquad \text{for all } u \in L^p(\Omega) \text{ with } \|u - \bar{u}\|_{L^p(\Omega)} \leq \rho. \tag{4.6}$$

We then consider the auxiliary optimal control problem

$$\min_{\overline{B}_{L^p(\Omega)}(\bar{u},\rho)} j_\varepsilon(u), \tag{$\mathrm{P}_\varepsilon^\rho$}$$

which by standard arguments admits at least one global minimizer $u_\varepsilon \in \overline{B}_{L^p(\Omega)}(\bar{u}, \rho) \subset U_0$. Now let $\varepsilon \to 0^+$. Then there exist a subsequence, denoted by the same symbol, and a function $\tilde{u} \in \overline{B}_{L^p(\Omega)}(\bar{u}, \rho)$ such that

$$u_\varepsilon \rightharpoonup \tilde{u} \quad \text{weakly in } L^p(\Omega). \tag{4.7}$$

Combining this with Proposition 3.15 yields that

$$y_\varepsilon \to S(\tilde{u}) =: \tilde{y} \quad \text{in } H_0^1(\Omega) \cap C(\overline{\Omega}), \tag{4.8}$$

where $y_\varepsilon := S_\varepsilon(u_\varepsilon)$. We now show that

$$\tilde{u} = \bar{u} \quad \text{and} \quad u_\varepsilon \to \bar{u} \text{ in } L^p(\Omega). \tag{4.9}$$

In fact, due to $\bar{u} \in \overline{B}_{L^p(\Omega)}(\bar{u}, \rho)$, we have from Proposition 3.15 that

$$j_\varepsilon(u_\varepsilon) \leq j_\varepsilon(\bar{u}) = J(S_\varepsilon(\bar{u}), \bar{u}) \to J(S(\bar{u}), \bar{u}) = j(\bar{u}). \tag{4.10}$$

Using the limits (4.7) and (4.8) as well as the weak lower semicontinuity of J and of the norm on $L^p(\Omega)$, we arrive at

$$
\begin{aligned}
(4.11)\qquad j(\bar{u}) \geq \limsup_{\varepsilon\to 0^+} j_\varepsilon(u_\varepsilon) &\geq \liminf_{\varepsilon\to 0^+} j_\varepsilon(u_\varepsilon) \\
&= \liminf_{\varepsilon\to 0^+} \left(J(y_\varepsilon, u_\varepsilon) + \frac{1}{p}\|u_\varepsilon - \bar{u}\|^p_{L^p(\Omega)} \right) \\
&\geq J(\tilde{y}, \tilde{u}) + \frac{1}{p}\|\tilde{u} - \bar{u}\|^p_{L^p(\Omega)} \\
&\geq j(\bar{u}) + \frac{1}{p}\|\tilde{u} - \bar{u}\|^p_{L^p(\Omega)}.
\end{aligned}
$$

Here we have just used (4.6) to obtain the last inequality in (4.11). We thus obtain $\tilde{u} = \bar{u}$ and $j_\varepsilon(u_\varepsilon) \to j(\bar{u})$. We then have

$$\limsup_{\varepsilon\to 0^+} \frac{1}{p}\|u_\varepsilon - \bar{u}\|^p_{L^p(\Omega)} = \limsup_{\varepsilon\to 0^+} (j_\varepsilon(u_\varepsilon) - J(y_\varepsilon, u_\varepsilon)) \leq 0.$$

Consequently, (4.9) holds. Moreover, the boundedness of $\{y_\varepsilon\}$ in $W^{1,\bar{s}}_0(\Omega)$ follows from the boundedness of $\{u_\varepsilon\}$ in $L^p(\Omega)$ and the a priori estimate (3.51).

It remains to show that $(y_\varepsilon, u_\varepsilon)$ is a local minimizer of (P_ε) for sufficiently small $\varepsilon > 0$. To this end, let $u \in L^p(\Omega)$ be arbitrary with $\|u - u_\varepsilon\|_{L^p(\Omega)} < \frac{\rho}{2}$. For ε small enough, we obtain

$$\|u - \bar{u}\|_{L^p(\Omega)} \leq \|u - u_\varepsilon\|_{L^p(\Omega)} + \|\bar{u} - u_\varepsilon\|_{L^p(\Omega)} < \frac{\rho}{2} + \frac{\rho}{2} = \rho,$$

which implies that u is a feasible point of problem (P^ρ_ε). The global optimality of u_ε for (P^ρ_ε) thus implies that $j_\varepsilon(u_\varepsilon) \leq j_\varepsilon(u)$. Hence $(y_\varepsilon, u_\varepsilon)$ is a local minimizer of problem (P_ε). □

By standard arguments using the continuous differentiability of J, the Fréchet differentiability of G, the Gâteaux differentiability of S_ε, and Corollary 3.14, we obtain necessary optimality conditions for the regularized problem (P_ε).

Proposition 4.4. *Assume that Assumptions (A1) to (A2) hold true and that the cost functional J is continuously Fréchet differentiable. Then, any local minimizer $(y_\varepsilon, u_\varepsilon)$, $u_\varepsilon \in U_0$, of problem (P_ε) fulfills together with the unique adjoint $w_\varepsilon \in H^1_0(\Omega)$ the optimality system*

$$
(4.12a)\qquad \begin{cases} -\operatorname{div}\left[\left(a_\varepsilon(y_\varepsilon)\operatorname{Id} + J_b(\nabla y_\varepsilon)^T\right)\nabla w_\varepsilon\right] + a'_\varepsilon(y_\varepsilon)\nabla y_\varepsilon \cdot \nabla w_\varepsilon = \partial_y J(y_\varepsilon, u_\varepsilon) & \textit{in } \Omega, \\ w_\varepsilon = 0 & \textit{on } \partial\Omega, \end{cases}
$$

$$(4.12b)\qquad w_\varepsilon + \partial_u J(y_\varepsilon, u_\varepsilon) + G'(u_\varepsilon - \bar{u}) = 0.$$

We now wish to pass to the limit in (4.12). While this is straightforward for (4.12b), it is difficult to do this in (4.12a) directly since we have only the boundedness of sequences $\{a'_\varepsilon(y_\varepsilon)\}$ and $\{\nabla w_\varepsilon\}$, respectively, in $L^\infty(\Omega)$ and $L^2(\Omega)$. Instead, we shall pass to the limit in the *adjoint* equation of (4.12a) – which coincides with the linearized equation (3.54) – and apply a duality argument. The presence of Clarke's generalized gradient ∂_C in the following conditions justifies the term *C-stationarity conditions.*

Theorem 4.5 (C-stationarity conditions). *Let Assumptions (A1) to (A3) hold and* $(\bar{y}, \bar{u}) \in H_0^1(\Omega) \times L^p(\Omega)$ *be a local minimizer of* (P). *Then there exist* $w \in H_0^1(\Omega)$ *and* $\chi \in L^\infty(\Omega)$ *such that*

$$\text{(4.13a)} \qquad \begin{cases} -\operatorname{div}\left[\left(a(\bar{y})\operatorname{Id} + J_b(\nabla\bar{y})^T\right)\nabla w\right] + \chi\nabla\bar{y}\cdot\nabla w = \partial_y J(\bar{y}, \bar{u}) & \textit{in } \Omega, \\ w = 0 & \textit{on } \partial\Omega, \end{cases}$$

$$\text{(4.13b)} \qquad \chi(x) \in \partial_C a(\bar{y}(x)) \quad \textit{for a.e. } x \in \Omega,$$

$$\text{(4.13c)} \qquad w + \partial_u J(\bar{y}, \bar{u}) = 0.$$

Proof. We first address (4.13c). Let $y_\varepsilon, u_\varepsilon$ and w_ε be given as in Propositions 4.3 and 4.4. From (4.12a), we have $w_\varepsilon = (T^{-1}_{y_\varepsilon,\varepsilon})^*\partial_y J(y_\varepsilon, u_\varepsilon)$. As a result of Proposition 4.3 and estimate (3.55), we obtain a constant $C > 0$ such that $\|w_\varepsilon\|_{H_0^1(\Omega)} \leq C$ for all $\varepsilon > 0$. Then there exists a subsequence, denoted in the same way, satisfying $w_\varepsilon \rightharpoonup w$ in $H_0^1(\Omega)$ as $\varepsilon \to 0^+$ for some $w \in H_0^1(\Omega)$. Now, letting $\varepsilon \to 0^+$ in (4.12b) and using limits (4.4) and (4.5), the continuity of $\partial_u J$ gives (4.13c).

To show (4.13b), we first see that there exists a constant $M > 0$ such that

$$\|y_\varepsilon\|_{C(\overline{\Omega})}, \|\bar{y}\|_{C(\overline{\Omega})} \leq M \quad \text{for all} \quad \varepsilon > 0.$$

Since a_ε is Lipschitz continuous on $[-M, M]$ with Lipschitz constant C_{M+1}, we have $|a_\varepsilon'(t)| \leq C_{M+1}$ for all $t \in [-M, M]$ and so

$$|a_\varepsilon'(y_\varepsilon(x))| \leq C_{M+1} \quad \text{for a.e. } x \in \Omega.$$

Furthermore, since a_ε is continuously differentiable, $a_\varepsilon'(y_\varepsilon(x)) \in \partial_C a_\varepsilon(y_\varepsilon(x))$ for almost every $x \in \Omega$. We can therefore extract a subsequence, denoted in the same way, such that

$$\text{(4.14)} \qquad a_\varepsilon'(y_\varepsilon) \rightharpoonup^* \chi \quad \text{in } L^\infty(\Omega) \quad \text{as } \varepsilon \to 0^+$$

for some $\chi \in L^\infty(\Omega)$. Combining this with the limit (4.5), we obtain from [25, Chap. I, Thm. 3.14] that $\chi(x) \in \partial_C a(\bar{y}(x))$ for a.e. $x \in \Omega$. This proves (4.13b).

It remains to show (4.13a). First, for any $\varphi \in H^{-1}(\Omega)$, we have from the fact that $w_\varepsilon = (T^{-1}_{y_\varepsilon,\varepsilon})^*\partial_y J(y_\varepsilon, u_\varepsilon)$ that

$$\text{(4.15)} \qquad \langle \varphi, w_\varepsilon \rangle = \left\langle \partial_y J(y_\varepsilon, u_\varepsilon), T^{-1}_{y_\varepsilon,\varepsilon}\varphi \right\rangle.$$

Thus, $z_\varepsilon := T^{-1}_{y_\varepsilon,\varepsilon}\varphi$ satisfies

$$\text{(4.16)} \qquad \begin{cases} -\operatorname{div}\left[(a_\varepsilon(y_\varepsilon)\operatorname{Id} + J_b(\nabla y_\varepsilon))\nabla z_\varepsilon + a_\varepsilon'(y_\varepsilon)z_\varepsilon\nabla y_\varepsilon\right] = \varphi & \text{in } \Omega, \\ z_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

The a priori estimate (3.53) (see also (3.55)) guarantees the boundedness of $\{z_\varepsilon\}$ in $H_0^1(\Omega)$. We can thus extract a subsequence, named also by $\{z_\varepsilon\}$, such that

$$\text{(4.17)} \qquad z_\varepsilon \rightharpoonup z_0 \quad \text{in } H_0^1(\Omega) \quad \text{and} \quad z_\varepsilon \to z_0 \quad \text{in } L^{s_1}(\Omega).$$

Letting $\varepsilon \to 0^+$ in (4.16) and using the fact that $a_\varepsilon(y_\varepsilon) \to a(\bar y)$ in $C(\bar\Omega)$, $\nabla y_\varepsilon \to \nabla \bar y$ in $L^2(\Omega)$, $J_b(\nabla y_\varepsilon) \to J_b(\nabla \bar y)$ in $L^m(\Omega)$ for all $m \geq 1$, as well as the limit (4.14), we obtain

$$\begin{cases} -\operatorname{div}\left[(a(\bar y)\operatorname{Id} + J_b(\nabla \bar y))\nabla z_0 + \chi z_0 \nabla \bar y\right] = \varphi & \text{in } \Omega, \\ z_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

We now define the operator $\hat T : H_0^1(\Omega) \to H^{-1}(\Omega)$ by

$$\hat T(z) := -\operatorname{div}\left[(a(\bar y)\operatorname{Id} + J_b(\nabla \bar y))\nabla z + \chi z \nabla \bar y\right], \quad z \in H_0^1(\Omega).$$

Arguing as in the proof of Proposition 3.13 shows that $\hat T$ is an isomorphism. In addition, we have $z_0 = \hat T^{-1}(\varphi)$. Letting $\varepsilon \to 0^+$, the right hand side of (4.15) tends to

$$\left\langle \partial_y J(\bar y, \bar u), z_0 \right\rangle = \left\langle \partial_y J(\bar y, \bar u), \hat T^{-1}\varphi \right\rangle = \left\langle \varphi, \left(\hat T^{-1}\right)^* \left(\partial_y J(\bar y, \bar u)\right) \right\rangle. \tag{4.18}$$

Here we have used the fact that $\partial_y J(y_\varepsilon, u_\varepsilon) \to \partial_y J(\bar y, \bar u)$ in $H^{-1}(\Omega)$. On the other hand, the left hand side of (4.15) converges to $\langle \varphi, w \rangle$ as $\varepsilon \to 0^+$. We thus have

$$\langle \varphi, w \rangle = \left\langle \varphi, \left(\hat T^{-1}\right)^* \left(\partial_y J(\bar y, \bar u)\right) \right\rangle.$$

Since φ is arbitrary in $H^{-1}(\Omega)$, we obtain $w = \left(\hat T^{-1}\right)^* \left(\partial_y J(\bar y, \bar u)\right)$, which yields (4.13a). □

Note that the multiplier $\chi \in L^\infty(\Omega)$ is not uniquely determined by (4.13), which is an obstacle for solving the latter using a Newton-type method. However, under additional assumptions on a, we can derive an equivalent optimality system for which this is possible; see Section 5.

4.2 STRONG STATIONARITY

The conditions (4.13) can be strengthened by including a pointwise generalized sign condition on the Lagrange multiplier w, which are typically referred to as *strong stationarity conditions.*

Theorem 4.6 (strong stationarity). *Let Assumptions (A1) to (A3) hold and $(\bar y, \bar u) \in H_0^1(\Omega) \times L^p(\Omega)$ be a local minimizer of (P). Then there exist $w \in H_0^1(\Omega)$ and $\chi \in L^\infty(\Omega)$ such that*

$$\begin{cases} -\operatorname{div}\left[\left(a(\bar y)\operatorname{Id} + J_b(\nabla \bar y)^T\right)\nabla w\right] + \chi \nabla \bar y \cdot \nabla w = \partial_y J(\bar y, \bar u) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.19a}$$

$$\chi(x) \in \partial_C a(\bar y(x)) \quad \text{for a.e. } x \in \Omega, \tag{4.19b}$$

$$w + \partial_u J(\bar y, \bar u) = 0, \tag{4.19c}$$

$$\left(a'(\bar y(x); \kappa) - \chi(x)\kappa\right)\nabla \bar y(x) \cdot \nabla w(x) \leq 0 \quad \text{for a.e. } x \in \Omega, \kappa \in \mathbb{R}. \tag{4.19d}$$

Proof. Due to Theorem 4.5, it only remains to show that (4.19d) holds. As a first step, we show that

$$\int_\Omega \left(a'(\bar y; z) - \chi z\right)\nabla \bar y \cdot \nabla w \, dx \leq 0 \quad \forall z \in L^2(\Omega) \tag{4.20}$$

by considering in turn the case of $z \in H_0^1(\Omega)$ with $z := S'(\bar{u}; h)$ for some $h \in L^p(\Omega)$, followed by the case of $z \in H_0^1(\Omega)$ arbitrary, and finally the case of $z \in L^2(\Omega)$.

(i) $z = S'(\bar{u}; h)$ with $h \in L^p(\Omega)$. In this case, z satisfies the equation

$$\begin{cases} -\operatorname{div}\left[(a(\bar{y})\operatorname{Id} + J_b(\nabla\bar{y}))\nabla z + a'(\bar{y}; z)\nabla\bar{y}\right] = h & \text{in } \Omega, \\ \hfill z = 0 & \text{on } \partial\Omega. \end{cases}$$

Multiplying the above equation by w, integrating over Ω, and using integration by parts, we deduce that

$$\int_\Omega (a(\bar{y})\operatorname{Id} + J_b(\nabla\bar{y}))\nabla z \cdot \nabla w dx + \int_\Omega a'(\bar{y}; z)\nabla\bar{y} \cdot \nabla w dx = \int_\Omega hw dx,$$

which is equivalent to

$$\left\langle -\operatorname{div}\left[\left(a(\bar{y})\operatorname{Id} + J_b(\nabla\bar{y})^T\right)\nabla w\right], z\right\rangle + \int_\Omega a'(\bar{y}; z)\nabla\bar{y} \cdot \nabla w dx = \int_\Omega hw dx.$$

Combining this with (4.19a) yields

$$\int_\Omega (a'(\bar{y}; z) - \chi z)\nabla\bar{y} \cdot \nabla w dx = -\left\langle \partial_y J(\bar{y}, \bar{u}), z\right\rangle + \int_\Omega hw dx.$$

From this and relation (4.19c) as well as the fact that $z = S'(\bar{u}; h)$, we arrive at

$$\int_\Omega (a'(\bar{y}; z) - \chi z)\nabla\bar{y} \cdot \nabla w dx = -\left\langle \partial_y J(\bar{y}, \bar{u}), S'(\bar{u}; h)\right\rangle - \langle \partial_u J(\bar{y}, \bar{u}), h\rangle \leq 0, \tag{4.21}$$

where the last inequality follows from Proposition 4.2.

(ii) $z \in H_0^1(\Omega)$. Note that $\bar{y} \in W_0^{1,\bar{s}}(\Omega) \hookrightarrow C(\overline{\Omega})$, $z \in L^{s_1}(\Omega)$, and so $a'(\bar{y}; z) \in L^{s_1}(\Omega)$ and $a'(\bar{y}; z)\nabla\bar{y} \in \left(L^2(\Omega)\right)^N$. Recall that $\bar{s}$ and s_1 are defined as in (3.21) and (3.22), respectively. Setting $h := -\operatorname{div}\left[(a(\bar{y})\operatorname{Id} + J_b(\nabla\bar{y}))\nabla z + a'(\bar{y}; z)\nabla\bar{y}\right]$, it holds that $h \in H^{-1}(\Omega)$. Since $L^p(\Omega)$ is dense in $H^{-1}(\Omega)$ with $p > N/2$, there exists a subsequence $\{h_n\} \subset L^p(\Omega)$ such that $h_n \to h$ in $H^{-1}(\Omega)$. This together with Theorem 3.7 implies that $S'(\bar{u}; h_n) \to z$ in $H_0^1(\Omega)$. Combining this with (4.21), we obtain that (4.20) holds for $z \in H_0^1(\Omega)$.

(iii) $z \in L^2(\Omega)$. In this case, (4.20) is a direct consequence of (ii) and the density of $H_0^1(\Omega)$ in $L^2(\Omega)$.

To conclude the proof, we show that (4.20) implies (4.19d). To this end, we assume in contrast that there exist a measurable set $\Omega_0 \subset \Omega$ with $|\Omega_0| > 0$ and a number $\kappa \in \mathbb{R}$ such that

$$(a'(\bar{y}(x); \kappa) - \chi(x)\kappa)\nabla\bar{y}(x) \cdot \nabla w(x) > 0 \quad \forall x \in \Omega_0. \tag{4.22}$$

Setting

$$z(x) := \begin{cases} \kappa & \text{if } x \in \Omega_0 \\ 0 & \text{otherwise,} \end{cases}$$

we have that $z \in L^2(\Omega)$ and $a'(\bar{y}(x); z(x)) = 0$ for a.e. $x \in \Omega \setminus \Omega_0$. Estimate (4.20) therefore leads to

$$\int_{\Omega_0} (a'(\bar{y};\kappa) - \chi\kappa)\,\nabla\bar{y}\cdot\nabla w\,dx = \int_{\Omega} (a'(\bar{y};z) - \chi z)\,\nabla\bar{y}\cdot\nabla w\,dx \le 0,$$

in contradiction to (4.22). Hence, (4.19d) holds. □

Obtaining more explicit sign conditions on the Lagrange multiplier requires additional assumptions on a (e.g., convexity, which implies regularity of a and hence that $\chi\kappa \le a'(\bar{y}(x);\kappa)$ for all $\kappa \in \mathbb{R}$, cf. [12, Thm. 4.12]) However, as we will show in Proposition 5.2 below, for a quite general and reasonable class of functions, condition (4.19d) holds with equality anyway.

5 PIECEWISE DIFFERENTIABLE NONLINEARITIES

We now consider the special case that the non-smooth nonlinearity a is differentiable apart from countably many points, where it is possible to reformulate the C-stationarity conditions (4.13) in a form that can be solved by a semi-smooth Newton method.

We first recall the following definition from, e.g. [24, Chap. 4] or [26, Def. 2.19]. Let V be an open subset of $\mathbb{R}$. A continuous function $g : V \to \mathbb{R}$ is said to be a PC^1*-function* if for each point $t_0 \in V$ there exist a neighborhood $W \subset V$ and a finite set of C^1-functions $g_i : W \to \mathbb{R}$, $i = 1, 2, \ldots, m$, such that

$$g(t) \in \{g_1(t), g_2(t), \ldots, g_m(t)\} \quad \text{for all} \quad t \in W.$$

For any continuous function $g : V \to \mathbb{R}$, we define the set

$$D_g := \{t \in V \mid g \text{ is not differentiable at } t\}.$$

We shall say that a PC^1 function g is *countably* PC^1 if the set D_g is countable, i.e., it can be represented as $D_g = \{t_i \mid i \in I_g\}$ with a countable set I_g.

5.1 OPTIMALITY CONDITIONS

Under the additional assumption that a is a countably PC^1 function, we can show that (4.13a) holds for *any* $\chi \in L^\infty(\Omega)$ satisfying (4.13b). For this purpose, we introduce for any $y \in H_0^1(\Omega)$ the measurable set

$$\Omega_y := \{y \in D_a\} = \{y = t_i, i \in I_a\}. \tag{5.1}$$

Theorem 5.1 (relaxed optimality system). *Assume that Assumptions (A1) to (A3) are satisfied and that a is a countably PC^1 function. Let $(\bar{y}, \bar{u}) \in H_0^1(\Omega) \times L^p(\Omega)$ be a local minimizer of (P). Then there exists a unique $w \in H_0^1(\Omega)$ satisfying*

$$\begin{cases} -\operatorname{div}\left[\left(a(\bar{y})\operatorname{Id} + J_b(\nabla\bar{y})^T\right)\nabla w\right] + \chi\nabla\bar{y}\cdot\nabla w = \partial_y J(\bar{y},\bar{u}) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.2a}$$

$$w + \partial_u J(\bar{y},\bar{u}) = 0 \tag{5.2b}$$

for any $\chi \in L^\infty(\Omega)$ with $\chi(x) \in \partial_C a(\bar{y}(x))$ a.e. $x \in \Omega$.

Proof. In view of Theorem 4.5, there exist $w \in H_0^1(\Omega)$ and $\chi_0 \in L^\infty(\Omega)$ with $\chi_0(x) \in \partial_C a\,(\bar{y}(x))$ for a.e. $x \in \Omega$ satisfying (5.2b) and

$$\begin{cases} -\operatorname{div}\left[\left(a(\bar{y})\operatorname{Id} + J_b(\nabla\bar{y})^T\right)\nabla w\right] + \chi_0 \nabla\bar{y}\cdot\nabla w = \partial_y J(\bar{y},\bar{u}) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.3}$$

It therefore suffices to prove that (5.3) holds for arbitrary $\chi \in L^\infty(\Omega)$ with $\chi(x) \in \partial_C a\,(\bar{y}(x))$ for almost every $x \in \Omega$.

We now proceed by pointwise almost everywhere inspection.

Case 1: $x \in \Omega \setminus \Omega_{\bar{y}}$. Then by definition, a is differentiable and hence even continuously differentiable in $\bar{y}(x)$ and hence $\partial_C a(\bar{y}(x)) = \{a'(\bar{y}(x))\}$, which implies that $\chi(x) = a'(\bar{y}(x)) = \chi_0(x)$.

Case 2: $x \in \Omega_{\bar{y}}$. Then $x \in \Omega_i := \{\bar{y} = t_i\}$ for some $i \in I_a$. Since $\bar{y} \in H_0^1(\Omega)$, this implies that $\nabla\bar{y}(x) = 0$ for almost every $x \in \Omega_{\bar{y}}$; see, e.g., [11, Rem. 2.6]. Hence,

$$\chi(x)\nabla\bar{y}(x)\cdot\nabla w(x) = 0 = \chi_0\nabla\bar{y}(x)\cdot\nabla w(x).$$

In both cases, we see that the left-hand side of (5.3) equals that of (5.2a), which yields the claim. □

The benefit of (5.2) is that we can fix and then eliminate χ from these optimality conditions such that the reduced system in $(\bar{y}, \bar{u}, w)$ has a unique solution, allowing application of a Newton-type method.

Before we turn to this, we note that a similar argument as in the proof of Theorem 5.1 shows that under this additional assumption on a, strong stationarity is in fact not stronger than C stationarity.

Proposition 5.2 (strong stationarity reduces to C-stationarity). *Assume that a is countably PC^1. Then, the system (4.19) is equivalent to the system (4.13).*

Proof. Clearly, (4.19) implies (4.13). It is therefore sufficient to show that if (4.13) holds, the sign condition (4.19d) is always satisfied. We again proceed by pointwise almost everywhere inspection.

Case 1: $x \in \Omega \setminus \Omega_{\bar{y}}$. Then by definition, a is differentiable and hence even continuously differentiable in $\bar{y}(x)$ and hence $\partial_C a(\bar{y}(x)) = \{a'(\bar{y}(x))\}$, which implies that $\chi(x)z(x) = a'(\bar{y}(x))z(x) = a'(\bar{y}(x); z(x))$.

Case 2: $x \in \Omega_{\bar{y}}$. Then $\nabla\bar{y}(x) = 0$ as above.

In both cases, we obtain that (4.19d) holds with equality. □

5.2 SEMI-SMOOTH NEWTON METHOD

For the sake of presentation, we consider problem (P) for $a(y) := 1 + |y|$, $b \equiv 0$, and

$$J(y,u) := \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)},$$

where Ω is a bounded convex domain in $\mathbb{R}^N$, $N \in \{2,3\}$, y_d is a given desired function in $L^\infty(\Omega)$, and $\alpha > 0$.

Let $(\bar y, \bar u) \in H^1_0(\Omega) \times L^2(\Omega)$ be a local minimizer for this instance of problem (P), and let $\bar\chi \in L^\infty(\Omega)$ with $\bar\chi(x) \in \partial_C a(\bar y(x)) = \operatorname{sign}(\bar y(x))$ for a.e. $x \in \Omega$ be arbitrary but fixed, e.g.,

$$\bar\chi(x) = \begin{cases} 1 & \text{if } \bar y(x) \geq 0, \\ -1 & \text{if } \bar y(x) < 0. \end{cases}$$

We then consider the system

$$\begin{cases} -\operatorname{div}\left[(1+|\bar y|)\nabla \bar y\right] = \bar u & \text{in } \Omega, \quad \bar y = 0 \quad \text{on } \partial\Omega, \\ -\operatorname{div}\left[(1+|\bar y|)\nabla w\right] + \bar\chi \nabla \bar y \cdot \nabla w = \bar y - y_d & \text{in } \Omega, \quad w = 0 \quad \text{on } \partial\Omega, \\ w + \alpha \bar u = 0, \end{cases} \tag{5.4}$$

which by Theorem 5.1 admits a solution $(\bar y, \bar u, w) \in H^1_0(\Omega) \times L^2(\Omega) \times H^1_0(\Omega)$. By virtue of Lemma A.2, it holds that $\bar y \in H^2(\Omega) \cap H^1_0(\Omega)$, which implies that $1 + |\bar y| \in W^{1,4}(\Omega)$ and $\bar\chi \nabla \bar y \in L^4(\Omega)^N$. From this and [7, Thm. 2.6], we deduce that $w \in H^2(\Omega) \cap H^1_0(\Omega)$ as well. The second equation in (5.4) is then equivalent to

$$\begin{cases} -\Delta w - \dfrac{\bar y - y_d}{1 + |\bar y|} = 0 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.5}$$

Setting $\psi := \bar y + \frac{\bar y |\bar y|}{2}$, a simple computation shows that

$$\bar y = \left(-1 + \sqrt{1 + 2\psi}\right)\mathbb{1}_{\{\psi \geq 0\}} + \left(1 - \sqrt{1 - 2\psi}\right)\mathbb{1}_{\{\psi < 0\}}. \tag{5.6}$$

By eliminating the control $\bar u$ using the third equation in (5.4) and using (5.5) together with (5.6), we see that (5.4) is equivalent to

$$\begin{cases} -\Delta \psi + \dfrac{1}{\alpha} w = 0 & \text{in } \Omega, \quad \psi = 0 \quad \text{on } \partial\Omega, \\ -\Delta w - \left[f_1(\psi) - y_d f_2(\psi)\right] = 0 & \text{in } \Omega, \quad w = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{5.7}$$

for

$$f_1(\psi) := \frac{-1 + \sqrt{1+2\psi}}{\sqrt{1+2\psi}}\mathbb{1}_{\{\psi \geq 0\}} + \frac{1 - \sqrt{1-2\psi}}{\sqrt{1-2\psi}}\mathbb{1}_{\{\psi < 0\}} = \frac{-1 + \sqrt{1+2|\psi|}}{\sqrt{1+2|\psi|}}\operatorname{sign}(\psi),$$

$$f_2(\psi) := \frac{1}{\sqrt{1+2\psi}}\mathbb{1}_{\{\psi \geq 0\}} + \frac{1}{\sqrt{1-2\psi}}\mathbb{1}_{\{\psi < 0\}} = \frac{1}{\sqrt{1+2|\psi|}}.$$

(Note that $f_1(0) = 0$ and hence is single-valued in spite of the occurrence of the set-valued sign.) Since f_1 and f_2 are globally Lipschitz continuous and PC^1-functions and $y_d \in L^\infty(\Omega)$, the corresponding superposition operators are semi-smooth as functions from $H_0^1(\Omega)$ to $L^2(\Omega)$; see, e.g., [26, Thm. 3.49]. From this and the continuous embedding $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$, we conclude that the system (5.7) is semi-smooth as an equation from $H_0^1(\Omega) \times H_0^1(\Omega)$ to $H^{-1}(\Omega) \times H^{-1}(\Omega)$

Introducing for $k \in \mathbb{N}$ the pointwise multiplication operator $D_k : H_0^1(\Omega) \to H^{-1}(\Omega)$ with

$$d^k := \frac{1}{\left(\sqrt{1+2\psi^k}\right)^3} \mathbb{1}_{\{\psi^k \geq 0\}}(1+y_d) + \frac{1}{\left(\sqrt{1-2\psi^k}\right)^3} \mathbb{1}_{\{\psi^k < 0\}}(1-y_d), \tag{5.8}$$

a semi-smooth Newton step thus consists in solving

$$\begin{pmatrix} \frac{1}{\alpha}\,\mathrm{Id} & -\Delta \\ -\Delta & -D_k \end{pmatrix} \begin{pmatrix} \delta w \\ \delta\psi \end{pmatrix} = -\begin{pmatrix} -\Delta\psi^k + \frac{1}{\alpha} w^k \\ -\Delta w^k - \left[f_1(\psi^k) - y_d f_2(\psi^k)\right] \end{pmatrix} \tag{5.9}$$

and setting $(w^{k+1}, \psi^{k+1}) := (w^k, \psi^k) + (\delta w, \delta\psi)$. To show the local superlinear convergence of this iteration, it is sufficient to prove the uniformly bounded invertibility of (5.9). To this end, we employ a technique as in [13].

Lemma 5.3. *Assume that $d \in L^\infty(\Omega)$ such that $\|d\|_{L^\infty(\Omega)} \leq M$ for some $M > 0$. If either $d \geq 0$ a.e. or $\alpha > 0$ is large enough, then the operator $B : H_0^1(\Omega)^2 =: E \to F := H^{-1}(\Omega)^2$ given by*

$$B := \begin{pmatrix} \frac{1}{\alpha}\,\mathrm{Id} & -\Delta \\ -\Delta & -D \end{pmatrix},$$

where D is the pointwise multiplication operator with d, is uniformly invertible and satisfies

$$\left\|B^{-1}\right\|_{\mathcal{L}(F,E)} \leq C$$

for some constant C depending only on α and M.

Proof. Let r_1 and r_2 be arbitrary in $H^{-1}(\Omega)$ and consider the equation $B\begin{pmatrix} \delta w \\ \delta\psi \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$, i.e.,

$$\begin{cases} -\Delta\delta\psi + \dfrac{1}{\alpha}\delta w = r_1 & \text{in } \Omega, \qquad \delta\psi = 0 \quad \text{on } \partial\Omega \\ -\Delta\delta w - d\delta\psi = r_2 & \text{in } \Omega, \qquad \delta w = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{5.10}$$

Obviously, $-\Delta$ is isomorphic as an operator from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$, and $(-\Delta)^{-1} : L^2(\Omega) \to L^2(\Omega)$ is self-adjoint. We now consider the continuous bilinear form $e : L^2(\Omega) \times L^2(\Omega) \to \mathbb{R}$ defined via

$$e(w,v) = (w,v)_{L^2(\Omega)} + \frac{1}{\alpha}\left(d(-\Delta)^{-1}w, (-\Delta)^{-1}v\right)_{L^2(\Omega)}, \quad \text{for all } w, v \in L^2(\Omega),$$

where $(\cdot,\cdot)_{L^2(\Omega)}$ stands for the inner product in $L^2(\Omega)$. We now show that e is coercive, i.e., there exists a constant $\lambda > 0$ such that

$$e(w,w) \geq \lambda \|w\|^2_{L^2(\Omega)} \quad \text{for all } w \in L^2(\Omega). \tag{5.11}$$

If $d \geq 0$ almost everywhere, then (5.11) holds with $\lambda = 1$. It therefore remains to prove (5.11) for the case where α is large enough. To this end, we observe for any $w \in L^2(\Omega)$ that

$$\begin{aligned} e(w,w) &\geq \|w\|^2_{L^2(\Omega)} - \frac{1}{\alpha}\|d\|_{L^\infty(\Omega)}\|(-\Delta)^{-1}w\|^2_{L^2(\Omega)} \\ &\geq \|w\|^2_{L^2(\Omega)}\left(1 - \frac{C_0^2 M}{\alpha}\right) \end{aligned}$$

with $C_0 := \|(-\Delta)^{-1}\|_{\mathcal{L}(L^2(\Omega))}$. This yields (5.11) provided that $\alpha > C_0^2 M$.

The Lax–Milgram theorem now implies that there exist a unique $\tilde{w} \in L^2(\Omega)$ and a constant $C = C(\alpha, M) > 0$ such that

$$e(\tilde{w}, v) = \left((-\Delta)^{-1}(r_2 + d(-\Delta)^{-1}r_1), v\right)_{L^2(\Omega)} \quad \text{for all} \quad v \in L^2(\Omega) \tag{5.12}$$

and

$$\|\tilde{w}\|_{L^2(\Omega)} \leq C\left(\|r_1\|_{H^{-1}(\Omega)} + \|r_2\|_{H^{-1}(\Omega)}\right). \tag{5.13}$$

Let $\delta\psi \in H^1_0(\Omega)$ be the solution to

$$-\Delta\delta\psi + \frac{1}{\alpha}\tilde{w} = r_1 \quad \text{in } \Omega, \qquad \delta\psi = 0 \quad \text{on } \partial\Omega, \tag{5.14}$$

and let $\delta w \in H^1_0(\Omega)$ be the corresponding solution to the second equation in (5.10). Then it follows from (5.13) that

$$\|\delta w\|_{H^1_0(\Omega)} + \|\delta\psi\|_{H^1_0(\Omega)} \leq C\left(\|r_1\|_{H^{-1}(\Omega)} + \|r_2\|_{H^{-1}(\Omega)}\right).$$

Finally, it follows from (5.14), the second equation in (5.10), and (5.12)

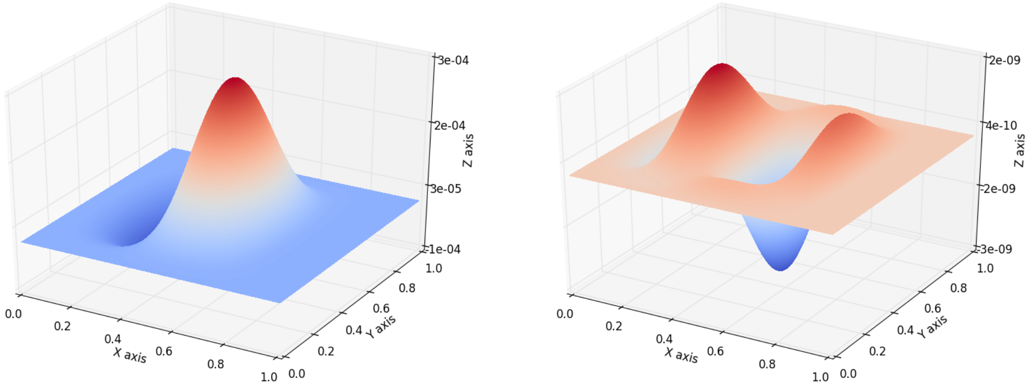
$$\begin{aligned} \delta w &= (-\Delta)^{-1}\left[r_2 + d\delta\psi\right] \\ &= (-\Delta)^{-1}\left[r_2 + d(-\Delta)^{-1}\left(r_1 - \frac{1}{\alpha}\tilde{w}\right)\right] \\ &= (-\Delta)^{-1}\left[r_2 + d(-\Delta)^{-1}r_1\right] - \frac{1}{\alpha}(-\Delta)^{-1}\left[d(-\Delta)^{-1}\tilde{w}\right] \\ &= \tilde{w}, \end{aligned}$$

which concludes the proof. □

We now arrive at the local convergence of the semi-smooth Newton iteration.

Theorem 5.4. *Let $y_d \in L^\infty(\Omega)$ and $\alpha > 0$ be such that either $\|y_d\|_{L^\infty(\Omega)} \leq 1$ or α is large enough. Assume that (w, ψ) is a solution to (5.7). Then there exists a constant $\rho > 0$ such that for $w^0 \in B_{H^1_0(\Omega)}(w, \rho)$ and $\psi^0 \in B_{H^1_0(\Omega)}(\psi, \rho)$, the semi-smooth Newton iteration (5.9) converges superlinearly in $H^1_0(\Omega) \times H^1_0(\Omega)$ to (w, ψ).*

Proof. From (5.8), we obtain that $\|d^k\|_{L^\infty(\Omega)} \leq 1 + \|y_d\|_{L^\infty(\Omega)}$ and $d^k \geq 0$ a.e. if $\|y_d\|_{L^\infty(\Omega)} \leq 1$. The claim then follows from Lemma 5.3 together with [18, Thm. 8.16]. □



(a) exact state $\bar{y}$ (b) exact adjoint $\bar{w} = -\alpha\bar{u}$

Figure 1: constructed exact solution for $\alpha = 10^{-7}$, $\beta = 0.85$

6 NUMERICAL EXPERIMENTS

We now illustrate the solvability of the relaxed optimality system (5.2) using the semi-smooth Newton method presented in Section 5.2 using a numerical example.

Specifically, we consider $\Omega = (0,1)^2 \subset \mathbb{R}^2$ and create a uniform triangular Friedrichs–Keller triangulation with $n_h \times n_h$ vertices which is the basis for a finite element discretization of the two elliptic equations in (5.7). We then compute a solution (w_h, ψ_h) from (5.7) by the presented semi-smooth Newton iteration, starting with $(w^0, \psi^0) = (0,0)$ and terminating whenever the number of iterations reaches 25 or the active sets $\{\psi^k \geq 0\}$ corresponding to two consecutive steps coincide. If the iteration is successful, we recover y_h via (5.6) and u_h via the third equation of (5.4). The Python implementation using DOLFIN [19, 20] that was used to generate the following results can be downloaded from https://github.com/clason/nonsmoothquasilinear.

We choose the target function y_d based on a constructed example, setting

$$\begin{aligned}
\bar{y} &= x_1^4\left[(x_1-\beta)^4 + 2(x_1-\beta)^5\right]\sin(\pi x_2)\mathbb{1}_{[0,\beta]}(x_1),\\
\bar{u} &= -(1+|\bar{y}|)\Delta\bar{y} - \operatorname{sign}(\bar{y})\left|\nabla\bar{y}\right|^2,\\
\bar{w} &= -\alpha\bar{u},\\
\bar{\psi} &= \bar{y}\left(1+\frac{1}{2}|\bar{y}|\right),\\
y_d &= \bar{y} + (1+|\bar{y}|)\Delta\bar{w},
\end{aligned}$$

for a parameter $\beta \in [0.5,1]$; see Figure 1 for $\alpha = 10^{-7}$ and $\beta = 0.85$. We point out that for $\beta \in (0.5,1)$, the sets on which the value of $\bar{y}$ is positive, negative, and zero all have positive measure. Moreover,

$$|\{\bar{y} = 0\}| = \left|\{\bar{\psi} = 0\}\right| = 1-\beta,$$

and $\|y_d\|_{L^\infty(\Omega)} \leq 1$ for any α small enough and for any $\beta \in [0.5,1]$. From this and Theorem 5.4, we can deduce that the semi-smooth Newton method for solving (5.7) will converge locally superlinearly.

The results for different values of n_h, α, and β are given in Table 1, where we list the relative H_0^1 errors for the computed state y_h and adjoint w_h as well as the number of semi-smooth Newton iterations. For the sake of completeness, we also give the L^∞ norm of y_d for the chosen parameters, verifying that $\|y_d\|_{L^\infty(\Omega)} \leq 1$ in all cases. We first address the dependence on n_h. As can be seen from Table 1a, the relative errors decrease linearly with increasing n_h, which matches the expected $O(h)$ convergence of the piecewise linear finite element approximation. Also, the number of semi-smooth Newton iterations (2–4) stays constant, demonstrating the mesh independence that is usually the consequence of an infinite-dimensional convergence result like Theorem 5.4. The dependence on α is shown in Table 1b. We point out that the Newton method is relatively robust with respect to this parameter with the required number of iterations only starting to increase from 3 to 25 for $\alpha < 10^{-6}$. Finally, we comment on the dependence of β shown in Table 1c. Since $\text{meas}\{\bar{y} = 0\} \to 0$ for $\beta \to 1$, it is not surprising that the relative errors for $\bar{y}$ decrease quickly as β increases. Here we observe only a slight increase in the number of semi-smooth Newton iterations from 3 to 6.

7 CONCLUSIONS

We have considered optimal control problems for a quasilinear elliptic differential equation with a nonlinear coefficient in the leading term that is Lipschitz continuous and directionally but not Gâteaux differentiable. By passing to the limit in a regularized equation, C- and strong stationarity conditions can be derived. If the nonlinear coefficient is a piecewise differentiable apart from a countable set of points, both stationarity conditions coincide and are equivalent to a relaxed optimality system that is amenable to numerical solution by a semi-smooth Newton method. This is illustrated by a numerical example.

This work can be extended in several directions. First, second-order sufficient optimality conditions can be considered based on the approach in [5] for an optimal control problem of non-smooth, semilinear parabolic equations. Furthermore, a practically relevant issue would be to derive error estimates for the finite element approximation of (P) as in the case of smooth settings [8, 10]. Finally, the results derived in this work can be used to study the Tikhonov regularization of parameter identification problems for non-smooth quasilinear elliptic equations.

APPENDIX A AUXILIARY LEMMAS

The first lemma about monotonicity of an auxiliary problem is needed in Theorem 3.2 to show higher regularity of the state equation (3.1).

Lemma A.1. *Assume that Assumption (A2) is fulfilled and $\lambda > 0$. Then the operator*

$$B : H_0^1(\Omega) \to H^{-1}(\Omega), \qquad B(z) := -\operatorname{div}\left[\lambda \nabla z + b(\nabla z)\right], \tag{A.1}$$

is maximally monotone.

Proof. Obviously, B is monotone since b is monotone. We now show that B is hemicontinuous, i.e., the mapping $[0, 1] \ni t \mapsto \langle B(z_1 + tz_2), z_3\rangle \in \mathbb{R}$ is continuous for all $z_1, z_2, z_3 \in H_0^1(\Omega)$. To

n_h	α	β	$\frac{\|y_h-\bar{y}\|_{H_0^1(\Omega)}}{\|\bar{y}\|_{H_0^1(\Omega)}}$	$\frac{\|w_h-\bar{w}\|_{H_0^1(\Omega)}}{\|\bar{w}\|_{H_0^1(\Omega)}}$	# SSN	$\|y_d\|_{L^\infty(\Omega)}$
100	$1\cdot 10^{-6}$	0.8	$3.27\cdot 10^{-3}$	$2.92\cdot 10^{-2}$	2	$2.07\cdot 10^{-4}$
200	$1\cdot 10^{-6}$	0.8	$1.66\cdot 10^{-3}$	$1.54\cdot 10^{-2}$	4	$2.07\cdot 10^{-4}$
400	$1\cdot 10^{-6}$	0.8	$8.36\cdot 10^{-4}$	$7.92\cdot 10^{-3}$	3	$2.07\cdot 10^{-4}$
800	$1\cdot 10^{-6}$	0.8	$4.19\cdot 10^{-4}$	$4.03\cdot 10^{-3}$	3	$2.07\cdot 10^{-4}$
1000	$1\cdot 10^{-6}$	0.8	$3.36\cdot 10^{-4}$	$3.24\cdot 10^{-3}$	3	$2.07\cdot 10^{-4}$

(a) dependence on n_h

n_h	α	β	$\frac{\|y_h-\bar{y}\|_{H_0^1(\Omega)}}{\|\bar{y}\|_{H_0^1(\Omega)}}$	$\frac{\|w_h-\bar{w}\|_{H_0^1(\Omega)}}{\|\bar{w}\|_{H_0^1(\Omega)}}$	# SSN	$\|y_d\|_{L^\infty(\Omega)}$
800	$1\cdot 10^{-2}$	0.8	$6.36\cdot 10^{-2}$	$1.36\cdot 10^{-2}$	4	$9.83\cdot 10^{-2}$
800	$1\cdot 10^{-4}$	0.8	$8.76\cdot 10^{-3}$	$7.32\cdot 10^{-3}$	3	$9.83\cdot 10^{-4}$
800	$1\cdot 10^{-6}$	0.8	$4.19\cdot 10^{-4}$	$4.03\cdot 10^{-3}$	3	$2.07\cdot 10^{-4}$
800	$1\cdot 10^{-8}$	0.8	$2.32\cdot 10^{-5}$	$2.19\cdot 10^{-3}$	25	$2.03\cdot 10^{-4}$

(b) dependence on α

n_h	α	β	$\frac{\|y_h-\bar{y}\|_{H_0^1(\Omega)}}{\|\bar{y}\|_{H_0^1(\Omega)}}$	$\frac{\|w_h-\bar{w}\|_{H_0^1(\Omega)}}{\|\bar{w}\|_{H_0^1(\Omega)}}$	# SSN	$\|y_d\|_{L^\infty(\Omega)}$
800	$1\cdot 10^{-5}$	0.5	$7.03\cdot 10^{-3}$	$1.20\cdot 10^{-2}$	3	$1.50\cdot 10^{-5}$
800	$1\cdot 10^{-5}$	0.7	$2.68\cdot 10^{-3}$	$7.11\cdot 10^{-3}$	3	$1.07\cdot 10^{-4}$
800	$1\cdot 10^{-5}$	0.9	$1.41\cdot 10^{-3}$	$4.27\cdot 10^{-3}$	4	$5.07\cdot 10^{-4}$
800	$1\cdot 10^{-5}$	1.0	$8.65\cdot 10^{-5}$	$3.39\cdot 10^{-3}$	6	$9.50\cdot 10^{-4}$

(c) dependence on β

Table 1: numerical results: number of Newton iterations and relative errors for state $\bar{y}$ and adjoint $\bar{w}$ in dependence of n_h, α, and β

this end, observe that for any $z_1, z_2, z_3 \in H_0^1(\Omega)$ and any $t \in [0,1]$,

$$\langle B(z_1 + tz_2), z_3\rangle = \lambda \int_\Omega (\nabla z_1 + t\nabla z_2)\cdot \nabla z_3 dx + \int_\Omega b\,(\nabla z_1 + t\nabla z_2)\cdot \nabla z_3 dx.$$

Together with the continuity of b, this implies the hemicontinuity of B. The maximal monotonicity of B then follows from [28, Prop. 32.7]. □

The next lemma shows H^2-regularity of the solutions to the state equation with a PC^1 nonlinearity and is needed in Section 5.2 to show Newton differentiability of the relaxed optimality system (5.4).

Lemma A.2. *Let Ω be a convex domain in $\mathbb{R}^N$ with $N \in \{2,3\}$. Assume that Assumption (A1) is* *valid. Assume furthermore that a is a PC^1-function. Then, for each $u \in L^2(\Omega)$, the equation*

$$\begin{cases} -\operatorname{div}[a(y)\nabla y] = u & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega \end{cases} \tag{A.2}$$

has a unique solution $y \in H^2(\Omega) \cap H_0^1(\Omega)$.

Proof. In view of Theorem 3.1, it suffices to prove the H^2-regularity of the unique solution $y \in H_0^1(\Omega) \cap C(\overline{\Omega})$ of (A.2). Setting

$$\theta := K(y) := \int_0^y a(t)dt,$$

equation (A.2) reduces to

$$\begin{cases} -\Delta\theta = u & \text{in } \Omega, \\ \theta = 0 & \text{on } \partial\Omega. \end{cases}$$

The regularity of solutions to Poisson's equation guarantees that $\theta \in H^2(\Omega)$; see e.g., [16, Thm. 3.2.1.2]. Since $y = K^{-1}(\theta)$, we have that

$$\frac{\partial y}{\partial x_i} = \frac{1}{K'(K^{-1}(\theta))} \frac{\partial \theta}{\partial x_i} = \frac{1}{a(y)} \frac{\partial \theta}{\partial x_i},$$

which implies that $\|y\|_{H_0^1(\Omega)} \leq \frac{1}{a_0} \|\theta\|_{H_0^1(\Omega)}$. We furthermore have that

$$\frac{\partial^2 y}{\partial x_j \partial x_i} = \frac{1}{a(y)} \frac{\partial^2 \theta}{\partial x_j \partial x_i} - \frac{1}{a^2(y)} \frac{\partial \theta}{\partial x_i} \frac{\partial a(y)}{\partial x_j}.$$

Note that, since $y \in C(\overline{\Omega})$, there exists a constant $M > 0$ such that $|y(x)| \leq M$ for all $x \in \overline{\Omega}$. Defining a PC^1-function

$$a_M : \mathbb{R} \to \mathbb{R}, \qquad a_M(t) = \begin{cases} a(2M) & \text{if } t > 2M, \\ a(t) & \text{if } |t| \leq 2M, \\ a(-2M) & \text{if } t < -2M. \end{cases}$$

Assumption (A1) implies that a_M is Lipschitz continuous with Lipschitz constant C_{2M}. We then have $\|\nabla a_M\|_{L^\infty(\mathbb{R})} \leq C_{2M}$, where ∇a_M is the weak derivative of a_M. From this and the chain rule (see, e.g. [15, Thm. 7.8]), we arrive at $a(y) \in H^1(\Omega)$ and

$$\frac{\partial a(y)}{\partial x_j}(x) = \begin{cases} a'(y(x)) \frac{\partial y}{\partial x_j}(x) & \text{if } y(x) \notin D_a, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, we have

$$\frac{\partial^2 y}{\partial x_j \partial x_i} = \frac{1}{a(y)} \frac{\partial^2 \theta}{\partial x_j \partial x_i} - \frac{\nabla a_M(y)}{a^3(y)} \frac{\partial \theta}{\partial x_i} \frac{\partial \theta}{\partial x_j} \mathbb{1}_{\{y \notin D_a\}}$$

and hence

$$\left\| \frac{\partial^2 y}{\partial x_j \partial x_i} \right\|_{L^2(\Omega)} \leq \frac{1}{a_0} \left\| \frac{\partial^2 \theta}{\partial x_j \partial x_i} \right\|_{L^2(\Omega)} + \frac{C_{2M}}{a_0^3} \left\| \frac{\partial \theta}{\partial x_i} \right\|_{L^4(\Omega)} \left\| \frac{\partial \theta}{\partial x_j} \right\|_{L^4(\Omega)}.$$

This together with the fact that $\theta \in H^2(\Omega) \hookrightarrow W^{1,6}(\Omega) \hookrightarrow W^{1,4}(\Omega)$ yields that $y \in H^2(\Omega)$. □

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