Deutsche Forschungsgemeinschaft
Priority Programme 1962

# *Optimal Control of ODEs with State Suprema*

Tobias Geiger, Daniel Wachsmuth, Gerd Wachsmuth

SPP 1962 | Non-smooth and Complementarity-based Distributed Parameter Systems: Simulation and Hierarchical Optimization

Preprint Number SPP1962-094

received on October 26, 2018

Edited by
SPP1962 at Weierstrass Institute for Applied Analysis and Stochastics (WIAS)
Leibniz Institute in the Forschungsverbund Berlin e.V.
Mohrenstraße 39, 10117 Berlin, Germany
E-Mail: spp1962@wias-berlin.de

World Wide Web: http://spp1962.wias-berlin.de/

# Optimal control of ODEs with state suprema

Tobias Geiger[*] Daniel Wachsmuth[*] Gerd Wachsmuth[†]

October 26, 2018

We consider the optimal control of a differential equation that involves the suprema of the state over some part of the history. In many applications, this non-smooth functional dependence is crucial for the successful modeling of real-world phenomena. We prove the existence of solutions and show that related problems may not possess optimal controls. Due to the non-smoothness in the state equation, we cannot obtain optimality conditions via standard theory. Therefore, we regularize the problem via a novel LogIntExp functional which generalizes the well-known LogSumExp. By passing to the limit with the regularization, we obtain an optimality system for the original problem. The theory is illustrated by some numerical experiments.

**Keywords:** functional differential equations, differential equations with state suprema, optimality conditions, maximum principle, LogIntExp
**MSC:** 49K21, 34K35, 49J21

## 1 Introduction

In this paper, we study optimal control problems with differential equations that involve the suprema of the state. To be precise, the state equation is given by

$$x'(t) = F\Big(x(t), \max_{s\in[t-\tau,t]} x(s), u(t)\Big), \qquad t \in (0,T) \tag{1.1}$$

with initial data

$$x(t) = \phi(t),\ t \in [-\tau, 0]. \tag{1.2}$$

[*]Institut für Mathematik, Universität Würzburg, 97074 Würzburg, Germany, tobias.geiger@mathematik.uni-wuerzburg.de and daniel.wachsmuth@mathematik.uni-wuerzburg.de

[†]Brandenburgische Technische Universität Cottbus-Senftenberg, Institute of Mathematics, Chair of Optimal Control, 03046 Cottbus, Germany, https://www.b-tu.de/fg-optimale-steuerung, gerd.wachsmuth@b-tu.de

Here, $T > 0$ is the final time, $\tau > 0$ is the parameter for the state suprema. The control $u$ has to be chosen optimal subject to an objective functional of Lagrange type

$$J(x, u) = \int_0^T j(t, x(t), u(t)) \, \mathrm{d}t \tag{1.3}$$

and the control has to obey the constraints

$$u(t) \in U, \qquad t \in (0, T). \tag{1.4}$$

For the precise assumptions, we refer to Section 2.3 below.

Control of such systems were proposed by [Azhmyakov et al., 2016; Verriest, Azhmyakov, 2017]. Differential equations with state suprema have an abundance of applications, we name only solar power plant control [Azhmyakov et al., 2016], population genetics [Wu, 1996], and quantum chemistry [Bainov, S. G. Hristova, 2011]. Input-to-output stability of such systems were addressed in [Dashkovskiy et al., 2017]. Systems with maximum are also popular in machine learning [Goodfellow et al., 2016], by means of the so-called max-pooling.

The evolution system (1.1) is inherently non-smooth due to the appearance of the max-term. This makes the development of necessary optimality conditions challenging. While the celebrated Pontryagin maximum principle can account for non-smooth functions of the control $u$, this is not the case for system with non-smooth functions applied to the state $x$. Therefore, the main contribution of the present work is the extension of [Banks, 1969] in which a Pontryagin maximum principle is derived for systems with a smooth functional dependence. In [Clarke, Wolenski, 1996], the authors studied the optimal control of a nonsmooth functional differential inclusion. After some tedious but straightforward transformations, the control problem (1.1)–(1.4) can be put in their framework and they obtain an optimality system similar to our Theorem 6.5. However, our regularization approach allows for a numerical realization and we also expect that our strategy for deriving optimality conditions can be transfered to control problems of partial differential equations involving state suprema.

The paper is structured as follows. In Section 2 we define some notation and fix the standing assumption (Assumption 2.2). The existence of solutions of the state equation (1.1) and of the optimal control problem is proven in Section 3. Moreover, we show in Section 3.3 that related optimal control problems may lack optimal solutions. In order to define a regularization of the state equation, we define a novel LogIntExp regularization by generalizing the well known LogSumExp function, see Section 4. In Section 5, we study the regularized problems and we pass to the limit with the regularization in Section 6. With this technique, we arrive at the optimality system in Theorem 6.5 and this allows us to derive jump conditions for the adjoint state in Corollary 6.7. Finally, we present some numerical examples in Section 7.

# 2 Notations, preliminaries and standing assumptions

## 2.1 Notation

Let us define abbreviations for time intervals

$$I := [0, T], \quad I_\tau := [-\tau, T].$$

For a time $t \in I$ and a function $x \in C(I_\tau; \mathbb{R}^d)$, we define $x_t \in C([0,\tau]; \mathbb{R}^d)$ via

$$x_t(s) := x(t-s) \qquad \forall s \in [0, \tau].$$

This notation implies

$$\max_{s \in [t-\tau, t]} x(s) = \max_{s \in [0,\tau]} x_t(s) = \max x_t.$$

Note that this maximum is evaluated component-wise. Let us define

$$C_\tau := C([0, \tau], \mathbb{R}^n),$$

which is supplied with the max-norm.

We will frequently use scalar functions applied to vectors. For vectors $v, w \in \mathbb{R}^n$, we denote by $\exp v$ and $\frac{v}{w}$ the component-wise exponentiation and division. Moreover, $v \odot w$ denotes the Hadamard (or component-wise) product.

## 2.2 Preliminaries

Due to the retarded structure of the state equation, we need a special integral inequality, which is very similar to Gronwall's lemma.

**Lemma 2.1.** *Let $x \in C(I_\tau; \mathbb{R})$ be a continuous function with $x \equiv 0$ on $[-\tau, 0]$. Suppose that there exist constants $k_1, k_2 \geq 0$ such that*

$$x(t) \leq k_1 + k_2 \int_0^t x(s) + \max x_s \,\mathrm{d}s$$

*holds for all $t \in I$. Then,*

$$x(t) \leq k_1 \exp(2\, k_2\, t)$$

*holds for all $t \in I$.*

*Proof.* We refer to [Bainov, S. G. Hristova, 2011, Theorem 2.1.1]. □

## 2.3 Standing assumptions

We fix the standing assumptions for the treatment of the optimal control problem (1.1)–(1.4). These assumptions shall hold throughout the paper.

**Assumption 2.2** (Standing assumptions)**.**

*(i) The function $F : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is affine in its third argument, i.e.,*

$$F(x, v, u) = F_0(x, v) + F_1(x, v)\, u \tag{2.1}$$

*for globally Lipschitz continuous and continuously differentiable functions $F_0 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $F_1 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$.*

*(ii) The initial datum $\phi$ belongs to $C([-\tau, 0]; \mathbb{R}^n)$.*

*(iii) The admissible set $U \subset \mathbb{R}^m$ is non-empty, convex and compact.*

*(iv) The integrand $j : I \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ is a normal integrand and convex in its third argument $u$, see [Ekeland, Temam, 1999, Chapter VIII, Section 2.1]. In addition, it is continuously differentiable w.r.t. the second argument $x$.*

Under these assumptions, a function $x \in C(I_\tau, \mathbb{R}^n) \cap W^{1,\infty}(I; \mathbb{R}^n)$ is a solution of (1.1) if and only if it satisfies the integral equation

$$x(t) = \phi(0) + \int_0^t F_0(x(s), \max x_s) + F_1(x(s), \max x_s)\, u(s)\, \mathrm{d}s \quad \forall t \in I \tag{2.2}$$

together with the initial condition (1.2).

**Lemma 2.3.** *If $x_k \to x$ in $L^1(I; \mathbb{R}^n)$ and $u_k \rightharpoonup u$ in $L^1(I; \mathbb{R}^m)$ with $u_k(t) \in U$ f.a.a. $t \in I$ then*

$$J(x, u) \le \liminf_{k\to\infty} J(x_k, u_k).$$

*Proof.* This follows from [Ekeland, Temam, 1999, Chapter VIII, Theorem 2.1]. Note that due to the boundedness of $u_k(t) \in U$ we do not need to impose the growth condition [Ekeland, Temam, 1999, Chapter VIII, (2.2)]. □

**Remark 2.4.** *Under standard modifications, the results of the paper are true for non-autonomous $F$, i.e., where $F$ is given by $F(t, x, v, u) = F_0(t, x, v) + F_1(t, x, v)\, u$.*

# 3 Existence of solutions

In this section, we study the properties of the following differential equation

$$x'(t) = F\Big(x(t), \max_{s\in[t-\tau,t]} x(s), u(t)\Big), \qquad t \in I \tag{3.1}$$

and the associated control problem.

## 3.1 Study of a nonlinear differential equation

Let us first study a more general equation than (3.1). We will investigate the solvability of

$$x'(t) = f(t, x(t), x_t, u(t)), \qquad t \in I. \tag{3.2}$$

subject to the initial conditions as above. A function $x \in C(I_\tau, \mathbb{R}^n) \cap W^{1,\infty}(I;\mathbb{R}^n)$ is called a solution of if (3.1) holds for almost all $t$ and the initial condition $x(t) = \phi(t)$ for all $t \in [-\tau, 0]$ is satisfied.

In order to prove solvability of (3.2) for all feasible controls $u$, we require the following assumption.

**Assumption 3.1.** *(i) The function $f : I \times \mathbb{R}^n \times C_\tau \times \mathbb{R}^m \to \mathbb{R}^n$ is measurable in the first, and continuous with respect to the other arguments.*

*(ii) For all $M > 0$ exists $L_M > 0$ such that*

$$|f(t, x_1, y_1, u) - f(t, x_2, y_2, u)| \le L_M(|x_1 - x_2| + \|y_1 - y_2\|_{C_\tau})$$

*for all $x_1, x_2 \in \mathbb{R}^n$, $y_1, y_2 \in C_\tau$, $u \in \mathbb{R}^m$ with $|u| \le M$, and almost all $t \in I$.*

*(iii) For all $M > 0$ exists $K_M > 0$ such that*

$$|f(t, 0, 0, u)| \le K_M$$

*$u \in \mathbb{R}^m$ with $|u| \le M$ and almost all $t \in I$.*

*(iv) The initial data satisfies $\phi \in C([-\tau, 0], \mathbb{R}^n)$.*

Since $x_t \mapsto \max_s x_t(s)$ is Lipschitz continuous, the original problem is covered by these assumptions. In addition, smooth regularizations of the max-functions are included as well.

**Theorem 3.2.** *Let Assumption 3.1 be satisfied. Let $u \in L^\infty(I, \mathbb{R}^m)$ be given. Then there exists a unique solution $x \in W^{1,\infty}(I;\mathbb{R}^n)$ of (1.1)–(1.2).*

*In addition, the mapping $u \mapsto x$ maps bounded sets in $L^\infty(I, \mathbb{R}^m)$ to bounded sets in $W^{1,\infty}(I;\mathbb{R}^n)$.*

*Proof.* We follow the proof of [Wu, 1996, Thm. 2.1.1], which uses a standard Picard-Lindelöf argument. Let us define the functions $x_k \in C(I_\tau, \mathbb{R}^n)$, $k = 0, 1, \dots,$ by

$$x_0(t) = \phi(\min(t, 0)), \qquad t \in I_\tau$$

and for $k \ge 0$

$$x_{k+1}(t) = \begin{cases} \phi(t), & \text{if } t \in [-\tau, 0], \\ \phi(0) + \int_0^t f(s, x_k(s), x_{k,s}, u(s))\mathrm{d}s, & \text{if } t \in (0, T]. \end{cases}$$

In the following, $t$ is taken from $I$. Let $M := \|u\|_{L^\infty(I;\mathbb{R}^m)}$. By Lipschitz continuity of $f$, we obtain

$$\begin{aligned} |x_{k+1}(t) - x_k(t)| &\le L_M \int_0^t |x_k(s) - x_{k-1}(s)| + \|x_{k,s} - x_{k-1,s}\|_{C_\tau}\mathrm{d}s \\ &\le 2L_M \int_0^t \|x_k - x_{k-1}\|_{C([0,s];\mathbb{R}^n)}\mathrm{d}s, \end{aligned}$$

which implies the estimate

$$\|x_{k+1}-x_k\|_{C([0,t];\mathbb{R}^n)} = \max_{s\in[0,t]}|x_{k+1}(t)-x_k(s)| \le 2L_M \int_0^t \|x_k - x_{k-1}\|_{C([0,s];\mathbb{R}^n)} \mathrm{d}s \quad (3.3)$$

Due to the assumptions on $f$, we obtain

$$\left|\int_0^t f(s,x_0(s),x_{0,s},u(s))\mathrm{d}s\right| \le \int_0^t L_M\,(|\phi(0)|+\|x_{0,s}\|_{C_\tau})+K_M\,\mathrm{d}s \le (2\,L_M\,\|\phi\|_{C_\tau}+K_M)\,t.$$

By definition of $x_0$ and $x_1$, this gives the estimate

$$|x_1(t)-x_0(t)| \le (2\,L_M\,\|\phi\|_{C_\tau}+K_M)\,t,$$

which implies

$$\|x_1-x_0\|_{C([0,t];\mathbb{R}^n)} \le (2\,L_M\,\|\phi\|_{C_\tau}+K_M)\,t =: K\,t. \quad (3.4)$$

By an induction argument based on (3.3) and (3.4), we obtain the estimate

$$\|x_k-x_{k-1}\|_{C([0,t];\mathbb{R}^n)} \le (2L_M)^{k-1}K\frac{t^k}{k!},$$

which implies

$$\|x_k-x_{k-1}\|_{C(I,\mathbb{R}^n)} \le (2L_M)^{k-1}K\,\frac{T^k}{k!}. \quad (3.5)$$

Since $\sum_{k=1}^{\infty}(2L_M)^{k-1}KT^k\,\frac{1}{k!}<\infty$, the sequence $(x_k)$ is a Cauchy sequence in $C(I,\mathbb{R}^n)$, hence convergent to some $x\in C(I,\mathbb{R}^n)$. It remains to show that $x$ solves (3.2). Let us estimate

$$\begin{aligned}\left|x(t)-\phi(0)-\int_0^t f(s,x(s),x_s,u(s))\mathrm{d}s\right| & \\ &\le |x(t)-x_{k+1}(t)| + \int_0^t |f(s,x_k(s),x_{k,s},u(s))-f(s,x(s),x_s,u(s))|\mathrm{d}s \\ &\le \|x-x_{k+1}\|_{C(I,\mathbb{R}^n)} + 2L_M T\|x-x_k\|_{C(I,\mathbb{R}^n)}.\end{aligned}$$

Passing to the limit $k\to\infty$, we find that $x$ solves the integral equation

$$x(t)=\phi(0)+\int_0^t f(s,x(s),x_s,u(s))\mathrm{d}s,$$

which implies that the weak derivative of $x$ satisfies

$$x'(t)=f(t,x(t),x_t,u(t))$$

for almost all $t$. This right-hand side is in $L^\infty(I;\mathbb{R}^n)$, hence $x\in W^{1,\infty}(I;\mathbb{R}^n)$ is proven.

Let $x_1,x_2\in W^{1,\infty}(I;\mathbb{R}^n)$ be two solutions. Then it holds

$$x_1(t)-x_2(t)=\int_0^t f(s,x_1(s),x_{1,s},u(s))-f(s,x_2(s),x_{2,s},u(s))\,\mathrm{d}s.$$

Arguing similarly as in the derivation of (3.3) above, we find

$$\|x_1 - x_2\|_{C([0,t];\mathbb{R}^n)} \le 2L_M \int_0^t \|x_1 - x_2\|_{C([0,s];\mathbb{R}^n)} \mathrm{d}s,$$

which implies $x_1 = x_2$ by Gronwall's lemma.

Let us prove the claimed boundedness result. By the construction of $x_0$ and the definition of $K$, we get $\|x_0\|_{C(I,\mathbb{R}^n)} \le K$. Summing inequality (3.5), gives

$$\|x\|_{C(I,\mathbb{R}^n)} \le K + \sum_{k=1}^{\infty} (2L_M)^{k-1} K T^k \, \frac{1}{k!} \le \max(1, (2L_M)^{-1}) K e^{2L_M T}.$$

In addition, we have

$$\|x'\|_{L^\infty(I,\mathbb{R}^n)} = \|f(\cdot, x, x_t, u)\|_{L^\infty(I,\mathbb{R}^n)} \le 2\, L_M \|x\|_{C(I,\mathbb{R}^n)} + K_M.$$

Let now $\tilde{U} \subset L^\infty(I, \mathbb{R}^m)$ be a bounded set with $\|u\|_{L^\infty(I,\mathbb{R}^m)} \le M$ for all $u \in \tilde{U}$. Then the estimate above is uniform with respect to controls $u \in \tilde{U}$, which proves the claim. □

Let us mention that the proof implies that the mapping $\phi \mapsto x$ is Lipschitz continuous.

**Corollary 3.3.** *Let Assumption 2.2 be satisfied. Let $u \in L^\infty(I, \mathbb{R}^m)$ be given. Then there exists a unique solution $x \in W^{1,\infty}(I; \mathbb{R}^n)$ of (3.1).*

*In addition, the mapping $u \mapsto x$ maps bounded sets in $L^\infty(I, \mathbb{R}^m)$ to bounded sets in $W^{1,\infty}(I; \mathbb{R}^n)$.*

*Proof.* The result follows directly from Theorem 3.2, as Assumption 2.2 implies Assumption 3.1. □

## 3.2 Existence of solutions of the optimal control problem

**Theorem 3.4.** *The optimal control problem (1.1)–(1.4) admits one solution.*

*Proof.* Due to the assumptions and Corollary 3.3, the feasible set is non-empty. Let $(x_k, u_k)$ be a minimizing sequence. By assumptions on $U$, the sequence $(u_k)$ is bounded in $L^\infty(I; \mathbb{R}^n)$, and by Theorem 3.2 the sequence $(x_k)$ is bounded in $W^{1,\infty}(I; \mathbb{R}^n)$. Hence, after possibly extracting subsequences, we have $u_k \rightharpoonup u$ in $L^2(I; \mathbb{R}^n)$ and $x_k \to x$ in $C(I_\tau; \mathbb{R}^n)$. Since the set of admissible controls is convex and closed in $L^2(I; \mathbb{R}^n)$, the feasibility of $u$ follows. Due to the special structure of $F$, we can pass to the limit in the integral equation (2.2). Hence, $x$ solves the integral equation, and consequently is a solution of (1.1). Due to Lemma 2.3, we obtain that $(x, u)$ is a solution of the considered optimal control problem. □

### 3.3 Non-existence of optimal controls for related problems

In this section, we are going to demonstrate that problem (1.1)–(1.4) may fail to possess optimal solutions, when the control function appears inside the maximum. Problems of this type were discussed in [Azhmyakov et al., 2016; Verriest, Azhmyakov, 2017].

We consider the problem with one-dimensional state and control

$$\begin{aligned} \text{Minimize} \quad & \int_0^1 |x(t) - 2\,t|^2 + u(t)\,\mathrm{d}t + 4\,|x(1) - 2| \\ \text{s.t.} \quad & x'(t) = \frac{1}{10} \max_{s \in [t-2,t]} (u(t)\,x(s)), \qquad t \in (0,1) \\ & x(t) = \phi(t), \qquad t \in [-2,0] \\ & -1 \le u \le 3. \end{aligned} \tag{3.6}$$

Moreover, $\phi : [-2,0] \to \mathbb{R}$ is a function such that the ranges of $\phi_{|[-2,-1]}$, $\phi_{|[-1,0]}$ are $[-10,10]$ and $\phi(0) = 0$.

Before we discussing the existence of controls, we are going to analyze the state equation. By splitting the max at $s = 0$, we find

$$|x'(t)| \le 3 + \frac{3}{10} \max_{s \in [0,t]} |x(s)|.$$

Integration yields

$$|x(t)| \le 3\,t + \frac{3}{10} \int_0^t \max_{s \in [0,r]} |x(s)|\,\mathrm{d}r.$$

Since this estimate is valid for all $t$ and since the right-hand side is monotone w.r.t. $t$, this implies

$$\max_{s \in [0,1]} |x(s)| \le 3 + \frac{3}{10} \int_0^1 \max_{s \in [0,r]} |x(s)|\,\mathrm{d}r \le 3 + \frac{3}{10} \max_{s \in [0,1]} |x(s)|.$$

Thus, $|x(t)| \le 30/7 \le 10$. This estimate allows us to evaluate the max in the state equation via

$$\max_{s \in [t-2,t]} (u(t)\,x(s)) = \max_{s \in [-1,0]} (u(t)\,x(s)) = 10\,|u(t)|.$$

Hence, we can simplify the state equation and obtain

$$x'(t) = |u(t)|.$$

Now, we can prove the main result of this section.

**Theorem 3.5.** *The control problem (3.6) does not possess a solution.*

*Proof. Step 1:* We show that the infimal value is at most 1. Let us define $u_k(t) = 1 + 2\,\mathrm{sign}(\sin(k\,t))$ and let $x_k$ be the associated state. Then, it is straightforward to check that $u_k \rightharpoonup \hat{u} \equiv 1$ in $L^2(0,1)$ and $x_k \to \hat{x}$ with $\hat{x}(t) = 2\,t$. This implies that the objective value of $(x_k, u_k)$ goes to $0 + 1 + 0 = 1$. Note that $\hat{x}$ is not the state corresponding to $\hat{u}$.

*Step 2:* We show that the objective value of all feasible $(x, t)$ is bigger than 1. We proceed by contradiction and assume that we have a feasible pair $(x, u)$ with objective value at most 1.

Let us denote

$$a := \int_{\{u<0\}} u \,\mathrm{d}t, \qquad b := \int_{\{u>0\}} u \,\mathrm{d}t.$$

By considering the control bounds and the length of the time interval, we obtain

$$1 = \int_{\{u<0\}} 1 \,\mathrm{d}t + \int_{\{u>0\}} 1 \,\mathrm{d}t \geq \int_{\{u<0\}} (-u) \,\mathrm{d}t + \int_{\{u>0\}} \frac{1}{3} u \,\mathrm{d}t = -a + \frac{b}{3}.$$

This inequality is equivalent to

$$-1 \geq 2(b - a - 2) - (a + b). \tag{3.7}$$

Since the objective is at most 1, we have

$$1 \geq \int_0^1 u \,\mathrm{d}x + 4\,|x(1) - 2| = \int_0^1 u \,\mathrm{d}t + 4 \left| \int_0^1 |u| \,\mathrm{d}t - 2 \right| = a + b + 4\,|b - a - 2|. \tag{3.8}$$

Adding the inequalities (3.7) and (3.8) yields $b - a - 2 = 0$, which in turn implies $\int_0^1 u \,\mathrm{d}t = a + b = 1$. By considering again the objective, we infer $x(t) = 2\,t$, i.e., $|u(t)| = x'(t) = 2$. Hence, $u(t) = 2$ and this is a contradiction. □

As a side result, this theorem also shows that it is not possible to relax the assumption that $F$ is affine in $u$, cf. (2.1).

## 4 LogIntExp as a generalization of LogSumExp

Let $x_i$, $i = 1, \ldots, n$ be given real numbers. It is well known that the maximum $\max(x_1, \ldots, x_n)$ depends in a non-smooth way on the parameters $x_i$. This has severe drawbacks in many applications. Therefore, a typical substitute for this hard maximum is the so-called LogSumExp function. For a given parameter $k > 0$, this function is defined via

$$\mathrm{LSE}_k(x_1, \ldots, x_n) := \frac{1}{k} \log\Bigl( \sum_{i=1}^n \exp(k\, x_i) \Bigr). \tag{4.1}$$

In the next lemma, we summarize some of the well-known properties of LogSumExp, see, e.g., [Rockafellar, Wets, 1998, Example 1.30], [Rockafellar, 1970, p. 325].

**Lemma 4.1.** *Let $k > 0$ be a given parameter. Then, the function $\mathrm{LSE}_k$ is convex, smooth, and Lipschitz continuous with rank 1. The estimate*

$$\max(x_1, \ldots, x_n) \leq \mathrm{LSE}_k(x_1, \ldots, x_n) \leq \max(x_1, \ldots, x_n) + \frac{\log(n)}{k}$$

*shows that $\mathrm{LSE}_k(x_1, \ldots, x_n) \to \max(x_1, \ldots, x_n)$ as $k \to \infty$. Concerning the derivatives, we have*

$$\frac{\mathrm{d}}{\mathrm{d}x_i} \mathrm{LSE}_k(x_1, \ldots, x_n) = \frac{\exp(k\, x_i)}{\sum_{j=1}^n \exp(k\, x_j)}.$$

*In particular,*

$$\frac{\mathrm{d}}{\mathrm{d}x_i}\mathrm{LSE}_k(x_1,\dots,x_n) \geq 0 \qquad \textit{and} \qquad \sum_{i=1}^{n}\frac{\mathrm{d}}{\mathrm{d}x_i}\mathrm{LSE}_k(x_1,\dots,x_n) = 1.$$

Due to these nice properties, the LogSumExp function has many applications, e.g., in machine learning [Goodfellow et al., 2016].

Currently there is no analogue smoothing for the essential supremum of measurable functions available. We are going to close this gap. To this end, we consider a measure space $(\Omega, \Sigma, \mu)$ with $\mu(\Omega) < \infty$. For convenience, integrals w.r.t. $\mu$ are indicated by $\int_\Omega \dots \mathrm{d}x$. Let $u : \Omega \to \mathbb{R}$ be a measurable function. We define the LogIntExp of $u$ with parameter $k > 0$ via

$$\mathrm{LIE}_k(u) := \frac{1}{k}\log\Big(\int_\Omega \exp(k\,u(x))\,\mathrm{d}x\Big). \tag{4.2}$$

First, we state some basic properties of LogIntExp.

**Lemma 4.2.** *For every measurable $u : \Omega \to \mathbb{R}$ and $k > 0$, the quantity $\mathrm{LIE}_k(u) \in \mathbb{R} \cup \{+\infty\}$ is well-defined and convex. Moreover, we have the estimates*

$$\operatorname*{ess\,inf}_{x\in\Omega} u(x) + \frac{\log(\mu(\Omega))}{k} \leq \mathrm{LIE}_k(u) \leq \operatorname*{ess\,sup}_{x\in\Omega} u(x) + \frac{\log(\mu(\Omega))}{k}.$$

*If for some $\delta \in \mathbb{R}$, the measure of the set $\{x \in \Omega \mid u(x) \geq \delta\}$ is at least $\varepsilon > 0$, then*

$$\delta + \frac{\log(\varepsilon)}{k} \leq \mathrm{LIE}_k(u).$$

*In particular,*

$$\mathrm{LIE}_k(u) \to \operatorname*{ess\,sup}_{x\in\Omega} u(x) \in \mathbb{R} \cup \{+\infty\} \qquad \textit{as } k \to \infty.$$

*Proof.* First, we comment on the well definedness. The function $x \mapsto \exp(k\,u(x))$ is measurable and positive, therefore the Lebesgue integral

$$\int_\Omega \exp(k\,u(x))\,\mathrm{d}x \in \mathbb{R} \cup \{+\infty\}$$

is well defined. Taking the logarithm (with the convention $\log(+\infty) = +\infty$) yields the claim.

The convexity of $\mathrm{LIE}_k$ is a simple application of Hölder's inequality. Indeed, for measurable functions $u, v$ satisfying $\mathrm{LIE}_k(u), \mathrm{LIE}_k(v) < \infty$ and $\lambda \in (0,1)$, we have

$$\begin{aligned}\mathrm{LIE}_k(\lambda\, u + (1-\lambda)\, v) &= \frac{1}{k}\,\log\Big(\int_\Omega \exp(\lambda\, k\, u + (1-\lambda)\, k\, v)\,\mathrm{d}x\Big)\\ &= \frac{1}{k}\,\log\Big(\int_\Omega \exp(\lambda\, k\, u)\,\exp((1-\lambda)\, k\, v)\,\mathrm{d}x\Big)\\ &\le \frac{1}{k}\,\log\Big(\Big[\int_\Omega \exp(\lambda\, k\, u)^{1/\lambda}\Big]^{\lambda}\,\Big[\int_\Omega \exp((1-\lambda)\, k\, v)^{1/(1-\lambda)}\Big]^{1-\lambda}\Big)\\ &= \frac{\lambda}{k}\,\log\Big(\int_\Omega \exp(k\, u)\,\mathrm{d}x\Big) + \frac{1-\lambda}{k}\,\log\Big(\int_\Omega \exp(k\, v)\,\mathrm{d}x\Big)\\ &= \lambda\,\mathrm{LIE}_k(u) + (1-\lambda)\,\mathrm{LIE}_k(v).\end{aligned}$$

The estimates for $\mathrm{LIE}_k(u)$ follow from

$$\mu(\Omega)\,\exp(k\,\operatorname*{ess\,inf}_{x\in\Omega} u(x)) \le \int_\Omega \exp(k\, u(x))\,\mathrm{d}x \le \mu(\Omega)\,\exp(k\,\operatorname*{ess\,sup}_{x\in\Omega} u(x))$$

and

$$\varepsilon\,\exp(k\,\delta) \le \int_\Omega \exp(k\, u(x))\,\mathrm{d}x.$$

□

On the space $L^\infty(\Omega)$, we have even nicer properties.

**Lemma 4.3.** *The function* $\mathrm{LIE}_k : L^\infty(\Omega) \to \mathbb{R}$ *is continuously differentiable with*

$$\mathrm{LIE}_k'(u)\, v = \frac{\int_\Omega \exp(k\, u(x))\, v(x)\,\mathrm{d}x}{\int_\Omega \exp(k\, u(\hat{x}))\,\mathrm{d}\hat{x}}.$$

*In particular,* $\mathrm{LIE}_k$ *is Lipschitz continuous with rank* $1$.

*Proof.* Standard results on Nemytski operators imply that $u \mapsto \exp(k\, u)$ is **continuously** Fréchet differentiable from $L^\infty(\Omega)$ to $L^1(\Omega)$ with derivative $k\,\exp(k\, u)$. Hence, the chain rule implies the announced formula for the derivative of $\mathrm{LIE}_k$. Finally, the function $x \mapsto \exp(k\, u(x))/\int_\Omega \exp(k\, u(\hat{x}))\,\mathrm{d}\hat{x}$ belongs to $L^1(\Omega)$ and has norm 1. Thus, the weak mean value theorem, see [Cartan, 1967, Proposition 3.3.1], implies the Lipschitz continuity of $\mathrm{LIE}_k$. □

This Lipschitz continuity implies that

$$\begin{aligned}\big|\mathrm{LIE}_k(u_k) - \operatorname{ess\,sup} u\big| &\le \big|\mathrm{LIE}_k(u_k) - \mathrm{LIE}_k(u)\big| + \big|\mathrm{LIE}_k(u) - \operatorname{ess\,sup} u\big|\\ &\le \|u_k - u\|_{L^\infty(\Omega)} + \big|\mathrm{LIE}_k(u) - \operatorname{ess\,sup} u\big| \to 0\end{aligned}$$

for $u_k \to u$ in $L^\infty(\Omega)$.

Finally, we provide an estimate specialized to our problem (3.1). From now on, the measure space $\Omega$ is just the interval $[0,\tau]$ (with the one-dimensional Lebesgue measure). Recall that $x_t(s) = x(t-s)$ for $s \in [0,\tau]$. In particular,

$$\mathrm{LIE}_k(x_t) = \frac{1}{k}\,\log\Big(\int_0^\tau \exp(k\, x_t(s))\,\mathrm{d}s\Big) = \frac{1}{k}\,\log\Big(\int_{t-\tau}^t \exp(k\, x(s))\,\mathrm{d}s\Big).$$

**Lemma 4.4.** *Let $x \in L^\infty(-\tau, T)$ be a given function. Then,*

$$\int_0^T \left|\operatorname{ess\,sup} x_t - \mathrm{LIE}_k(x_t)\right| \mathrm{d}t \to 0.$$

*Proof.* The convergence result from Lemma 4.2 implies the pointwise convergence

$$\left|\operatorname{ess\,sup} x_t - \mathrm{LIE}_k(x_t)\right| \to 0$$

for all $t \in I$. Moreover, we have the integrable bound

$$\left|\operatorname{ess\,sup} x_t - \mathrm{LIE}_k(x_t)\right| \le 2\,\|x\|_{L^\infty(-\tau,T)} + \frac{|\log(\tau)|}{k} \le 2\,\|x\|_{L^\infty(-\tau,T)} + |\log(\tau)|,$$

cf. Lemma 4.2. Thus, the dominated convergence theorem yields the claim. □

# 5 Regularization

## 5.1 Regularized state equation

As a regularization of (3.1), we use

$$x'(t) = F(x(t), \mathrm{LIE}_k(x_t), u(t)), \qquad t \in I. \tag{5.1}$$

Note that $\mathrm{LIE}_k(x_t)$ is understood component-wise. The existence of a unique solution follows directly from Theorem 3.2, due to the Lipschitz continuity of $\mathrm{LIE}_k$.

The convergence of the solutions of (5.1) towards the solution of (3.1) is made precise in the next result. Let us emphasize that the proof heavily relies on the affine structure of $F$.

**Lemma 5.1.** *Let the sequence $(u_k)$ be bounded in $L^\infty(I;\mathbb{R}^m)$. Let $x_k$ be the solution of*

$$x_k'(t) = F(x_k(t), \mathrm{LIE}_k(x_{k,t}), u_k(t)), \qquad t \in I,$$

*with initial data $x_k(t) = \phi(t)$ for $t \in [-\tau, 0]$. If $u_k \rightharpoonup u$ in $L^1(I;\mathbb{R}^m)$ then $x_k \to x$ in $C(I;\mathbb{R}^n)$.*

*Proof.* Let $M > 0$ such that $\|u_k\|_{L^\infty(I;\mathbb{R}^m)} \le M$, which implies $\|u\|_{L^\infty(I;\mathbb{R}^m)} \le M$. Since $\mathrm{LIE}_k$ is Lipschitz continuous uniformly with respect to $k$, the solutions $(x_k)$ are bounded in $W^{1,\infty}(I;\mathbb{R}^n)$ by Theorem 3.2. Let $L > 0$ denote the maximum of the Lipschitz moduli of $F_0$ and $F_1$.

First, we investigate the difference of the equations

$$\begin{aligned} x'(t) - x_k'(t) &= F_0(x(t), \max x_t) - F_0(x_k(t), \mathrm{LIE}_k(x_{k,t})) \\ &\quad + u(t)\, F_1(x(t), \max x_t) - u_k(t)\, F_1(x_k(t), \mathrm{LIE}_k(x_{k,t})) \\ &= F_0(x(t), \max x_t) - F_0(x_k(t), \mathrm{LIE}_k(x_{k,t})) \\ &\quad + (u(t) - u_k(t))\, F_1(x(t), \max x_t) \\ &\quad + u_k(t)\, \big(F_1(x(t), \max x_t) - F_1(x_k(t), \mathrm{LIE}_k(x_{k,t}))\big). \end{aligned}$$

By Lipschitz continuity, we have

$$
\begin{aligned}
&|F_0(x(t), \max x_t) - F_0(x_k(t), \mathrm{LIE}_k(x_{k,t}))| \\
&\qquad \le L\left(|x(t) - x_k(t)| + |\max x_t - \mathrm{LIE}_k(x_{k,t})|\right) \\
&\qquad \le L\left(|x(t) - x_k(t)| + |\max x_t - \mathrm{LIE}_k(x_t)| + |\mathrm{LIE}_k(x_t) - \mathrm{LIE}_k(x_{k,t})|\right) \\
&\qquad \le L\left(|x(t) - x_k(t)| + |\max x_t - \mathrm{LIE}_k(x_t)| + \max|x_t - x_{k,t}|\right).
\end{aligned}
$$

Similarly, we can estimate

$$
\begin{aligned}
&|u_k(t)\left(F_1(x(t), \max x_t) - F_1(x_k(t), \mathrm{LIE}_k(x_{k,t}))\right)| \\
&\qquad \le LM\left(|x(t) - x_k(t)| + |\max x_t - \mathrm{LIE}_k(x_t)| + \max|x_t - x_{k,t}|\right).
\end{aligned}
$$

Integration over $t$ and using the Lipschitz estimates above, we have

$$
\begin{aligned}
|x(t) - x_k(t)| \le L(M+1) &\int_0^t |x(s) - x_k(s)| + \max|x_s - x_{k,s}| \,\mathrm{d}s \\
&+ \left|\int_0^t (u(s) - u_k(s))\, F_1(x(s), \max x_s)\,\mathrm{d}s\right| \\
&+ L(M+1)\int_0^t |\max x_s - \mathrm{LIE}_k(x_s)|\,\mathrm{d}s
\end{aligned}
$$

for all $t \in I$. Since $(x_k)$ is bounded in $W^{1,\infty}(I;\mathbb{R}^n)$, there exists a subsequence (without relabeling), such that $x_k \to \tilde{x}$ in $C(I;\mathbb{R}^n)$. Passing to the limit in the above inequality yields

$$
|x(t) - \tilde{x}(t)| \le L\,(M+1)\int_0^t |x(s) - \tilde{x}(s)| + \max|x_s - \tilde{x}_s|\,\mathrm{d}s
$$

for all $t \in I$. The integral inequality from Lemma 2.1 implies $x = \tilde{x}$. Thus, a standard subsequence-subsequence argument implies $x_k \to x$ in $C(I;\mathbb{R}^n)$ for the entire sequence $x_k$. □

**Corollary 5.2.** *Under the assumptions of Lemma 5.1, we have $x_k \rightharpoonup x$ in $W^{1,1}(I;\mathbb{R}^n)$.*

*Proof.* Using the result of Lemma 5.1 and Assumption 2.2, it is easy to see that the mappings $t \mapsto F(x_k(t), \mathrm{LIE}_k(x_{k,t}), u_k(t))$ converge weakly in $L^1(I;\mathbb{R}^n)$ to $t \mapsto F(x(t), \max x_t, u(t))$. □

**Remark 5.3.** *If $F$ is supposed to be nonlinear with respect to the control, then the result of the above Lemma 5.1 is no longer true. In order to obtain convergence of the states, one has to assume strong convergence $u_k \to u$. However, in Section 5.2 below, we have to deal with weakly converging sequences of controls $(u_k)$, see, e.g., the proof of Theorem 5.5. Hence, we are restricted to the affine setting.*

### 5.2 Regularized optimal control problem

In the following, let $(x^*, u^*)$ be a local solution of the original problem. Then there is $\delta > 0$ such that $J(x^*, u^*) \le J(x, u)$ for all feasible controls $u$ with associated states $x$ satisfying $\|u - u^*\|_{L^2(I;\mathbb{R}^m)} \le \delta$.

Let us consider the following regularized optimal control problem: Minimize

$$J(x,u) + \frac{1}{2}\,\|u - u^*\|_{L^2(I;\mathbb{R}^m)} = \int_0^T j(t, x(t), u(t)) + \frac{1}{2}\,|u(t) - u^*(t)|^2 \,\mathrm{d}t \tag{5.2}$$

subject to the non-linear equation (5.1), the initial condition (1.2), the control constraints (1.4), and the auxiliary constraints

$$\|u - u^*\|_{L^2(I;\mathbb{R}^m)} \le \delta. \tag{5.3}$$

**Theorem 5.4.** *For every $k$, the regularized optimal control problem admits a global solution $(x_k, u_k)$.*

*Proof.* The proof can be carried out using similar arguments as in the proof of Theorem 3.4. □

**Theorem 5.5.** *Let $(x_k, u_k)$ be a sequence of global solutions of the regularized optimal control problem. Then we have the convergence $x_k \to x^*$ and $u_k \to u^*$ in $C(I, \mathbb{R}^n)$ and $L^2(I; \mathbb{R}^m)$, respectively.*

*Proof.* Let $x_k^*$ denote the solution of the regularized differential equation (5.1) to the control $u^*$. Then Lemma 5.1 implies $x_k^* \to x^*$ in $C(I, \mathbb{R}^n)$. The integrand $j$ is continuous with respect to $x$, which implies $J(x_k^*, u^*) \to J(x^*, u^*)$. Since $(u_k)$ is a bounded sequence in $L^2(I; \mathbb{R}^m)$, we obtain after extracting a subsequence $u_{k_n} \rightharpoonup \bar{u}$ in $L^2(I; \mathbb{R}^m)$. Then $\bar{u}$ satisfies the control constraints as well as $\|\bar{u} - u^*\|_{L^2(I;\mathbb{R}^m)} \le \delta$. Using Lemma 5.1 again, we find $x_{k_n} \to \bar{x}$ in $C(I, \mathbb{R}^n)$, which solves (1.1) to $\bar{u}$. In addition, Lemma 2.3 yields $J(\bar{x}, \bar{u}) \le \liminf_{n\to\infty} J(x_{k_n}, u_{k_n})$. By global optimality, we have

$$J(x_k, u_k) + \frac{1}{2}\|u_k - u^*\|^2_{L^2(I;\mathbb{R}^m)} \le J(x_k^*, u^*).$$

Passing to the limit along the subsequence yields

$$\begin{aligned} J(\bar{x}, \bar{u}) + \frac{1}{2}\|\bar{u} - u^*\|^2_{L^2(I;\mathbb{R}^m)} &\le \liminf_{n\to\infty} \left( J(x_{k_n}, u_{k_n}) + \frac{1}{2}\|u_{k_n} - u^*\|^2_{L^2(I;\mathbb{R}^m)} \right) \\ &\le \limsup_{n\to\infty} \left( J(x_{k_n}, u_{k_n}) + \frac{1}{2}\|u_{k_n} - u^*\|^2_{L^2(I;\mathbb{R}^m)} \right) \le J(x^*, u^*), \end{aligned}$$

which implies $(\bar{x}, \bar{u}) = (x^*, u^*)$. Hence, the above chain of inequalities are equalities, which imply the strong convergence $u_{k_n} \to u^*$ in $L^2(I; \mathbb{R}^m)$. Since the limit is independent of the chosen subsequence, we obtain convergence of the whole sequence. □

Let $(x_k, u_k)$ be locally optimal for the regularized problem. For abbreviation, let us define

$$F^k(t) := F(x_k(t), \mathrm{LIE}_k(x_{k,t}), u_k(t)).$$

Similarly, we define $F^k_x(t)$, $j^k_x$, and $F^k_y(t)$ to be the derivatives of $F$ and $j$ with respect to the first and second argument, respectively.

**Theorem 5.6.** *Let $(x_k, u_k)$ be locally optimal for the regularized problem with $\|u_k - u^*\|_{L^2(I)} < \delta$. Then there exists $\lambda_k \in W^{1,\infty}(I, \mathbb{R}^n)$ with $\lambda_k(T) = 0$ such that*

$$\lambda_k'(s)^\intercal + \lambda_k(s)^\intercal F^k_x(s) + \int_s^{\min(s+\tau,T)} \lambda_k(t)^\intercal F^k_y(t)\, \mathrm{diag}\left(\frac{\exp(k\, x_k(s))}{\int_{t-\tau}^t \exp(k\, x_k(\hat{s}))\, \mathrm{d}\hat{s}}\right) \mathrm{d}t = j^k_x(s) \tag{5.4}$$

*is satisfied for almost all $s \in (0, T)$. Here, $\mathrm{diag}\,(v)$ denotes the diagonal matrix with diagonal entries taken from the vector $v$. The division and exponentiation has to be understood component-wise.*

*Moreover, the maximum principle in integrated form holds*

$$\int_0^T \lambda_k(t)^\intercal F(x_k(t), \mathrm{LIE}_k(x_{k,t}), u_k(t)) - j(t, x_k(t), u_k(t)) - (u_k(t) - u^*(t))^\intercal u_k(t)\, \mathrm{d}t \geq$$
$$\int_0^T \lambda_k(t)^\intercal F(x_k(t), \mathrm{LIE}_k(x_{k,t}), u(t)) - j(t, x_k(t), u(t)) - (u_k(t) - u^*(t))^\intercal u(t) \mathrm{d}t \tag{5.5}$$

*for all $u$ satisfying the control constraint (1.4).*

*Proof.* Due to the assumptions, the control $u_k$ is a local solution of an optimal control problem without the constraint (5.3). We are going to apply [Banks, 1969, Theorem 1]. Due to the standing Assumption 2.2, the requirements on the problem are fulfilled. Hence, there exists multipliers $\lambda_0$ and $\lambda$, satisfying a system that constitutes the optimality conditions of the regularized problem. We will develop this system in the course of the proof using the notation of [Banks, 1969].

Since the control problem does not include constraints on $x(T)$, we can set $\lambda_0 = -1$. By this theorem, there exists an adjoint state $\lambda : I \to \mathbb{R}^n$ of bounded variation, such that

$$\lambda(s)^\intercal + \int_s^T \lambda_0 \eta_0(t, s) + \lambda(t)^\intercal \eta(t, s) \mathrm{d}t = \lambda^\intercal(T) = 0 \quad \forall s \in [0, T). \tag{5.6}$$

Here, the matrix-valued quantities $\eta(s, t)$ and $\eta_0(s, t)$ are defined by the equations[1]

$$F^k_x(t)\xi(t) + F^k_y(t)\mathrm{LIE}_k'(x_{k,t})\xi_t = \int_{-\tau}^t \mathrm{d}_s \eta(t, s)\xi(s) \quad \forall \xi \in C(I_\tau, \mathbb{R}^n), t \in I,$$

where the latter integral denotes the Lebesgue-Stieltjes integral with respect to the integration variable $s$, and

$$j^k_x(t)\xi(t) = \int_{-\tau}^t \mathrm{d}_s \eta_0(t, s)\xi(s) \quad \forall \xi \in C(I_\tau, \mathbb{R}), t \in I.$$

[1] The $i$-th component of the integral on the right-hand side is defined as $\sum_{j=1}^n \int_{-\tau}^t \xi_j(s) \mathrm{d}_s \eta_{i,j}(t, s)$.

In order to investigate the adjoint equation, we have to calculate an explicit expression of $\eta$. First, it is not difficult to check, see also [Banks, 1969, Section 4], that it holds

$$F_x^k(t)\xi(t) = \int_{-\tau}^{t} \mathrm{d}_s\eta_1(t,s)\xi(s) \quad \forall \xi \in C(I_\tau, \mathbb{R}^n), t \in I$$

for $\eta_1$ defined by

$$\eta_1(t,s) = -\chi_{[-\tau,t)}(s) \cdot F_x^k(t),$$

which directly implies

$$\int_s^T \lambda(t)^\mathsf{T}\eta_1(t,s)\mathrm{d}t = -\int_s^T \lambda(t)^\mathsf{T} F_x^k(t)\mathrm{d}t. \tag{5.7}$$

Analogously, we get for $\eta_0(t,s) = -\chi_{[-\tau,t)}(s) \cdot j_x^k(t)$

$$\int_s^T \lambda_0\eta_0(t,s)\mathrm{d}t = -\int_s^T \eta_0(t,s)\mathrm{d}t = \int_s^T j_x^k(t)\mathrm{d}t. \tag{5.8}$$

Second, we find by elementary calculations

$$\begin{aligned} F_y^k(t)\mathrm{LIE}_k'(x_{k,t})\xi_t &= F_y^k(t)\frac{\int_{t-\tau}^t \exp(k\,x_k(s)) \odot \xi(s)\,\mathrm{d}s}{\int_{t-\tau}^t \exp(k\,x_k(\hat{s}))\,\mathrm{d}\hat{s}} \\ &= F_y^k(t)\int_{-\tau}^t \chi_{(t-\tau,t)}(s)\operatorname{diag}\left(\frac{\exp(k\,x_k(s))}{\int_{t-\tau}^t \exp(k\,x_k(\hat{s}))\,\mathrm{d}\hat{s}}\right)\xi(s)\,\mathrm{d}s. \end{aligned}$$

With the choice

$$\eta_2(t,s) := -F_y^k(t)\int_s^t \chi_{(t-\tau,t)}(s')\operatorname{diag}\left(\frac{\exp(k\,x_k(s'))}{\int_{t-\tau}^t \exp(k\,x_k(\hat{s}))\,\mathrm{d}\hat{s}}\right)\mathrm{d}s'$$

we get the identity

$$F_y^k(t)\mathrm{LIE}_k'(x_{k,t})\xi_t = \int_{-\tau}^t \mathrm{d}_s\eta_2(t,s)\xi(s).$$

In addition, we find

$$\begin{aligned} \int_s^T \lambda(t)^\mathsf{T}\eta_2(t,s)\mathrm{d}t &= -\int_s^T \lambda(t)^\mathsf{T}F_y^k(t)\int_s^t \chi_{(t-\tau,t)}(s')\operatorname{diag}\left(\frac{\exp(k\,x_k(s'))}{\int_{t-\tau}^t \exp(k\,x_k(\hat{s}))\,\mathrm{d}\hat{s}}\right)\mathrm{d}s'\,\mathrm{d}t \\ &= -\int_s^T\int_{s'}^{\min(s'+\tau,T)} \lambda(t)^\mathsf{T}F_y^k(t)\operatorname{diag}\left(\frac{\exp(k\,x_k(s'))}{\int_{t-\tau}^t \exp(k\,x_k(\hat{s}))\,\mathrm{d}\hat{s}}\right)\mathrm{d}t\,\mathrm{d}s'. \end{aligned} \tag{5.9}$$

Using $\eta := \eta_1 + \eta_2$, (5.7)–(5.9) in the adjoint equation (5.6) yields

$$\begin{aligned} &\lambda(s)^\mathsf{T} - \int_s^T \lambda(t)^\mathsf{T}F_x^k(t)\mathrm{d}t + \int_s^T j_x^k(t)\mathrm{d}t \\ &\qquad - \int_s^T\int_{s'}^{\min(s'+\tau,T)} \lambda(t)^\mathsf{T}F_y^k(t)\operatorname{diag}\left(\frac{\exp(k\,x_k(s'))}{\int_{t-\tau}^t \exp(k\,x_k(\hat{s}))\,\mathrm{d}\hat{s}}\right)\mathrm{d}t\,\mathrm{d}s' = 0. \end{aligned}$$

Since $\lambda$ is of bounded variation, the integrands are bounded functions, which implies that $\lambda \in W^{1,\infty}(I;\mathbb{R}^n)$. In addition, the differential equation

$$\lambda'(s)^\mathsf{T} + \lambda(s)^\mathsf{T} F_x^k(s) + \int_s^{\min(s+\tau,T)} \lambda(t)^\mathsf{T} F_y^k(t) \operatorname{diag}\left(\frac{\exp(k\, x_k(s))}{\int_{t-\tau}^t \exp(k\, x_k(\hat{s}))\,\mathrm{d}\hat{s}}\right) \mathrm{d}t = j_x^k(s)$$

is satisfied for almost all $s \in I$ with the terminal value $\lambda(T) = 0$. In addition, the result of [Banks, 1969] includes the maximum principle

$$\int_0^T \lambda_k(t)^\mathsf{T} F(x_k(t), \mathrm{LIE}_k(x_{k,t}), u_k(t)) - j(t, x_k(t), u_k(t)) - \frac{1}{2}\left|u_k(t) - u^*(t)\right|^2 \mathrm{d}t \geq$$
$$\int_0^T \lambda_k(t)^\mathsf{T} F(x_k(t), \mathrm{LIE}_k(x_{k,t}), u(t)) - j(t, x_k(t), u(t)) - \frac{1}{2}\left|u(t) - u^*(t)\right|^2 \mathrm{d}t$$

for admissible $u$. As the mapping $u \mapsto \int_0^T -\lambda_k^\mathsf{T} F(x_k, \mathrm{LIE}_k(x_{k,t}), u) + j(t, x_k, u)\mathrm{d}t$ is convex, this implies (5.5). ☐

Testing the adjoint equation (5.4) with a test function $v \in L^\infty(I_\tau, \mathbb{R}^n)$ with $v(t) = 0$ for $t \in (-\tau, 0)$ and undoing the interchanging of integration order in the previous proof, we arrive at the following weak formulation of the adjoint equation,

$$\int_0^T \lambda_k'(t)^\mathsf{T} v(t) + \lambda_k(t)^\mathsf{T} F_x^k(t) v(t) + \lambda_k(t)^\mathsf{T} F_y^k(t) \frac{\int_{t-\tau}^t \exp(k\, x_k(s)) \odot v(s)\,\mathrm{d}s}{\int_{t-\tau}^t \exp(k\, x_k(\hat{s}))\,\mathrm{d}\hat{s}} \mathrm{d}t$$
$$= \int_0^T j_x^k(t) v(t) \mathrm{d}t, \quad (5.10)$$

which is more suitable for studying the limit $k \to \infty$.

**Theorem 5.7.** *The sequence* $(\lambda_k)$ *is bounded in* $L^\infty(I;\mathbb{R}^n)$ *and* $W^{1,1}(I;\mathbb{R}^n)$.

*Proof.* Since $(x_k)$ and $(u_k)$ are bounded in $C(I;\mathbb{R}^n)$ and $L^\infty(I;\mathbb{R}^m)$, respectively, we find that $(F_x^k)$, $(F_y^k)$, and $(j_x^k)$ are bounded in $L^\infty(I;\mathbb{R}^{n,n})$ and $L^\infty(I;\mathbb{R}^n)$, respectively. Setting $v(s) = \chi_{(t',T)}(s)\frac{\lambda_k(s)}{|\lambda_k(s)|}$ in (5.10), we obtain

$$|\lambda_k(t')| \leq C \int_{t'}^T |\lambda_k(t)| + |\lambda_k(t)| \cdot \left|\frac{\int_{\max(t-\tau,t')}^t \exp(k\, x_k(s)) \odot \lambda_k(s) \cdot |\lambda_k(s)|^{-1}\,\mathrm{d}s}{\int_{t-\tau}^t \exp(k\, x_k(\hat{s}))\,\mathrm{d}\hat{s}}\right| + 1\,\mathrm{d}t$$
$$\leq C \int_{t'}^T |\lambda_k(t)| + 1\,\mathrm{d}t$$

with some constant $C$ independent of $k$. Here, the usage of $\frac{\lambda_k(s)}{|\lambda_k(s)|}$ can be justified by first testing with $\frac{\lambda_k(s)}{\sqrt{|\lambda_k(s)|^2+\epsilon}}$, and then passing to the limit $\epsilon \searrow 0$. By Gronwall's inequality, we find that $(\lambda_k)$ is bounded in $L^\infty(I;\mathbb{R}^n)$.

Now let $v \in L^\infty(I;\mathbb{R}^n)$ be arbitrary. Then we get from (5.10) the estimate

$$\int_0^T \lambda_k'(t)^T v(t)\mathrm{d}t \le C \int_0^T |\lambda_k(t)| \cdot |v(t)| + (|\lambda_k(t)| + 1) \cdot \|v\|_{L^\infty(I;\mathbb{R}^n)}\mathrm{d}t$$
$$\le C(\|\lambda_k\|_{L^\infty(I;\mathbb{R}^n)} + 1)\|v\|_{L^\infty(I;\mathbb{R}^n)}.$$

with some constant $C > 0$ independent of $k$. By the uniform boundedness principle, $(\lambda_k')$ is a bounded sequence in $L^1(I;\mathbb{R}^n)$. □

**Lemma 5.8.** *The sequence of functions* $(t,s) \mapsto \mathrm{LIE}_k'(x_{k,t})(s)$ *is uniformly bounded in* $L^\infty(I;L^1(I_\tau))^n$*, where we used the notation*

$$\mathrm{LIE}_k'(x_t)(s) = \frac{\chi_{(t-\tau,t)}(s)\exp(k\,x(s))}{\int_{t-\tau}^t \exp(k\,x(\hat{s}))\,\mathrm{d}\hat{s}}.$$

*Proof.* Obviously, this function is non-negative. Let $t \in I$ be given. Then it holds

$$\|\mathrm{LIE}_k'(x_t)\|_{L^1(I_\tau)} = \int_{-\tau}^T \mathrm{LIE}_k'(x_t)(s)\mathrm{d}s = \int_{-\tau}^T \frac{\chi_{(t-\tau,t)}(s)\exp(k\,x(s))}{\int_{t-\tau}^t \exp(k\,x(\hat{s}))\,\mathrm{d}\hat{s}}\mathrm{d}s = e.$$

Here, the $e$ is the vector in $\mathbb{R}^n$ with all entries equal to 1. □

# 6 Passing to the limit in the optimality system

In this section, we are going to pass to the limit $k \to \infty$ in the optimality system provided by Theorem 5.6. The main work is to understand the behaviour of the expression $\mathrm{LIE}_k'(x_{k,t})(s)$ which appears in the adjoint equation (5.10). We define $\mu_k \in L^\infty(I;L^1(I_\tau))^n$ via

$$\mu_k(t,s) := \mathrm{LIE}_k'(x_{k,t})(s), \tag{6.1}$$

cf. Lemma 5.8. We have seen in Lemma 5.8 that $\mu_k$ is bounded in the space $L^\infty(I;L^1(I))^n$. Since this space is neither a dual space nor a reflexive space, we cannot extract a subsequence which is weak($\star$)ly convergent in this space. Therefore, we embed this space into a suitable dual space, see Section 6.1. Finally, Section 6.2 contains the necessary optimality system.

## 6.1 The dual space of $L^1(I;C(I_\tau))$

It is well known that $L^1(I_\tau)$ is not a dual space. Therefore, $L^\infty(I;L^1(I_\tau))$ cannot be a dual space as well. A typical remedy is to embed $L^1(I_\tau)$ into $\mathcal{M}(I_\tau)$, where $\mathcal{M}(I_\tau)$ is the space of regular signed Borel measures on $I_\tau$ equipped with the total variation norm, since this is the dual space of $C(I_\tau)$, see [Rudin, 1987, Theorem 6.19]. We will see below, that the dual space of $L^1(I;C(I_\tau))$ will be useful in our situation. In order to characterize this dual space, we have to introduce a space of weak-$\star$ measurable functions. We follow the presentation in [Papageorgiou, Kyritsi-Yiallourou, 2009, Section 10.1].

**Definition 6.1.** *A function $\nu : I \to \mathcal{M}(I_\tau)$ is said to be weak-$\star$ measurable if the function*

$$t \mapsto \langle \nu(t), z\rangle$$

*is measurable for all $z \in C(I_\tau)$. If $\nu_1, \nu_2$ are weak-$\star$ measurable, we define the equivalence relation $\sim$ via*

$$\nu_1 \sim \nu_2 \quad \Leftrightarrow \quad \langle \nu_1(t) - \nu_2(t), z\rangle = 0 \text{ f.a.a. } t \in I \text{ for all } z \in C(I_\tau).$$

*Note that the null set may depend on $z$. Finally, the space $L^\infty(I; \mathcal{M}_w(I_\tau))$ consists of all equivalence classes $[\nu]$ of weak-$\star$ measurable functions $\nu : I \to \mathcal{M}(I_\tau)$ satisfying*

$$\left|\langle \nu(t), z\rangle\right| \leq c\, \|z\|_{C(I_\tau)} \qquad \text{f.a.a. } t \in I \text{ for all } z \in C(I_\tau).$$

*for some $c \geq 0$. Again, the null set may depend on $z \in C(I_\tau)$. The infimum of all these constants $c \geq 0$ is denoted by $\|\nu\|_{L^\infty(0,T;\mathcal{M}_w(I_\tau))}$. This is a norm on $L^\infty(0, T; \mathcal{M}_w(I_\tau))$.*

**Theorem 6.2.** *The dual space of $L^1(0, T; C(I_\tau))$ is isometrically isomorphic to the space $L^\infty(I; \mathcal{M}_w(I_\tau))$ via the duality pairing*

$$\langle \nu, z\rangle := \int_0^T \langle \nu(t), z(t)\rangle \, \mathrm{d}t$$

*for all $\nu \in L^\infty(I; \mathcal{M}_w(I_\tau))$ and $z \in L^1(I; C(I_\tau))$.*

This result can be found in [Papageorgiou, Kyritsi-Yiallourou, 2009, Theorem 10.1.16], see also [Edwards, 1995, Theorem 8.18.2]. We also note that the measurability of

$$t \mapsto \langle \nu(t), z(t)\rangle$$

(which is essential for the definition of the above duality pairing) is proven in the first part of the proof of [Edwards, 1995, Theorem 8.18.2].

In the sequel, it will be useful to define a function $f \otimes y \in L^1(I; C(I_\tau))$ for $f \in L^1(I)$ and $y \in C(I_\tau)$ via

$$(f \otimes y)(t, s) := f(t)\, y(s)$$

for all $t \in I$ and $s \in I_\tau$. In addition, we need the following notation for the component-wise application of $\mu \in \mathcal{M}_w(I_\tau)^n$ to $v \in C(I_\tau)^n$:

$$\langle \mu \odot v\rangle := \big( \langle \mu_i, v_i\rangle \big)_{i=1\ldots n}. \tag{6.2}$$

An analogue notation is used for $\mu \in L^\infty(I; \mathcal{M}_w(I_\tau))^n$ and $v \in L^1(I; C(I_\tau))^n$.

Now we are faced with the following situation: $\mu_k$ is a bounded sequence in the space $L^\infty(I; L^1(I_\tau))^n$ (Lemma 5.8) and this space is isometrically embedded into the dual space $L^\infty(I; \mathcal{M}_w(I_\tau))^n = (L^1(I; C(I_\tau))^n)^\star$. This leads to the following result.

**Lemma 6.3.** *We can extract a subsequence of* $(\mu_k)$ *(without relabeling) such that*

$$\mu_k \overset{\star}{\rightharpoonup} \mu \quad \text{in } (L^1(I; C(I_\tau))^n)^\star = L^\infty(I; \mathcal{M}_w(I_\tau))^n.$$

*The limit* $\mu$ *satisfies*

$$\mu(t) \in \partial \max x_t$$

*for a.a.* $t \in I$*. Here,* $\partial \max x_t$ *is the (coefficientwise) (convex) subdifferential of the function* $C(I_\tau)^n \ni x \mapsto \max x_t$ *at the optimal state* $x$*.*

*Proof.* The first claim follows from the Banach-Alaoglu-Bourbaki theorem.

The definition (6.1) of $\mu_k$ implies

$$\mu_k(t, \cdot) \in \partial \mathrm{LIE}_k(x_{k,t}).$$

Here, $\partial \mathrm{LIE}_k(x_{k,t})$ is the coefficientwise convex subdifferential of $C(I_\tau)^n \ni x_k \mapsto \mathrm{LIE}_k(x_{k,t})$ at $x_k$. Thus, for arbitrary $\varphi \in L^\infty(I)$, $\varphi \geq 0$ and $z \in C(I_\tau)^n$, we have

$$\mathrm{LIE}_k(z_t) \geq \mathrm{LIE}_k(x_{k,t}) + \int_{-\tau}^{T} \mu_k(t, s) \odot z(s) \, \mathrm{d}s,$$

thus,

$$\begin{aligned} \int_0^T \varphi(t) \, \mathrm{LIE}_k(z_t) \, \mathrm{d}t &\geq \int_0^T \varphi(t) \, \mathrm{LIE}_k(x_{k,t}) \, \mathrm{d}t + \int_0^T \int_{-\tau}^T \varphi(t) \, \mu_k(t, s) \odot z(s) \, \mathrm{d}s \, \mathrm{d}t \\ &\geq \int_0^T \varphi(t) \, \mathrm{LIE}_k(x_{k,t}) \, \mathrm{d}t + \langle \mu_k \odot (\varphi \otimes z) \rangle. \end{aligned}$$

By using Lemma 4.4 and the Lipschitz continuity of $\mathrm{LIE}_k$, we can pass to the limit $k \to \infty$. This yields

$$\begin{aligned} \int_0^T \varphi(t) \, \max z_t \, \mathrm{d}t &\geq \int_0^T \varphi(t) \, \max x_t \, \mathrm{d}t + \langle \mu \odot (\varphi \otimes z) \rangle \\ &= \int_0^T \varphi(t) \, \max x_t \, \mathrm{d}t + \int_0^T \varphi(t) \, \langle \mu(t) \odot z \rangle \, \mathrm{d}t. \end{aligned}$$

Since $\varphi \geq 0$ is arbitrary, this yields

$$\max z_t \geq \max x_t + \langle \mu(t) \odot z \rangle$$

for a.a. $t \in I$. Note that the null set may depend on $z \in C(I_\tau)^n$. By using the separability of $C(I_\tau)^n$, we can show that the null set can be chosen independently of $z$. Thus,

$$\max z_t \geq \max x_t + \langle \mu(t) \odot z \rangle \qquad \forall z \in C(I_\tau)^n$$

holds for a.a. $t \in I$. This shows the claim. □

The standard characterization of the subdifferential of the maximum function yields the following properties of $\mu$.

**Corollary 6.4.** *The limit $\mu$ from Lemma 6.3 satisfies $\mu \geq 0$,*

$$\|\mu(t)_i\|_{\mathcal{M}(I_\tau)} = 1, \qquad \forall i = 1, \dots, n,$$

*and*

$$\operatorname{supp}(\mu(t)_i) \subset \arg\max_{s \in [t-\tau, t]} x_i(s), \qquad i = 1, \dots, n$$

*for almost all $t \in I$.*

Here, supp denotes the support of a measure. Thus, $\mu(t)_i$ is a probability measure supported at the maximizers of $x$ on the interval $[t - \tau, t]$.

## 6.2 Necessary optimality conditions

For abbreviation, let us define

$$F^*(t) := F(x^*(t), \max x_t^*, u^*(t)).$$

Similarly, we define $F_x^*(t)$, $F_y^*(t)$, $j_x^*$ to denote the derivatives of $F$ and $j$ with respect to the first and second argument, respectively, evaluated along the optimal state and control.

**Theorem 6.5.** *Let $(x^*, u^*)$ be a local solution of the original problem. Then there is $\lambda \in BV(I; \mathbb{R}^n)$ and $\mu \in L^\infty(I; \mathcal{M}_w(I_\tau))^n$ such that the following optimality system is satisfied:*

*(i) (Adjoint equation) For all $v \in C(I_\tau, \mathbb{R}^n)$ with $v(s) = 0$ for all $s \in [-\tau, 0]$ it holds*

$$\int_0^T \mathrm{d}\lambda(t)^\intercal v(t) + \int_0^T \lambda(t)^\intercal F_x^*(t) v(t) + \lambda(t)^\intercal F_y^*(t) \langle \mu(t) \odot v \rangle \mathrm{d}t = \int_0^T j_x^*(t) v(t) \mathrm{d}t. \tag{6.3}$$

*Here, $\langle \mu(t) \odot v \rangle$ denotes the vector with entries $\langle \mu_i(t), v_i \rangle$, see (6.2).*

*(ii) (Maximum principle) The inequality*

$$\int_0^T \lambda(t)^\intercal F(x^*(t), \max x_t^*, u^*(t)) - j(t, x^*(t), u^*(t)) \mathrm{d}t \geq \int_0^T \lambda(t)^\intercal F(x^*(t), \max x_t^*, u(t)) - j(t, x^*(t), u(t)) \mathrm{d}t \tag{6.4}$$

*holds for all feasible controls $u$.*

*(iii) (Subdifferential condition) The measure-valued function $\mu$ satisfies*

$$\mu(t) \in \partial \max x_t^*$$

*for almost all $t \in I$.*

*Proof.* Let $(x_k, u_k)$ be a sequence of global solutions of the regularized optimal control problem as considered in Section 5.2. Then $x_k \to x^*$ in $C(I;\mathbb{R}^n)$ and $u_k \to u \in L^2(I;\mathbb{R}^m)$. For sufficiently large $k$ the requirements of Theorem 5.6 are satisfied. Hence, there is a sequence $(\lambda_k)$ in $W^{1,\infty}(I,\mathbb{R}^n)$ such that (5.4)–(5.5) is satisfied. In addition, the weak formulation of the adjoint equation (5.10) holds. By Theorem 5.7, the sequence $(\lambda_k)$ is bounded in $W^{1,1}(I;\mathbb{R}^n)$. Let us define

$$\mu_k(t,s) := \mathrm{LIE}_k'(x_{k,t})(s), s \in I_\tau,\ t \in I.$$

Then by Lemma 5.8, the sequence $(\mu_k)$ is bounded in $L^\infty(I;L^1(I_\tau))^n$. Using Helly's selection theorem and Lemma 6.3, there is (after extracting subsequences if necessary) $\lambda \in BV(I;\mathbb{R}^n)$ and $\mu$ such that

$$\lambda_k \to \lambda \ \text{ in } L^p(I;\mathbb{R}^n) \quad \forall p < \infty, \tag{6.5}$$

$$\lambda_k' \overset{\star}{\rightharpoonup} \lambda' \text{ in } \mathcal{M}(I;\mathbb{R}^n) = C(I;\mathbb{R}^n)^\star, \tag{6.6}$$

$$\mu_k \overset{\star}{\rightharpoonup} \mu \ \text{ in } L^1(I;C(I_\tau))^\star = L^\infty(I;\mathcal{M}_w(I_\tau)). \tag{6.7}$$

Passing to the limit in the maximum principle (5.5) to get (6.4) is straightforward. In the weak formulation of the adjoint equation (5.10), let us argue the convergence of the third term. To this end, let $v \in C(I_\tau,\mathbb{R}^n)$ be given with $v(s) = 0$ for all $s \in [-\tau, 0]$. Due to the definition of $\mu_k$, we have

$$\int_0^T \lambda_k(t)^T F_y^k(t) \frac{\int_{t-\tau}^t \exp(k\, x_k(s)) \odot v(s)\, \mathrm{d}s}{\int_{t-\tau}^t \exp(k\, x_k(\hat{s}))\, \mathrm{d}\hat{s}} \mathrm{d}t = \int_0^T \lambda_k(t)^T F_y^k(t) \langle \mu_k(t) \odot v \rangle \mathrm{d}t$$

By the convergence properties above, we have that the functions $t \mapsto \langle \mu_k(t) \odot v \rangle$ converge weak-$\star$ in $L^\infty(I)$ to $t \mapsto \langle \mu(t) \odot v \rangle$. This allows the passage to the limit in the adjoint equation to obtain (6.3). The subdifferential condition is a consequence Lemma 6.3. □

## 6.3 Continuity properties of the adjoint

In this subsection, we analyze the continuity of the adjoint state $\lambda$. Our first result gives an expression for the difference between limits from the left and right. Recall that $\lambda$ has bounded variation and this ensures the existence of these one-sided limits.

**Theorem 6.6.** *Let $s_0 \in (0,T)$ be given. Then it holds*

$$\lambda_i(s_0+) - \lambda_i(s_0-) = \int_0^T \lambda(t)^T F_y^*(t) e_i \cdot \mu_i(t)(\{s_0\})\, \mathrm{d}t$$

*where $\lambda_i(s_0+)$ and $\lambda_i(s_0-)$ denote the limits from the right and from the left of $\lambda_i$ at $s_0$, respectively.*

*Proof.* Let $(s_1^j)$, $(s_2^j)$, $(\epsilon_j)$ be sequences such that $s_1^j, s_2^j \in I$ for all $j$, $s_1^j \nearrow s_0$, $s_2^j \searrow s_0$, and $\epsilon_j \searrow 0$. Let $(v_j)$ be the sequence of piecewise linear functions in $C(I)$ defined by

$\operatorname{supp} v_j = [s_1^j - \epsilon_j, s_2^j + \epsilon_j]$ and $v_j(s_1^j) = v_j(s_2^j) = 1$ for all $j$, and with kinks in $s_1^j - \epsilon_j$, $s_1^j$, $s_2^j$ and $s_2^j + \epsilon_j$. Testing (6.3) with $v_j \cdot e_i$, where $e_i$ is the $i$-th unit vector, yields

$$\int_0^T \mathrm{d}\lambda_i(t) v_j(t) = \int_0^T (j_x^*(t)^T - \lambda(t)^T F_x^*(t)) e_i v_j(t)\,\mathrm{d}t - \int_0^T \lambda(t)^T F_y^*(t) \langle \mu(t) \odot e_i v_j \rangle\,\mathrm{d}t.$$

Here, the second integral vanishes for $j \to 0$. For the left-hand side we have

$$\begin{aligned}\int_0^T \mathrm{d}\lambda_i(t) v_j(t) &= -\int_0^T \lambda_i(t) v_j'(t)\,\mathrm{d}t = -\frac{1}{\varepsilon_j}\int_{s_1^j-\varepsilon_j}^{s_1^j} \lambda_i(t)\,\mathrm{d}t + \frac{1}{\varepsilon_j}\int_{s_2^j}^{s_2^j+\varepsilon_j} \lambda_i(t)\,\mathrm{d}t \\ &\xrightarrow{j\to 0} \lambda_i(s_0+) - \lambda_i(s_0-).\end{aligned}$$

It remains to prove convergence for the third integral. We will use dominated convergence theorem. First, we have the integrable bound

$$\left|\lambda(t)^T F_y^*(t) \langle \mu(t) \odot e_i v_j \rangle\right| \le C \|\mu(t)\|_{\mathcal{M}(I_\tau)}$$

since $\lambda^*$ and $F_y^*$ are bounded, and $\|v_j\|_{C(I)} \le 1$. In addition,

$$\mu_i(t)(\{s_0\}) \le \langle \mu_i(t), v_j \rangle \le \mu_i(t)([s_1^j - \epsilon, s_2^j + \epsilon])$$

by the non-negativity of $\mu(t)$ and $\chi_{\{s_0\}} \le v_j \le \chi_{[s_1^j-\epsilon, s_2^j+\epsilon]}$. This proves the pointwise convergence $\langle \mu_i(t), v_j \rangle \to \mu_i(t)(\{s_0\})$. The dominated convergence theorem yields

$$\lim_{j\to\infty} \int_0^T \lambda(t)^T F_y^*(t) \langle \mu(t) \odot e_i v_j \rangle\,\mathrm{d}t = \int_0^T \lambda(t)^T F_y^*(t) e_i \mu_i(t)(\{s_0\})\,\mathrm{d}t,$$

and the claim is proven. □

This shows that $\lambda_i$ is can be discontinuous only for those $s_0$, for which $\mu_i(t)(\{s_0\})$ is non-zero on a set of positive measure. Due to the subdifferential condition in Theorem 6.5, this is equivalent to the statement that $\max x_{i,t}^* = x_i^*(s_0)$ for $t$ in a set of positive measure.

**Corollary 6.7.** *(i) Let $s_0 \in (0, T)$ be such that $s_0 \notin \operatorname{argmax} x_{i,t}^*$ for almost all $t \in I$. Then $\lambda_i$ is continuous in $s_0$.*

*(ii) Let $s_0 \in (0, T)$ be not a local maximum of $x_i^*$. Then $\lambda_i$ is continuous in $s_0$.*

*(iii) Let $s_0 \in I$ be a strict local maximum of $x_i^*$. Assume that there exists a closed interval $[s_1, s_2] \subset [s_0, s_0 + \tau]$ such that $s_0$ is the unique maximum of $x_{i,t}^*$ for all $t \in [s_1, s_2]$, and $\max x_{i,t}^* > x_i^*(s_0)$ for all $t \in [s_0, s_0 + \tau] \setminus [s_1, s_2]$, then it holds*

$$\lambda_i(s_0-) - \lambda_i(s_0+) = -\int_{s_1}^{s_2} \lambda(t)^T F_y^*(t) e_i\,\mathrm{d}t.$$

*Proof.* (i) By assumption, the support of $\mu_i(t)$ does not contain $s_0$ for almost all $t \in I$. Hence, it holds $\mu_i(t)(\{s_0\}) = 0$ for almost all $t \in I$, and we obtain the continuity of $\lambda_i$ at $s_0$ by Theorem 6.6.

(ii) Due to the assumption, there exists a sequence $(t^j)_j \subset I$ with $t^j \to s_0$ and $x_i^*(t^j) > x_i^*(s_0)$. Suppose that there is a subsequence such that (without relabelling) $t^j > s_0$ for all $j \in \mathbb{N}$. Then it follows $\max x_t > x(s_0)$ and $\mu_i(t)(\{s_0\}) = 0$ for all $t > s_0$, which implies $\mu(t)(\{s_0\}) = 0$ for all $t \neq s_0$. If there is a subsequence such that (without relabelling) $t^j < s_0$ for all $j \in \mathbb{N}$, then $\mu_i(t)\{s_0\} = 0$ for all $t < s_0 + \tau$, which implies $\mu(t)(\{s_0\}) = 0$ for all $t \neq s_0 + \tau$. Hence, the claim follows from (i).

(iii) Due to the assumptions, it holds $\mu_i(t) = \delta_{s_0}$ for all $t \in [s_1, s_2]$. In addition, $\mu_i(t)(\{s_0\}) = 0$ for all $t \notin [s_1, s_2]$. The claim now follows from Theorem 6.6.

□

## 7 Numerical experiments

In this final section, we present some numerical results for the regularized optimal control problems

$$
\begin{aligned}
\min \quad & \frac{\alpha}{2}\|x - x_d\|^2_{L^2(I)} + \frac{\beta}{2}\|u\|^2_{L^2(I)} \\
\text{s.t.} \quad & \dot{x}(t) = x(t) - 2\,\mathrm{LIE}_k(x_t) + u(t), \qquad t \in I, \\
& x|_{[-\tau,0]} = 0, \\
& -5 \le u(t) \le 5, \qquad t \in I.
\end{aligned}
$$

In particular, we use the parameters

$$\alpha = 100, \quad \beta = 0.1, \quad T = 3, \quad \tau = 0.2.$$

State and adjoint equation were solved with the explicit Euler method. The optimization problem is solved by the projected gradient method using the Armijo step size rule. As initial guess for the control, we choose the zero function. We present numerical results for certain desired states $x_d$ for several time discretizations $\Delta t$ and regularizations parameters $k$. In the plots in Figs. 7.1 and 7.2 one can see the graphs of $u^*, x^*, x_d, \lambda, \lambda', \mathrm{LIE}_k$ and the gradient of the reduced objective function.

(i) Let $x_d(t) = \sin(\frac{6\pi}{T}t)$. According to Corollary 6.7, the adjoint state $\lambda$ can only have discontinuities in the local maxima of $x^*$. These jumps can be seen in the plot of $\lambda'$ for $k = 100\,000$.

(ii) We now consider a piecewise linear function $x_d$, which is defined via

$$
x_d(t) := \begin{cases} \frac{T}{2} - |t - \frac{T}{2}| & 0 \le t \le \frac{1}{3}T, \\ 0 & \frac{1}{3}T < t < \frac{2}{3}T, \\ |t - \frac{5}{6}T| - \frac{5}{6}T & \frac{2}{3}T \le t \le T. \end{cases}
$$

Here, the plots for $k = 100$ might suggest possible discontinuities of the adjoint in the non-differentiable points of $x_d$. In contrast, for $k = 100\,000$ the adjoint seems to have discontinuities at the strict local maximum of $x_d$ at $t = 0.5$ and $t \approx 1.87$.

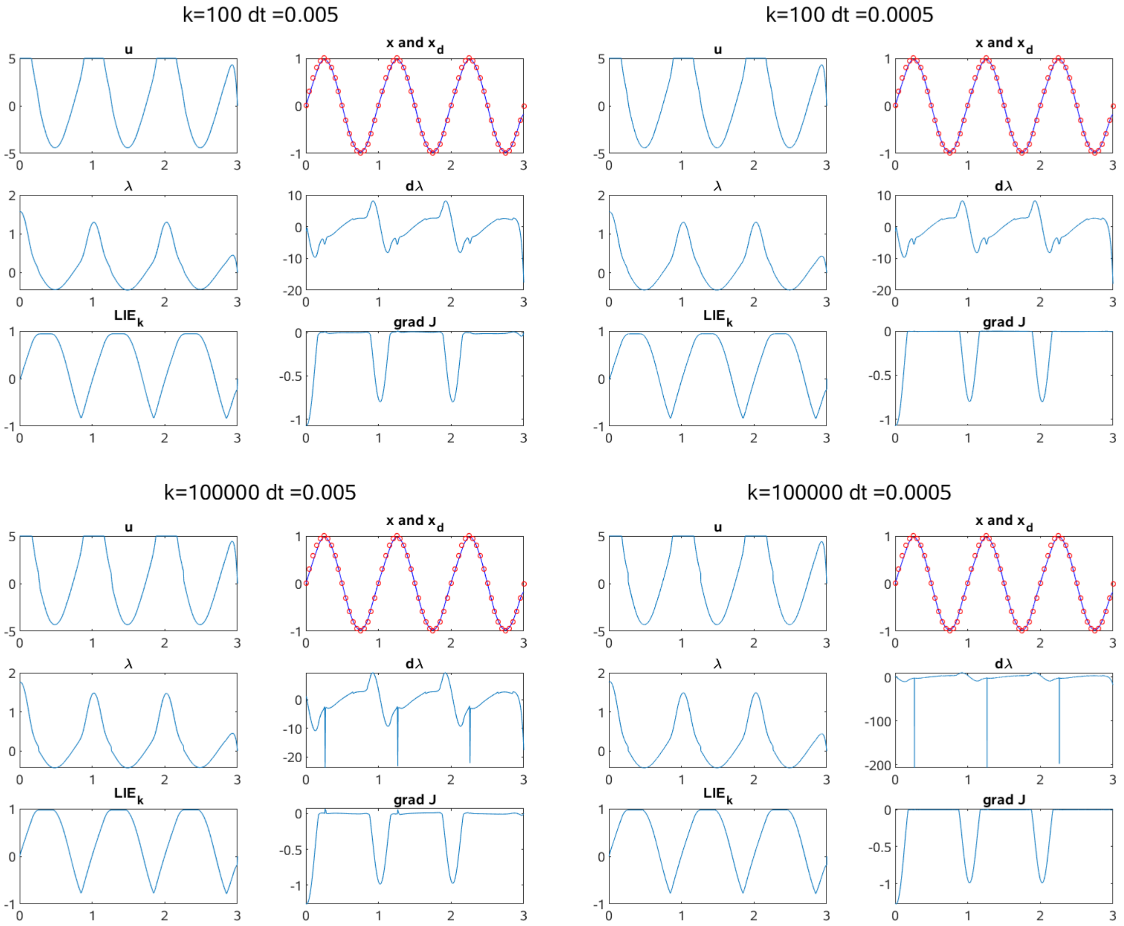


Figure 7.1: Plots for $x_d(t) = \sin(\frac{6\pi}{T}t)$

## References


Azhmyakov, V.; Ahmed, A.; Verriest, E. I. (2016). "On the optimal control of systems evolving with state suprema". In: *2016 IEEE 55th Conference on Decision and Control (CDC)*, pp. 3617–3623. DOI: 10.1109/CDC.2016.7798813.

Bainov, D. D.; Hristova, S. G. (2011). *Differential equations with maxima*. Boca Raton, FL: CRC Press, pp. x + 302.

Banks, H. (1969). "Variational Problems Involving Functional Differential Equations". In: *SIAM Journal on Control* 7.1, pp. 1–17. DOI: 10.1137/0307001.

Cartan, H. (1967). *Calcul différentiel*. Paris: Hermann, p. 178.

Clarke, F. H.; Wolenski, P. R. (1996). "Necessary conditions for functional differential inclusions". In: *Applied Mathematics & Optimization* 34.1, pp. 51–78. DOI: 10.1007/bf01182473.

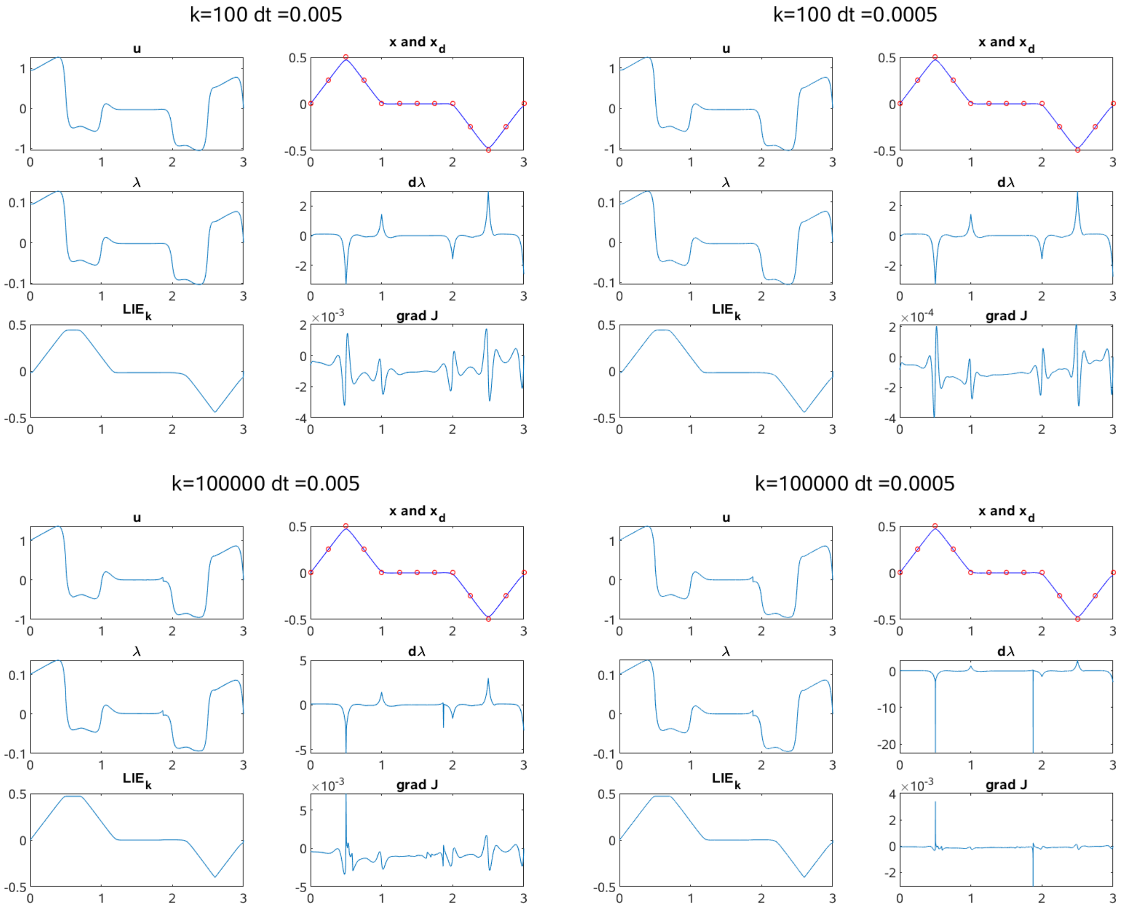


Figure 7.2: Plots for piecewise linear $x_d$

Dashkovskiy, S.; Hristova, S. G.; Kichmarenko, O.; Sapozhnikova, K. (2017). “Behavior of solutions to systems with maximum”. In: *IFAC-PapersOnLine* 50.1. 20th IFAC World Congress, pp. 12925–12930. ISSN: 2405-8963. DOI: 10.1016/j.ifacol.2017.08.1790.

Edwards, R. E. (1995). *Functional Analysis: Theory and Applications*. Dover books on mathematics. Dover Publications.

Ekeland, I.; Temam, R. (1999). *Convex Analysis and Variational Problems*. Vol. 28. Classics in Applied Mathematics. Philadelphia: SIAM.

Goodfellow, I.; Bengio, Y.; Courville, A. (2016). *Deep Learning*. MIT Press. URL: http://www.deeplearningbook.org.

Papageorgiou, N. S.; Kyritsi-Yiallourou, S. T. (2009). *Handbook of applied analysis*. Vol. 19. Advances in Mechanics and Mathematics. Springer, New York, pp. xviii+793. DOI: 10.1007/b120946.

Rockafellar, R. T. (1970). *Convex analysis*. Princeton Mathematical Series, No. 28. Princeton University Press, Princeton, N.J., pp. xviii+451.

Rockafellar, R. T.; Wets, R. J.-B. (1998). *Variational analysis*. Vol. 317. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Berlin: Springer-Verlag, pp. xiv+733. DOI: 10.1007/978-3-642-02431-3.

Rudin, W. (1987). *Real and Complex Analysis*. 3rd ed. New York, NY, USA: McGraw-Hill, Inc.

Verriest, E. I.; Azhmyakov, V. (2017). "Advances in optimal control of differential systems with the state suprema". In: *2017 IEEE 56th Annual Conference on Decision and Control (CDC)*, pp. 739–744. DOI: 10.1109/CDC.2017.8263748.

Wu, J. (1996). *Theory and Applications of Partial Functional Differential Equations*. Springer New York. DOI: 10.1007/978-1-4612-4050-1.