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AN OPTIMAL CONTROL PROBLEM GOVERNED BY A REGULARIZED PHASE-FIELD FRACTURE PROPAGATION MODEL. PART II THE REGULARIZATION LIMIT

I. NEITZEL*, T. WICK†, AND W. WOLLNER‡

Abstract. We consider an optimal control problem of tracking type governed by a time-discrete regularized phase-field fracture or damage propagation model. The energy minimization problem describing the fracture process is described by the corresponding Euler-Lagrange-equations that contain a regularization term that penalizes the violation of the irreversibility condition in the evolution of the fracture. We prove convergence of solutions of the regularized problem when taking the limit with respect to the penalty term, and obtain an estimate for the constraint violation in terms of the penalty parameter γ. To this end, we make use of convexity of the energy functional due to a viscous regularization which corresponds to a time-step restriction in the temporal discretization of the problem. Numerical experiments underline our theoretical findings.

Key words. optimal control, regularized fracture model, phase-field, regularization limit

AMS subject classifications. 49J21, 49K21, 74R10

1. Introduction.

This paper considers an optimal control setting for a regularized fracture propagation problem utilizing a phase-field description of the fracture set. Such phase-field formulations belong to variational fracture approaches, which were first proposed in [8, 3, 4]. In contrast to papers on optimization problems involving fractures where the analysis considers fractures of fixed length, see, [13], or of variable length, but with prescribed path, see [16], such a phase-field approach allows for a convenient treatment of arbitrary fracture paths, branching, crack nucleation, and complex fracture networks. In [19], we have shown that such a problem is well-posed. However, to assert differentiability properties of the control-to-state coupling, we had relaxed the irreversibility condition on the fracture by a penalty approach. Within this paper, we will show, that taking the limit with respect to this penalty allows the recovery of all local minimizers of the unregularized problem as long as they posses certain properties. In this respect, we also mention the closely related simultaneous work of [6, 7] where control of a viscous-damage model in a time continuous setting was considered.

In [19], we considered an optimal control problem for a phase-field damage model with regularized irreversibility constraint. After replacing the time-discrete energy minimization for the lower level phase-field model by its first order necessary conditions, this problem read as follows: Let $M \in \mathbb{N}$ and let u_d^i be a given displacement field at the loading step i. Find $\mathbf{u} = (u, \varphi) \in V^M := (H_D^1(\Omega; \mathbb{R}^2) \times H^1(\Omega))^M$ and $q \in Q^M := L^2(\Gamma_N)^M$ solving

$$\min_{q,\mathbf{u}} J(q, \mathbf{u}) := \frac{1}{2} \sum_{i=1}^{M} \|u^i - u_d^i\|^2 + \frac{\alpha}{2} \sum_{i=1}^{M} \|q^i\|_{\Gamma_N}^2 \tag{NLP$^\gamma$}$$

s.t. $\mathbf{u} = (u, \varphi)$ solves (EL$^\gamma$) given the data q, for each $i = 1, \ldots, M$,

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where the regularized Euler-Lagrange equation reads as

$$\begin{aligned}\Big(g(\varphi^i)\mathbb{C}e(u^i), e(v)\Big) - (q^i, v)_{\Gamma_N} &= 0,\\ \varepsilon(\nabla\varphi^i, \nabla\psi) - \frac{1}{\varepsilon}(1-\varphi^i, \psi)&\\ +(1-\kappa)(\varphi^i\mathbb{C}e(u^i) : e(u^i), \psi)&\\ +\gamma([(\varphi^i - \varphi^{i-1})^+]^3, \psi) &= 0\end{aligned} \tag{EL$^\gamma$}$$

for any $(v, \psi) \in V$ and $i = 1, \dots, M$. Here g is given as

$$g(x) := (1-\kappa)x^2 + \kappa,$$

$\kappa, \varepsilon, \gamma > 0$ are given parameters, the term $\gamma([(\varphi^i - \varphi^{i-1})^+]^3, \psi)$ penalizes the violation of the irreversibility condition. Next, $\mathbb{C}$ is the rank-4 elasticity tensor with the usual properties; see for instance [20][Theorem 3.1]. Moreover, the domain $\Omega \subset \mathbb{R}^2$ has the boundary $\partial\Omega = \Gamma_D \cup \Gamma_N$, each non trivial. Further, we assume that $\Omega \cup \Gamma_N$ is regular in the sense of Gröger, cf. [10], compare [12, Remark 1.6] for a characterization in the case $\Omega \subset \mathbb{R}^2$ considered here. The set of admissible displacements is

$$H^1_D(\Omega; \mathbb{R}^2) := \{v \in H^1(\Omega; \mathbb{R}^2) \,|\, v = 0 \text{ on } \Gamma_D\}.$$

Here and throughout the paper, we employ the usual notation for Sobolev spaces and denote the corresponding norms and inner product by associated indices, which are omitted whenever referring to L^2.

It was shown in [19], that this problem admits at least one global solution and given that the kernel of the linearized equation to (EL$^\gamma$) is trivial, any local solution can be characterized by a system of KKT-like conditions.

In the current paper, we will consider the limit $\gamma \to \infty$ in view of this optimization problem, see our main result in Theorem 5.2. To this end, we will add an additional term to the equation, namely a term

$$\eta(\varphi^i - \varphi^{i-1}, \psi) \tag{1.1}$$

for some fixed $\eta \geq 0$. This term corresponds to a potential viscous regularization of a rate-independent damage model, see [14]. This will be advantageous, as it can be shown that for sufficiently large η the control-to-state mapping induced by (EL$^\gamma$) is single valued due to strict convexity of the energy corresponding to the equation. The assumption of η being sufficiently large is reasonable as it corresponds to a restriction on the time-step size in the temporal discretization of the time-continuous problem, where we refer the reader again to [14].

Our analysis includes a careful tracking of all constants appearing for instance in stability estimates. From [19, Corollary 4.2], we can already observe that there is some $p > 2$, depending on the domain, such that any solution of (EL$^\gamma$) satisfies

$$\|u^i\|_{1,p} \leq c\|q^i\|, \qquad 0 \leq \varphi^i \leq 1$$

where the constant c depends on $\kappa > 0$ and Ω, only. Moreover, the calculations in [11, Corollary 20] show that if Ω is $W^{2,q}$ regular, with $q = p/2 > 1$, for the homogeneous Neumann-problem $-\varepsilon\Delta\varphi + \varepsilon^{-1}\varphi = f$, then

$$\|\varphi^i\|_{2,q} \leq c\Big(1 + \|q^i\|^2 + \gamma\|((\varphi^i - \varphi^{i-1})^+)^3\|_q + \eta\|\varphi^i - \varphi^{i-1}\|_q\Big),$$

with a constant depending on ε and Ω, only. From this it follows that there exists some $s \in (0, 1/2)$ and $c > 0$, depending on the norm bound of $\|\varphi^i\|_{2,q}$, s, and $\kappa > 0$, only, such that

$$\|u^i\|_{1+s} \leq c\|q^i\|_{-1+s} \leq c\|q^i\|$$

holds. We point out in particular, that the appearing constants are independent of the penalization parameter γ. We will also need and show boundedness estimates for the approximate Lagrange multiplier defined with the help of the penalty term to obtain our convergence results.

The rest of the paper is structured as follows. Section 2 is devoted to a discussion of the Euler-Lagrange equations (EL^γ) amended by the term (1.1) yielding ($\mathrm{EL}^{\gamma,\eta}$) defined below. We introduce convenient notation to describe this so-called forward problem and collect some known properties from the literature. In Section 3, we will investigate the convergence behavior of this lower level problem with respect to the limit in the penalty parameter for the irreversibility constraint for only one time-step of the fracture model. Specifically, we obtain an estimate for the constraint violation in Lemma 3.11. The next Section 4 collects these estimates for the time-discrete fracture evolution. The main result is then obtained in Section 5, where we consider convergence of solutions of the regularized optimal control problem

$$\begin{aligned} &\min_{q,\mathbf{u}} J(q,\mathbf{u}) := \frac{1}{2}\sum_{i=1}^{M} \|u^i - u_d^i\|^2 + \frac{\alpha}{2}\sum_{i=1}^{M}\|q^i\|^2_{\Gamma_N} \\ &\quad \text{s.t. } \mathbf{u} = (u,\varphi) \text{ solves } (\mathrm{EL}^{\gamma,\eta}) \text{ given the data } q, \text{ for each } i = 1,\dots,M, \end{aligned} \tag{$\mathrm{NLP}^{\gamma,\eta}$}$$

where the regularized Euler-Lagrange equation with viscous term reads as

$$\begin{aligned} \Big(g(\varphi^i)\mathbb{C}e(u^i), e(v)\Big) - (q^i, v)_{\Gamma_N} &= 0, \\ \varepsilon(\nabla\varphi^i, \nabla\psi) - \frac{1}{\varepsilon}(1-\varphi^i,\psi) + \eta(\varphi^i - \varphi^{i-1},\psi)& \\ +(1-\kappa)(\varphi^i\mathbb{C}e(u^i) : e(u^i), \psi)& \\ +\gamma([(\varphi^i - \varphi^{i-1})^+]^3, \psi) &= 0, \end{aligned} \tag{$\mathrm{EL}^{\gamma,\eta}$}$$

with respect to $\gamma \to \infty$. To be precise, we show in Theorem 5.2 that any isolated minimizer of the unregularized optimization problem

$$\begin{aligned} &\min_{q,\mathbf{u}} J(q,\mathbf{u}) := \frac{1}{2}\sum_{i=1}^{M} \|u^i - u_d^i\|^2 + \frac{\alpha}{2}\sum_{i=1}^{M}\|q^i\|^2_{\Gamma_N} \\ &\quad \text{s.t. } \mathbf{u} = (u,\varphi) \text{ solves } (\mathrm{EL}^{\eta}) \text{ given the data } q, \text{ for each } i = 1,\dots,M, \end{aligned} \tag{NLP^{η}}$$

with the Euler-Lagrange equation

$$\begin{aligned} \Big(g(\varphi^i)\mathbb{C}e(u^i), e(v)\Big) - (q^i, v)_{\Gamma_N} &= 0, \\ \varepsilon(\nabla\varphi^i, \nabla\psi) - \frac{1}{\varepsilon}(1-\varphi^i,\psi) + \eta(\varphi^i - \varphi^{i-1},\psi)& \\ +(1-\kappa)(\varphi^i\mathbb{C}e(u^i) : e(u^i), \psi)& \\ +(\lambda^i, \psi) &= 0, \\ \varphi^i &\leq \varphi^{i-1}, \\ \lambda^i &\geq 0, \\ (\lambda^i, \varphi^i - \varphi^{i-1}) &= 0, \end{aligned} \tag{EL^{η}}$$

can be approximated by our penalty method; provided that the control-to-state map is single valued in the selected minimizer. We end this paper with Section 6, where numerical examples underline our theoretical results.

2. Problem & Notation. We will start by analyzing the γ-limit in the forward problem ($\mathrm{EL}^{\gamma,\eta}$). Corresponding to the brief discussion in the introduction, let us assume throughout that Ω is $W^{2,q}$ regular for the homogeneous Neumann-problem $-\varepsilon\Delta\varphi + \frac{1}{\varepsilon}\varphi = f$. To simplify notation, we consider only one time-step, i.e., $M = 1$, and a given previous phase-field $\hat{\varphi}$, which we assume to satisfy $0 \le \hat{\varphi} \le 1$ as well as the regularity $\hat{\varphi} \in W^{2,q}(\Omega)$ with $q = p/2 > 1$. Solutions corresponding to the regularized problem will be denoted by an index γ for ease of representation, whereas limit functions carry the index ∞.

With this simplification, let us consider the first-order necessary conditions to the energy minimization problem, given by the Euler-Lagrange equations

$$\begin{aligned} (g(\varphi_\gamma)\mathbb{C}e(u_\gamma), e(v)) - (q, v)_{\Gamma_N} = 0, \quad \forall\, v \in H^1_D(\Omega), \\ \varepsilon(\nabla\varphi_\gamma, \nabla\psi) - \frac{1}{\varepsilon}(1 - \varphi_\gamma, \psi) + \eta(\varphi_\gamma - \hat{\varphi}, \psi) \\ +\gamma(((\varphi_\gamma - \hat{\varphi})^+)^3, \psi) + (1-\kappa)(\varphi_\gamma e(u_\gamma), e(u_\gamma)\psi) = 0 \quad \forall\, \psi \in H^1(\Omega) \end{aligned} \tag{2.1}$$

with parameters $\varepsilon, \kappa, \gamma > 0$ and $\eta \ge 0$.

From our introductory exposition, we know, that (2.1) has at least one solution $\mathbf{u}_\gamma = (u_\gamma, \varphi_\gamma)$ and that any solution $(u_\gamma, \varphi_\gamma) \in V$ satisfies the additional regularity

$$u_\gamma \in W^{1,p}(\Omega) \cap H^{1+s}(\Omega), \quad \varphi_\gamma \in W^{2,q}(\Omega), \quad 0 \le \varphi_\gamma \le 1$$

for some $p > 2$, $s > 0$, and $q = p/2$, as well as the stability properties

$$\begin{aligned} \|u_\gamma\|_{1,p} &\le c\|q\|, \\ \|\varphi_\gamma\|_{2,q} &\le c\Big(1 + \|q\|^2 + \gamma\|((\varphi_\gamma - \hat{\varphi})^+)^3\|_q + \eta\|\varphi_\gamma - \hat{\varphi}\|_q\Big), \\ \|u_\gamma\|_{1+s} &\le c_\varphi\|q\|. \end{aligned} \tag{2.2}$$

Here and throughout the paper $c > 0$ denotes a generic constant independent of all functions shown in the inequality as well as the parameter γ. The constant c_φ depends on the bound of $\|\varphi\|_{2,q}$. We immediately notice, that the bound on $\|\varphi_\gamma\|_{2,q}$ is problematic, in that it induces a potential γ dependency. We will see later, in Corollary 3.9, that $\gamma\|((\varphi_\gamma - \hat{\varphi})^+)^3\|_q$ is indeed bounded independently of γ. Since we do not consider limits with respect to η, any η-dependence will not be tracked. We point out again, that for η large enough, only one solution to (2.1) exists, and will make use of this implicit assumption throughout the following.

To deal with the irreversibility condition, we define the approximate Lagrange-multiplier for the inequality constraint $\varphi \le \hat{\varphi}$ by

$$\lambda_\gamma := \gamma((\varphi_\gamma - \hat{\varphi})^+)^3,$$

where $(\cdot)^+$ is the usual abbreviation for the nonnegative part.

For convenience, let the left-hand-side of (2.1), without the penalty term, define an operator $A\colon (W^{1,p}_D(\Omega) \times H^1(\Omega)) \times (H^1_D(\Omega) \times H^1(\Omega)) \to \mathbb{R}$ such that the solution $(u_\gamma, \varphi_\gamma)$ of (2.1) is equivalently given by

$$A[u_\gamma, \varphi_\gamma; v, \psi] + (\lambda_\gamma, \psi) = (q, v)_{\Gamma_N} \quad \forall (v, \psi) \in (H^1_D(\Omega) \times H^1(\Omega)).$$

We will subsequently show, that any (weak) limit $(u_\infty, \varphi_\infty, \lambda_\infty)$ of solutions to (2.1), for $\gamma \to \infty$ satisfies the following system

$$\begin{aligned} A[u_\infty, \varphi_\infty; v, \psi] + (\lambda_\infty, \psi) &= (q, v)_{\Gamma_N} && \forall\, (v, \psi) \in (H^1_D(\Omega) \times H^1(\Omega)),\\ \varphi_\infty &\le \hat\varphi && \text{a.e. in } \Omega,\\ \lambda_\infty &\ge 0,\\ (\lambda_\infty, \varphi_\infty - \hat\varphi) &= 0, \end{aligned} \tag{2.3}$$

and thus the variational inequality

$$\begin{aligned} A[u_\infty, \varphi_\infty; v, \psi - \varphi_\infty] &\ge (q, v)_{\Gamma_N} && \forall\, (v, \psi) \in (H^1_D(\Omega) \times H^1(\Omega)), \psi \le \hat\varphi,\\ \varphi_\infty &\le \hat\varphi && \text{a.e. in } \Omega. \end{aligned} \tag{2.4}$$

3. Stability Properties. We will now investigate the limit in equation (2.1) as $\gamma \to \infty$. As usually, see, e.g., [15, 17, 21], the proof will rely on several stability estimates implying the convergence of the feasibility violation. In contrast to the cited works, the nonlinearity involved will provide some additional difficulties, similar to those encountered in the well-posedness analysis of pressurized phase-field fractures [18]. We will divide this section into a part that deals with basic estimates that are obtained similar to the linear setting, and a section where we improve these estimates for our later analysis.

3.1. Basic Estimates. Let us first show elementary stability estimates for the regularized phase field φ_γ, as well as the associated displacement u_γ.

LEMMA 3.1. *Under the assumptions of Section 2, we obtain that any solution $(u_\gamma, \varphi_\gamma)$ to (2.1) satisfies*

$$\varepsilon\|\nabla\varphi_\gamma\|^2 + \frac{1}{2\varepsilon}\|\varphi_\gamma\|^2 + \frac{\eta}{2}\|\varphi_\gamma\|^2 \le \frac{1}{2\varepsilon}\|1\|^2 + \frac{\eta}{2}\|\hat\varphi\|^2$$

and

$$\|u_\gamma\|_{1,p} \lesssim c\|q\|$$

with a constant independent of γ and φ_γ.

Proof. We test the second equation in (2.1) with $\psi = \varphi_\gamma$, and obtain

$$\begin{aligned} \varepsilon\|\nabla\varphi_\gamma\|^2 + \frac{1}{\varepsilon}\|\varphi_\gamma\|^2 + \eta\|\varphi_\gamma\|^2 &+ \gamma(((\varphi_\gamma - \hat\varphi)^+)^3, \varphi_\gamma)\\ &+ (1-\kappa)(\varphi_\gamma e(u_\gamma), e(u_\gamma)\varphi_\gamma)\\ &= \frac{1}{\varepsilon}(1, \varphi_\gamma) + \eta(\hat\varphi, \varphi_\gamma)\\ &\le \frac{1}{2\varepsilon}\|1\|^2 + \frac{1}{2\varepsilon}\|\varphi_\gamma\|^2 + \frac{\eta}{2}\|\hat\varphi\|^2 + \frac{\eta}{2}\|\varphi_\gamma\|^2. \end{aligned}$$

This shows the assertion for φ_γ noting that all terms on the left are non-negative using $\varphi_\gamma \in [0, 1]$. By [12] the assertion on u_γ follows. □

In a second step, we show boundedness of the approximate multiplier λ_γ.

LEMMA 3.2. *Under the assumptions of Section 2 the multiplier λ_γ satisfies $\lambda_\gamma \in L^1(\Omega) \cap H^1(\Omega)^*$ with the bounds*

$$\begin{aligned} \|\lambda_\gamma\|_{L^1} &\le c,\\ \|\lambda_\gamma\|_{(H^1)^*} &\le c \end{aligned}$$

for a constant c independent of γ.

Proof. For the L^1-bound, we choose $\psi \equiv 1$ in (2.1), and get

$$\|\lambda_\gamma\|_{L^1} = \int_\Omega \gamma((\varphi_\gamma - \hat\varphi)^+)^3 \cdot 1 \,\mathrm{d}x \le \frac{1}{\varepsilon}\|1\|^2 + \eta(\hat\varphi, 1) \le c$$

analogous to the proof of Lemma 3.1. For the H^{-1}-bound, we test the equation with $\psi \in H^1(\Omega)$ and $\|\psi\|_{H^1} = 1$ to get

$$\begin{aligned}\int_\Omega \lambda_\gamma \psi \,\mathrm{d}x = &-\varepsilon(\nabla\varphi_\gamma, \nabla\psi) + \frac{1}{\varepsilon}(1-\varphi_\gamma, \psi) - \eta(\varphi_\gamma - \hat\varphi, \psi)\\ &- (1-\kappa)(\varphi_\gamma e(u_\gamma), e(u_\gamma)\psi).\end{aligned}$$

The desired bound follows by Hölder's-inequality together with the H^1-bound on φ_γ and the $W^{1,p}$ bound on u_γ from Lemma 3.1. ∎

With these preparations, we directly obtain the following weak convergence result.

Corollary 3.3. *There exists $\lambda_\infty \in H^1(\Omega)^*$, $\varphi_\infty \in H^1(\Omega)$, and $u_\infty \in W^{1,p}(\Omega)$ and a sequence $\gamma \to \infty$ such that*

$$\begin{aligned}\lambda_\gamma &\rightharpoonup \lambda_\infty \quad \text{in } H^1(\Omega)^*,\\ \varphi_\gamma &\rightharpoonup \varphi_\infty \quad \text{in } H^1(\Omega),\\ u_\gamma &\rightharpoonup u_\infty \quad \text{in } W^{1,p}(\Omega).\end{aligned}$$

In particular any such weak limit satisfies for any $r < \infty$

$$\begin{aligned}\varphi_\gamma &\to \varphi_\infty \quad \text{in } L^r(\Omega),\\ \varphi_\infty(x) &\in [0,1] \quad \text{a.e in } \Omega.\end{aligned}$$

Proof. This is an immediate consequence of Lemmas 3.1 and 3.2, the reflexivity of the spaces, and compact embeddings. ∎

Corollary 3.4. *Any such limit triplet $(u_\infty, \varphi_\infty, \lambda_\infty)$ from Corollary 3.3 solves* (2.3).

Proof. Picking $r < \infty$ such that $\frac{1}{2} = \frac{1}{p} + \frac{1}{r}$, Corollary 3.3 and the definition of g assert

$$\begin{aligned}g(\varphi_\gamma) &\to g(\varphi_\infty) && \text{in } L^r,\\ e(u_\gamma) &\rightharpoonup e(u_\infty) && \text{in } L^p,\end{aligned}$$

and hence the first PDE in (2.3) is satisfied as $g(\varphi_\gamma)\mathbb{C}e(u_\gamma) \rightharpoonup g(\varphi_\infty)\mathbb{C}e(u_\infty)$ in L^2.

The weak continuity of the second equation follows by [14, Corollary 2.1].

Moreover, boundedness of λ_γ in $L^1(\Omega)$ together with strong convergence $\varphi_\gamma \to \varphi_\infty$ in L^2 implies $\varphi_\infty \le \hat\varphi$. The sign of λ_∞ follows immediately from the sign of λ_γ. Finally, the complementarity is a consequence of

$$(\lambda_\infty, \varphi_\infty - \hat\varphi) \leftarrow (\lambda_\gamma, \varphi_\gamma - \hat\varphi) = (\lambda_\gamma, (\varphi_\gamma - \hat\varphi)^+) \ge 0$$

and the fact, that $\varphi_\infty - \hat\varphi \le 0$ and thus

$$(\lambda_\infty, \varphi_\infty - \hat\varphi) \le 0$$

due to the sign of λ_∞. ∎

3.2. Improved Estimates. We will now move from basic estimates, which have been almost identical in their proofs to the case of a linear equation as in [15, 17, 21], to improved convergence estimates, showing higher regularity of the limits as well as improved convergence orders of the approximating sequences.

We start with an auxiliary result implying weak lower semicontinuity of the nonlinear term.

LEMMA 3.5. *For any (weakly) convergent sequence, given by Corollary 3.3, it holds*

$$\begin{aligned}\liminf_{\gamma\to\infty}(g(\varphi_\gamma)\mathbb{C}e(u_\gamma),e(u_\gamma)) &\geq (g(\varphi_\infty)\mathbb{C}e(u_\infty),e(u_\infty)),\\ \liminf_{\gamma\to\infty}(\varphi_\gamma\mathbb{C}e(u_\gamma),e(u_\gamma)\varphi_\gamma) &\geq (\varphi_\infty\mathbb{C}e(u_\infty),e(u_\infty)\varphi_\infty).\end{aligned}$$

Proof. We consider the first inequality, only, the second one follows by analogous arguments.

We calculate, with a suitable $r\in(1,\infty)$,

$$\begin{aligned}(g(\varphi_\gamma)\mathbb{C}e(u_\gamma),e(u_\gamma)) &= ((g(\varphi_\gamma)-g(\varphi_\infty))\mathbb{C}e(u_\gamma),e(u_\gamma)) + (g(\varphi_\infty))\mathbb{C}e(u_\gamma),e(u_\gamma))\\ &\geq -c\|g(\varphi_\gamma)-g(\varphi_\infty)\|_r\|e(u_\gamma)\|_p^2 + (g(\varphi_\infty))\mathbb{C}e(u_\gamma),e(u_\gamma)).\end{aligned}$$

Strong convergence of $\varphi_\gamma\to\varphi_\infty$ in L^r from Corollary 3.3, boundedness of $\|e(u_\gamma)\|_p$, and convexity, and hence weak lower semicontinuity, of the last summand, with respect to u_γ, show the assertion. ◻

LEMMA 3.6. *Under the conditions of Corollary 3.3 it holds for any (weakly) convergent sequence $\varphi_\gamma\rightharpoonup\varphi_\infty$ in $H^1(\Omega)$ that*

$$\varphi_\gamma\to\varphi_\infty$$

strongly in $H^1(\Omega)$.

Proof. The proof follows the ideas of [17, Lemma 3.3.]. However, the proof is complicated by the fact that A is nonlinear and that $A(u,\varphi;u,\varphi)$ does not define a norm.

We consider the difference between (2.1) and (2.3) tested with their respective solutions. We obtain

$$\begin{aligned}A(u_\gamma,\varphi_\gamma;u_\gamma,\varphi_\gamma) &- A(u_\infty,\varphi_\infty;u_\infty,\varphi_\infty)\\ &= \varepsilon\Big(\|\nabla\varphi_\gamma\|^2-\|\nabla\varphi_\infty\|^2\Big)\\ &\quad-\frac{1}{\varepsilon}\Big((1-\varphi_\gamma,\varphi_\gamma)-(1-\varphi_\infty,\varphi_\infty)\Big)+\eta\Big((\varphi_\gamma-\hat\varphi,\varphi_\gamma)-(\varphi_\infty-\hat\varphi,\varphi_\infty)\Big)\\ &\quad+(g(\varphi_\gamma)\mathbb{C}e(u_\gamma),e(u_\gamma))-(g(\varphi_\infty)\mathbb{C}e(u_\infty),e(u_\infty))\\ &\quad+(\varphi_\gamma\mathbb{C}e(u_\gamma),e(u_\gamma)\varphi_\gamma)-(\varphi_\infty\mathbb{C}e(u_\infty),e(u_\infty)\varphi_\infty).\end{aligned}$$

Taking the limit inferior, the third line from below vanishes by strong convergence from Corollary 3.3 and the last two lines have a sign due to Lemma 3.5. Thus we obtain

$$\begin{aligned}\liminf_{\gamma\to\infty}&\Big(A(u_\gamma,\varphi_\gamma;u_\gamma,\varphi_\gamma)-A(u_\infty,\varphi_\infty;u_\infty,\varphi_\infty)\Big)\\ &\geq\varepsilon\liminf_{\gamma\to\infty}\Big(\|\nabla\varphi_\gamma\|^2-\|\nabla\varphi_\infty\|^2\Big)\geq 0\end{aligned}\tag{3.1}$$

by weak lower semicontinuity of the norm. We continue by noting that

$$
\begin{aligned}
A(u_\gamma,\varphi_\gamma;u_\gamma,\varphi_\gamma) &- A(u_\infty,\varphi_\infty;u_\infty,\varphi_\infty)\\
&= A(u_\infty,\varphi_\infty;u_\gamma-u_\infty,\varphi_\gamma-\varphi_\infty) + A(u_\gamma,\varphi_\gamma;u_\gamma-u_\infty,\varphi_\gamma-\varphi_\infty)\\
&\quad + A(u_\gamma,\varphi_\gamma;u_\infty,\varphi_\infty) - A(u_\infty,\varphi_\infty;u_\gamma,\varphi_\gamma).
\end{aligned}
\tag{3.2}
$$

The first line can be treated as in the linear case, noting that

$$A(u_\gamma,\varphi_\gamma;v,\psi) = (q,v)_{\Gamma_N} - (\lambda_\gamma,\psi).$$

For the special choice $\psi = \varphi_\gamma - \varphi_\infty$ we obtain

$$(\lambda_\gamma,\psi) = \int_{\varphi_\gamma>\hat\varphi} \lambda_\gamma(\varphi_\gamma-\varphi_\infty)\,\mathrm{d}x \ge 0$$

since $\varphi_\gamma \ge \hat\varphi \ge \varphi_\infty$ as shown in Corollary 3.4. Hence

$$A(u_\gamma,\varphi_\gamma;u_\gamma-u_\infty,\varphi_\gamma-\varphi_\infty) \le (q,u_\gamma-u_\infty)_{\Gamma_N}. \tag{3.3}$$

For linear and symmetric A, the last line of (3.2) would be zero. Here we need to calculate

$$
\begin{aligned}
A(u_\gamma,\varphi_\gamma;u_\infty,\varphi_\infty) &- A(u_\infty,\varphi_\infty;u_\gamma,\varphi_\gamma)\\
&= \frac{1}{\varepsilon}(1,\varphi_\gamma-\varphi_\infty) + \eta(\hat\varphi,\varphi_\infty-\varphi_\gamma)\\
&\quad + (1-\kappa)\Big((\varphi_\gamma\mathbb{C}e(u_\gamma),e(u_\gamma)\varphi_\infty) - (\varphi_\infty\mathbb{C}e(u_\infty),e(u_\infty)\varphi_\gamma)\Big)\\
&\quad + \Big((g(\varphi_\gamma)\mathbb{C}e(u_\gamma),e(u_\infty)) - (g(\varphi_\infty)\mathbb{C}e(u_\infty),e(u_\gamma))\Big).
\end{aligned}
$$

Taking the limit, the last and third to last line converge to zero by the strong and weak convergence result given in Corollary 3.3. The second to last line vanishes due to [14, Corollary 2.1].

Combining this last estimate with (3.1), (3.2), and (3.3), we conclude

$$
\begin{aligned}
0 &\le \varepsilon \liminf_{\gamma\to\infty}\Big(\|\nabla\varphi_\gamma\|^2 - \|\nabla\varphi_\infty\|^2\Big)\\
&\le \limsup_{\gamma\to\infty}\Big(A(u_\gamma,\varphi_\gamma;u_\gamma,\varphi_\gamma) - A(u_\infty,\varphi_\infty;u_\infty,\varphi_\infty)\Big)\\
&\le \lim_{\gamma\to\infty}\Big(A(u_\infty,\varphi_\infty;u_\gamma-u_\infty,\varphi_\gamma-\varphi_\infty) + (q,u_\gamma-u_\infty)_{\Gamma_N}\Big)\\
&= 0
\end{aligned}
$$

by weak convergence of $(u_\gamma,\varphi_\gamma)$. Hence we have seen

$$\|\nabla\varphi_\gamma\|_{H^1} \to \|\nabla\varphi_\infty\|_{H^1}$$

and combined with the weak convergence in H^1 in Corollary 3.3 the assertion is shown. □

By similar calculations, we obtain strong convergence of the displacement in $H^1(\Omega)$.

Lemma 3.7. *Under the conditions of Corollary 3.3 it holds for any (weakly) convergent sequence $u_\gamma \rightharpoonup u_\infty$ in $H^1(\Omega)$ that*

$$u_\gamma \to u_\infty$$

strongly in $H^1(\Omega)$.

Proof. The proof is analogous to Lemma 3.6 noting that $\kappa \le g(\varphi)$ for all $\varphi \in H^1$. $\square$

LEMMA 3.8. *Assuming that* $\hat\varphi \in W^{2,q}$ *for some* $q > 1$ *it holds* $\lambda_\gamma \in L^q$ *with*

$$\|\lambda_\gamma\|_q \le \left\|\frac{1}{\varepsilon} + \varepsilon\Delta\hat\varphi\right\|_q$$

for a constant c independent of γ.

Proof. To see this, we first notice that for any $s > 0$ we can estimate

$$\begin{aligned}(\nabla(\varphi_\gamma - \hat\varphi), \nabla\lambda_\gamma^s) &= (\nabla(\varphi_\gamma - \hat\varphi), s\lambda_\gamma^{s-1}\nabla\lambda_\gamma)\\ &= \frac{1}{\gamma}\int_{\varphi_\gamma \ge \hat\varphi} s\lambda_\gamma^{s-1}\nabla(\varphi_\gamma - \hat\varphi)\nabla\lambda_\gamma \ge 0.\end{aligned}$$

Thus we obtain, testing (2.1) with $\psi = \lambda_\gamma^s$

$$\begin{aligned}\int_\Omega \lambda_\gamma^{1+s}\,\mathrm{d}x &\le \int_\Omega \lambda_\gamma^{1+s}\,\mathrm{d}x + \varepsilon(\nabla(\varphi_\gamma - \hat\varphi), \nabla\lambda_\gamma^s) + \frac{1}{\varepsilon}(\varphi_\gamma, \lambda_\gamma^s) + \eta(\varphi_\gamma - \hat\varphi, \lambda_\gamma^s)\\ &\quad + (1-\kappa)(\varphi_\gamma e(u_\gamma), e(u_\gamma)\lambda_\gamma^s)\\ &= \frac{1}{\varepsilon}(1, \lambda_\gamma^s) - \varepsilon(\nabla\hat\varphi, \nabla\lambda_\gamma^s)\\ &= \left(\frac{1}{\varepsilon} + \varepsilon\Delta\hat\varphi, \lambda_\gamma^s\right)\\ &\le \left\|\frac{1}{\varepsilon} + \varepsilon\Delta\hat\varphi\right\|_q \|\lambda_\gamma^s\|_{q^*},\end{aligned}$$

where $\frac{1}{q} + \frac{1}{q^*} = 1$. Using generalized Hölder's-inequality, we obtain

$$\int_\Omega \lambda_\gamma^{1+s}\,\mathrm{d}x \le \frac{\left\|\frac{1}{\varepsilon} + \varepsilon\Delta\hat\varphi\right\|_q^q}{q} + \frac{\|\lambda_\gamma^s\|_{q^*}^{q^*}}{q^*}.$$

Selecting $s = (q^* - 1)^{-1}$, i.e.,

$$(1+s) = 1 + \frac{1}{q^*-1} = \frac{q^*}{q^*-1} = sq^*,$$

we notice that

$$\frac{\|\lambda_\gamma^s\|_{q^*}^{q^*}}{q^*} = \frac{\int_\Omega \lambda_\gamma^{sq^*}\,\mathrm{d}x}{q^*} = \frac{\int_\Omega \lambda_\gamma^{1+s}\,\mathrm{d}x}{q^*}.$$

Since $\frac{1}{q^*} < 1$ we obtain

$$\int_\Omega \lambda_\gamma^{1+s}\,\mathrm{d}x \le \left(1 - \frac{1}{q^*}\right)^{-1} \frac{\left\|\frac{1}{\varepsilon} + \varepsilon\Delta\hat\varphi\right\|_q^q}{q} = \left\|\frac{1}{\varepsilon} + \varepsilon\Delta\hat\varphi\right\|_q^q.$$

By definition of s it holds

$$1 + s = sq^* = \frac{q^*}{q^* - 1} = q.$$

This shows

$$\|\lambda_\gamma\|_q \le \left\|\frac{1}{\varepsilon} + \varepsilon\Delta\hat{\varphi}\right\|_q.$$

□

We collect the results obtained so far and derive a boundedness result for the phase field even in $W^{2,q}(\Omega)$.

COROLLARY 3.9. *Under the assumptions of this section, let $q = p/2 > 1$ and $\hat{\varphi} \in W^{2,q}(\Omega)$. Then we obtain*

$$\|\varphi_\gamma\|_{2,q} \le c\Big(1 + \|q\|^2 + \|\frac{1}{\varepsilon} + \varepsilon\Delta\hat{\varphi}\|_q\Big)$$

independent of γ.

Proof. This is an immediate consequence of the stability estimate (2.2) together with regularity of u_γ, φ_γ from Lemma 3.1 and the multiplier bound on λ_γ from Lemma 3.8, noting that both φ_γ and $\hat{\varphi}$ are $[0,1]$-valued, and hence uniformly bounded.
□

With the boundedness results at hand, we are finally in the position to prove our improved convergence result.

COROLLARY 3.10. *Any weakly convergent sequence $(u_\gamma, \varphi_\gamma, \lambda_\gamma)$ from Corollary 3.3 satisfies*

$$u_\gamma \to u_\infty$$

strongly in $W^{1,p}(\Omega) \cap H^{1+s}(\Omega)$ and

$$\varphi_\gamma \to \varphi_\infty$$

strongly in $C^{0,\alpha}(\Omega)$, where $p > 2$, $s > 0$ are as in Section 2, and any $\alpha > 0$ satisfying the compact embedding $W^{2,q}(\Omega) \subset C^{0,\alpha}(\Omega)$.

Proof. From Corollary 3.9, we obtain, that φ_γ is uniformly bounded in $W^{2,q}(\Omega)$ and hence converges weakly in this space; to the same limit. By compact embeddings the strong convergence in $C^{0,\alpha}(\Omega)$ is obtained.

The displacements now satisfy

$$(g(\varphi_\infty)\mathbb{C}e(u_\infty), e(v)) = (q, v)_{\Gamma_N} = (g(\varphi_\gamma)\mathbb{C}e(u_\gamma), e(v))$$

for all $v \in H^1_D(\Omega)$. From [11], we obtain the improved regularity $u_\gamma, u_\infty \in H^{1+s}(\Omega)$ together with the bound

$$\|u_\gamma\|_{1+s} \le c_\varphi\|q\|$$

with a constant depending on the $C^{0,\alpha}$ norm bound of φ_γ, only. Further, the error satisfies

$$(g(\varphi_\infty)\mathbb{C}e(u_\infty - u_\gamma), e(v)) = ((g(\varphi_\gamma) - g(\varphi_\infty))\mathbb{C}e(u_\gamma), e(v)).$$

Due to

$$|((g(\varphi_\gamma) - g(\varphi_\infty))\mathbb{C}e(u_\gamma), e(v))| \le c\|(g(\varphi_\gamma) - g(\varphi_\infty))\|_{C^{0,\alpha/2}}\|e(u_\gamma)\|_s\|v\|_{1-s} \to 0$$

for any $v \in H_D^{1-s}(\Omega)$ the right hand side of this equation converges to zero as an element of $H^{-1+s}(\Omega) = (H_D^{1-s})^*(\Omega)$ and due to [11] this shows

$$\|u_\gamma - u_\infty\|_{1+s} \le c_\varphi\|(g(\varphi_\gamma) - g(\varphi_\infty))\|_{C^{0,\alpha/2}}\|e(u_\gamma)\|_s \to 0.$$

□

We conclude this section by showing that the constraint violation not only converges pointwise but also in $H^1(\Omega)$.

LEMMA 3.11. *For $\hat\varphi \in W^{2,q}(\Omega)$, with $q = p/2$, let q^* be given such that $\frac{1}{q^*} + \frac{2}{p} = 1$. Then the constraint violation of φ_γ can be estimated by*

$$\|\min(\varphi_\gamma, \hat\varphi) - \varphi_\gamma\|_{H^1} \lesssim \gamma^{-\frac{1}{2q^*}}.$$

Proof. Setting $w := \varphi_\gamma - \min(\varphi_\gamma, \hat\varphi) = (\varphi_\gamma, \hat\varphi)^+$ we obtain, with a suitable constant $c > 0$

$$\begin{aligned}
c\|w\|_{H^1}^2 \le& \varepsilon(\nabla w, \nabla w) + \frac{1}{\varepsilon}(w, w) + \eta(w, w)\\
=& \varepsilon(\nabla(\varphi_\gamma - \hat\varphi), \nabla w) + \frac{1}{\varepsilon}(\varphi_\gamma - \hat\varphi, w) + \eta(\varphi_\gamma - \hat\varphi, w)\\
=& \varepsilon(\nabla\varphi_\gamma, \nabla w) + \frac{1}{\varepsilon}(\varphi_\gamma, w) + \eta(\varphi_\gamma - \hat\varphi, w) - \varepsilon(\nabla\hat\varphi, \nabla w) + \frac{1}{\varepsilon}(\hat\varphi, w)\\
=& \frac{1}{\varepsilon}(1, w) - (1-\kappa)\int_\Omega \varphi_\gamma \mathbb{C}e(u_\gamma) \colon e(u_\gamma)\,dx\\
&- \gamma\int_\Omega ((\varphi_\gamma - \hat\varphi)^+)^4 dx - \big(\varepsilon(\nabla\hat\varphi, \nabla w) + \frac{1}{\varepsilon}(\hat\varphi, w)\big),
\end{aligned}$$

where the last equality follows from testing the second equation in (2.1) with $\psi = w$. By nonnegativity of φ_γ and the definition of w we can neglect nonpositive terms on the right-hand-side and obtain, after integration by parts

$$\begin{aligned}
c\|w\|_{H^1}^2 \le& (\frac{1}{\varepsilon} + \varepsilon\Delta\hat\varphi, w) - \gamma\|(\varphi_\gamma, \hat\varphi)^+\|_{L^4}^4\\
\le& \int_\Omega |\frac{1}{\varepsilon} + \varepsilon\Delta\hat\varphi|(\varphi_\gamma - \hat\varphi)^+\gamma^{\frac{1}{q^*}}\gamma^{-\frac{1}{q^*}}\,dx - \gamma\|(\varphi_\gamma, \hat\varphi)^+\|_{L^4}^4\\
\le& \|\gamma^{-\frac{1}{q^*}}(\frac{1}{\varepsilon} + \varepsilon\Delta\hat\varphi)\|_{L^q}\|\gamma^{\frac{1}{q^*}}(\varphi_\gamma - \hat\varphi)^+\|_{L^{q^*}} - \gamma\|(\varphi_\gamma, \hat\varphi)^+\|_{L^4}^4.
\end{aligned}$$

Recall that $q^* = \frac{p}{2}$ and note that for $q^* \le 4$ we find

$$\|\gamma^{\frac{1}{q^*}}(\varphi_\gamma - \hat\varphi)^+\|_{L^{q^*}} \le c\|\gamma^{\frac{1}{q^*}}(\varphi_\gamma - \hat\varphi)^+\|_{L^4},$$

whereas for $q^* < 4$ we can estimate

$$\|\gamma^{\frac{1}{q^*}}(\varphi_\gamma - \hat\varphi)^+\|_{L^{q^*}} \le \|\gamma^{\frac{1}{4}}(\varphi_\gamma - \hat\varphi)^+\|_{L^4}^{\frac{4}{q^*}}.$$

Hence, w.l.o.g. let $q^* \geq 4$. Then, estimating the right-hand-side by means of the generalized Hölder's inequality we finally obtain

$$\|w\|_{H^1}^2 \leq c\gamma^{-\frac{1}{q^*}} \|\frac{1}{\varepsilon} + \varepsilon\Delta\hat{\varphi}\|_{L^q}^q \pm \gamma\|(\varphi_\gamma - \hat{\varphi})^+\|_{L^4}^4 = c\gamma^{-\frac{1}{q^*}} \|\frac{1}{\varepsilon} + \varepsilon\Delta\hat{\varphi}\|_{L^q}^q.$$

This finally yields

$$\|\min(\varphi_\gamma, \hat{\varphi}) - \varphi_\gamma\|_{H^1} \leq c\gamma^{-\frac{1}{2q^*}}.$$

□

4. The Time Discretized Lower Level Fracture Problem. Based on our findings in Section 3, we can now collect the results for the time-dependent problem (EL$^\gamma$) with possible viscous regularization. Let us recall the regularized problem formulation. For given $\varphi^0 \in W^{2,q}(\Omega)$, $0 \leq \varphi^0 \leq 1$ and $q \in L^2(\Gamma_N)^M$ it consists of finding $\mathbf{u}_\gamma = (u_\gamma, \varphi_\gamma) \in V^M$ solving

$$\begin{aligned} \Big(g(\varphi^i)\mathbb{C}e(u^i), e(v)\Big) - (q^i, v)_{\Gamma_N} &= 0, \\ \varepsilon(\nabla\varphi^i, \nabla\psi) - \frac{1}{\varepsilon}(1-\varphi^i, \psi) + \eta(\varphi^i - \varphi^{i-1}, \psi)& \\ +(1-\kappa)(\varphi^i\mathbb{C}e(u^i) : e(u^i), \psi)& \\ +\gamma([(\varphi^i - \varphi^{i-1})^+]^3, \psi) &= 0 \end{aligned} \tag{EL$^{\gamma,\eta}$}$$

for any $(v, \psi) \in V$ and $i = 1, \ldots, M$.

Recall also that its limit as $\gamma \to \infty$ is given by finding $\mathbf{u}_\infty = (u_\infty, \varphi_\infty) \in V^M$ and $\lambda_\infty \in (H^1(\Omega)^*)^M$ solving

$$\begin{aligned} \Big(g(\varphi^i)\mathbb{C}e(u^i), e(v)\Big) - (q^i, v)_{\Gamma_N} &= 0, \\ \varepsilon(\nabla\varphi^i, \nabla\psi) - \frac{1}{\varepsilon}(1-\varphi^i, \psi) + \eta(\varphi^i - \varphi^{i-1}, \psi)& \\ +(1-\kappa)(\varphi^i\mathbb{C}e(u^i) : e(u^i), \psi)& \\ +(\lambda^i, \psi) &= 0, \\ \varphi^i &\leq \varphi^{i-1}, \\ \lambda^i &\geq 0, \\ (\lambda^i, \varphi^i - \varphi^{i-1}) &= 0, \end{aligned} \tag{EL$^{\eta}$}$$

for any $(v, \psi) \in V$ and $i = 1, \ldots, M$.

For this we have the following stability and regularity result:

COROLLARY 4.1. *Any solution $\mathbf{u}_\gamma$ to (EL$^{\gamma,\eta}$) satisfies the additional regularity*

$$\varphi_\gamma \in W^{2,q}(\Omega)^M, \quad u_\gamma \in (H^{1+s}(\Omega))^M$$

with the stability estimate

$$\begin{aligned} \|\varphi_\gamma^i\|_{2,q} &\leq c\Big(i + \sum_{j=1}^{i} \|q^j\|^2 + \|\varphi^0\|_{2,q}\Big) \\ &\leq c\Big(M + \sum_{j=1}^{M} \|q^j\|^2 + \|\varphi^0\|_{2,q}\Big) =: R_q \\ \|u^i\|_{1+s} &\leq c_q\|q^i\| \end{aligned}$$

where c_q depends on R_q, κ, only.

Proof. The regularity and stability estimate for φ_γ^i are an immediate consequence of Corollary 3.9. The bound for u_γ^i then follows from [11]. ☐

COROLLARY 4.2. *Let*

$$(u_\gamma, \varphi_\gamma, \gamma((\varphi_\gamma^i - \varphi_\gamma^{i-1})^+)^3) \rightharpoonup (u_\infty, \varphi_\infty, \lambda_\infty)$$

be any weakly convergent sequence of solutions to ($\mathrm{EL}^{\gamma,\eta}$) *in* $V^M \times (H^1(\Omega)^*)^M$ *with limit* $(u_\infty, \varphi_\infty, \lambda_\infty)$ *as* $\gamma \to \infty$*. Then* $(u_\infty, \varphi_\infty, \lambda_\infty)$ *solves* (EL^η)*, satisfies the regularity*

$$(u_\infty, \varphi_\infty, \lambda_\infty) \in H^{1+s}(\Omega)^M \times W^{2,q}(\Omega)^M \times L^q(\Omega)^M$$

and it holds

$$\begin{aligned} u_\gamma &\to u_\infty && \text{in } H^{1+s}(\Omega)^M,\\ \varphi_\gamma &\to \varphi_\infty && \text{in } C^{0,\alpha}(\Omega)^M,\\ \gamma((\varphi_\gamma^i - \varphi_\gamma^{i-1})^+)^3 &\rightharpoonup \lambda_\infty && \text{in } L^q(\Omega)^M. \end{aligned}$$

Proof. The proof is analogously to Corollary 3.10 for u and φ and Lemma 3.8 for λ. ☐

5. The Limit in the Optimization Problem. We are now in the position to consider the convergence of solutions of the problem ($\mathrm{NLP}^{\gamma,\eta}$) towards solutions of the corresponding limit problem

$$\begin{aligned} &\min_{q,\mathbf{u}} J(q,\mathbf{u}) := \frac{1}{2}\sum_{i=1}^M \|u^i - u_d^i\|^2 + \frac{\alpha}{2}\sum_{i=1}^M \|q^i\|_{\Gamma_N}^2\\ &\text{s.t. } \mathbf{u} = (u,\varphi) \text{ solves } (\mathrm{EL}^\eta) \text{ given the data } q, \text{ for each } i = 1,\dots,M. \end{aligned} \tag{NLP^η}$$

For this we need to extend the convergence analysis of the previous sections slightly to cope with the changes in the respective optimal controls.

LEMMA 5.1. *Let $q_\gamma \in Q^M$ be a given sequence with $\gamma \to \infty$ with weak limit $q \in Q^M$. Further, let $\mathbf{u}_\gamma = (u_\gamma, \varphi_\gamma) \in V^M$ be a sequence of solutions to* ($\mathrm{EL}^{\gamma,\eta}$) *with data q_γ and set $\lambda_\gamma^i = \gamma((\varphi_\gamma^i - \varphi_\gamma^{i-1})^+)^3$ for all $i = 1,\dots,M$. Then it holds*

$$\sum_{i=1}^M \Big(\|u_\gamma^i\|_{1+s} + \|\varphi_\gamma^i\|_{2,q} + \|\lambda_\gamma^i\|_q \Big) \le c$$

independent of γ and each weakly convergent subsequence $(\mathbf{u}_\gamma, \lambda_\gamma)$ with limit $\mathbf{u} = (u,\varphi) \in V^M$ and $\lambda \in L^q(\Omega)^M$ satisfies

$$\mathbf{u}_\gamma \to \mathbf{u} \text{ strongly in } (W^{1,p}(\Omega) \times C^{0,\alpha}(\Omega))^M,$$

for some $p > 2$ and $\alpha > 0$ as given in Corollary 4.2. Further, any such limit point $\mathbf{u}$ solves (EL^η) *for the data q.*

Proof. The stability estimates of Corollary 4.1 together with the bound on q_γ due to its weak convergence imply the norm bound. Compact embeddings imply the strong convergence in $W^{1,p}(\Omega)$ and $C^{0,\alpha}(\Omega)$. The bound on λ_γ follows from Lemma 3.8.

With these convergences, passing to the limit from $(\mathrm{EL}^{\gamma,\eta})$ to (EL^{η}) is easy. □

We can now show the convergence result for the optimization problems where we need to assume that the constraint system (EL^{η}) is uniquely solvable in the considered minimizer.

THEOREM 5.2. *Let $\bar q \in Q^M$ be an isolated local minimizer of* (NLP^{η}) *and assume that the corresponding state $\bar{\mathbf{u}}$, $\bar\lambda$ is the unique solution of* (EL^{η}). *Then, for γ sufficiently large, there exists a sequence $q_\gamma, \mathbf{u}_\gamma$ of local minimizers of* (NLP^{γ}) *such that*

$$\begin{aligned} q_\gamma &\to \bar q && \text{in } Q^M\\ \mathbf{u}_\gamma &\to \bar{\mathbf{u}} && \text{in } V^M \cap (H^{1+s}(\Omega)\times C^{0,\alpha}(\Omega))^M. \end{aligned}$$

Proof. Since $\bar q$ is an isolated minimizer, there exists $\delta > 0$ such that

$$J(q,u) > J(\bar q, \bar{\mathbf{u}})$$

for all $q \in \operatorname{cl} B_\delta(\bar q) \setminus \{\bar q\}$ and any u solving (EL^{η}).

We now define q_γ and u_γ as global minimizers of $(\mathrm{NLP}^{\gamma,\eta})$ with additional constraint $q_\gamma \in \operatorname{cl} B_\delta(\bar q)$, having no influence on the existence of such minimizers, cf. the proof of [19, Theorem 4.3].

Now, clearly q_γ is bounded in Q^M and hence there is a weakly convergent subsequence, denoted the same, with limit $q \in \operatorname{cl} B_\delta(\bar q)$. By Lemma 5.1, we may assume that the corresponding sequence $\mathbf{u}_\gamma$, again up to subsequences, converges in $(W^{1,p}(\Omega)\times C^{0,\alpha}(\Omega))^M$ to a limit $\mathbf{u}$ and $\lambda_\gamma \rightharpoonup \lambda$ in $L^q(\Omega)^M$. Again, by Lemma 5.1, this limit solves (EL^{η}) and thus by weak lower-semicontinuity of J

$$J(\bar q, \bar{\mathbf{u}}) \le J(q,\mathbf{u}) \le \liminf_{\gamma\to\infty} J(q_\gamma, \mathbf{u}_\gamma) \le \limsup_{\gamma\to\infty} J(q_\gamma,\mathbf{u}_\gamma).$$

On the other hand, let $\tilde{\mathbf{u}}_\gamma$ be a solution to $(\mathrm{EL}^{\gamma,\eta})$ with data $\bar q$. Then, by Corollary 4.2 $\tilde{\mathbf{u}}_\gamma \to \bar{\mathbf{u}}$ as $\bar{\mathbf{u}}$ is the only solution to (EL^{η}) for the data $\bar q$. Since $(q_\gamma, \mathbf{u}_\gamma)$ are global minimizers of the localized (NLP^{γ}), it holds

$$J(q_\gamma, \mathbf{u}_\gamma) \le J(\bar q, \tilde{\mathbf{u}}_\gamma) \to J(\bar q, \bar{\mathbf{u}})$$

and thus we conclude

$$J(\bar q, \bar{\mathbf{u}}) = J(q,\mathbf{u}) = \lim_{\gamma\to\infty} J(q_\gamma,\mathbf{u}_\gamma).$$

Since we assumed the minimizer to be isolated, we conclude

$$\begin{aligned} q_\gamma &\rightharpoonup q = \bar q && \text{in } Q^M,\\ \mathbf{u}_\gamma &\to \mathbf{u} = \bar{\mathbf{u}} && \text{in } (W^{1,p}(\Omega)\times C^{0,\alpha}(\Omega))^M. \end{aligned}$$

For the strong convergence, we utilize Clarkson's inequality to obtain

$$\sum_{i=1}^M \Big(\frac{1}{2}\|\bar u^i - u^i_\gamma\|^2 + \frac{\alpha}{2}\|\bar q^i - q^i_\gamma\|^2\Big) \le \frac{1}{2} J(\bar q, \bar{\mathbf{u}}) + \frac{1}{2} J(q_\gamma, \mathbf{u}_\gamma) - J(\tilde q, \tilde{\mathbf{u}}) \tag{5.1}$$

with $\tilde q = \frac{1}{2}(\bar q + q_\gamma)$, $\tilde{\mathbf{u}} = \frac{1}{2}(\bar{\mathbf{u}} + \mathbf{u}_\gamma)$. Now, let $\hat{\mathbf{u}}_\gamma$ be any solution to $(\mathrm{EL}^{\gamma,\eta})$ corresponding to $\tilde q$. Then, since $\tilde q \rightharpoonup \bar q$, Lemma 5.1 asserts $\hat{\mathbf{u}}_\gamma \to \bar{\mathbf{u}}$. Analogous $\tilde{\mathbf{u}} \to \bar{\mathbf{u}}$. Using that $(q_\gamma, \mathbf{u}_\gamma)$ is a global minimizer of the localized (NLP^{γ}), we get

$$J(q_\gamma, \mathbf{u}_\gamma) \le J(\tilde q, \hat{\mathbf{u}}_\gamma)$$

and we conclude from (5.1)

$$\begin{aligned}\sum_{i=1}^{M}\Big(\frac{1}{2}\|\bar u^i - u^i_\gamma\|^2 + \frac{\alpha}{2}\|\bar q^i - q^i_\gamma\|^2\Big) &\le \frac{1}{2}J(\bar q,\bar{\mathbf{u}}) + \frac{1}{2}J(q_\gamma,\mathbf{u}_\gamma) - J(\tilde q,\hat{\mathbf{u}}_\gamma)\\ &\quad + \frac{1}{2}\sum_{i=1}^{M}\Big(\|\hat u^i_\gamma - u^i_d\|^2 - \|\tilde u^i - u^i_d\|^2\Big)\\ &\le \frac{1}{2}J(\bar q,\bar{\mathbf{u}}) - \frac{1}{2}J(q_\gamma,\mathbf{u}_\gamma) + \frac{1}{2}\sum_{i=1}^{M}\Big(\|\hat u^i_\gamma - u^i_d\|^2 - \|\tilde u^i - u^i_d\|^2\Big)\\ &\to 0.\end{aligned}$$

This proves the strong convergence of $q_\gamma \to \bar q$.

From strong convergence of q_γ the strong convergence of $\mathbf{u}_\gamma \in (H^{1+s}(\Omega) \times C^{0,\alpha}(\Omega))^M$ follows analogously as in Corollary 3.10. ☐

We note that it is sufficient, to assert that the set of solutions $\mathbf{u}$ to (EL$^\eta$) is single-valued for the optimal data $\bar q$ while no such assumption was needed for other data.

6. Numerical tests. In this last section, some numerical tests are presented. The goal is to show that our findings from Section 5 work in practice and confirm our theoretical results for $\gamma \to \infty$. To this end, we choose large, increasing, penalizations γ in our computations yielding a convergent optimization algorithm. Moreover, we investigate the convergence rates of the feasibility violation with respect to γ derived in Lemma 3.11. The computations are performed with DOpElib [9, 5] utilizing the deal.II finite element library [2, 1]. The spatial discretization is based on usual conforming Q_1 finite elements for all unknowns.

Setup. The setup is to employ a control q on the top boundary of a two-dimensional square domain, acting in normal direction only, in order to steer the solution towards a manufactured solution u_d defined in the entire domain. The (spatially) discretized optimization problem (NLP$^{\gamma,\eta}$) is solved by a globalized Newton's method for the reduced optimization problem, i.e., the optimization problem is transformed into an unconstrained problem via elimination of the equality constraint due to the choice of a specific solution of the discretized fracture problem. In our numerical example this solution was found utilizing a globalized Newton's method for the discretized Euler-Lagrange equation (EL$^{\gamma,\eta}$) (with $\eta = 0$).

As it is well-known, too large γ yield ill-conditioned systems. For this reason, we iterate by starting with a reasonable value for γ and increase its value by one order of magnitude once the reduced optimization algorithm converged.

Domain. The domain is given by $\Omega := (-1,1)^2$ in which a horizontal fracture is prescribed. The initial value for φ^0 is taken such that $\varphi^0 = 0$ on $(-0.1-h, 0.1+h) \times (-h,h) \subset \Omega$, where h denotes the diameter of the elements. The boundary is divided into three parts $\partial\Omega := \Gamma_N \cup \Gamma_D \cup \Gamma_{\text{free}}$ corresponding to the control boundary Γ_N, the Dirichlet boundary Γ_D, and the rest, where natural boundary conditions for the displacement are attained. These boundary parts are given by

$$\Gamma_N = \{(x,1) \mid -1 \le x \le 1\} \quad \text{and} \quad \Gamma_{\text{free}} = \{(x,y) \mid x \in \{\pm 1\}; -1 \le y \le 1\}.$$

On Γ_D, we prescribe the Dirichlet values $u = 0$.

We consider different spatial meshes obtained by uniform mesh refinement:

- 6 times uniform refinement yielding 12675 DoFs;
- 7 times uniform refinement yielding 49923 DoFs;
- 8 times uniform refinement yielding 198147 DoFs;

Cost functional. The cost functional is given by

$$J(q,\mathbf{u}) := \frac{1}{2}\sum_{i=1}^{M} \|u^i - u_d^i\|^2 + \frac{\alpha}{2}\|q-q_d\|^2_{\Gamma_N}$$
$$\text{s.t. } (q,\mathbf{u}) \text{ satisfying } (\mathrm{EL}^{\gamma,\eta}),$$

with $\eta = 0$, $u_d^i = 0.001(y+1)$ for all $i = 1,\dots,M$, $\alpha = 10^{-10}$ and a control acting on Γ_N but being the same in all time-steps, i.e, $q^i = q$ for all $i = 1,\dots,M$, and $q_d \equiv 50$. Moreover, $\mathbf{u}^0 = (0;\varphi^0)$ with φ^0 as described above.

Model and material parameters. The phase-field regularization parameter is fixed with $\varepsilon = 0.088$. This value corresponds to $\varepsilon = 2h$ on the coarsest refinement level 6. The bulk regularization parameter is $\kappa = 10^{-10}$.

The fracture energy release rate is $G_c = 1.0$, Young's modulus is $E = 10^6$ and Poisson's ratio is $\nu = 0.2$. We consider 5 loading steps, i.e., $M = 5$, are performed.

As previously mentioned, the penalization parameter is variably adjusted from $\gamma = 10^8$ to $\gamma = 10^{14}$.

Numerical results for $\gamma \to \infty$. We present results for three different meshes levels and seven iterations on the penalization parameter γ. We observe first a significant reduction of the initial cost functional, which then only undergoes slight variations for larger γ. As expected, the reduced Newton algorithm requires some iterations at the beginning and using the obtained control for $\gamma = 10^8$, yields an excellent initial value for the next Newton iteration resulting in much less iterations, sometimes even no additional iterations are needed. However we notice, that this still means that the fracture model and the corresponding adjoint are recomputed for the updated values giving a correct value for the residual.

Table 6.1
Results on refinement level 6 with initial cost functional value $1.248e-05$. The initial control is $q^0 \equiv 10$ and the final control $\max(q) = 84.36$.

γ	$J[\times 10^{-5}]$	Iter.	Residual	$\|\lambda_\gamma\|_1$	$\|\lambda_\gamma\|_2^2$	$\|\max(\varphi_\gamma^i,\varphi^{i-1}) - \varphi_\gamma^i\|^2_{H^1(\Omega)}$
10^8	1.0533	4	$8.7\cdot 10^{-13}$	1.1	244	$6\cdot 10^{-3}$
10^9	1.0531	1	$9.4\cdot 10^{-13}$	1.1	254	$2\cdot 10^{-3}$
10^{10}	1.0531	1	$4.6\cdot 10^{-13}$	1.1	258	$7\cdot 10^{-4}$
10^{11}	1.0532	1	$3.4\cdot 10^{-13}$	1.1	260	$3\cdot 10^{-4}$
10^{12}	1.0532	0	$4.0\cdot 10^{-13}$	1.1	262	$6\cdot 10^{-5}$
10^{13}	1.0532	0	$9.6\cdot 10^{-13}$	1.1	262	$2\cdot 10^{-5}$
10^{14}	1.0532	1	$6.4\cdot 10^{-14}$	1.1	262	$4\cdot 10^{-6}$

Table 6.2

Results on refinement level 7 *with initial cost functional value* 1.248e − 05*. The initial control is* $q^0 = 10$ *and the final control* $\max(q) = 50.01$.

γ	$J[\times 10^{-5}]$	Iter.	Residual	$\Vert\lambda_\gamma\Vert_1$	$\Vert\lambda_\gamma\Vert_2^2$	$\Vert \max(\varphi_\gamma^i, \varphi^{i-1}) - \varphi_\gamma^i\Vert_{H^1(\Omega)}^2$
10^8	1.0832	1	$1.1 \cdot 10^{-13}$	0.7	406	$1 \cdot 10^{-2}$
10^9	1.0832	0	$1.1 \cdot 10^{-13}$	0.7	432	$3 \cdot 10^{-3}$
10^{10}	1.0831	0	$1.1 \cdot 10^{-13}$	0.8	445	$2 \cdot 10^{-3}$
10^{11}	1.0831	0	$1.1 \cdot 10^{-13}$	0.8	449	$1 \cdot 10^{-3}$
10^{12}	1.0831	0	$1.1 \cdot 10^{-13}$	0.8	450	$1 \cdot 10^{-3}$
10^{13}	1.0831	0	$1.1 \cdot 10^{-13}$	0.8	451	$1 \cdot 10^{-3}$
10^{14}	1.0831	0	$1.1 \cdot 10^{-13}$	0.7	452	$1 \cdot 10^{-3}$

Table 6.3

Results on refinement level 8 *with initial cost functional value* 1.248e − 05*. The initial control is* $q^0 = 10$ *and the final control* $\max(q) = 50.03$.

γ	$J[\times 10^{-5}]$	Iter.	Residual	$\Vert\lambda_\gamma^i\Vert_1$	$\Vert\lambda_\gamma^i\Vert_2^2$	$\Vert \max(\varphi_\gamma^i, \varphi^{i-1}) - \varphi_\gamma^i\Vert_{H^1(\Omega)}^2$
10^8	1.0844	1	$8.7 \cdot 10^{-14}$	0.6	178	$8 \cdot 10^{-3}$
10^9	1.0843	0	$8.7 \cdot 10^{-14}$	0.6	194	$3 \cdot 10^{-3}$
10^{10}	1.0842	0	$8.7 \cdot 10^{-14}$	0.6	205	$7 \cdot 10^{-4}$
10^{11}	1.0842	0	$8.7 \cdot 10^{-14}$	0.6	211	$2 \cdot 10^{-4}$
10^{12}	1.0842	0	$8.7 \cdot 10^{-14}$	0.6	214	$6 \cdot 10^{-5}$
10^{13}	1.0842	0	$8.7 \cdot 10^{-14}$	0.6	217	$2 \cdot 10^{-5}$
10^{14}	1.0842	0	$8.7 \cdot 10^{-14}$	0.6	217	$4 \cdot 10^{-6}$

Our findings are provided in the Tables 6.1, 6.2, and 6.3 for different numbers of unknowns.

Computing the norm of Lemma 3.11. In the last three columns of the provided tables, norms of λ_γ and the feasibility violation $\max(\varphi_\gamma^i, \varphi^{i-1}) - \varphi_\gamma^i$ are depicted. In these we only show the largest value over all time-steps. We notice that the computation over increasing γ is rather efficient, as for large values already a good enough starting point is presented to require no further optimization iterations; although the fracture forward and adjoint problem need to be solved once to verify this. As proven in our theoretical sections, λ_γ remains bounded in $L^1(\Omega)$ but appears unbounded in $L^2(\Omega)$ indicating that the difficulties taken in the proofs for $\lambda_\gamma \in L^q(\Omega)$ for some $q > 1$ but close to one can not be avoided. Finally, we observe a mild decay of the feasibility violation in $H^1(\Omega)$ as established.

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