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# A PRIORI ERROR ESTIMATES FOR A LINEARIZED FRACTURE CONTROL PROBLEM*

MASOUMEH MOHAMMADI† AND WINNIFRIED WOLLNER ‡

**Abstract.** A control problem for a linearized time-discrete fracture propagation process is considered. The discretization of the problem is done using a conforming finite element method. In contrast to many works on discretization of PDE constrained optimization problems, the particular setting has to cope with the fact that the linearized fracture equation is not necessarily coercive. A quasi-best approximation result will be shown in the case of an invertible, though not necessarily coercive, linearized fracture equation. Based on this a priori error estimates for the control, state, and adjoint variables will be derived.

**Key words.** optimal control, linearized fracture model, finite element method, a priori error estimate

**AMS subject classifications.** 65N12, 65N15, 65N30, 49M25, 74S05

## 1. Introduction.

Modeling, predicting, and control of fracture or damage in solid materials are of great technical importance for the safety requirements of structures in various fields of engineering, e.g., automobile components, aerospace, and marine industries. Therefore, developing a comprehensive model of fracture propagation has long been a challenge in physics, mechanics, and material sciences [20, 22]. The classical method for modeling the fracture propagation is to consider a sharp interface in order to separate the structure explicitly into a fully broken part and a fully intact one. This approach implies tracking the exact position of the interface to be able to follow the propagation of the fracture. Therefore, in finite element settings for fracture description, the numerical implementation requires handling of the discontinuities. To overcome the problem of explicit interface tracking, the phase-field method, going back to [2], is recently widely used for the description of fracture phenomena. This method is also attractive because of the high ability of simulating the fracture initiation, propagation, merging, and branching. The phase-field approach to model the fracture, as a two-phase discontinuous model with a sharp interface, consists in introducing a continuous field variable in order to approximate the sharp fracture discontinuity. The field variable smoothly differentiates between the two phases. In fact, the fracture phase-field represents the smooth transition from the fully destroyed phase to the fully intact part. The fracture propagation is tracked by the evolution of the phase field.

In this paper, our linearized model is based upon classical Griffiths theory of fracture [15] which was rewritten as a variational model in [11]. For some overview and summary of the obtained results, see, e.g., [6, 25, 1]. As the variational inequality, resulting from the fracture irreversibility, is sometimes undesirable when working in optimization, we allow for a regularization of the irreversibility analyzed in [27].

The approximation error analysis for finite element simulations of fractures is only considered in simple situations. See, e.g., [26] for fracture propagation without a

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phase-field, or [8, 9] for a phase-field fracture model. However, even in this literature only qualitative convergence has been shown. To the best of our knowledge, no quantitative convergence explicitly clarifying the dependence of the convergence speed on the mesh-size and the problem data can be found in studies on FE-analysis of fracture propagation. Since in optimization problems the problem data vary, this quantitative dependence is crucial in the discretization error analysis of optimization problems.

To deal with this lack of analysis, we provide analysis for a linearized fracture model considered within an optimization problem. While this equation is linear, it does not correspond to a positive definite bilinear form. Furthermore, the regularity in the linear equation is severely limited, due to the non-smooth coefficients induced by the known regularity of the nonlinear fracture problem. These two points are in contrast to the usual assumptions made in the discretization error analysis of optimization problems. In fact, known regularity results for the coefficients allow for $W^{1,p}$ regularity of the solutions, only, thereby prohibiting quantitative rates of convergence in $W^{1,2}$ as would be needed for the standard error analysis of elliptic optimization problems, compare, e.g., [10, 13, 21, 3, 28, 23, 24, 18] to name only a few.

To circumvent the problems coming from indefinite bilinear forms, we will utilize an approach proposed by [29], to show that if the continuous linearized fracture model admits a unique solution, the discretized equation does so too, for sufficiently small mesh size. We can then expand the technique of [12], to assert that the same asymptotic error estimates holds for the adjoint problem as well.

Finally, the lack in regularity can be avoided, as [17] have shown that a slightly improved differentiability may be assumed for solutions to fracture problems. This will be crucial for our work in obtaining quantitative estimates for the discretization error.

While we do not tackle the nonlinear fracture problem, the obtained discretization errors can then be utilized in SQP-type methods applied to the control of nonlinear fracture problems to efficiently couple discretization error and progress in the optimization variable, see, e.g., [31].

This paper is organized as follows. In Section 2, we review the linearized model of the fracture control problem which was discussed in [27]. Section 3 contains the finite element discretization of the considered model, and a priori error estimate for the discretized model. The section is subdivided, for better clarity, into two parts.

First, in Section 3.1, we consider the case when the linearized equation is an isomorphism, but the corresponding bilinear form is not positive definite. Based upon an approach by [29], we will utilize compactness to show that for sufficiently small mesh sizes the discretized equation remains an isomorphism, and that the error satisfies a quasi-best approximation property. Based upon this result, and an improved differentiability result by [17], we can utilize standard techniques to derive a posteriori error estimates for the optimization problem using a discretization approach suggested by [18]. Second, in Section 3.3, we will extend a new technique, developed in [12] for the case of an isomorphism, to show that a similar estimate also holds for the adjoint variable. We will place particular emphasis on the stability of the estimates with respect to variations of the linearization point as it is needed for the inexact iterative solution of a corresponding nonlinear optimization problem via, e.g., an SQP algorithm.

Section 4 presents the numerical test highlighting the reduced rates of convergence compared to the standard setting where smooth coefficients are considered.

Throughout, $c$ denotes a generic constant which may be different at each instance.

**2. The Linearized Fracture Control Problem.** Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, with boundary $\partial\Omega$ consisting of $\Gamma_D$ and $\Gamma_N$ with

$$\mathcal{H}^{d-1}(\Gamma_D) \neq 0 \quad \text{and} \quad \mathcal{H}^{d-1}(\Gamma_N) \neq 0,$$

where $H^{d-1}$ is the $d-1$-dimensional Hausdorff-measure, and $\Gamma_D$ and $\Gamma_N$ are the parts where Dirichlet and Neumann boundary conditions are imposed, respectively. We assume that $\Omega \cup \Gamma_N$ is regular in the sense of Gröger [16], and the fracture propagation is controlled by the traction force $q$ acting on the boundary $\Gamma_N$.

By $u$ we represent the vector-valued displacement field, in the space of admissible displacements $H^1_D(\Omega;\mathbb{R}^2) := \{v \in H^1(\Omega;\mathbb{R}^2) \,|\, v = 0 \text{ on}\, \Gamma_D\}$. The usual $L^2$-scalar product, and the corresponding norm, are denoted by $(\cdot,\cdot)$ and $\|\cdot\|$ respectively. We use appropriate subscripts $1, p$ or $r$ in the norms in corresponding Sobolev spaces $W^{1,p}(\Omega)$ or $H^r(\Omega) = W^{r,2}(\Omega)$. The $L^2(\Gamma_N)$-norm will be indicated by a subscript $\Gamma_N$.

Following [27], the fracture $\mathcal{C}$ is initially modeled by Griffith's criterion for brittle fracture, which assumes the fracture propagates when the elastic energy restitution rate reaches its critical value $G_c$. It is then regularized by a phase-field approach. The phase-field variable $\varphi$, represents the fracture region by $\varphi = 0$, and the non-fractured part by $\varphi = 1$. The values in between, $0 < \varphi < 1$, correspond to a transition zone with width $\varepsilon$ on each side of the fracture path. The problem is then to find $u(t), \varphi(t)$ minimizing the energy of the system subject to the irreversibility constraint

$$\varphi(t_2) \leq \varphi(t_1), \quad \forall\, t_1 \leq t_2.$$

After introducing a time partition, the time evolution of the fracture is given by a sequence of problems associated to each time step. As the error estimate, which is the scope of this work, remains invariant for any time level, we ignore the time discretization, and provide the argument only for one time step.

In order to avoid degeneracy in the elastic energy, the model is further regularized by the parameter $\kappa > 0$, $\kappa \ll \varepsilon$, and the coefficient function $g(\varphi)$. To guarantee the irreversibility of the fracture as well as the differentiablity, the regularized fracture model is relaxed by a penalization term with some positive factor $\gamma$. Letting $\mathbb{C}$ represent the elasticity tensor, and $e(u) = \frac{1}{2}(\nabla u + \nabla u^T)$ the symmetric gradient, the fracture model presented in [27] asserts that any energy minimizer $\mathbf{u} = (u, \varphi)$ satisfies the Euler-Lagrange equations

$$\begin{aligned} \Big(g(\varphi)\mathbb{C}e(u), e(v)\Big) - (q, v)_{\Gamma_N} &= 0 \\ G_c\varepsilon(\nabla\varphi, \nabla\psi) - \frac{G_c}{\varepsilon}(1-\varphi, \psi) & \\ +(1-\kappa)(\varphi\mathbb{C}e(u) : e(u), \psi) & \\ +\gamma([(\varphi - \varphi^0)^+]^3, \psi) &= 0 \end{aligned} \tag{1}$$

for a given $\varphi^0 \in H^1(\Omega)$, $0 \leq \varphi^0 \leq 1$, a given $q \in Q$, and any $(v, \psi) \in V$. Here

$$Q := L^2(\Gamma_N), \qquad V := H^1_D(\Omega;\mathbb{R}^2) \times H^1(\Omega),$$

and the coefficient function is given by

$$g(\varphi) := (1-\kappa)\varphi^2 + \kappa.$$

It is further shown in [27] that there exists at least one solution $\mathbf{u} = (u, \varphi)$ of (1) in $V$, while any solution of (1) satisfies the additional regularity

$$\mathbf{u} \in W^{1,p}(\Omega; \mathbb{R}^2) \times L^\infty(\Omega)$$

for some $p > 2$, depending only on $\kappa$ and $\Omega$. More precisely, it holds $0 \le \varphi \le 1$, and there exists a constant $c_\kappa$ depending on $\kappa$ such that

$$\|u\|_{1,p} \le c_\kappa \|q\|.$$

Very recently, a higher regularity of the solution is derived in [17]. In fact, the estimate

$$\|u\|_{1+s} \le c\|q\| \tag{2}$$

holds true for a constant $c = c(\|q\|^2, \gamma, \varepsilon)$, and a sufficiently small positive $s$, depending only on $\kappa$ and $\Omega$.

Since the fracture is modeled in order to finally propagate subject to an optimal control, it is required to provide an appropriate means for discussing first order necessary optimality conditions, as well as the potential approximation of the nonlinear optimization problem by a sequence of linear-quadratic problems. Therefore, the model is then linearized at a given point $(q_k, \mathbf{u}_k) = (q_k, u_k, \varphi_k)$. It is shown in [27] that we can assume the regularity $(u_k, \varphi_k) \in (V \cap (W^{1,p}(\Omega; \mathbb{R}^2) \times L^\infty(\Omega)))$ for $p > 2$. In combination with (2), we consider the following regularity, throughout the paper, on the point where the model is linearized about.

Assumption 1. *We assume the existence of constants $s > 0$ and $p > 2$ and $C$ such that*

$$(q_k, \mathbf{u}_k) = (q_k, u_k, \varphi_k) \in Q \times (V \cap H^{1+s}(\Omega) \cap (W^{1,p}(\Omega; \mathbb{R}^2) \times L^\infty(\Omega)))$$

*with*

$$\|q_k\|_{\Gamma_N}, \|u_k\|_{1+s}, \|u_k\|_{1,p}, \|\varphi_k\|_{1,p} \le C.$$

*Further, we assume that the linearized operator $A$ given by (4) has trivial kernel.*

Then the linearized model reads as follows. For given $q \in Q$ and $\varphi^0 \in H^{1+s}(\Omega)$, $0 \le \varphi^0 \le 1$, find $\mathbf{u} = (u, \varphi)$ such that for any $(v, \psi) \in V$

$$\begin{aligned}
&\Big(g(\varphi_k)\mathbb{C}e(u), e(v)\Big) + 2(1-\kappa)(\varphi_k \mathbb{C}e(u_k)\varphi, e(v)) = (q, v)_{\Gamma_N} \\
&G_c\varepsilon(\nabla\varphi, \nabla\psi) + \frac{G_c}{\varepsilon}(\varphi, \psi) + (1-\kappa)(\varphi\mathbb{C}e(u_k) : e(u_k), \psi) \\
&+3\gamma([(\varphi_k - \varphi^0)^+]^2\varphi, \psi) + 2(1-\kappa)(\varphi_k\mathbb{C}e(u_k) : e(u), \psi) = 0.
\end{aligned} \tag{3}$$

Denote the dual space of $V$ by $V^*$, and define the bilinear form $a : V \times V$, and the linear operator $A\colon V \to V^*$ by

$$\begin{aligned}
a(u, \varphi; v, \psi) &= \langle A(u, \varphi), (v, \psi)\rangle_{V^*,V} \\
&= \Big(g(\varphi_k)\mathbb{C}e(u), e(v)\Big) + 2(1-\kappa)(\varphi_k\mathbb{C}e(u_k)\varphi, e(v)) \\
&\quad + G_c\varepsilon(\nabla\varphi, \nabla\psi) + \frac{G_c}{\varepsilon}(\varphi, \psi) + (1-\kappa)(\varphi\mathbb{C}e(u_k) : e(u_k), \psi) \\
&\quad + 3\gamma([(\varphi_k - \varphi_k^0)^+]^2\varphi, \psi) + 2(1-\kappa)(\varphi_k\mathbb{C}e(u_k) : e(u), \psi).
\end{aligned} \tag{4}$$

Defining further the compact operator $B\colon Q \to V^*$ by

$$\langle Bq, (v,\psi)\rangle_{V^*,V} := (q,v)_{\Gamma_N} \tag{5}$$

for all $v, \psi \in V$, the linearized Euler-Lagrange equations (3) can be expressed as

$$A\mathbf{u} = Bq.$$

It is worthwhile to mention that the operator $A\colon V \to V^*$ is Fredholm of index zero, see [27], which plays an important role in adjoint error analysis, providing us with

LEMMA 2.1. *The variational form $a(\cdot,\cdot)$ is continuous on $V \times V$, and satisfies a Gårding-like inequality. Namely, there exists constants $c_c, c_1, c_2$ depending on $C$ in Assumption 1, and some $r \in (0,1)$ such that*

$$a(\mathbf{u};\mathbf{v}) \le c_c \|\mathbf{u}\|_V \|\mathbf{v}\|_V,$$

*and*

$$a(\mathbf{u};\mathbf{u}) \ge c_1 \|\mathbf{u}\|_V^2 - c_2 \|\varphi\|_r^2.$$

With this, we consider the following optimal control problem for fracture propagation, where the displacement $u$ is forced to be as close as possible to a desired displacement $u^{\mathrm{d}} \in L^2(\Omega)$, by the action of the control variable $q$.

Find $(q,\mathbf{u}) = (q,(u,\varphi)) \in (Q \times V)$ solving

$$\begin{aligned} \min_{q,\mathbf{u}}\ J(q,\mathbf{u}) :&= \frac{1}{2}\|u - u_{\mathrm{d}}\|^2 + \frac{\alpha}{2}\|q\|^2_{L^2(\Gamma_N)} \\ \text{s.t. } A\mathbf{u} &= Bq, \end{aligned} \tag{6}$$

in which the parameter $\alpha > 0$ scales the cost of the control.

According to [27], the problem (6) admits a unique solution. In contrast to standard analysis for (6), even if it is assumed that $A$ is an isomorphism, $A$ is usually not coercive, cf. Lemma 2.1. Based on (2), we consider the regularity of any solution to $A\mathbf{u} = Bq$ in $H^{1+s}(\Omega)$ for some $s > 0$.

**3. A priori finite element error estimate.** This section is devoted to discretization setting and derivation of estimate of corresponding error. We consider a conforming finite element method (FEM) to discretize the problem (6) in space. Let $\{\mathcal{T}_h\}$ be a sequence of meshes with mesh size $h > 0$, $h \to 0$. The mesh $\mathcal{T}_h$ consists of open cells $T$ which provide a decomposition of $\overline{\Omega}$, that is

$$\overline{\Omega} = \bigcup_{T\in\mathcal{T}_h} \overline{T}$$

such that the mesh matches the splitting of the boundary into $\Gamma_D$ and $\Gamma_N$. The mesh size $h$ is defined by $h := \max_{T\in\mathcal{T}_h} \operatorname{diam}(T)$, and $\mathcal{T}_h$ satisfies the standard quasi-uniform mesh properties in the sense of [7]. With this setting, we consider a conforming finite dimensional space $V_h \subset V$, with piecewise linear test- and ansatz functions, over the decomposed domain $\mathcal{T}_h$, and discretize the following variational formulation.

$$\begin{aligned} &\text{Find} \quad \mathbf{u} = (u,\varphi) \in V \\ &\text{such that} \quad a(\mathbf{u};\mathbf{v}) = (q,v)_{\Gamma_N} \quad \forall \mathbf{v} \in V. \end{aligned} \tag{7}$$

in which $\mathbf{v} = (v, \psi)$. That is, we look for a solution $\mathbf{u}_h = (u_h, \varphi_h)$ in the ansatz space $V_h$, such that the following discrete problem is satisfied in the test space $V_h$ with the elements $\mathbf{v}_h = (v_h, \psi_h)$.

$$\begin{aligned} &\text{Find} \quad \mathbf{u}_h = (u_h, \varphi_h) \in V_h \\ &\text{such that} \quad a(\mathbf{u}_h; \mathbf{v}_h) = (q, v_h)_{\Gamma_N} \quad \forall \mathbf{v}_h \in V_h. \end{aligned} \tag{8}$$

Assuming that the mapping $A : V \to V^*$, given by $\mathbf{u} \mapsto A\mathbf{u} = a(\mathbf{u}; \cdot)$, is an isomorphism we are now ready to provide the error estimate between the solution $\mathbf{u} \in V$ and $\mathbf{u}_h \in V_h$, respectively to the problem (6) and its discretization (8), in the following subsection. The error estimate of the optimal control problem is presented afterwards. The adjoint error will be studied in the second subsection.

We notice, that by [17], the operator $A$ is in fact $H^{1+s}$ regular for some sufficiently small $s > 0$ which we fix from now on.

**3.1. Forward problem.** In this section, we first provide the error analysis for finite element approximations of the state variable $\mathbf{u} = (u, \varphi)$, and then the control variable $q$. To this end, let us $I_h : H^{1+s} \to V_h$ be an interpolation operator satisfying the interpolation error estimate

$$\|w - I_h w\|_V \leq c_I h^s \|w\|_{1+s}$$

for any $w \in H^{1+s}$. Taking this into account, we infer the following theorem.

THEOREM 3.1. *Let Assumption 1 hold and assume that the mapping $A : V \to V^*$, given by (4) is an isomorphism, then there are constants $h_0$ and $c$, such that for all $h \leq h_0$, the problem (8) admits a unique solution, and the solutions $\mathbf{u} \in V$ and $\mathbf{u}_h \in V_h$ of the problems (7) and (8) satisfy the following quasi best-approximation property*

$$\|\mathbf{u} - \mathbf{u}_h\|_V \leq c \inf_{\mathbf{v} \in V_h} \|\mathbf{u} - \mathbf{v}\|_V.$$

*Moreover, $h_0$ is independent of the linearization point, and only depends on $C$ in Assumption 1.*

*Proof.* Following the technique by [29], see also [7], we first show that any solution of the problem (8), if any exists, satisfies the quasi best-approximation error estimate. Next, with the help of the obtained estimation result, we provide the argument for existence of a unique solution to (8). To this end, let us assume that $\mathbf{u}_h$ is such a solution. Furthermore, for compactness of notation, let us denote the error by

$$e_{\mathbf{u}} = (e_u, e_\varphi) := \mathbf{u} - \mathbf{u}_h = (u - u_h, \varphi - \varphi_h).$$

Based on Lemma 2.1, we have

$$c_1 \|\mathbf{v}\|_V^2 \leq a(\mathbf{v}; \mathbf{v}) + c_2 \|\psi\|_r^2, \quad \forall\, \mathbf{v} = (v, \psi) \in V.$$

Letting $\mathbf{v} = e_{\mathbf{u}}$ in the inequality above, based on the Galerkin orthogonality and continuity of the bilinear form $a$, we obtain the following for all $\mathbf{v}_h \in V_h$.

$$\begin{aligned} c_1 \|e_{\mathbf{u}}\|_V^2 &\leq a(e_{\mathbf{u}}; e_{\mathbf{u}}) + c_2 \|e_\varphi\|_r^2 \\ &= a(e_{\mathbf{u}}; \mathbf{u} - \mathbf{v}_h) + c_2 \|e_\varphi\|_r^2 \\ &\leq c_c \|e_{\mathbf{u}}\|_V \|\mathbf{u} - \mathbf{v}_h\|_V + c_2 \|e_\varphi\|_r^2. \end{aligned} \tag{9}$$

Next, we consider that since $A : V \to V^*$ is an isomorphism, where

$$\langle A\mathbf{u}; \mathbf{v}\rangle = a(\mathbf{u}; \mathbf{v}),$$

the mapping $A^* : V^* \to V$ is an isomorphism too.

Noting that

$$\left(\frac{e_u}{\|e_u\|_r}, v\right)_r + \left(\frac{e_\varphi}{\|e_\varphi\|_r}, \psi\right)_r$$

is an element in $(H^r)^*$, and considering the fact that $(H^r)^*$ is embedded in $(H^1_D)^*$, we observe that the adjoint equation

$$\begin{aligned} a(\mathbf{v}; \lambda) &= \langle A^*\lambda; \mathbf{v}\rangle \\ &= \left(\frac{e_u}{\|e_u\|_r}, v\right)_r + \left(\frac{e_\varphi}{\|e_\varphi\|_r}, \psi\right)_r \end{aligned} \tag{10}$$

has a unique solution $\lambda = (\lambda_u, \lambda_\varphi)$ in $H^1_D$.

Without loss of generality, let $s > 0$ in Assumption 1 coincide with the regularity of $A$ in $H^{1+s}$ and be such that $r \leq 1 - s$. Then

$$\begin{aligned} \left(\frac{e_u}{\|e_u\|_r}, v\right)_r + \left(\frac{e_\varphi}{\|e_\varphi\|_r}, \psi\right)_r &\leq \|v\|_r + \|\psi\|_r \\ &\leq \|v\|_{1-s} + \|\psi\|_{1-s}, \end{aligned}$$

which implies that the right hand side of equation (10) is an element of $(H^{1-s})^* = H^{-1+s}$. Therefore, by elliptic regularity based on Lemma (2.1), the solution $\lambda$ of the adjoint equation (10) belongs also to the space $H^{1+s}$, with $\|\lambda\|_{1+s} \leq c_z$. That is, $\lambda \in H^1_D \cap H^{1+s}$.

Now, we can employ the Aubin-Nitsche duality argument, along with the Galerkin orthogonality and the continuity of the bilinear form $a$, to obtain for arbitrary $\lambda_h \in V_h$,

$$\begin{aligned} \|e_u\|_r + \|e_\varphi\|_r &= \left(\frac{e_u}{\|e_u\|_r}, e_u\right)_r + \left(\frac{e_\varphi}{\|e_\varphi\|_r}, e_\varphi\right)_r \\ &= a(e_\mathbf{u}; \lambda) \\ &= a(e_\mathbf{u}; \lambda - \lambda_h) \\ &\leq c_c\|e_\mathbf{u}\|_V\|\lambda - \lambda_h\|_V \\ &\leq c_c c_I h^s\|e_\mathbf{u}\|_V\|\lambda\|_{1+s} \\ &\leq c_c c_I c_z h^s\|e_\mathbf{u}\|_V, \end{aligned}$$

using the previously defined interpolation operator to bound the best approximation error. Consequently, we obtain

$$\|e_\varphi\|_r \leq c_0 h^s\|e_\mathbf{u}\|_V. \tag{11}$$

with $c_0 = c_c c_I c_z$. Combining (9) and (11) we obtain

$$c_1\|e_\mathbf{u}\|_V \leq c_c\|\mathbf{u} - \mathbf{v}\|_V + c_0 c_2 h^s\|e_\mathbf{u}\|_V, \tag{12}$$

which implies that for $h \leq h_0$, where $h_0 = \frac{1}{2}(\frac{c_1}{c_0 c_2})^{1/s}$, the following desired quasi best-approximation property holds:

$$\|\mathbf{u} - \mathbf{u}_h\|_V \leq \frac{2c_c}{c_1}\inf_{\mathbf{v}\in V_h}\|\mathbf{u} - \mathbf{v}\|_V. \tag{13}$$

To complete the proof, it is left to show the existence of $\mathbf{u}_h$ as a unique solution to (8), when $h \le h_0$. Since (8) describes a finite dimensional linear system, it suffices to show that the bilinear form

$$a(\mathbf{u}_h; \mathbf{v}_h) = (q, v_h)_{\Gamma_N} \quad \forall \mathbf{v}_h \in V_h$$

has a trivial kernel for $h \le h_0$. This is clear by noting that $q = 0$ implies $\mathbf{u} = 0$, since $A$ is an isomorphism. Then the error estimate (13) allows us to conclude that $q = 0$ implies $\mathbf{u}_h = 0$ for $h \le h_0$. □

As an immediate consequence, we obtain the following quantitative convergence rate.

COROLLARY 3.2. *Let $u$ and $q$ be the state and the control solutions of the model problem (6). Then there exist positive constants $h_0$, $c$ and $s$ such that*

$$\|e_u\|_V \le ch^s \|q\|_{L^2(\Gamma_N)}$$

*for all $h \le h_0$.*

*Proof.* This is an immediate consequence of combining the regularity estimate (2) with the quasi best-approximation of Theorem 3.1. □

**3.2. The Control Problem.** The result obtained in Theorem 3.1 provides a means to estimate the error in approximating the solution $(q, \mathbf{u})$ of the optimal control problem (6) by a conforming finite element method. Following the idea of [18], the control space $Q$ does not need to be discretized as the optimality conditions imply a variational discretization of $Q$. Let us consider the variational form of the optimization problem

$$\begin{aligned} &\min_{q,\mathbf{u}} \ J(q, \mathbf{u}) = \frac{1}{2}\|u - u^{\mathrm{d}}\|^2 + \frac{\alpha}{2}\|q\|^2_{L^2(\Gamma_N)} \\ &\text{s.t.} \quad a(\mathbf{u}; \mathbf{v}) = (q, v)_{\Gamma_N} \quad \forall\, \mathbf{v} = (v, \psi) \in V, \end{aligned} \tag{14}$$

and the corresponding discretized model

$$\begin{aligned} &\min_{q_h,\mathbf{u}_h} \ J(q_h, \mathbf{u}_h) = \frac{1}{2}\|u_h - u^{\mathrm{d}}\|^2 + \frac{\alpha}{2}\|q_h\|^2_{L^2(\Gamma_N)} \\ &\text{s.t.} \quad a(\mathbf{u}_h; \mathbf{v}_h) = (q_h, v_h)_{\Gamma_N} \quad \forall\, \mathbf{v}_h = (v_h, \varphi_h) \in V_h. \end{aligned} \tag{15}$$

The error estimate can now be derived.

THEOREM 3.3. *Let $(\bar{q}, \bar{\mathbf{u}}) = (\bar{q}, (\bar{u}, \bar{\varphi}))$ be the solution to the problem (14), and $(\bar{q}_h, \bar{\mathbf{u}}_h) = (\bar{q}_h, (\bar{u}_h, \bar{\varphi}_h))$ be the solution to the problem (15), with $h \le h_0$; $h_0$ being the constant introduced in Theorem 3.1. Then we have the following estimate for some positive $c$ and $s$.*

$$\alpha\|\bar{q} - \bar{q}_h\|^2 + \|\bar{u} - \bar{u}_h\|^2 \le c(1 + \frac{1}{\alpha})h^{2s}.$$

*Proof.* With most of the work done in Theorem 3.1 the proof is now standard. We recall that the variational form $a(\cdot, \cdot)$ defines the operator $A$, such that $A\mathbf{u} = Bq$, with $A$ and $B$ introduced in (4) and (5). Since $A$ is an isomorphism, by $A\mathbf{u} = Bq$ we can define the solution operator $S \in \mathcal{L}(Q, V)$, $S = A^{-1}B$, such that $\mathbf{u} = Sq$. Analogously, the discrete operator $S_h \in (Q, V_h)$ is defined corresponding to (8). As the phase-field variable $\varphi$ does not play a role directly in the objective function $J$, we may ignore $\varphi$, and consider $u = Sq$ and $u_h = S_h q_h$ to construct the reduced objective functions

$$j(q) := \frac{1}{2}\|Sq - u^{\mathrm{d}}\|^2 + \frac{\alpha}{2}\|q\|^2_{L^2(\Gamma_N)}$$

and

$$j(q_h) := \frac{1}{2}\|S_h q_h - u^{\mathrm{d}}\|^2 + \frac{\alpha}{2}\|q_h\|^2_{L^2(\Gamma_N)}$$

for the problems (14) and (15), respectively. Denoting the adjoint of $S$ by $S^*$, the necessary optimality conditions for $\bar{q}$ and $\bar{q}_h$ read as follows.

$$(S^*(S\bar{q} - u^{\mathrm{d}}) + \alpha\bar{q}, q - \bar{q}) = 0, \tag{16}$$

and

$$(S_h^*(S_h\bar{q}_h - u^{\mathrm{d}}) + \alpha\bar{q}_h, q - \bar{q}_h) = 0. \tag{17}$$

Noting that $\bar{q}_h$ is a feasible test function for (16), and $\bar{q}$ is a feasible test function for (17), we obtain

$$\begin{aligned} 0 &= (S^*(S\bar{q} - u^{\mathrm{d}}) - S_h^*(S_h\bar{q}_h - u^{\mathrm{d}}) + \alpha\bar{q} - \alpha\bar{q}_h, \bar{q}_h - \bar{q}) \\ &= -\alpha\|\bar{q}_h - \bar{q}\|^2 + (S\bar{q} - u^{\mathrm{d}}, S(\bar{q}_h - \bar{q})) - (S_h\bar{q}_h - u^{\mathrm{d}}, S_h(\bar{q}_h - \bar{q})), \end{aligned}$$

and consequently, after some more manipulation,

$$\alpha\|\bar{q} - \bar{q}_h\|^2 + \|\bar{u} - \bar{u}_h\|^2 = (S\bar{q} - S_h\bar{q}_h, (S - S_h)\bar{q}) + ((S - S_h)^*(S\bar{q} - u^{\mathrm{d}}), \bar{q}_h - \bar{q}). \tag{18}$$

By Corollary 3.2 we have

$$\|(S - S_h)q\| \le ch^s\|q\|_{L^2(\Gamma_N)}, \tag{19}$$

for some $c$ and $s > 0$. Applying (19) together with the Young's inequality to the right hand side of (18) we obtain

$$\alpha\|\bar{q} - \bar{q}_h\|^2 + \|\bar{u} - \bar{u}_h\|^2 \le ch^{2s} + \frac{c}{\alpha}h^{2s} + \frac{\alpha}{2}\|\bar{q} - \bar{q}_h\|^2,$$

and thus the assertion. □

**3.3. Adjoint error estimate.** In this subsection, considering the first-order optimality system corresponding to the optimal control problem (6), with the help of some techniques based on the results obtained previously we analyze the error estimate. It will be shown that the quasi-best approximation holds also for the adjoint solution of the optimality system, following [12].

The following theorem, which is proven in [27], introduces the related optimality system.

THEOREM 3.4. *Let Assumption 1 be given, and let $(\bar{q}, \bar{\mathbf{u}}) \in Q \times V$ be a solution to (6). Then there exists a Lagrange multiplier $\bar{\mathbf{z}} = (\bar{z}, \bar{\zeta}) \in V$ such that the system*

$$\begin{array}{llll} A\bar{\mathbf{u}} & = B\bar{q} & in & V^* \\ A^*\bar{\mathbf{z}} & = \bar{\mathbf{u}} - \mathbf{u}^{\mathrm{d}} & in & V^* \\ \alpha\bar{q} & = -B^*\bar{\mathbf{z}} & on & \Gamma_N \end{array} \tag{20}$$

*is satisfied. Moreover, because of the convexity of (6), any triplet $(\bar{q}, \bar{\mathbf{u}}, \bar{\mathbf{z}}) \in Q \times V \times V$ solving (20) gives rise to a solution of (6).*

We notice that, by replacing $\bar{q}$ with $-\frac{1}{\alpha}B^*\bar{\mathbf{z}}$, the reduced form of the optimality system (20) can be written in the following matrix form for $(\bar{\mathbf{u}}, \bar{\mathbf{z}}) \in V \times V$,

$$\begin{pmatrix} A & \frac{1}{\alpha}BB^* \\ -I & A^* \end{pmatrix} \begin{pmatrix} \bar{\mathbf{u}} \\ \bar{\mathbf{z}} \end{pmatrix} = \begin{pmatrix} 0 \\ -\mathbf{u}^{\mathrm{d}} \end{pmatrix}. \tag{21}$$

The operator matrix in (21) possesses a special property which leads to important consequences. In order to analyze that, we formulate the corresponding bilinear form. To this end, let us consider the variational form of the reduced optimality system (21) with $\mathbf{v} = (v, \psi) \in V$.

$$\begin{cases} (A\bar{\mathbf{u}}, \mathbf{v}) + \frac{1}{\alpha}(B^*\bar{\mathbf{z}}, B^*\mathbf{v}) = 0, & \forall \mathbf{v} \in V \\ (A\mathbf{v}, \bar{\mathbf{z}}) - (\bar{\mathbf{u}}, \mathbf{v}) = -(\mathbf{u}^{\mathrm{d}}, \mathbf{v}) & \forall \mathbf{v} \in V \end{cases} \tag{22}$$

We define the normed space $X := V \times V$, in which any element $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2) \in X$ with $\mathbf{x}_1 = (x_{1,1}, x_{1,2})$ and $\mathbf{x}_2 = (x_{2,1}, x_{2,2})$ has the norm

$$\|\mathbf{x}\|_X := (\|\mathbf{x}_1\|_V^2 + \|\mathbf{x}_2\|_V^2)^{1/2}.$$

Summing up the two variational equations presented in (22), allows us to introduce the bilinear form $b : X \times X \to \mathbb{R}$, associated with the matrix operator $M$, defined by

$$b(\mathbf{\Phi}, \mathbf{\Psi}) := a(\mathbf{\Phi}_1, \mathbf{\Psi}_1) + a(\mathbf{\Psi}_2, \mathbf{\Phi}_2) + \left( \frac{1}{\alpha}(B^*\mathbf{\Phi}_2, B^*\mathbf{\Psi}_1) - (\mathbf{\Phi}_1, \mathbf{\Psi}_2) \right) \tag{23}$$

for any $\mathbf{\Phi}_1, \mathbf{\Phi}_2, \mathbf{\Psi}_1, \mathbf{\Psi}_2 \in V$, with $\mathbf{\Phi} := (\mathbf{\Phi}_1, \mathbf{\Phi}_2), \mathbf{\Psi} := (\mathbf{\Psi}_1, \mathbf{\Psi}2) \in X$. Then, with $\bar{\mathbf{x}} = (\bar{\mathbf{u}}, \bar{\mathbf{z}})$, the variational formulation (22) reads as

$$b(\bar{\mathbf{x}}, \mathbf{\Psi}) = -(\mathbf{u}^{\mathrm{d}}, \mathbf{\Psi}_2) \quad \forall \mathbf{\Psi} \in X.$$

We are then able to conclude the following lemma.

Lemma 3.5. *For any given $(q_k, \mathbf{u}_k)$ satisfying Assumption 1, the linear operator $M : V \times V \to V^* \times V^*$ corresponding to (20) defined by*

$$M := \begin{pmatrix} A & \frac{1}{\alpha}BB^* \\ -I & A^* \end{pmatrix}$$

*is Fredholm of index zero. Moreover, the associated bilinear form $b : X \times X$, defined by (23), is continuous on $X \times X$, and satisfies a Gårding-like inequality; more precisely, there exists constants $\tilde{c}_c, \tilde{c}_1, \tilde{c}_2$ and some $r \in (0, 1)$, depending on $C$ in Assumption 1, such that*

$$|b(\mathbf{\Phi}, \mathbf{\Psi})| \le \tilde{c}_c \|\mathbf{\Phi}\|_X \|\mathbf{\Psi}\|_X$$

*and*

$$b(\mathbf{\Phi}, \mathbf{\Phi}) + \tilde{c}_2 \|\mathbf{\Phi}\|_r^2 \ge \tilde{c}_1 \|\mathbf{\Phi}\|_X^2$$

*for suitable constants $c$.*

*Proof.* With the help of Lemma 2.1, and considering the boundedness of the compact operator $B^*$, it is straightforward to observe that $b$ satisfies the mentioned Gårding's inequality and the continuity relation. Then by applying the Lax-Milgram lemma, and considering the compactness of $H^1(\Omega) \subset H^r(\Omega)$, the matrix operator $M$ can be observed as a summation of a Fredholm operator of index zero, and a compact one. Therefore, based on Theorem 12.8 in [30], we deduce that $M$ is a Fredholm operator of index zero. ∎

Having introduced the bilinear form $b : X \times X \to \mathbb{R}$, we notice that the variational formulation of the reduced optimality system (21) reads as

$$\begin{aligned} &\text{Find} \quad \mathbf{x} = (\mathbf{u}, \mathbf{z}) \in X \\ &\text{such that} \quad b(\mathbf{x}, \mathbf{\Psi}) = -(\mathbf{u}^{\mathrm{d}}, \mathbf{\Psi}_2) \quad \forall \mathbf{\Psi} = (\mathbf{\Psi}_1, \mathbf{\Psi}_2) \in X. \end{aligned} \tag{24}$$

Correspondingly, defining $X_h := V_h \times V_h$, we consider the following discretized problem.

$$\begin{aligned} &\text{Find} \quad \mathbf{x}_h = (\mathbf{u}_h, \mathbf{z}_h) \in X_h \\ &\text{such that} \quad b(\mathbf{x}_h, \mathbf{\Psi}_h) = -(\mathbf{u}^{\mathrm{d}}, \mathbf{\Psi}_{h,2}) \quad \forall \mathbf{\Psi}_h = (\mathbf{\Psi}_{h,1}, \mathbf{\Psi}_{h,2}) \in X_h. \end{aligned} \tag{25}$$

To estimate the error between the solutions $\mathbf{x} \in X$ and $\mathbf{x}_h \in X_h$, to the problems (24) and (25) respectively, let us define $e_{\mathbf{x}} := \mathbf{x} - \mathbf{x}_h$; that is $e_{\mathbf{x}} = (e_{\mathbf{u}}, e_{\mathbf{z}}) = (\mathbf{u} - \mathbf{u}_h, \mathbf{z} - \mathbf{z}_h)$.

We would need the following intermediate estimate for our final argument.

Lemma 3.6. *Let $e_{\mathbf{x}} = (e_{\mathbf{u}}, e_{\mathbf{z}}) = ((e_u, e_\varphi), (e_z, e_\zeta))$, and $s > 0$ be such that $r \leq 1 - s$ for some $0 < r < 1$, and Assumption 1 hold. Then there is a constant $h_0$ such that the following estimate holds true*

$$\|e_{\mathbf{x}}\|_r \leq c h^s \|e_{\mathbf{x}}\|_X$$

*for some constant $c$, and for all $h \leq h_0$.*

*Proof.* Based on Lemma 3.5, the matrix operator $M : V \times V \to V^* \times V^*$ is Fredholm of index zero, and it is straightforward to show that kern $M = \{0\}$. Therefore, $M$ is an isomorphism, hence the same argument presented in the proof of Theorem 3.1 can be applied to obtain

$$\|e_\varphi\|_r + \|e_\zeta\|_r \leq c h^s \|e_{\mathbf{x}}\|_X,$$

and consequently the desired bound.

It remains to show that ker $M = \{0\}$. Let $(\mathbf{u}, \mathbf{z}) \in \ker M$. By definition, this implies

$$\begin{cases} (A\mathbf{u}, \mathbf{v}) + \frac{1}{\alpha}(B^*\mathbf{z}, B^*\mathbf{v}) = 0, & \forall \mathbf{v} \in V, \\ (A\mathbf{v}, \mathbf{z}) - (\mathbf{u}, \mathbf{v}) = 0, & \forall \mathbf{v} \in V. \end{cases} \tag{26}$$

Testing the first equation in (26) with $\mathbf{v} := \mathbf{z}$, and the second one with $\mathbf{v} := \mathbf{u}$, and then subtracting the second equation from the first one we get

$$\frac{1}{\alpha}\|B^*\mathbf{z}\|^2 + \|\mathbf{u}\|^2 = 0.$$

This simply implies $\mathbf{u} = 0$. Noting that $A^*$ is an isomorphism, it is then an immediate result from the second equation in (26) that $\mathbf{z} = 0$. ∎

We are now in the place to present the final result.

Theorem 3.7. *Given Assumption 1, let $h \leq h_1$ for some sufficiently small $h_1 \leq h_0$. Then there is a constant $c > 0$ such that the solutions $\mathbf{x} = (\mathbf{u}, \mathbf{z}) \in X$ and $\mathbf{x}_h = (\mathbf{u}_h, \mathbf{z}_h) \in X_h$ to the problems (24) and (25) satisfy the quasi best-approximation property*

$$\|\mathbf{x} - \mathbf{x}_h\|_X \leq c \inf_{\mathbf{\Psi}_h \in X_h} \|\mathbf{x} - \mathbf{\Psi}_h\|_X.$$

*Proof.* Analogous to Theorem 3.1, we get from Lemma 3.6 that for any $h \leq h_0$ the estimate

$$\tilde{c}_1 \|e_{\mathbf{x}}\|_V^2 \leq \tilde{c}_c \|e_{\mathbf{x}}\|_V \|\mathbf{x} - \mathbf{\Psi}_h\|_V + \tilde{c}_2 \|e_{\mathbf{x}}\|_r^2 \leq \tilde{c}_c \|e_{\mathbf{x}}\|_V \|\mathbf{x} - \mathbf{\Psi}_h\|_V + ch^{2s} \|e_{\mathbf{x}}\|_V^2$$

for arbitrary $\mathbf{\Psi}_h \in V_h$. This shows the result once $h \leq h_1$ with

$$ch_1^{2s} \leq \frac{\tilde{c}_1}{2}.$$

□

**4. Numerical experiment.** In this section, we present the numerical implementation to simulate the fracture problem (3). The aim is to demonstrate the validity of the error estimates we have obtained in previous section. Let us consider the two-dimensional square domain $\Omega = [-1, 1]^2$, with the boundary $\partial\Omega = \Gamma_N \cup \Gamma_D \cup \Gamma_{\text{free}}$ consists of three different parts, where

$$\Gamma_N = \{(x, 1) | -1 \leq x \leq 1\}, \qquad \Gamma_D = \{(x, -1) | -1 \leq x \leq 1\},$$

and $\Gamma_{\text{free}}$ represents the rest of the boundary. On the boundary piece $\Gamma_N$ a control $q$ is applied in normal direction, whereas on the Dirichlet boundary $\Gamma_D$ the displacement vector is prescribed by $u = 0$. $\Gamma_{\text{free}}$ is a free boundary over which a natural boundary condition for the displacement is employed.

The fixed parameters at the linearized model (3) are set as follows. The control acting on $\Gamma_N$ is $q = 10$, the penalization parameter is $\gamma = 10^8$, the fracture energy release rate is $G_c = 1$, the bulk regularization parameter is $\kappa = 10^{-10}$, and the phase-field regularization parameter is $\varepsilon = 0.088$. We linearize the fracture model at point $\mathbf{u}_k = (u_k, \varphi_k)$, with

$$u_k(x, y) = (0, (1 + y) \times 10^{-5}), \quad (x, y) \in \Omega,$$

and $\varphi_k = \varphi^0$. The initial fracture $\varphi^0$ is imposed to the problem variously through Examples 4.1 to 4.3. By this choice, the penalty term vanishes. Notice, that the functions $\varphi^0$ are not $W^{1,p}$ but still $g(I_h\varphi^0)$ is regular enough to assert $H^{1+s}$ regularity of $(u, \varphi)$.

To approximate the solution of (3), we discretize the model by choosing standard $Q_1$ finite elements for the displacement $u$ and the phase-field $\varphi$. The implementation is performed with the help of software DOpElib [14], which is developed based on the finite element software library deal.II [5, 4]. We start the calculations on a twice globally refined quadrilateral mesh of the domain, i.e., $h_0 = \sqrt{1/2} \approx 0.707107$. The fines level is given by globally refining this initial mesh eight times. Since the exact solution of the problem (3) is not available, to investigate the impact of mesh refinement on the accuracy of the approximated solution, we follow two strategies to estimate the order of convergence:

$I$) We compare the solutions at coarser meshes with the solution at the finest mesh. According to Corollary 3.2 we expect that

$$\|u - u_{h_i}\|_V \approx c\, h_i^s$$

where $h_i = h_0/2^i$, $i = 0, 1, 2, \cdots$, is the element size on level $i$. We denote the solution at the finest mesh by $\hat{u} = u_8$, and approximate $u$ by $\hat{u}$, and estimate

$$s \approx \log_2 \left( \frac{\|\hat{u} - u_{h_{i-1}}\|_V}{\|\hat{u} - u_{h_i}\|_V} \right), \quad i = 1, 2, \cdots.$$

*II*) It should be noted, that strategy I) is known to provide bad estimates for $s$ if the reference solution is not good enough compared to the level $i$. Thus we perform a second test for the convergence order:

$$s \approx \log_2 \left( \frac{\|u_{h_{i-1}} - u_{h_i}\|_V}{\|u_{h_i} - u_{h_{i+1}}\|_V} \right), \quad i = 1, 2, \cdots .$$

**4.1. Example 1.** Inside the domain $\Omega$, we initially consider a horizontal fracture represented by

$$\varphi^0 = \begin{cases} 0, & \text{on } (-0.1 - h, 0.1 + h) \times (-h, h) \\ 1, & \text{o.w.} \end{cases}$$

with $h$ being the diameter of the mesh elements. Figure 1 illustrate the resulting numerical solutions of the phase-field $\varphi$ and the displacement $u$. The mesh refinement result is displayed in Table 1, where it is observable that by refining the mesh, test I) suggests $s \approx 0.3$, but as test II) indicates these values are potentially not very reliable. Nonetheless, it can be seen, that the convergence rate is far from the optimal regularity case.

Table 1
*Example 1; Mesh refinement result and the order of convergence.*

| $i$ | DOF | $\|\hat{u} - u_{h_i}\|_V$ | $s$ by I) | $\|u_{h_{i+1}} - u_{h_i}\|_V$ | $s$ by II) |
|---|---|---|---|---|---|
| 0 | 25 | 0.000483385 | - | 0.00048419 | - |
| 1 | 81 | 0.000110988 | 2.12276 | 8.72039e-05 | 2.47311 |
| 2 | 289 | 6.53677e-05 | 0.763759 | 4.69515e-05 | 0.893222 |
| 3 | 1089 | 3.98234e-05 | 0.714961 | 2.14839e-05 | 1.12791 |
| 4 | 4225 | 2.79684e-05 | 0.509818 | 1.48783e-05 | 0.530052 |
| 5 | 16641 | 2.1567e-05 | 0.374975 | 1.10487e-05 | 0.429333 |
| 6 | 66049 | 1.86941e-05 | 0.206243 | 1.23807e-05 | -0.164216 |
| 7 | 263169 | 1.45926e-05 | 0.357348 | 1.45926e-05 | -0.237144 |

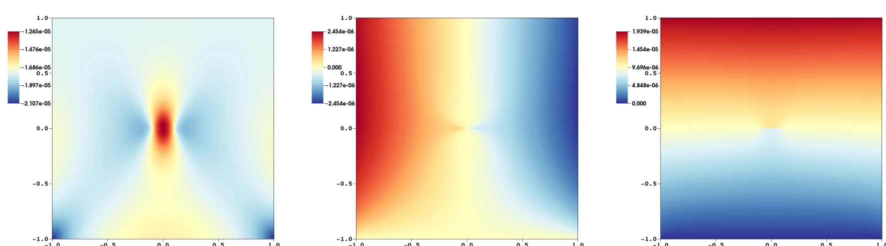

Fig. 1. *Example 1; approximated phase-field $\varphi$ (left), x-component of approximated displacement $u$ (middle), and y-component of approximated displacement $u$ (right) at the finest mesh level.*

**4.2. Example 2.** Although the first test is a standard setup for phase-field fracture simulations, it involves an initial value $\varphi^0$ depending on the chosen mesh. To avoid this additional $h$ coupling, we impose the following initial fracture to the problem.

$$\varphi^0 = \begin{cases} 0, & \text{on } (-0.1-d, 0.1+d) \times (-d, d) \\ 1, & \text{o.w.} \end{cases}$$

where $d = h_2 = h_0/4$. The value for $d$ is chosen small enough to have a reasonable shape of fracture. It leads to the expense that the first two rows in Table 2 are valueless, as the mesh size is not yet fine enough for representing the fracture. As can be seen in Table 2, both strategies provide a value $s \approx 0.5$. The initial fracture and resulting approximated phase-field and displacement solutions are depicted in Figures 2–3.

TABLE 2
*Example 2; Mesh refinement result and the order of convergence.*

| $i$ | DOF | $\|\hat{u} - u_{h_i}\|_V$ | $s$ by I) | $\|u_{h_{i+1}} - u_{h_i}\|_V$ | $s$ by II) |
|---|---|---|---|---|---|
| 0 | 25 | 6.34608e-05 | - | 3.64033e-05 | - |
| 1 | 81 | 6.051e-05 | 0.0686912 | 5.59011e-05 | -0.618807 |
| 2 | 289 | 3.05078e-05 | 0.987998 | 2.99038e-05 | 0.902548 |
| 3 | 1089 | 2.7183e-05 | 0.16647 | 2.60363e-05 | 0.199805 |
| 4 | 4225 | 3.20916e-05 | -0.23949 | 2.52668e-05 | 0.04328 |
| 5 | 16641 | 2.76929e-05 | 0.212678 | 2.57842e-05 | -0.0292417 |
| 6 | 66049 | 1.92512e-05 | 0.524573 | 1.87135e-05 | 0.462407 |
| 7 | 263169 | 1.27844e-05 | 0.590565 | 1.27844e-05 | 0.549699 |

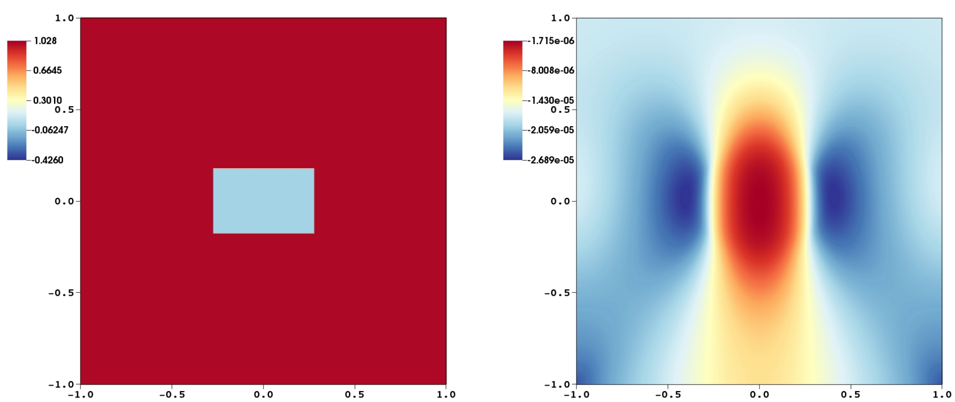


FIG. 2. *Example 2; Initial fracture $\varphi^0$ (left) and approximated phase-field $\varphi$ (right) at the finest mesh level.*

**4.3. Example 3.** Finally, we examine the convergence results by considering the example introduced in [19] for our initial fracture, since it has been the subject of attention in many papers concerning singularities of interface problems. Because of singularities, it is difficult to obtain accurate numerical approximations to interface

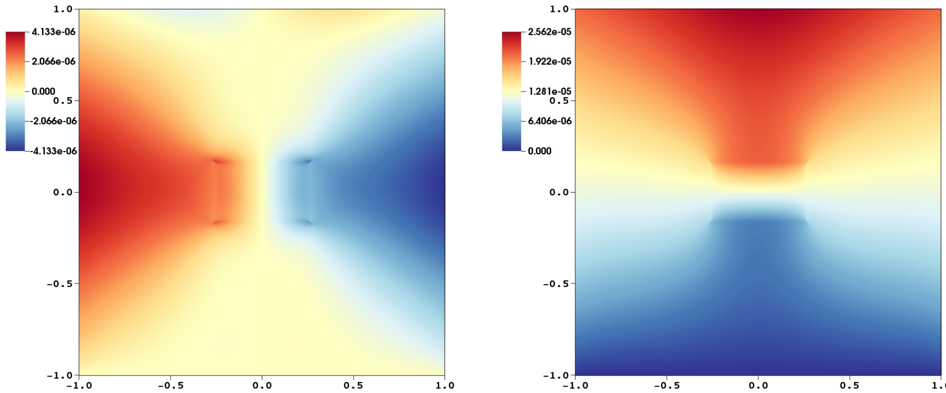


Fig. 3. *Example 2; The $x$-component (left) and the $y$-component (right) of approximated displacement $u$ at the finest mesh level.*

problems by standard finite element methods. Starting with the fracture

$$\varphi^0 = \begin{cases} 0, & x.y < 0 \\ 1, & \text{o.w.} \end{cases}$$

illustrated in Figure 4 (left), the finite element scheme yields large errors as presented in Table 3. Yet the mesh refinement strategy confirms the theoretical arguments that after some steps the error decreases in such a way that the parameters $s$ tend to converge to a positive value for both strategies. However, as in this test the considered solution $\hat{u}$ is very far from the unknown exact solution, we would expect to obtain better results by comparing successive numerical solutions instead of comparing with $\hat{u}$. The last column of Table 3 fulfills our expectation, where strategy II) shows a more reasonable convergence behavior. Figures 4 (right) and 5 display approximated solutions $\varphi$ and $u$, respectively.

Table 3
*Example 3; Mesh refinement result and the order of convergence.*

| $i$ | DOF | $\|\hat{u} - u_{h_i}\|_V$ | $s$ by I) | $\|u_{h_{i+1}} - u_{h_i}\|_V$ | $s$ by II) |
|---|---|---|---|---|---|
| 0 | 25 | 116614 | - | 0.0728649 | - |
| 1 | 81 | 116614 | 2.94568e-07 | 1235 | -14.0485 |
| 2 | 289 | 116419 | 0.00241352 | 52165 | -5.40097 |
| 3 | 1089 | 101565 | 0.19693 | 62230 | -0.254515 |
| 4 | 4225 | 76827 | 0.402705 | 50969 | 0.287989 |
| 5 | 16641 | 53423 | 0.524152 | 37539 | 0.441217 |
| 6 | 66049 | 34311 | 0.638787 | 26617 | 0.496032 |
| 7 | 263169 | 18615 | 0.88223 | 18615 | 0.515917 |

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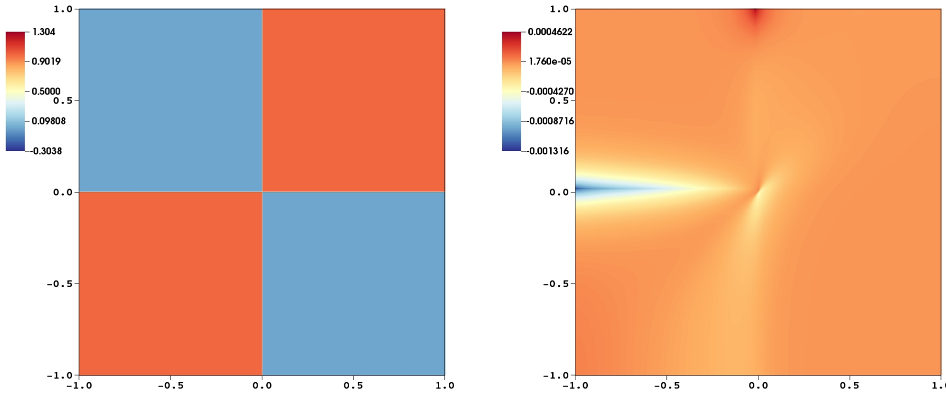


FIG. 4. *Example 3; Initial fracture $\varphi^0$ (left) and approximated phase-field $\varphi$ (right) at the finest mesh level.*

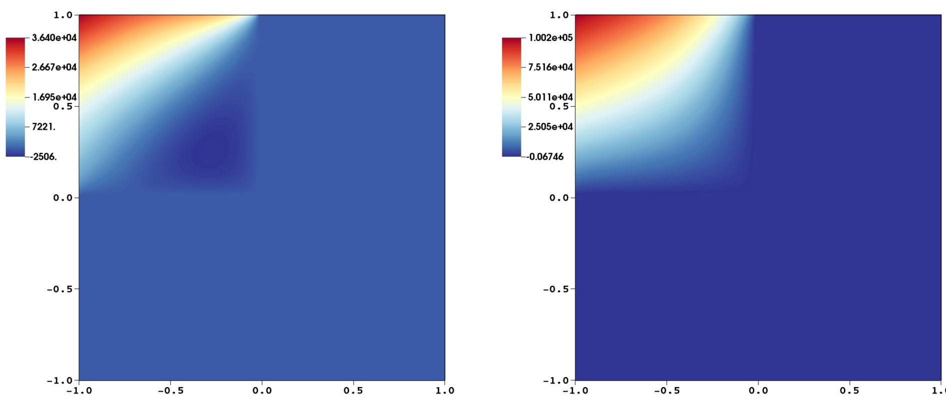


FIG. 5. *Example 3; The $x$-component (left) and the $y$-component (right) of approximated displacement $u$ at the finest mesh level.*

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