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CONSISTENT FINITE-DIMENSIONAL APPROXIMATION OF PHASE-FIELD MODELS OF FRACTURE

STEFANO ALMI AND SANDRO BELZ

Abstract. In this paper we focus on the finite-dimensional approximation of quasi-static evolutions of critical points of the phase-field model of brittle fracture. In a space discretized setting, we first discuss an alternating minimization scheme which, together with the usual time-discretization procedure, allows us to construct such finite-dimensional evolutions. Then, passing to the limit as the space discretization becomes finer and finer, we prove that any limit of a sequence of finite-dimensional evolutions is itself a quasi-static evolution of the phase-field model of fracture. Our proof shows for the first time the consistency of a numerical scheme for evolutions of fractures along critical points.

1. Introduction

In this paper we are interested in the study of convergence of numerical schemes for quasi-static evolution of brittle fractures in elastic bodies. We focus on the phase-field (or damage) approximation of fracture studied by Bourdin, Francfort, and Marigo in [8, 11, 14], and first introduced by Ambrosio and Tortorelli in [2, 3] in the framework of image processing.

In a planar setting, given an open bounded subset Ω of $\mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$, we deal with an energy functional of the form

$$\mathcal{J}_\varepsilon(u,v) := \frac{1}{2}\int_\Omega (v^2+\eta_\varepsilon)|\nabla u|^2\,\mathrm{d}x + \kappa\int_\Omega \varepsilon|\nabla v|^2\,\mathrm{d}x + \kappa\int_\Omega \frac{(1-v)^2}{4\varepsilon}\,\mathrm{d}x\,, \tag{1.1}$$

where ε and η_ε are two small positive parameters, $u \in H^1(\Omega)$ stands for the displacement field, $v \in H^1(\Omega;[0,1])$ denotes the damage variable, and the positive constant κ may be interpreted as the toughness of the material, which we assume to be equal to one for the following discussion. From a physical point of view, the variable v in (1.1) takes into account how damaged the elastic body is, so that, for $x \in \Omega$, $v(x) = 0$ means that the damage is complete (fracture) at x, while $v(x) = 1$ means that the material is perfectly intact at x.

In [2, 3] it has been shown that choosing $0 < \eta_\varepsilon \ll \varepsilon$ and letting $\varepsilon \to 0$, the functional $\mathcal{J}_\varepsilon$ Γ-converges to

$$\mathcal{G}(u) := \frac{1}{2}\int_\Omega |\nabla u|^2\,\mathrm{d}x + \mathcal{H}^1(S_u)\,, \tag{1.2}$$

defined on $GSBV(\Omega)$, the space of generalized special function of bounded variation (for the theory of such spaces see, for instance, [1]). In (1.2), $\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure and S_u stands for the approximate discontinuity set of u. In the mathematical model of fracture (see, e.g., [14]), the functional (1.2) represents the energy of an elastic body Ω subject to an antiplanar displacement u and with a crack S_u.

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In view of such a convergence result, the phase-field functional (1.1) has been widely and successfully used in numerical simulations of crack growth processes (see, for instance [6, 8, 11, 14, 18, 19]).

In the framework of numerical approaches to fracture mechanics, our interest is in the proof of consistency (or analysis of convergence) of some numerical schemes used in the study of the crack growth process. In particular, our goal is to prove the existence of quasi-static evolutions for the phase-field model (1.1) as limits of evolutions obtained in a space-discretized setting. It is indeed clear that any numerical simulation based on (1.1) gives as an outcome only *finite-dimensional* approximations of an evolution of the phase-field variable (see, e.g., [8, 11, 14]). This is due to the fact that, in order to implement some kind of algorithm, a discretization of the functional space $H^1(\Omega)$ is needed. Having this in mind, our contribution is, roughly speaking, the following: we show that we can construct a quasi-static evolution of *critical points* of the phase-field model of fracture (1.1) as a limit of *finite-dimensional quasi-static evolutions* obtained in a discretized H^1-framework. Clearly, the limit process is performed as the function space discretization becomes finer and finer.

To our knowledge, this paper provides the first proof of consistency of a numerical method for such evolutions, going beyond the empirical consistency checks even recently done, for instance, in [7] and related literature. To obtain the result, we needed to fuse classical methods of PDE discretization, such as FEM (Finite Element Method) and their typical quasi-interpolating estimates, together with variational techniques to handle nonlinearities, going far beyond the usual linear setting where, e.g., FEM are employed.

Concerning the variational methodology, we innovate over [26], or even over the more general framework of [35], where only variational limits of evolutions along *global minimizers* were developed and analyzed, essentially, by means of Γ-convergence techniques (see also [13, 20, 21, 27, 28]). Instead, here, perhaps more closely to the work of Braides and coauthors in [16, 17], we develop results of consistency for evolutions along *critical points*, which are more realistic.

We anticipate here that all the results we are going to discuss are still valid in the vectorial case, i.e., when considering the functional

$$\mathcal{I}_\varepsilon(u,v) := \frac{1}{2}\int_\Omega (v^2+\eta_\varepsilon)\mathbb{C}\mathrm{E}u\cdot\mathrm{E}u\,\mathrm{d}x + \int_\Omega \varepsilon|\nabla v|^2\,\mathrm{d}x + \int_\Omega \frac{(1-v)^2}{4\varepsilon}\,\mathrm{d}x\,,$$

where $u \in H^1(\Omega;\mathbb{R}^2)$, $v \in H^1(\Omega)$, and $\mathbb{C}$ is the usual elasticity tensor. For the sake of simplicity, we decided to present here in details only the scalar setting (1.1).

In order to be more precise in the discussion of our result, let us briefly present the quasi-static evolution problem we want to tackle in this work. For notational convenience, let us fix the parameters $\varepsilon = \frac{1}{2}$ and $\eta_\varepsilon = \eta > 0$ and let us drop the subscript ε in (1.1), so that we consider the functional

$$\mathcal{J}(u,v) := \frac{1}{2}\int_\Omega (v^2+\eta)|\nabla u|^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega \big(|\nabla v|^2 + (1-v)^2\big)\,\mathrm{d}x \tag{1.3}$$

for $u, v \in H^1(\Omega)$. Given $T > 0$, we assume that the evolution of the elastic body Ω is driven by the energy functional (1.3) and by a time-dependent Dirichlet boundary datum $w \in W^{1,2}([0,T]; H^1(\Omega))$. In this context, a quasi-static evolution is described by the pair of functions $(u,v)\colon [0,T] \to H^1(\Omega)\times H^1(\Omega)$ standing for displacement and damage, respectively, and satisfying the following conditions (we refer to Definition 2.3 for a precise statement):

(1) *Irreversibility*: $0 \le v(t) \le v(\tau) \le 1$ a.e. in Ω for every $0 \le \tau \le t \le T$;

(2) *Stability*: for every $t \in [0,T]$, the pair $(u(t), v(t))$ is a "critical point" of the energy functional $\mathcal{J}$ in the class of pairs $(u,v) \in H^1(\Omega) \times H^1(\Omega)$ such that $u = w(t)$ on $\partial\Omega$ and $v \leq v(t)$ a.e. in Ω;

(3) *Energy-dissipation inequality*: for every $t \in [0,T]$

$$\mathcal{J}(u(t), v(t)) \leq \mathcal{J}(u(0), v(0)) + \int_0^t \int_\Omega (v^2(\tau) + \eta) \nabla u(\tau) \cdot \nabla \dot{w}(\tau) \, \mathrm{d}x \, \mathrm{d}\tau \,,$$

where $\dot{w}$ denotes the time derivative of w.

We mention that this notion of evolution is often referred to as *local energetic evolution.* See, e.g., [32, 34] for further discussions on the topic.

The irreversibility property (1) means that the damage process is unidirectional, in the sense that once the elastic body Ω is damaged, i.e., $v < 1$ in a subset of Ω, it can not be repaired, not even partially. We notice that this is the natural counterpart of the irreversibility of brittle fracture, which states that once a crack is created, it can not be closed anymore during the evolution process.

The stability condition (2), discussed in details in Section 2, can be mathematically rephrased, roughly speaking, as

$$\partial_{(u,v)} \mathcal{J}(u(t), v(t)) \leq 0 \,, \tag{1.4}$$

where $\partial_{(u,v)}$ denotes the partial derivative w.r.t. the pair of variables (u,v), and the inequality is due to the irreversibility constraint discussed above. As we will see in Section 2, inequality (1.4) can be splitted in

$$\partial_u \mathcal{J}(u(t), v(t)) = 0 \qquad \text{and} \qquad \partial_v \mathcal{J}(u(t), v(t)) \leq 0 \,, \tag{1.5}$$

or, which is equivalent because of the separate convexity of $\mathcal{J}$ w.r.t. u and v,

$$\mathcal{J}(u(t), v(t)) \leq \mathcal{J}(u, v(t)) \qquad \text{for every } u \in H^1(\Omega) \text{ with } u = w(t) \text{ on } \partial\Omega, \tag{1.6}$$

$$\mathcal{J}(u(t), v(t)) \leq \mathcal{J}(u(t), v) \qquad \text{for every } v \in H^1(\Omega) \text{ with } v \leq v(t) \text{ a.e. in } \Omega. \tag{1.7}$$

We notice that conditions (1.6)–(1.7) are not equivalent to the global stability property

$$\mathcal{J}(u(t), v(t)) \leq \mathcal{J}(u, v)$$

for every pair $(u,v) \in H^1(\Omega) \times H^1(\Omega; [0,1])$ such that $u = w(t)$ on $\partial\Omega$ and $v \leq v(t)$ a.e. in Ω. For this reason, (1.4)–(1.7) could be referred to as *local stability* properties, since they involve the local behavior of the energy functional $\mathcal{J}$ close to the pair $(u(t), v(t))$.

Finally, the energy-dissipation inequality (3) is due to the lack of 1-homogeneous term in the original Francfort-Marigo model. In [31], the authors have been able to recover an energy balance for the continuous phase-field model described by (1.3) thanks to a time reparametrization technique (see also [33]). As remarked below, we succeeded in adapting the strategy of [31] in the finite-dimensional setting (see Sections 2-4), but it resulted to be difficult to obtain, in our finite-dimensional to continuum limit, the convergences necessary to preserve an energy-dissipation balance.

Following the main steps of numerical schemes, in order to construct a quasi-static evolution satisfying (1)–(3) we first discretize the function space $H^1(\Omega)$ and define the discrete counterpart of the functional $\mathcal{J}$. More precisely, for every value of the mesh parameter $h > 0$ we consider a triangulation $\mathcal{T}_h$ of Ω satisfying the standard requirements arising from interpolation estimates (see (2.14) and [37] for more details), we define the finite-dimensional space

$$\mathcal{F}_h := \{ u \in H^1(\Omega) : u \text{ is affine on } K \text{ for every } K \in \mathcal{T}_h \} \,, \tag{1.8}$$

and we set, for every $u, v \in \mathcal{F}_h$,

$$\mathcal{J}_h(u,v) := \frac{1}{2}\int_\Omega \big(\mathrm{P}_h(v^2)+\eta\big)|\nabla u|^2\,\mathrm{d}x+\frac{1}{2}\int_\Omega |\nabla v|^2\,\mathrm{d}x+\frac{1}{2}\int_\Omega \mathrm{P}_h\big((1-v)^2\big)\,\mathrm{d}x\,, \tag{1.9}$$

where $\mathrm{P}_h\colon C(\overline{\Omega}) \to \mathcal{F}_h$ is the Lagrangian interpolation operator.

In the finite-dimensional framework described above, we are able to construct a *finite-dimensional quasi-static evolution* driven by the energy functional $\mathcal{J}_h$ in (1.9) and satisfying better conditions than (1)–(3) (see Definition 2.8). More precisely, the finite-dimensional quasi-static evolution is represented by a triple (t_h, u_h, v_h), where t_h is a suitable Lipschitz reparametrization of time t. With this notation, the triple (t_h, u_h, v_h) satisfies the discrete counterpart of (1)-(2) and an energy-dissipation balance of the form

$$\begin{aligned}\mathcal{J}_h(u_h(s), v_h(s)) =& \mathcal{J}_h(u_{0,h}, v_{0,h}) - \int_0^s |\partial_u \mathcal{J}_h|(u_h(\sigma), v_h(\sigma))\|u_h'(\sigma)\|_{h,v_h(\sigma)}\,\mathrm{d}\sigma \\ & - \int_0^s |\partial_v \mathcal{J}_h|(u_h(\sigma), v_h(\sigma))\|v_h'(\sigma)\|_{h,u_h(\sigma)}\,\mathrm{d}\sigma \\ & + \int_0^s\int_\Omega (\mathrm{P}_h(v_h^2(\sigma)) + \eta)\nabla u_h(\sigma)\cdot\nabla \dot{w}_h(t_h(\sigma))t_h'(\sigma)\,\mathrm{d}x\,\mathrm{d}\sigma\,,\end{aligned} \tag{1.10}$$

where $|\partial_u \mathcal{J}_h|$ and $|\partial_v \mathcal{J}_h|$ are the slope of the functional $\mathcal{J}_h$ w.r.t. the displacement u and the phase-field v, respectively, $\|\cdot\|_{h,v}$ and $\|\cdot\|_{h,u}$ denotes suitable weighted norms on $\mathcal{F}_h$, and $'$ stands for the derivative w.r.t. s.

The algorithm used to detect such a triple (t_h, u_h, v_h) is a fusion of the one developed in [4] together with the alternating minimization of [8, 31]. In particular, besides the usual time-discretization procedure, typical in the study of many rate-independent processes (see, for instance, [32, 34]), at each time step $t_i^k := \frac{iT}{k}$, $k \in \mathbb{N}\setminus\{0\}$, $i \in \{1, \dots, k\}$, we construct a critical point of the energy $\mathcal{J}_h$ at time t_i^k by solving the incremental minimum problems

$$\min\{\mathcal{J}_h(u, v_{j-1}) : u \in \mathcal{F}_h, u = w(t_i^k) \text{ on } \partial\Omega\}\,, \tag{1.11}$$

$$\min\{\mathcal{J}_h(u_j, v) : v \in \mathcal{F}_h, v \le v_{j-1} \text{ in } \Omega\}\,, \tag{1.12}$$

where we have set, as initial conditions, $u_0 := u_k^h(t_{i-1}^k) + w(t_i^k) - w(t_{i-1}^k)$ and $v_0 := v_k^h(t_{i-1}^k)$. Denoting by u_j and v_j the solutions to (1.11) and (1.12), respectively, we show in Proposition 3.4 that the pair (u_j, v_j) converges in $\mathcal{F}_h \times \mathcal{F}_h$ to a critical point of $\mathcal{J}_h$, which we denote by $(u_k^h(t_i^k), v_k^h(t_i^k))$. The second step is to define an arc-length parametrization of time based on the distance between two subsequent steps of the minimization scheme (1.11)-(1.12). This leads us to a discrete in time energy balance that we are then able to keep in the time-continuous limit. We refer to Theorem 2.9 and to its proof in Section 4 for more details about this scheme.

As we have already mentioned, we are not able to pass to the limit as the mesh parameter h tends to 0 showing that the parametric finite-dimensional quasi-static evolutions (t_h, u_h, v_h) converge to a continuous in space quasi-static evolution satisfying, besides (1) and (2), also the continuous form of the energy equality (1.10). To be precise, the main reason is that we could not prove the right h-independent estimates on the triple (t_h, u_h, v_h) that would allow us to keep the equality in (1.10) as $h \to 0$. For this reason, we have based our "finite-dimensional to continuum" limit on the weaker notion of evolution (1)-(3). In particular, it can be easily proven (see Corollary 2.10) that from the time-parametrized triple (t_h, u_h, v_h) it is possible to go back to the real time t obtaining, with abuse of notation, a pair $(u_h, v_h)\colon [0,T] \to \mathcal{F}_h\times\mathcal{F}_h$ satisfying only an energy inequality of the form (3) holds.

The last step of our construction is then the passage to the limit as the triangulation $\mathcal{T}_h$ becomes finer and finer. This is indeed the subject of the proof of

Theorem 2.4, where we show that any limit of a sequence of finite-dimensional quasi-static evolutions (u_h, v_h) (in non-parametrized time) is a quasi-static evolution in the sense of (1)–(3) above (see also Definition 2.3). From a numerical viewpoint, this shows that the numerical results, obtained through a sort of finite-dimensional implementation of the damage model (1.1) and (1.3), are actually close to the "theoretical" quasi-static evolutions $(u(t), v(t))$ given by (1)–(3). Moreover, we notice that the method we exploit to prove Theorems 2.4 and 2.9 is also suitable for applications and numerical simulations, which, in particular, will be performed in Section 6.

In conclusion, we stress once again that the problem of existence of a quasi-static evolution for the phase-field model (1.1) has been already tackled in various papers (see, for instance, [26, 31, 36]). In particular, in [31] an existence result of quasi-static evolution for the damage model via critical points of the energy functional (1.1) has been achieved using, in a space-continuous setting, an alternate minimization scheme similar to the one described above. In [36], instead, the convergence scheme is based on a local minimization procedure w.r.t. the damage variable v. In [26] the evolution problem has been addressed in the setting of global minimizers, giving particular emphasis to the connection between the notions of quasi-static evolution in the phase-field model and in the variational "sharp interface" model of fracture (see, e.g., [25]). In view of these previous works, what we claim is new in our paper is not the existence result itself, but rather the technique used to construct an evolution, which is based on the algorithm given by (1.11) and (1.12) and which has been frequently used in numerical implementations (see [6, 8, 11, 12]).

Plan of the paper. The paper is organized as follows: in Section 2 we present the evolution problem in full details, giving the definition of quasi-static evolution for the phase-field model (see Definition 2.3) and stating the main result (Theorem 2.4). Then, we start discussing our discretization algorithm and, eventually, in Section 3 we discuss the alternate minimization scheme which is at the core of our approximation. In Sections 4 and 5 we prove Theorems 2.9 and 2.4, respectively. Finally, in Section 6 we present some numerical simulations which exploit the alternate minimization algorithm discussed in this paper.

2. Setting of the problem

In this section, we describe the problem setting and introduce the main notation of the paper. We first start with the space-continuous notion, and in the second part of the section we discuss the space-discrete setting.

Space-continuous setting. As already mentioned in the Introduction, we are studying quasi-static evolutions in the framework of phase-field approximation of brittle fractures in elastic bodies (for more details see, e.g., [2, 3, 24, 25]). Since the aim of this paper is to show a new constructive approach to the evolution problem based on a space discretization procedure, in order to keep the notations as simple as possible we focus here on a two dimensional model. In particular, we consider as a reference configuration the unit square $\Omega := (0,1)^2$ in $\mathbb{R}^2$. We believe that this is not a serious restriction and also evolutions in three dimensions can be similarly approached.

Once some $\eta > 0$ is fixed, we define the *phase-field stored elastic energy* as

$$\mathcal{E}(u, v) := \frac{1}{2}\int_\Omega (v^2 + \eta)|\nabla u|^2 \,\mathrm{d}x\,, \tag{2.1}$$

where $u \in H^1(\Omega)$ denotes the antiplanar displacement and $v \in H^1(\Omega)$ stands for the phase-field (or damage) variable. In particular, from (2.1) we deduce that the elastic behavior of Ω depends pointwise on how damaged the body is, and, due to the presence of the positive parameter η, the damage is never complete, in the sense that the elastic body Ω is always able to store a positive amount of elastic energy depending on the displacement u. We also recall that the phase-field v is usually constrained to take values in the interval $[0,1]$, where, for $x \in \Omega$, $v(x) = 0$ means that the elastic body Ω is experiencing a maximal damage in x, while $v(x) = 1$ means that the material is perfectly sound at x. In order to avoid some technical issues related to the discrete setting described in the second part of this section, we simply assume v to belong to $H^1(\Omega)$. We will see how the above constraint can be naturally enforced in the space-discrete approximation of the evolution problem. We refer to Proposition 3.1 for more details.

As usual in the phase-field approximation, we add to the stored elastic energy (2.1) a dissipative term $\mathcal{D}(v)$ which depends only on the damage $v \in H^1(\Omega)$, namely,

$$\mathcal{D}(v) := \frac{1}{2} \int_\Omega \big(|\nabla v|^2 + (1-v)^2 \big) \, \mathrm{d}x \,. \tag{2.2}$$

In the sense of Γ-convergence, the dissipation functional $\mathcal{D}$ approximates, in the language of fracture mechanics, the energy dissipated by the crack production, as it has been shown in [2, 3].

We are now in a position to introduce the *total phase-field energy* of the system as the sum of (2.1) and (2.2): for every $u, v \in H^1(\Omega)$, we simply set

$$\mathcal{J}(u,v) := \mathcal{E}(u,v) + \mathcal{D}(v) \,. \tag{2.3}$$

As usual, the evolution problem will be driven by a time-dependent forcing term. In this case, given a time horizon $T > 0$, we assume that the elastic body Ω is subject to a Dirichlet boundary datum $w \in W^{1,2}([0,T]; H^1(\Omega))$, so that, for every $t \in [0,T]$, the set of admissible displacement $\mathcal{A}(w(t))$ is defined by

$$\mathcal{A}(w(t)) := \{ u \in H^1(\Omega) : u = w(t) \text{ on } \partial\Omega \} \,, \tag{2.4}$$

where the equality has to be intended in the trace sense. The notation (2.4) will be adopted also for functions $w \in H^1(\Omega)$ not depending on time.

In this context, a quasi-static evolution for the damage model is expressed by a pair displacement-damage $(u,v) \colon [0,T] \to H^1(\Omega) \times H^1(\Omega; [0,1])$. The first natural condition we want to impose is the so-called *irreversibility* of the phase-field variable. Namely, the function $t \mapsto v(t)$ has to be non-increasing. This means that once the elastic body Ω is damaged, it can not be repaired, not even partially.

The second property a quasi-static evolution has to satisfy is a *stability* condition. In our case, to be stable at time t means that the pair $(u(t), v(t))$ is a critical point of the energy (2.3) in the class of pairs $(u,v) \in H^1(\Omega) \times H^1(\Omega; [0,1])$ with $u \in \mathcal{A}(w(t))$ and $v \le v(t)$ a.e. in Ω. Since $\mathcal{J}$ is Fréchet differentiable on $H^1(\Omega) \times (H^1(\Omega) \cap L^\infty(\Omega))$ (see [18, Proposition 1.1]) with

$$\begin{aligned} \partial_{(u,v)} \mathcal{J}(u,v)[\varphi, \psi] = & \int_\Omega (v^2 + \eta) \nabla u \cdot \nabla \varphi \, \mathrm{d}x + \int_\Omega v \psi |\nabla u|^2 \, \mathrm{d}x \\ & + \int_\Omega \nabla v \cdot \nabla \psi \, \mathrm{d}x - \int_\Omega (1-v) \psi \, \mathrm{d}x \,, \end{aligned} \tag{2.5}$$

for every $u \in H^1(\Omega)$, $v \in H^1(\Omega) \cap L^\infty(\Omega)$, $\varphi \in C_c^\infty(\Omega)$, $\psi \in C^\infty(\overline{\Omega})$, the stability condition can be written as

$$\partial_{(u,v)} \mathcal{J}\big(u(t), v(t)\big)[\varphi, \psi] \ge 0 \tag{2.6}$$

for every $\varphi \in C_c^\infty(\Omega)$ and every $\psi \in C^\infty(\overline{\Omega})$ with $\psi \le 0$.

Remark 2.1. We notice that the inequality in (2.6) and the restriction to test functions $\psi \le 0$ arise from the irreversibility condition of the damage variable $v(t)$ discussed above.

By the structure of the derivative of $\mathcal{J}$ (2.5), inequality (2.6) can be simply rephrased in terms of the following inequalities:

$$0 = \partial_u \mathcal{J}\big(u(t), v(t)\big)[\varphi] = \int_\Omega \big(v^2(t) + \eta\big)\nabla u(t) \cdot \nabla \varphi \,\mathrm{d}x \tag{2.7}$$

$$\begin{aligned} &0 \le \partial_v \mathcal{J}\big(u(t), v(t)\big)[\psi] \\ &= \int_\Omega v(t)\psi \big|\nabla u(t)\big|^2 \,\mathrm{d}x + \int_\Omega \nabla v(t) \cdot \nabla \psi \,\mathrm{d}x - \int_\Omega \big(1 - v(t)\big)\psi \,\mathrm{d}x\,, \end{aligned} \tag{2.8}$$

for every $\varphi \in C_c^\infty(\Omega)$ and $\psi \in C^\infty(\overline{\Omega})$ with $\psi \le 0$.

Remark 2.2. The right-hand sides of (2.7) and (2.8) represent the Gateaux derivatives in the direction of u and v, respectively.

We also notice that once we know that inequalities (2.7) and (2.8) are satisfied for every test functions $\varphi \in C_c^\infty(\Omega)$ and $\psi \in C^\infty(\overline{\Omega})$ with $\psi \le 0$ in Ω, by density and truncation argument it is easy to see that they hold also for $\varphi \in H_0^1(\Omega)$ and $\psi \in H^1(\Omega)$, $\psi \le 0$ a.e. in Ω.

Finally, by the separate convexity of $\mathcal{J}$ w.r.t. the variables u and v, from formulas (2.7) and (2.8) we derive the actual stability condition, given in terms of minimum problems: for every $t \in [0, T]$, $u(t)$ minimizes $\mathcal{J}(\cdot, v(t))$ in the class $\mathcal{A}(w(t))$, while $v(t)$ minimizes $\mathcal{J}(u(t), \cdot)$ in the class of functions $v \in H^1(\Omega)$ such that $v \le v(t)$ a.e. in Ω.

This leads us to the following definition of quasi-static evolution for the phase-field model via critical points of the energy $\mathcal{J}$ in (2.1)–(2.3).

Definition 2.3. Let $T > 0$ and $w \in W^{1,2}([0, T]; H^1(\Omega))$. We say that a pair $(u, v)\colon [0, T] \to H^1(\Omega) \times H^1(\Omega)$ is a *quasi-static evolution (of critical points)* if the following conditions are satisfied:

(1) *Irreversibility*: $0 \le v(t) \le v(\tau) \le 1$ a.e. in Ω for every $0 \le \tau \le t \le T$;
(2) *Stability*: for every $t \in [0, T]$

$$\mathcal{J}\big(u(t), v(t)\big) \le \mathcal{J}\big(u, v(t)\big) \qquad \text{for all } u \in \mathcal{A}\big(w(t)\big), \tag{2.9}$$

$$\mathcal{J}\big(u(t), v(t)\big) \le \mathcal{J}\big(u(t), v\big) \qquad \text{for all } v \in H^1(\Omega), v \le v(t) \text{ a.e. in } \Omega\,. \tag{2.10}$$

(3) *Energy-dissipation inequality*: for every $t \in [0, T]$

$$\mathcal{J}\big(u(t), v(t)\big) \le \mathcal{J}(u(0), v(0)) + \int_0^t \int_\Omega (v^2(\tau) + \eta)\nabla u(\tau) \cdot \nabla \dot{w}(\tau) \,\mathrm{d}x \,\mathrm{d}\tau\,. \tag{2.11}$$

From now on, the dot represents the derivative w.r.t. time t.

We can now state the main existence result of the paper, which will be proved in Section 5.

Theorem 2.4. *Let $T > 0$, $w \in W^{1,2}([0, T]; H^1(\Omega))$, and u_0, $v_0 \in H^1(\Omega)$ be such that $u_0 \in \mathcal{A}(w(0))$ and $0 \le v_0 \le 1$ a.e. in Ω. Assume that the pair (u_0, v_0) satisfies the stability conditions at time $t = 0$:*

$$\mathcal{J}(u_0, v_0) \le \mathcal{J}(u, v_0) \qquad \textit{for all } u \in \mathcal{A}(w(0)), \tag{2.12}$$

$$\mathcal{J}(u_0, v_0) \le \mathcal{J}(u_0, v) \qquad \textit{for all } v \in H^1(\Omega), \textit{ such that } v \le v_0 \textit{ a.e. in } \Omega. \tag{2.13}$$

Then, there exists a quasi-static evolution $(u, v)\colon [0, T] \to H^1(\Omega) \times H^1(\Omega)$ with $u(0) = u_0$ and $v(0) = v_0$.

Remark 2.5. We again stress that the study of existence of quasi-static evolution for the phase-field model based on the Ambrosio-Tortorelli functional (1.1) is not a novelty. For instance, such a problem has been tackled in [26, 31, 36] using different convergence schemes, always in a continuous-space setting.

What we claim is the main contribution of our paper is the technique used to construct such an evolution, which is based on the algorithm introduced in [6]. In particular, the method we exploit here is suitable for applications and numerical simulations, since, as we explain in the second part of this section, we first show the existence of a finite-dimensional quasi-static evolution in a discretized H^1-space (see (2.15)), and then pass to the limit as the discretization becomes finer and finer. We show that in the limit we recover a quasi-static evolution in the sense of Definition 2.3. By this argument we deduce that the numerical scheme used to construct approximate solutions for the evolution problem guarantees convergence to a suitable quasi-static evolution.

2.1. **Space-discrete setting.** Let us now describe the space-discrete counterpart of the above setting. We first want to discretize the domain Ω following the basic ideas of the finite element method (for more details on the theory see, e.g., [37]). Let us fix $\lambda \in (0, +\infty)$. Given the mesh parameter $h > 0$, we consider a triangulation $\mathcal{T}_h$ such that $\mathrm{diam}(K) \leq h$ for every $K \in \mathcal{T}_h$ and such that

$$\frac{R_h}{\rho_h} \leq \lambda \qquad \text{uniformly for } h > 0\,, \tag{2.14}$$

where ρ_h is the minimum of the radii of the incircles of the triangulation $\mathcal{T}_h$, and R_h is the maximum of the radii of the excircles. The above condition guarantees the usual piecewise affine interpolation estimates. We refer to [37] for more details.

Once we are given the triangulation $\mathcal{T}_h$, we need to discretize the function space $H^1(\Omega)$. Thus, we define the finite-dimensional function space $\mathcal{F}_h$ as the set of continuous functions on $\overline{\Omega}$ that are affine on each triangle $K \in \mathcal{T}_h$. More precisely, we set

$$\mathcal{F}_h := \{u \in C(\overline{\Omega}) \cap H^1(\Omega) : \nabla u \text{ is constant a.e. on } K \text{ for every } K \in \mathcal{T}_h\}\,. \tag{2.15}$$

Denoting with Δ_h the set of all the vertices of $\mathcal{T}_h$ and setting $N_h := \#\Delta_h$, a basis $\{\xi_l\}_{l=1}^{N_h}$ of $\mathcal{F}_h$ can be defined in the following natural way: for every $l = 1, \dots, N_h$, the element $\xi_l \in \mathcal{F}_h$ is such that

$$\xi_l(x_m) = \delta_{lm} \qquad \text{for every } x_m \in \Delta_h, \tag{2.16}$$

where δ_{lm} is the Kronecker delta. We further assume that the basis $\{\xi_l\}_{l=1}^{N_h}$ satisfies the stiffness condition

$$\int_\Omega \nabla\xi_l \cdot \nabla\xi_m \,\mathrm{d}x \leq 0 \qquad \text{for every } l, m \in \{1, \dots, N_h\},\ l \neq m, \tag{2.17}$$

which is fulfilled, e.g., if the angles of the triangles are smaller or equal to $\frac{\pi}{2}$ (see [22]).

Clearly, the space $\mathcal{F}_h$ can be endowed with the usual H^1-norm. In the sequel, we will also use the following:

$$\|\varphi\|_{\mathcal{F}_h} := \left(\int_\Omega \mathrm{P}_h(\varphi^2)\,\mathrm{d}x\right)^{1/2} \qquad \text{for every } \varphi \in \mathcal{F}_h\,, \tag{2.18}$$

where $\mathrm{P}_h \colon C(\overline{\Omega}) \to \mathcal{F}_h$ is the Lagrangian interpolant onto the space $\mathcal{F}_h$, i.e., the unique operator defined on $C(\overline{\Omega})$ with values in $\mathcal{F}_h$ such that

$$\mathrm{P}_h(\varphi)(x_l) = \varphi(x_l) \qquad \text{for every } \varphi \in C(\overline{\Omega}) \text{ and every } x_l \in \Delta_h. \tag{2.19}$$

It can be easily checked that formula (2.18) defines a norm in $\mathcal{F}_h$.

In this framework, we introduce the discrete counterpart of the stored elastic energy (2.1) and of the dissipated energy (2.2): for every $u, v \in \mathcal{F}_h$, we set

$$\mathcal{E}_h(u,v) := \frac{1}{2}\int_\Omega \big(\mathrm{P}_h(v^2)+\eta\big)|\nabla u|^2 \,\mathrm{d}x\,, \tag{2.20}$$

$$\mathcal{D}_h(v) := \frac{1}{2}\int_\Omega |\nabla v|^2 \,\mathrm{d}x + \frac{1}{2}\int_\Omega \mathrm{P}_h\big((1-v)^2\big)\,\mathrm{d}x\,. \tag{2.21}$$

As in (2.3), the *discrete total energy* is the sum of $\mathcal{E}_h$ and $\mathcal{D}_h$. Hence,

$$\mathcal{J}_h(u,v) := \mathcal{E}_h(u,v) + \mathcal{D}_h(v)\,. \tag{2.22}$$

We note that, thanks to [37], we can also approximate the Dirichlet boundary datum in $\mathcal{F}_h$. More precisely, there exists a sequence $w_h \in W^{1,2}([0,T];\mathcal{F}_h)$ such that $w_h \to w$ in $W^{1,2}([0,T];H^1(\Omega))$ as $h \to 0$. In particular, this implies that $w_h(t) \to w(t)$ in $H^1(\Omega)$ for every $t \in [0,T]$ and $\dot w_h(t) \to \dot w(t)$ in $H^1(\Omega)$ for a.e. $t \in [0,T]$. Hence, the quasi-static evolution in the space-discrete setting (see Definition 2.6) will be driven by the approximate boundary datum w_h, and, as in (2.4), for every h and every $t \in [0,T]$ we restrict the set of admissible displacements to

$$\mathcal{A}_h(w_h(t)) := \{u \in \mathcal{F}_h : u = w_h(t) \text{ on } \partial\Omega\}\,. \tag{2.23}$$

Analogously to Definition 2.3, the notion of finite-dimensional quasi-static evolution reads as follows:

Definition 2.6. Let $T > 0$ and $h > 0$ be fixed. Let $w_h \in W^{1,2}([0,T];\mathcal{F}_h)$. We say that a pair of functions $(u_h, v_h)\colon [0,T] \to \mathcal{F}_h \times \mathcal{F}_h$ is a *finite-dimensional quasi-static evolution* if it satisfies the following conditions:

(1) *Irreversibility*: $0 \le v_h(t) \le v_h(\tau) \le 1$ in Ω for every $0 \le \tau \le t \le T$;
(2) *Stability*: for every $t \in (0,T]$ we have

$$\mathcal{J}_h\big(u_h(t), v_h(t)\big) \le \mathcal{J}_h\big(u, v_h(t)\big) \qquad \text{for every } u \in \mathcal{A}_h(w_h(t))\,, \tag{2.24}$$

$$\mathcal{J}_h\big(u_h(t), v_h(t)\big) \le \mathcal{J}_h\big(u_h(t), v\big) \qquad \text{for every } v \in \mathcal{F}_h, v \le v_h(t) \text{ in } \Omega\,; \tag{2.25}$$

(3) *Energy-dissipation inequality*: for every $t \in [0,T]$

$$\begin{aligned}\mathcal{J}_h\big(u_h(t), v_h(t)\big) &\le \mathcal{J}_h\big(u_h(0), v_h(0)\big)\\ &\quad + \int_0^t \int_\Omega \big(\mathrm{P}_h(v_h^2(\tau)) + \eta\big)\nabla u_h(\tau)\cdot\nabla \dot w_h(\tau)\,\mathrm{d}x\,\mathrm{d}\tau\,.\end{aligned} \tag{2.26}$$

Remark 2.7. Let us briefly comment on the stability condition (2) of Definition 2.6. As in the space-continuous setting, being the functional $\mathcal{J}_h$ separately convex w.r.t. the variables u and v, inequalities (2.24) and (2.25) are equivalent to

$$\begin{aligned}0 &= \partial_u \mathcal{J}_h(u_h(t), v_h(t))[\varphi] = \int_\Omega (\mathrm{P}_h(v_h^2(t)) + \eta)\nabla u_h(t)\cdot\nabla\varphi\,\mathrm{d}x\,,\\ 0 &\le \partial_v \mathcal{J}_h(u_h(t), v_h(t))[\psi]\\ &= \int_\Omega \mathrm{P}_h(v_h(t)\psi)|\nabla u_h(t)|^2\,\mathrm{d}x + \int_\Omega \nabla v_h(t)\cdot\nabla\psi\,\mathrm{d}x - \int_\Omega \mathrm{P}_h\Big(\big(1 - v_h(t)\big)\psi\Big)\,\mathrm{d}x\,,\end{aligned}$$

for every $\varphi \in \mathcal{A}_h(0)$ and $\psi \in \mathcal{F}_h$ with $\psi \le 0$ in Ω.

Moreover, we want property (2) to be satisfied only in the interval $(0,T]$. The motivation of this choice is the following: in Theorem 2.4 (see also Section 5) we aim to construct a quasi-static evolution in the space-continuous setting as limit of finite-dimensional quasi-static evolutions. For this reason, as it will be shown in the proof of Theorem 2.4, we need to find *ad hoc* approximations of the initial conditions u_0, v_0 in the space $\mathcal{F}_h$. In doing this, we can not guarantee to keep track of the stability properties (2.12)–(2.13) of the pair (u_0, v_0) (see Theorem 2.4).

Therefore, at this stage it is enough for us to have stability for strictly positive time, while in the space-continuous limit we will recover it also for $t = 0$.

In the finite-dimensional case, the existence of a finite-dimensional quasi-static evolution in the sense of Definition 2.6 will actually be a consequence of a stronger result (see Theorem 2.9) which allows us to show, in a suitable time parametrized setting, the validity of an energy-dissipation equality, instead of the simpler inequality stated in (2.26). In order to state precisely the existence result, we need to introduce some further notation.

Following the lines of [31], we introduce two new weighted norms on $\mathcal{F}_h$, which could be referred to as *energy norms*, in the sense that they resemble the energy functional $\mathcal{J}_h$ in (2.22): for every $u, v \in \mathcal{F}_h$ and every $\varphi \in \mathcal{A}_h(0)$ we set

$$\|\varphi\|^2_{h,v} := \int_\Omega (\mathrm{P}_h(v^2) + \eta)|\nabla\varphi|^2 \,\mathrm{d}x\,, \tag{2.27}$$

$$\|v\|^2_{h,u} := \int_\Omega \mathrm{P}_h(v^2)|\nabla u|^2 \,\mathrm{d}x + \int_\Omega |\nabla v|^2 \,\mathrm{d}x + \int_\Omega \mathrm{P}_h(v^2)\,\mathrm{d}x\,. \tag{2.28}$$

We notice that $\|\cdot\|_{h,v}$ in (2.27), which is a norm on $\mathcal{A}_h(0)$, actually fails to be a norm on the whole $\mathcal{F}_h$. Furthermore, we denote by $\langle\cdot,\cdot\rangle_{h,v}$ and by $\langle\cdot,\cdot\rangle_{h,u}$ the scalar products inducing the norms (2.27) and (2.28), respectively. We refer to Appendix A for more properties of $\|\cdot\|_{h,v}$ and $\|\cdot\|_{h,u}$.

Finally, we introduce the slope functions: for every $u, v \in \mathcal{F}_h$ we define

$$|\partial_u\mathcal{J}_h|(u,v) := \max\{-\partial_u\mathcal{J}_h(u,v)[\varphi] : \varphi \in \mathcal{A}_h(0),\ \|\varphi\|_{h,v} \le 1\}\,, \tag{2.29}$$

$$|\partial_v\mathcal{J}_h|(u,v) := \max\{-\partial_v\mathcal{J}_h(u,v)[\psi] : \psi \in \mathcal{F}_h,\ \psi \le 0,\ \|\psi\|_{h,u} \le 1\}\,. \tag{2.30}$$

We notice that the sign restriction in (2.30) is again due to the irreversibility condition on the phase-field variable. We refer to Appendix A for some properties of the slopes (2.29)-(2.30).

With this notation, we can give the definition of *parametrized finite-dimensional quasi-static evolution*.

Definition 2.8. Let $T > 0$ and $h > 0$ be fixed. Let $w_h \in W^{1,2}([0,T];\mathcal{F}_h)$. We say that a triple $(t_h, u_h, v_h)\colon [0,S] \to [0,T] \times \mathcal{F}_h \times \mathcal{F}_h$, $S \in (0,+\infty)$, is a *parametrized finite-dimensional quasi-static evolution* if it satisfies the following conditions:

(1) *Time regularity*: $t_h \in W^{1,\infty}([0,S];[0,T])$, $u_h \in W^{1,2}([0,S];\mathcal{F}_h)$, and $v_h \in W^{1,\infty}([0,S];\mathcal{F}_h)$;
(2) *Time parametrization*: $0 \le t_h(s) \le t_h(\sigma) \le T$ for every $0 \le s \le \sigma \le S$ and $t_h(0) = 0$, $t_h(S) = T$;
(3) *Irreversibility*: $0 \le v_h(\sigma) \le v_h(s) \le 1$ in Ω for every $0 \le s \le \sigma \le S$;
(4) *Stability*: for every $s \in (0,S]$ such that $t'_h(s) > 0$ we have

$$\mathcal{J}_h\big(u_h(s), v_h(s)\big) \le \mathcal{J}_h\big(u, v_h(s)\big) \qquad \text{for every } u \in \mathcal{A}_h(w_h(t_h(s)))\,, \tag{2.31}$$

$$\mathcal{J}_h\big(u_h(s), v_h(s)\big) \le \mathcal{J}_h\big(u_h(s), v\big) \qquad \text{for every } v \in \mathcal{F}_h, v \le v_h(s) \text{ in } \Omega\,; \tag{2.32}$$

(5) *Energy-dissipation equality*: for every $s \in [0,S]$

$$\begin{aligned}\mathcal{J}_h\big(u_h(s), v_h(s)\big) = \mathcal{J}_h\big(u_h(0), v_h(0)\big) &- \int_0^s |\partial_u\mathcal{J}_h|(u_h(\sigma), v_h(\sigma))\|u'_h(\sigma)\|_{h,v_h(\sigma)}\,\mathrm{d}\sigma \\ &- \int_0^s |\partial_v\mathcal{J}_h|(u_h(\sigma), v_h(\sigma))\|v'_h(\sigma)\|_{h,u_h(\sigma)}\,\mathrm{d}\sigma \\ &+ \int_0^s\int_\Omega \big(\mathrm{P}_h(v_h^2(\sigma)) + \eta\big)\nabla u_h(\sigma)\cdot\nabla\dot{w}_h(t_h(\sigma))t'_h(\sigma)\,\mathrm{d}x\,\mathrm{d}\sigma\,.\end{aligned} \tag{2.33}$$

From now on, the symbol $'$ denotes the derivative w.r.t. s.

We state here the main existence result for the finite-dimensional setting. Before showing the proof, we need some auxiliary results. Here, we only mention that the construction of a parametrized finite-dimensional quasi-static evolution is based on the incremental procedure described in Section 3 and presented in a different context in [4, 31].

Theorem 2.9. *Let $h > 0$, $w_h \in W^{1,2}([0,T];\mathcal{F}_h)$, and $u_{0,h}, v_{0,h} \in \mathcal{F}_h$ be such that $u_{0,h} \in \mathcal{A}_h(w_h(0))$ and $0 \le v_{0,h} \le 1$ in Ω. Then, there exists a parametrized finite-dimensional quasi-static evolution $(t_h, u_h, v_h)\colon [0,S] \to [0,T] \times \mathcal{F}_h \times \mathcal{F}_h$, $S \in (0,+\infty)$, with $(u_h(0), v_h(0)) = (u_{0,h}, v_{0,h})$.*

As a direct consequence of Theorem 2.9, we get the existence of a finite-dimensional quasi-static evolution in the sense of Definition 2.6.

Corollary 2.10. *Let $h > 0$, $w_h \in W^{1,2}([0,T];\mathcal{F}_h)$, and $u_{0,h}, v_{0,h} \in \mathcal{F}_h$ be such that $u_{0,h} \in \mathcal{A}_h(w_h(0))$ and $0 \le v_{0,h} \le 1$ in Ω. Then, there exists a finite dimensional quasi-static evolution $(u_h, v_h)\colon [0,T] \to \mathcal{F}_h \times \mathcal{F}_h$ such that $(u_h(0), v_h(0)) = (u_{0,h}, v_{0,h})$.*

Proof. Let w_h, $u_{0,h}$, and $v_{0,h}$ be as in the statement of the corollary, and let $(t_h, \tilde{u}_h, \tilde{v}_h)\colon [0,S] \to [0,T] \times \mathcal{F}_h \times \mathcal{F}_h$, $S \in (0,+\infty)$, be a parametrized finite-dimensional quasi-static evolution with initial conditions $(\tilde{u}_h(0), \tilde{v}_h(0)) = (u_{0,h}, v_{0,h})$. For every $t \in [0,T]$, we set

$$s(t) := \min\{s \in [0,S] : t_h(s) = t\} \quad \text{and} \quad (u_h(t), v_h(t)) := (\tilde{u}_h(s(t)), \tilde{v}_h(s(t)))\,.$$

Clearly, v_h satisfies condition (1) of Definition 2.6. The energy-dissipation inequality (2.26) follows by the simple inequality

$$\mathcal{J}_h(\tilde{u}_h(s(t)), \tilde{v}_h(s(t))) \le \mathcal{J}_h(u_{0,h}, v_{0,h}) + \int_0^{s(t)} \int_\Omega (\mathrm{P}_h(\tilde{v}_h^2(\sigma)) + \eta) \nabla \tilde{u}_h(\sigma) \cdot \nabla \dot{w}_h(\sigma) t_h'(\sigma)\, \mathrm{d}x\, \mathrm{d}\sigma$$

and by the change of variable $\tau = t_h(\sigma)$ in the last integral.

Finally, in order to deduce (2.24)-(2.25) from (2.31)-(2.32), we distinguish two cases: $t_h'(s(t)) > 0$ and $t_h'(s(t)) = 0$. In the former case, (2.24)-(2.25) and (2.31)-(2.32) coincide. In the latter case, we know that, by definition of $s(t)$, there exists a sequence $\sigma_k \in [0,S]$ such that $\sigma_k \nearrow s(t)$ with $t_h'(\sigma_k) > 0$. Noticing that $\tilde{u}_h(\sigma_k) \to u_h(t)$ in $\mathcal{F}_h$, that $\tilde{v}_h(\sigma_k) \to v_h(t)$ in $\mathcal{F}_h$, and that $\tilde{v}_h(\sigma_k) \ge v_h(t)$ for every k, applying Lemma 3.3 below we deduce (2.24)-(2.25). □

Remark 2.11. We stress here that we are not able to prove, passing to the limit as the mesh parameter h tends to 0, that a sequence of parametrized finite-dimensional quasi-static evolutions converges to some parametric quasi-static evolution in the space-continuous setting. In particular, we are not able to guarantee the right h-independent estimates on u_h and v_h that would allow us to show an energy-dissipation equality similar to (2.33) for a space-continuous limit (u, v). For this reason, in Section 5 we will only be able to show that any limit as $h \to 0$ of a sequence of finite-dimensional quasi-static evolution is itself a quasi-static evolution in the sense of Definition 2.3.

We refer the interested reader to [31] for more technical details concerning the proof of existence of a parametrized quasi-static evolution in the pure space-continuous setting, regardless of the approximability of such an evolution through space-discrete ones.

Notation. From now on, we will denote by $\|\cdot\|_{H^1}$ the H^1-norm and by $\|\cdot\|_p$ the usual L^p-norm for $p \in [1,+\infty]$.

3. The alternate minimization scheme

In this section we describe the core of our convergence algorithm, which will allow us to construct critical points of the energy functional $\mathcal{J}_h$ satisfying proper boundary and irreversibility conditions. Such a scheme will be exploited in the proof of Theorem 2.9. More precisely, for a given mesh parameter $h > 0$ we fix two functions $v_0, w \in \mathcal{F}_h$, with $v_0 \geq 0$ in Ω and we show a constructive way to find a critical point $(\bar{u}, \bar{v}) \in \mathcal{F}_h \times \mathcal{F}_h$ of the functional $\mathcal{J}_h$ under the constraints $\bar{u} = w$ on $\partial\Omega$ and $\bar{v} \leq v_0$ in Ω.

The recursive scheme we adopt here is a modification of the strategy presented in [6] in a finite-dimensional setting, and similar to the alternate minimization procedure used in [31] for the continuous phase-field model. What we have to take care of in our context is the presence of the operator P_h of (2.19) in the definition of the functional $\mathcal{J}_h$ (see (2.20)–(2.22)).

For every $j \in \mathbb{N}\backslash\{0\}$ we define the functions u_j and v_j in $\mathcal{F}_h$ as follows:

$$u_j := \arg\min\{\mathcal{J}_h(u, v_{j-1}) : \, u \in \mathcal{A}_h(w)\}\,, \tag{3.1}$$

$$v_j := \arg\min\{\mathcal{J}_h(u_j, v) : v \in \mathcal{F}_h,\, v \leq v_{j-1} \text{ in } \Omega\}\,. \tag{3.2}$$

The existence of minimizers of (3.1) is standard. The uniqueness follows by the strict convexity of the functional $\mathcal{J}_h(\cdot, v)$ for $v \in \mathcal{F}_h$.

In the following proposition, we briefly discuss the existence and uniqueness of v_j. We also show the usual bound $0 \leq v_j \leq 1$, which does not follow by simple truncation argument because of the nature of the function space $\mathcal{F}_h$ and of the presence of the interpolation operator $\mathrm{P}_h\colon C(\overline{\Omega}) \to \mathcal{F}_h$. The proof is contained in the Appendix A.

Proposition 3.1. *The minimum problem* (3.2) *admits a unique solution. Moreover, the solution $v_j \in \mathcal{F}_h$ satisfies $0 \leq v_j \leq 1$ for every $j \in \mathbb{N}\backslash\{0\}$.*

Proof. See Appendix A. □

In the following lemma we show a "one-step" energy balance involving the pairs (u_j, v_j) and (u_{j+1}, v_{j+1}) constructed in (3.1)-(3.2).

Lemma 3.2. *Let $j \in \mathbb{N}$. For every $r \in [0, 1]$ let*

$$u(r) := (1 - r)u_j + r u_{j+1} \qquad \textit{and} \qquad v(r) := (1 - r)v_j + r v_{j+1}\,.$$

Then, for every $\overline{r} \in [0, 1]$ the following equalities hold:

$$\mathcal{J}_h(u(\overline{r}), v_j) = \mathcal{J}_h(u_j, v_j) - \int_0^{\overline{r}} |\partial_u \mathcal{J}_h|(u(r), v_j) \|u'(r)\|_{h,v_j} \,\mathrm{d}r\,, \tag{3.3}$$

$$\mathcal{J}_h(u_{j+1}, v(\overline{r})) = \mathcal{J}_h(u_{j+1}, v_j) - \int_0^{\overline{r}} |\partial_v \mathcal{J}_h|(u_{j+1}, v(r)) \|v'(r)\|_{h,u_{j+1}} \,\mathrm{d}r\,, \tag{3.4}$$

where $'$ denotes the derivative w.r.t. r.

Proof. By the minimality of u_{j+1}, for every $\varphi \in \mathcal{A}_h(0)$ we have

$$\begin{aligned}\partial_u \mathcal{J}_h(u_j, v_j)[\varphi] &= \partial_u \mathcal{J}_h(u_{j+1}, v_j)[\varphi] + \int_\Omega (\mathrm{P}_h(v_j^2) + \eta)\nabla(u_j - u_{j+1}) \cdot \nabla\varphi \,\mathrm{d}x \\ &= \int_\Omega (\mathrm{P}_h(v_j^2) + \eta)\nabla(u_j - u_{j+1}) \cdot \nabla\varphi \,\mathrm{d}x = \langle u_j - u_{j+1}, \varphi\rangle_{h,v_j}\,.\end{aligned}$$

Hence, by definition of the slope (2.29) we deduce that

$$|\partial_u \mathcal{J}_h|(u_j, v_j) = -\partial_u \mathcal{J}_h(u_j, v_j) \frac{[u_{j+1} - u_j]}{\|u_{j+1} - u_j\|_{h,v_j}}\,. \tag{3.5}$$

Let us now fix $r \in (0,1]$. By linearity, for every $\varphi \in \mathcal{A}_h(0)$ we have

$$\begin{aligned}\partial_u \mathcal{J}_h(u(r), v_j)[\varphi] &= r\partial_u \mathcal{J}_h(u_{j+1}, v_j)[\varphi] + (1-r)\partial_u \mathcal{J}_h(u_j, v_j)[\varphi] \\ &= (1-r)\partial_u \mathcal{J}_h(u_j, v_j)[\varphi]\,.\end{aligned}$$

In view of (3.5) we get that

$$|\partial_u \mathcal{J}_h|(u(r), v_j)\|u'(r)\|_{h,v_j} = -\partial_u \mathcal{J}_h(u(r), v_j)[u'(r)]\,. \tag{3.6}$$

Combining the chain-rule and (3.6) we obtain (3.3).

As for (3.4), for every $\varphi \in \mathcal{F}_h$ with $\varphi \le 0$ we have

$$\begin{aligned}\partial_v \mathcal{J}_h(u_{j+1}, v_j)[\varphi] &= \partial_v \mathcal{J}_h(u_{j+1}, v_{j+1})[\varphi] + \int_\Omega \mathrm{P}_h((v_j - v_{j+1})\varphi)|\nabla u_{j+1}|^2 \,\mathrm{d}x \\ &\qquad + \int_\Omega \nabla(v_j - v_{j+1})\cdot\nabla\varphi \,\mathrm{d}x + \int_\Omega \mathrm{P}_h((v_j - v_{j+1})\varphi)\,\mathrm{d}x \\ &= \partial_v \mathcal{J}_h(u_{j+1}, v_{j+1})[\varphi] + \langle v_j - v_{j+1}, \varphi\rangle_{h,u_{j+1}}\,.\end{aligned} \tag{3.7}$$

Since, by definition of v_{j+1} in (3.2),

$$\begin{aligned}&\partial_v \mathcal{J}_h(u_{j+1}, v_{j+1})[\varphi] \ge 0 \qquad \text{for every } \varphi \in \mathcal{F}_h,\ \varphi \le 0 \text{ in } \Omega\,,\\ &\partial_v \mathcal{J}_h(u_{j+1}, v_{j+1})[v_{j+1} - v_j] = 0\,,\end{aligned}$$

inequality (3.7) implies that

$$|\partial_v \mathcal{J}_h|(u_{j+1}, v_j) = -\partial_v \mathcal{J}_h(u_{j+1}, v_j)\frac{[v_{j+1} - v_j]}{\|v_{j+1} - v_j\|_{h,u_{j+1}}}\,. \tag{3.8}$$

For $r \in (0,1]$, again by linearity we can write

$$\partial_v \mathcal{J}_h(u_{j+1}, v(r))[\varphi] = (1-r)\partial_v \mathcal{J}_h(u_{j+1}, v_j)[\varphi] + r\partial_v \mathcal{J}_h(u_{j+1}, v_{j+1})[\varphi]\,,$$

which implies, together with (3.8), that

$$|\partial_v \mathcal{J}_h|(u_{j+1}, v(r))\|v'(r)\|_{h,u_{j+1}} = -\partial_v \mathcal{J}_h(u_{j+1}, v(r))[v'(r)]\,. \tag{3.9}$$

Again, combining the chain-rule and (3.9) we get (3.4), and the proof is thus concluded. □

We now want to show that any limit $(\bar u, \bar v)$ of the sequence (u_j, v_j) defined in (3.1)–(3.2) is a critical point of the functional $\mathcal{J}_h$ satisfying $\bar u \in \mathcal{A}_h(w)$ and $\bar v \le v_0$ in Ω. To do so, we first show a stability property of the minimum problems (3.1) and (3.2). This is the aim of the following lemma, which is stated in a more general setting than the one needed in this section, since it will be useful also in the proof of Theorem 2.9.

Lemma 3.3. *Let $u_k, v_k, w_k, z_k \in \mathcal{F}_h$ be such that $u_k \in \mathcal{A}_h(w_k)$ and*

$$\mathcal{J}_h(u_k, z_k) \le \mathcal{J}_h(u, z_k) \qquad \text{for every } u \in \mathcal{A}_h(w_k), \tag{3.10}$$

$$\mathcal{J}_h(u_k, v_k) \le \mathcal{J}_h(u_k, v) \qquad \text{for every } v \in \mathcal{F}_h \text{ such that } v \le z_k \text{ in } \Omega. \tag{3.11}$$

Assume that there exist $\bar u, \bar v, \bar w, \bar z \in \mathcal{F}_h$ such that $u_k \to \bar u$, $v_k \to \bar v$, $w_k \to \bar w$, and $z_k \to \bar z$ in $\mathcal{F}_h$ as $k \to +\infty$. Then $\bar u \in \mathcal{A}_h(\bar w)$ and

$$\mathcal{J}_h(\bar u, \bar z) \le \mathcal{J}_h(u, \bar z) \qquad \text{for every } u \in \mathcal{A}_h(\bar w), \tag{3.12}$$

$$\mathcal{J}_h(\bar u, \bar v) \le \mathcal{J}_h(\bar u, v) \qquad \text{for every } v \in \mathcal{F}_h \text{ such that } v \le \bar z \text{ in } \Omega\,. \tag{3.13}$$

Proof. Let us prove (3.12). For every $k \in \mathbb{N}$ and every $u \in \mathcal{A}_h(\bar w)$ we have

$$\mathcal{J}_h(u_k, z_k) \le \mathcal{J}_h(u + w_k - \bar w, z_k)\,. \tag{3.14}$$

Since $u + w_k - \bar w \to u$ in $\mathcal{F}_h$ as $k \to +\infty$, passing to the limit in (3.14) we get (3.12).

As for (3.13), for every $v \in \mathcal{F}_h$ such that $v \le \bar z$ in Ω we have that $z_k + v - \bar z \le z_k$. Hence, by (3.11),

$$\mathcal{J}_h(u_k, v_k) \le \mathcal{J}_h(u_k, z_k + v - \bar z)\,.$$

Passing to the limit in the previous inequality we get (3.13). □

We are now ready to show the convergence of the sequence (u_j, v_j), defined by (3.1) and (3.2), to a critical point of $\mathcal{J}_h$.

Proposition 3.4. *Let $v_0, w \in \mathcal{F}_h$ with $v_0 \geq 0$, and let u_j, v_j be defined by* (3.1) *and* (3.2)*, respectively. Then the following facts hold:*

(1) *there exist $\bar{u}, \bar{v} \in \mathcal{F}_h$ such that $u_j \to \bar{u}$ and $v_j \to \bar{v}$ in $\mathcal{F}_h$ as $j \to +\infty$;*
(2) *the limit function $\bar{v}$ satisfies $0 \leq \bar{v} \leq 1$;*
(3) *the limit functions $\bar{u}, \bar{v} \in \mathcal{F}_h$ satisfy*

$$\mathcal{J}_h(\bar{u}, \bar{v}) \leq \mathcal{J}_h(u, \bar{v}) \qquad \text{for every } u \in \mathcal{A}_h(w)\,, \tag{3.15}$$

$$\mathcal{J}_h(\bar{u}, \bar{v}) \leq \mathcal{J}_h(\bar{u}, v) \qquad \text{for every } v \in \mathcal{F}_h \text{ with } v \leq \bar{v}\,. \tag{3.16}$$

Proof. By definition of u_j and v_j, for every $j \geq 2$ we have

$$\mathcal{J}_h(u_j, v_j) \leq \mathcal{J}_h(u_j, v_{j-1}) \leq \mathcal{J}_h(u_{j-1}, v_{j-1})\,. \tag{3.17}$$

Iterating inequality (3.17), we obtain

$$\mathcal{J}_h(u_j, v_j) \leq \mathcal{J}_h(u_1, v_1) \leq \mathcal{J}_h(w, v_0) < +\infty\,,$$

from which we deduce that the sequences u_j and v_j are bounded in $\mathcal{F}_h$. Being v_j a decreasing sequence with values in the interval $[0,1]$ and being $\mathcal{F}_h$ finite-dimensional, we deduce that there exists $\bar{v} \in \mathcal{F}_h$ such that $v_j \to \bar{v}$ in $\mathcal{F}_h$ and $0 \leq \bar{v} \leq 1$ in Ω, so that property (2) holds. Moreover, by compactness, there exists $\bar{u} \in \mathcal{F}_h$ such that, up to a subsequence, $u_j \to \bar{u}$.

Property (3) results from Lemma 3.3 applied to the sequences u_j, v_j, with fixed boundary datum $w \in \mathcal{F}_h$. By uniqueness of solution to (3.15), we also deduce that the whole sequence u_j converges to $\bar{u}$ in $\mathcal{F}_h$, and the proof is thus concluded. □

We conclude this section proving a continuity property of the minimum problem (3.1) w.r.t. the phase-field v and the boundary data w. This result will be useful in the construction of a suitable time parametrization for the discrete solutions (see Section 4).

Proposition 3.5. *Let $M > 0$ and $u_1, u_2, v_1, v_2, w_1, w_2 \in \mathcal{F}_h$ be such that, for $i = 1, 2$, $\|v_i\|_\infty \leq M$ and*

$$u_i = \arg\min\,\{\mathcal{J}_h(u, v_i) :\, u \in \mathcal{A}_h(w_i)\}\,.$$

Then, there exists a positive constant $C = C(M, w_i, h)$ such that

$$\|u_1 - u_2\|_{H^1} \leq C(\|w_1 - w_2\|_{H^1} + \|v_1 - v_2\|_{\mathcal{F}_h})\,. \tag{3.18}$$

Proof. Let us consider $u_* = \arg\min\,\{\mathcal{J}_h(u, v_1) :\, u \in \mathcal{A}_h(w_2)\}$. Then, it is easy to see that there exists $C > 0$ such that

$$\|u_1 - u_*\|_{H^1} \leq C\|w_1 - w_2\|_{H^1}\,.$$

In order to estimate $\|u_* - u_2\|_{H^1}$, for every $\varphi \in \mathcal{A}_h(0)$ we write

$$\int_\Omega (\mathrm{P}_h(v_1^2) + \eta)\nabla(u_2 - u_*) \cdot \nabla\varphi \,\mathrm{d}x = \int_\Omega \mathrm{P}_h(v_1^2 - v_2^2)\nabla u_2 \cdot \nabla\varphi \,\mathrm{d}x\,. \tag{3.19}$$

Testing $\varphi = u_2 - u_*$ in (3.19) and recalling the hypothesis $\|v_i\|_\infty \leq M$, we deduce that

$$\eta\|u_2 - u_*\|^2_{H^1} \leq 2M \int_\Omega |\mathrm{P}_h(v_1 - v_2)||\nabla u_2||\nabla(u_2 - u_*)| \,\mathrm{d}x\,.$$

From the previous inequality we deduce (3.18) by Hölder inequality, using the fact that $\mathcal{F}_h$ is finite dimensional, so that all the norms are equivalent, and $\|u_2\|_{H^1} \leq c\|w_2\|_{H^1}$ for some positive c. The dependence of C in (3.18) from the mesh parameter h is due to the h-dependent equivalence of norms in $\mathcal{F}_h$. □

4. Construction of parametrized finite-dimensional quasi-static evolutions

We are now ready to prove Theorem 2.9, that is, the existence of a parametrized finite-dimensional quasi-static evolution in the sense of Definition 2.8. The strategy of the proof is based on a time discretization procedure, typical of many rate-independent processes (see, e.g., [32, 34]) and on an arc-length reparametrization of time similar to the one used in [31].

Let $h > 0$. For every $k \in \mathbb{N}$, we consider the uniform subdivision of the time interval $[0, T]$ given by $t^k_i := \frac{iT}{k}$, $i = 0, \dots, k$. In order to construct a discrete in time evolution in the finite-dimensional setting we follow the algorithm proposed in [6]: for $i = 0$ we set $u^{k,h}_0 := u_{0,h}$ and $v^{k,h}_0 := v_{0,h}$. For $i \geq 1$, at the instant t^k_i we construct a critical point $(u^{k,h}_i, v^{k,h}_i) \in \mathcal{F}_h \times \mathcal{F}_h$ of $\mathcal{J}_h$ as limit of the alternating minimization process described in Section 3. More precisely, let us set

$$u^{k,h}_{i,0} := u^{k,h}_{i-1} + w_h(t^k_i) - w_h(t^k_{i-1}) \qquad \text{and} \qquad v^{k,h}_{i,0} := v^{k,h}_{i-1}\,.$$

For $j \in \mathbb{N}$, $j \geq 1$, we define iteratively two sequences of functions $u^{k,h}_{i,j}$ and $v^{k,h}_{i,j}$ as

$$u^{k,h}_{i,j} := \arg\min \{\mathcal{J}_h(u, v^{k,h}_{i,j-1}) : u \in \mathcal{A}_h(w_h(t^k_i))\}\,, \tag{4.1}$$

$$v^{k,h}_{i,j} := \arg\min \{\mathcal{J}_h(u^{k,h}_{i,j}, v) : v \in \mathcal{F}_h,\ v \leq v^{k,h}_{i,j-1}\}\,. \tag{4.2}$$

We notice that, since by assumption $v^{k,h}_0 \geq 0$, combining Propositions 3.1 and 3.4 we deduce that (4.1) and (4.2) always admit unique solutions and, for every $k \in \mathbb{N}$ and every $i \in \{1, \dots, k\}$, there exist $u^{k,h}_i, v^{k,h}_i \in \mathcal{F}_h$ such that $u^{k,h}_{i,j} \to u^{k,h}_i$ and $v^{k,h}_{i,j} \to v^{k,h}_i$ in $\mathcal{F}_h$ as $j \to +\infty$. Moreover, $0 \leq v^{k,h}_i \leq v^{k,h}_{i,j} \leq v^{k,h}_{i-1} \leq 1$ in Ω and, again thanks to Proposition 3.4,

$$\mathcal{J}_h(u^{k,h}_i, v^{k,h}_i) \leq \mathcal{J}_h(u, v^{k,h}_i) \qquad \text{for every } u \in \mathcal{A}_h(w_h(t^k_i)), \tag{4.3}$$

$$\mathcal{J}_h(u^{k,h}_i, v^{k,h}_i) \leq \mathcal{J}_h(u^{k,h}_i, v) \qquad \text{for every } v \in \mathcal{F}_h,\ v \leq v^k_i \text{ in } \Omega. \tag{4.4}$$

In the following proposition we prove a finite-length property of the sequences $\{u^{k,h}_{i,j}\}$ and $\{v^{k,h}_{i,j}\}$ for $j \in \mathbb{N}$ and $i \in \{1, \dots, k\}$.

Proposition 4.1. *There exists $C_h \in (0, +\infty)$ such that for every $k \in \mathbb{N}$*

$$\sum_{i=1}^{k} \sum_{j=1}^{\infty} \|u^{k,h}_{i,j} - u^{k,h}_{i,j-1}\|_{H^1} + \|v^{k,h}_{i,j} - v^{k,h}_{i,j-1}\|_{H^1} \leq C_h\,. \tag{4.5}$$

Proof. In this proof, C denotes a generic positive constant, which could change from line to line.

In view of (4.1)-(4.2) and of Proposition 3.5, we have that, for every $i \in \{1, \dots, k\}$ and every $j \geq 2$

$$\|u^{k,h}_{i,j} - u^{k,h}_{i,j-1}\|_{H^1} \leq C\|v^{k,h}_{i,j-1} - v^{k,h}_{i,j-2}\|_{\mathcal{F}_h} \leq C\|v^{k,h}_{i,j-1} - v^{k,h}_{i,j-2}\|_1\,, \tag{4.6}$$

where the second inequality is due to the equivalence of norms in finite dimension. For $j = 1$, instead, by definition of $u^{k,h}_{j,0}$ and of $v^{k,h}_{i,0}$ we get

$$\|u^{k,h}_{i,1} - u^{k,h}_{i,0}\|_{H^1} \leq C\|w_h(t^k_i) - w_h(t^k_{i-1})\|_{H^1}\,. \tag{4.7}$$

By definition (4.2) of $v^{k,h}_{i,j}$, we have that

$$\sum_{j=1}^{\infty} \|v^{k,h}_{i,j} - v^{k,h}_{i,j-1}\|_1 = \sum_{j=1}^{\infty} \int_\Omega (v^{k,h}_{i,j-1} - v^{k,h}_{i,j})\,\mathrm{d}x = \int_\Omega (v^{k,h}_{i-1} - v^{k,h}_i)\,\mathrm{d}x\,. \tag{4.8}$$

Thus, collecting inequalities (4.6)-(4.8) we obtain

$$\sum_{i=1}^{k}\sum_{j=1}^{\infty}\|u^{k,h}_{i,j}-u^{k,h}_{i,j-1}\|_{H^1}\leq C\sum_{i=1}^{k}\int_{\Omega}(v^{k,h}_{i-1}-v^{k,h}_{i})\,\mathrm{d}x+C\sum_{i=1}^{k}\|w_h(t^k_i)-w_h(t^k_{i-1})\|_{H^1}$$
$$\leq C\int_{\Omega}(v^{k,h}_{0}-v^{k,h}_{k})\,\mathrm{d}x+C\int_0^T\|\dot w_h(t)\|_{H^1}\,\mathrm{d}t\leq C|\Omega|+C\int_0^T\|\dot w_h(t)\|_{H^1}\,\mathrm{d}t\,.$$

By equivalence of the norms in finite dimension, we obtain a similar inequality also in terms of phase-field v. Hence, being $w_h\in W^{1,2}([0,T];\mathcal{F}_h)$, (4.5) follows. □

Following the lines of [31], we now construct a suitable arc-length parametrization of time and the functions u^h_k, v^h_k interpolating between the values $u^{k,h}_{i-1}$, $u^{k,h}_{i}$ and $v^{k,h}_{i-1}$, $v^{k,h}_{i}$, respectively.

Let $s^k_0:=0$. For $i\geq 1$ and $j\in\mathbb{N}$, let

$$s^k_{i,-1}:=s^k_{i-1}\,,\qquad s^k_{i,0}:=s^k_{i-1}+\tau_k\,,$$
$$s^k_{i,j+1/2}:=s^k_{i,j}+\|u^{k,h}_{i,j+1}-u^{k,h}_{i,j}\|_{h,v^{k,h}_{i,j}}\,,\qquad s^k_{i,j+1}:=s^k_{i,j+1/2}+\|v^{k,h}_{i,j+1}-v^{k,h}_{i,j}\|_{h,u^{k,h}_{i,j+1}}\,.$$

We define $s^k_i:=\lim_j s^k_{i,j}$, which exists finite in view of Proposition 4.1 and Lemma A.1.

We now define the interpolating functions t^h_k, u^h_k, and v^h_k, distinguishing between the three intervals $[s^k_{i,-1},s^k_{i,0}]$, $[s^k_{i,j},s^k_{i,j+1/2}]$, and $[s^k_{i,j+1/2},s^k_{i,j+1}]$. For $s\in[s^k_{i,-1},s^k_{i,0}]$ we set

$$\begin{aligned}&t^h_k(s):=t^k_{i-1}+s-s^k_{i,-1}\,,\\&u^h_k(s):=u^{k,h}_{i-1}+w_h(t^h_k(s))-w_h(t^k_{i-1})\,,\\&v^h_k(s):=v^k_{i-1}\,.\end{aligned}\tag{4.9}$$

For $s\in[s^k_{i,j},s^k_{i,j+1/2}]$ we define

$$\begin{aligned}&t^h_k(s):=t^k_i\,,\\&u^h_k(s):=\begin{cases}u^{k,h}_{i,j}+(s-s^k_{i,j})\dfrac{u^{k,h}_{i,j+1}-u^{k,h}_{i,j}}{s^k_{i,j+1/2}-s^k_{i,j}} & \text{if } s^k_{i,j+1/2}\neq s^k_{i,j}\,,\\ u^{k,h}_{i,j}=u^{k,h}_{i,j+1} & \text{if } s^k_{i,j+1/2}=s^k_{i,j}\,,\end{cases}\\&v^h_k(s):=v^{k,h}_{i,j}\,.\end{aligned}\tag{4.10}$$

Finally, for $s\in[s^k_{i,j+1/2},s^k_{i,j+1}]$ we set

$$\begin{aligned}&t^h_k(s):=t^k_i\,,\\&u^h_k(s):=u^{k,h}_{i,j+1}\,,\\&v^h_k(s):=\begin{cases}v^{k,h}_{i,j}+(s-s^k_{i,j+1/2})\dfrac{v^{k,h}_{i,j+1}-v^{k,h}_{i,j}}{s^k_{i,j+1}-s^k_{i,j+1/2}} & \text{if } s^k_{i,j+1}\neq s^k_{i,j+1/2}\,,\\ v^k_{i,j}=v^k_{i,j+1} & \text{if } s^k_{i,j+1}=s^k_{i,j+1/2}\,.\end{cases}\end{aligned}\tag{4.11}$$

In this way we have constructed a sequence of interpolating functions $(t^h_k,u^h_k,v^h_k)\colon[0,S^h_k]\to[0,T]\times\mathcal{F}_h\times\mathcal{F}_h$, where S^h_k is the maximal value of the arc-length parameter s. By Proposition 4.1 and Lemma A.1, S^h_k is uniformly bounded w.r.t. k. Moreover, we notice that

$$t^h_k(s^k_i)=t^k_i\,,\qquad u^h_k(s^k_i)=u^{k,h}_i,\qquad v^h_k(s^k_i)=v^{k,h}_i,\tag{4.12}$$

and that there exists a positive constant C independent of k such that

$$\|t^h_k\|_{W^{1,\infty}([0,S^h_k];[0,T])}+\|u^h_k\|_{W^{1,2}([0,S^h_k];\mathcal{F}_h)}+\|v^h_k\|_{W^{1,\infty}([0,S^h_k];\mathcal{F}_h)}\leq C\,.\tag{4.13}$$

In the next proposition we collect the stability properties and the energy balance satisfied by the interpolation functions t^h_k, u^h_k, and v^h_k.

Proposition 4.2. *For every $k \in \mathbb{N}$ and every $i \in \{1, \dots, k\}$ it holds*

$$\mathcal{J}_h(u^h_k(s^k_i), v^h_k(s^k_i)) \leq \mathcal{J}_h(u, v^h_k(s^k_i)) \qquad \text{for every } u \in \mathcal{A}_h(w_h(t^k_h(s^k_i)))\,, \tag{4.14}$$

$$\mathcal{J}_h(u^h_k(s^k_i), v^h_k(s^k_i)) \leq \mathcal{J}_h(u^h_k(s^k_i), v) \qquad \text{for every } v \in \mathcal{F}_h,\ v \leq v^h_k(s^k_i)\,. \tag{4.15}$$

Moreover, for every $s \in [0, S^h_k]$ we have

$$\begin{aligned} \mathcal{J}_h(u^h_k(s), v^h_k(s)) = \mathcal{J}_h(u_{0,h}, v_{0,h}) &- \int_0^s |\partial_u \mathcal{J}_h|(u^h_k(\sigma), v^h_k(\sigma)) \|(u^h_k)'(\sigma)\|_{h, v^h_k(\sigma)} \,\mathrm{d}\sigma \\ &- \int_0^s |\partial_u \mathcal{J}_h|(u^h_k(\sigma), v^h_k(\sigma)) \|(v^h_k)'(\sigma)\|_{h, u^h_k(\sigma)} \,\mathrm{d}\sigma \\ &+ \int_0^s \int_\Omega (\mathrm{P}_h(v^h_k(\sigma)^2) + \eta) \nabla u^h_k(\sigma) \cdot \nabla \dot{w}(t^h_k(\sigma))(t^h_k)'(\sigma) \,\mathrm{d}x \,\mathrm{d}\sigma\,. \end{aligned} \tag{4.16}$$

Proof. In view of (4.12), the equilibrium conditions (4.14)-(4.15) are equivalent to (4.3) and (4.4).

As for (4.16), we need to show the energy balance in each interval of the form $[s^k_{i-1}, s^k_{i,0}]$, $[s^k_{i,j}, s^k_{i,j+1/2}]$, and $[s^k_{i,j+1/2}, s^k_{i,j+1}]$. For every $\bar{s} \in [s^k_{i-1}, s^k_{i,0}]$ we have, by chain-rule,

$$\begin{aligned} \mathcal{J}_h(u^h_k(\bar{s}), v^h_k(\bar{s})) &= \mathcal{J}_h(u^h_k(s^k_{i-1}), v^h_k(s^k_{i-1})) \\ &\quad + \int_{s^k_{i-1}}^{s^k_{i,0}} \int_\Omega (\mathrm{P}_h(v^h_k(\sigma)^2) + \eta) \nabla u^h_k(\sigma) \cdot \nabla \dot{w}_h(t^h_k(\sigma))(t^h_k)'(\sigma) \,\mathrm{d}x \,\mathrm{d}\sigma\,. \end{aligned} \tag{4.17}$$

For $\bar{s} \in [s^k_{i,j}, s^k_{i,j+1/2}]$ we recall the discrete energy balance (3.3) proved in Lemma 3.2, to which we apply the change of variable $r = \frac{\sigma - s^k_{i,j}}{s^k_{i,j+1/2} - s^k_{i,j}}$. Being t^h_k and v^h_k constant on the whole interval $[s^k_{i,j}, s^k_{i,j+1/2}]$, we get

$$\begin{aligned} \mathcal{J}_h(u^h_k(\bar{s}), v^h_k(\bar{s})) &= \mathcal{J}_h(u^h_k(s^k_{i,j}), v^h_k(s^k_{i,j})) \\ &\quad - \int_{s^k_{i,j}}^{s^k_{i,j+1/2}} |\partial_u \mathcal{J}_h|(u^h_k(\sigma), v^h_k(\sigma)) \|(u^h_k)'(\sigma)\|_{h, v^h_k(\sigma)} \,\mathrm{d}\sigma\,. \end{aligned} \tag{4.18}$$

In a similar way, we can show that for every $\bar{s} \in [s^k_{i,j+1/2}, s^k_{i,j+1}]$

$$\begin{aligned} \mathcal{J}_h(u^h_k(\bar{s}), v^h_k(\bar{s})) &= \mathcal{J}_h(u^h_k(s^k_{i,j+1/2}), v^h_k(s^k_{i,j+1/2})) \\ &\quad - \int_{s^k_{i,j+1/2}}^{s^k_{i,j+1}} |\partial_v \mathcal{J}_h|(u^h_k(\sigma), v^h_k(\sigma)) \|(v^h_k)'(\sigma)\|_{h, u^h_k(\sigma)} \,\mathrm{d}\sigma\,. \end{aligned} \tag{4.19}$$

Iterating equalities (4.17)-(4.19) we obtain (4.16). □

We are now in a position to conclude the proof of Theorem 2.9.

Proof of Theorem 2.9. In view of inequality (4.13), we have that there exist $S_h \in (0, +\infty)$, $t_h \in W^{1,\infty}([0, S_h]; [0, T])$, $u_h \in W^{1,2}([0, S_h]; \mathcal{F}_h)$, and $v_h \in W^{1,\infty}([0, S_h]; \mathcal{F}_h)$ such that, up to a subsequence, $S^h_k \to S_h$, $t^h_k \rightharpoonup t_h$ weakly* in $W^{1,\infty}([0, S_h]; [0, T])$, $u^h_k \rightharpoonup u_h$ weakly in $W^{1,2}([0, S_h]; \mathcal{F}_h)$, and $v^h_k \rightharpoonup v_h$ weakly* in $W^{1,\infty}([0, S_h]; \mathcal{F}_h)$. In particular, we have that for every $s \in [0, S_h]$ and every sequence $S^h_k \to s$ it holds

$$t^h_k(S^h_k) \to t_h(s)\,, \quad u^h_k(S^h_k) \to u_h(s) \text{ in } \mathcal{F}_h\,, \quad v^h_k(S^h_k) \to v_h(s) \text{ in } \mathcal{F}_h\,. \tag{4.20}$$

Moreover, $u'_h(\cdot) = \dot{w}_h(t_h(\cdot))t'_h(\cdot)$ in $L^2([0, S_h]; \mathcal{F}_h)$.

Let us prove the stability conditions (2.31)-(2.32). Let $s \in (0, S_h)$ be such that $t'_h(s) > 0$. This implies that there exists a sequence of indeces $i_k \in \{1, \dots, k\}$ such that $s^k_{i_k} \to s$. Since the pair $(u^h_k(s^k_{i_k}), v^h_k(s^k_{i_k}))$ satisfies (4.14)-(4.15) for every k, Lemma 3.3 implies (2.31)-(2.32). For $s = S_h$ the stability follows in the same way from the stability in S^h_k.

Finally, we have to show the energy equality. In order to do this, we pass to the limit in equality (4.16). Fix $s \in [0, S_h]$. In view of the convergences discussed above, we have that

$$\begin{aligned}\lim_k \int_0^s \int_\Omega (\mathrm{P}_h(v^h_k(\sigma)^2) + \eta)\nabla u^h_k(\sigma) \cdot \nabla \dot{w}(t^h_k(\sigma))(t^h_k)'(\sigma)\,\mathrm{d}x\,\mathrm{d}\sigma \\ = \int_0^s \int_\Omega (\mathrm{P}_h(v_h(\sigma)^2) + \eta)\nabla u_h(\sigma) \cdot \nabla \dot{w}(t_h(\sigma))t'_h(\sigma)\,\mathrm{d}x\,\mathrm{d}\sigma\,.\end{aligned}$$

As for the other two terms in the right-hand side of (4.16), we apply Lemma A.2 and [9, Theorem 3.1], which guarantees that

$$\begin{aligned}&\int_0^s |\partial_u \mathcal{J}_h|(u_h(\sigma), v_h(\sigma))\|u'_h(\sigma)\|_{h, v_h(\sigma)}\,\mathrm{d}\sigma \\ &\qquad \le \liminf_k \int_0^s |\partial_u \mathcal{J}_h|(u^h_k(\sigma), v^h_k(\sigma))\|(u^h_k)'(\sigma)\|_{h, v^h_k(\sigma)}\,\mathrm{d}\sigma\,,\end{aligned}$$

$$\begin{aligned}&\int_0^s |\partial_v \mathcal{J}_h|(u_h(\sigma), v_h(\sigma))\|v'_h(\sigma)\|_{h, u_h(\sigma)}\,\mathrm{d}\sigma \\ &\qquad \le \liminf_k \int_0^s |\partial_v \mathcal{J}_h|(u^h_k(\sigma), v^h_k(\sigma))\|(v^h_k)'(\sigma)\|_{h, u^h_k(\sigma)}\,\mathrm{d}\sigma\,.\end{aligned}$$

Combining the previous inequalities and passing to the limsup in (4.16) as $k \to +\infty$ we deduce that

$$\begin{aligned}\mathcal{J}_h(u_h(s), v_h(s)) \le \mathcal{J}_h(u_{0,h}, v_{0,h}) &- \int_0^s |\partial_u \mathcal{J}_h|(u_h(\sigma), v_h(\sigma))\|u'_h(\sigma)\|_{h, v_h(\sigma)}\,\mathrm{d}\sigma \\ &- \int_0^s |\partial_v \mathcal{J}_h|(u_h(\sigma), v_h(\sigma))\|v'_h(\sigma)\|_{h, u_h(\sigma)}\,\mathrm{d}\sigma \\ &+ \int_0^s \int_\Omega (\mathrm{P}_h(v_h(\sigma)^2) + \eta)\nabla u_h(\sigma) \cdot \nabla \dot{w}(t_h(\sigma))t'_h(\sigma)\,\mathrm{d}x\,\mathrm{d}\sigma\,.\end{aligned} \tag{4.21}$$

In order to show the opposite inequality, we first apply the chain-rule to the energy function $\sigma \mapsto \mathcal{J}_h(u_h(\sigma), v_h(\sigma))$, obtaining

$$\begin{aligned}\mathcal{J}_h(u_h(s), v_h(s)) = \mathcal{J}_h(u_{0,h}, v_{0,h}) &+ \int_0^s \partial_u \mathcal{J}_h(u_h(\sigma), v_h(\sigma))[u'_h(\sigma)]\,\mathrm{d}\sigma \\ &+ \int_0^s \partial_v \mathcal{J}_h(u_h(\sigma), v_h(\sigma))[v'_h(\sigma)]\,\mathrm{d}\sigma\,.\end{aligned} \tag{4.22}$$

Being $\sigma \mapsto v_h(\sigma)$ non-increasing, we have that $\dot{v}_h(\sigma) \le 0$ in Ω. Hence, by definition of the slope (2.30) w.r.t. the phase-field v, we estimate the last term in the right-hand side of (4.22) with

$$\int_0^s \partial_v \mathcal{J}_h(u_h(\sigma), v_h(\sigma))[v'_h(\sigma)]\mathrm{d}\sigma \ge -\int_0^s |\partial_v \mathcal{J}_h|(u_h(\sigma), v_h(\sigma))\|v'_h(\sigma)\|_{h, u_h(\sigma)}\mathrm{d}\sigma\,. \tag{4.23}$$

As for the first integral term in (4.22), we rewrite it as

$$
\begin{aligned}
&\int_0^s \partial_u \mathcal{J}_h(u_h(\sigma), v_h(\sigma))[u_h'(\sigma)]\, \mathrm{d}\sigma \\
&\quad = \int_0^s \partial_u \mathcal{J}_h(u_h(\sigma), v_h(\sigma))[u_h'(\sigma) - \dot{w}_h(t_h(\sigma))t_h'(\sigma) + \dot{w}_h(t_h(\sigma))t_h'(\sigma)]\, \mathrm{d}\sigma \\
&\quad = \int_0^s \partial_u \mathcal{J}_h(u_h(\sigma), v_h(\sigma))[u_h'(\sigma) - \dot{w}_h(t_h(\sigma))t_h'(\sigma)]\, \mathrm{d}\sigma \\
&\qquad + \int_0^s \int_\Omega (\mathrm{P}_h(v_h(\sigma)^2) + \eta)\nabla u_h(\sigma) \cdot \nabla \dot{w}(t_h(\sigma))t_h'(\sigma)\, \mathrm{d}x\, \mathrm{d}\sigma\,,
\end{aligned}
\tag{4.24}
$$

and we notice that $u_h'(\sigma) - \dot{w}_h(t_h(\sigma))t_h'(\sigma) = 0$ on $\partial\Omega$ for $\sigma \in [0, S_h]$. In particular, whenever $t_h'(\sigma) > 0$, (4.14) holds, so that

$$
\begin{aligned}
&\partial_u \mathcal{J}_h(u_h(\sigma), v_h(\sigma))[u_h'(\sigma) - \dot{w}_h(t_h(\sigma))t_h'(\sigma)] \\
&\qquad\qquad = |\partial_u \mathcal{J}_h|(u_h(\sigma), v_h(\sigma)) \|u_h'(\sigma)\|_{h, v_h(\sigma)} = 0\,,
\end{aligned}
$$

where the second equality is due to the definition (2.29) of the slope w.r.t. the displacement u. On the other hand, when $t_h'(\sigma) = 0$, by (2.29) we have

$$
\begin{aligned}
&\partial_u \mathcal{J}_h(u_h(\sigma), v_h(\sigma))[u_h'(\sigma) - \dot{w}_h(t_h(\sigma))t_h'(\sigma)] \\
&\qquad \geq -|\partial_u \mathcal{J}_h|(u_h(\sigma), v_h(\sigma)) \|u_h'(\sigma) - \dot{w}_h(t_h(\sigma))t_h'(\sigma)\|_{h, v_h(\sigma)} \\
&\qquad = -|\partial_u \mathcal{J}_h|(u_h(\sigma), v_h(\sigma)) \|u_h'(\sigma)\|_{h, v_h(\sigma)}\,.
\end{aligned}
$$

Combining these two inequalities with (4.24) we deduce that

$$
\begin{aligned}
&\int_0^s \partial_u \mathcal{J}_h(u_h(\sigma), v_h(\sigma))[u_h'(\sigma)]\, \mathrm{d}\sigma \\
&\qquad \geq -\int_0^s |\partial_u \mathcal{J}_h|(u_h(\sigma), v_h(\sigma)) \|u_h'(\sigma)\|_{h, v_h(\sigma)}\, \mathrm{d}\sigma \\
&\qquad\quad + \int_0^s \int_\Omega (\mathrm{P}_h(v_h(\sigma)^2) + \eta)\nabla u_h(\sigma) \cdot \nabla \dot{w}(t_h(\sigma))t_h'(\sigma)\, \mathrm{d}x\, \mathrm{d}\sigma\,.
\end{aligned}
\tag{4.25}
$$

Finally, inserting (4.23) and (4.25) in (4.22) we get (2.33), and the proof is thus concluded. □

M-step algorithm. We conclude this section discussing a variant of the above construction. Namely, we modify the infinite minimization algorithm (4.1)-(4.2) by stopping it after M steps, with $M \in \mathbb{N}$ fixed a priori. Then, we define

$$
u_i^{k,h} := u_{i,M}^{k,h} \qquad \text{and} \qquad v_i^{k,h} := v_{i,M}^{k,h}\,.
$$

As in Proposition 4.1, we can show that there exists $C_h \in (0, +\infty)$ such that

$$
\sum_{i=1}^{k} \sum_{j=1}^{M} \|u_{i,j}^{k,h} - u_{i,j-1}^{k,h}\|_{H^1} + \|v_{i,j}^{k,h} - v_{i,j-1}^{k,h}\|_{H^1} \leq C_h
\tag{4.26}
$$

uniformly w.r.t. k. This allows us to construct a time reparametrization as in (4.9)-(4.11) with $s_0^k := 0$ and, for $i \in \{1, \dots, k\}$ and $j \in \{0, \dots, M-1\}$,

$$
s_{i,-1}^k := s_{i-1}^k\,, \quad s_{i,0}^k := s_{i-1}^k + \tau_k\,,
$$

$$
s_{i,j+1/2}^k := s_{i,j}^k + \|u_{i,j+1}^{k,h} - u_{i,j}^{k,h}\|_{h, v_{i,j}^{k,h}}\,, \quad s_{i,j+1}^{k,h} := s_{i,j+1/2}^{k,h} + \|v_{i,j+1}^{k,h} - v_{i,j}^{k,h}\|_{h, u_{i,j+1}^{k,h}}\,.
$$

We set $s_i^k := s_{i,M}^k$. In this way, we still obtain, for h fixed, a sequence of triples $(t_k^h, u_k^h, v_k^h)\colon [0, S_k^h] \to [0, T] \times \mathcal{F}_h \times \mathcal{F}_h$ satisfying the uniform bound (4.13) and the discrete energy balance (4.16). Moreover, thanks to (4.26), we have that S_k^h is uniformly bounded.

Up to a subsequence, we have that $S_k^h \to S_h$, $t_k^h \rightharpoonup t_h$ weakly* in $W^{1,\infty}([0,S_h];[0,T])$, $u_k^h \rightharpoonup u_h$ weakly in $W^{1,2}([0,S_h];\mathcal{F}_h)$, and $v_k^h \rightharpoonup v_h$ weakly* in $W^{1,\infty}([0,S_h];\mathcal{F}_h)$. As in the proof of Theorem 2.9, we can show that (t_h,u_h,v_h) satisfies the energy balance (2.33).

In order to prove that (t_h,u_h,v_h) is a parametrized finite-dimensional quasi-static evolution, we have to take care of the stability conditions (2.31)-(2.32), taking into account that this time the interpolation functions t_k^h, u_k^h, v_k^h do not satisfy (4.14)-(4.15) at time $s_i^k = s_{i,M}^k$. However, given $s \in [0,S_h]$ such that $t_h'(s) > 0$, for every k there exists an index $i_k \in \{1,\dots,k\}$ such that $s \in [s_{i_k-1}^k, s_{i_k}^k]$. Up to subsequence, we have that $s_{i_k-1}^k \to \underline{s}$ and $s_{i_k}^k \to \overline{s}$ for some $0 \le \underline{s} \le \overline{s} \le S_h$. Moreover, $s_{i_k,0}^k = s_{i_k-1}^k + \tau_k \to \underline{s}$. We claim that $\underline{s} = \overline{s}$. Indeed, if by contradiction $\underline{s} < \overline{s}$, being t_k^h constant in the interval $[s_{i_k,0}^k, s_{i_k}^k]$, we get that t_h is constant on $[\underline{s},\overline{s}]$, which implies $t_h'(s) = 0$. Therefore $s = \underline{s} = \overline{s}$, $s_{i_k,j}^k \to s$ for every $j \in \{-1,\dots,M\}$, and the stability conditions (2.31)-(2.32) follows by Lemma 3.3.

Remark 4.3. We finally notice that the same construction can be applied to a slightly more general scheme, in which the number of steps in the alternate minimization algorithm is not fixed a priori for every $i \in \{1,\dots,k\}$, but depends on the time node t_i^k. This fact is very important from a numerical point of view (see Section 6), since in general the algorithm (4.1)-(4.2) has to be artificially stopped according to some criterion, which, at time t_i^k, is satisfied after a certain number M_i^k of iterations. Thanks to the analysis described above, we are able to include all the possible stopping criteria.

5. From space-discrete to space-continuous evolution

This section is devoted to the proof of Theorem 2.4. In particular, we show that any limit of a sequence $(u_h,v_h)\colon [0,T] \to \mathcal{F}_h \times \mathcal{F}_h$ of finite-dimensional quasi-static evolutions is a quasi-static evolution in the sense of Definition 2.3.

Before proving Theorem 2.4, we show two useful properties. Firstly, we state a uniform estimate on the operator P_h. Secondly, we prove a stability property of the functionals $\mathcal{J}_h$ and $\mathcal{J}$ analogous to Lemma 3.3, and which takes into account the "convergence" of the finite dimensional spaces $\mathcal{F}_h$ to $H^1(\Omega)$ as $h \to 0$.

Although the following result is standard, we provide its proof in Appendix A for the sake of completeness.

Lemma 5.1. *Let $h > 0$, let $\mathrm{P}_h\colon C(\overline{\Omega}) \to H^1(\Omega)$ be the operator defined by (2.19), and let $g \in C^2(\mathbb{R})$. Then, for every $M > 0$ there exists a positive constant $C = C(g,M)$ depending only on g and M such that for every $v \in \mathcal{F}_h$ with $\|v\|_\infty \le M$*

$$\big\|(g\circ v) - \mathrm{P}_h(g\circ v)\big\|_1 \le Ch^2 \|\nabla v\|_2^2\,. \tag{5.1}$$

Proof. See Appendix A. □

Lemma 5.2. *For every $h > 0$, let $w_h, u_h, v_h \in \mathcal{F}_h$ be such that $u_h \in \mathcal{A}_h(w_h)$, $0 \le v_h \le 1$ in Ω, and*

$$\mathcal{J}_h(u_h,v_h) \le \mathcal{J}_h(u,v_h) \qquad \text{for every } u \in \mathcal{A}_h(w_h), \tag{5.2}$$

$$\mathcal{J}_h(u_h,v_h) \le \mathcal{J}_h(u_h,v) \qquad \text{for every } v \in \mathcal{F}_h \text{ such that } v \le v_h. \tag{5.3}$$

Assume that $w_h \to \bar w$ in $H^1(\Omega)$, $u_h \rightharpoonup \bar u$ and $v_h \rightharpoonup \bar v$ weakly in $H^1(\Omega)$ as $h \to 0$. Then $\bar u \in \mathcal{A}(\bar w)$ and

$$\mathcal{J}(\bar u,\bar v) \le \mathcal{J}(u,\bar v) \qquad \text{for every } u \in \mathcal{A}(\bar w), \tag{5.4}$$

$$\mathcal{J}(\bar u,\bar v) \le \mathcal{J}(\bar u,v) \qquad \text{for every } v \in H^1(\Omega) \text{ with } v \le \bar v. \tag{5.5}$$

Moreover, $u_h \to \bar u$ strongly in $H^1(\Omega)$.

Proof. Let us prove (5.4). Let $u \in H^1(\Omega)$ be such that $u = \bar{w}$ on $\partial\Omega$. Thanks to (2.14) and to the interpolation error estimates in, e.g., [37, Theorem 3.4.2], for every $\varphi \in C_c^\infty(\Omega)$ there exists a sequence $\varphi_h \in \mathcal{F}_h$ such that $\varphi_h = 0$ on $\partial\Omega$ and $\varphi_h \to \varphi$ in $H^1(\Omega)$ as $h \to 0$. Let us consider as a competitor in (5.2) the function $\psi_h := \varphi_h + w_h$. For such a ψ_h we have

$$\int_\Omega \big(\mathrm{P}_h(v_h^2) + \eta\big)|\nabla u_h|^2 \,\mathrm{d}x \le \int_\Omega \big(\mathrm{P}_h(v_h^2) + \eta\big)|\nabla \psi_h|^2 \,\mathrm{d}x \,. \tag{5.6}$$

It is clear that $\psi_h \to \psi := \varphi + \bar{w}$ in $H^1(\Omega)$ and, by Lemma 5.1, $\mathrm{P}_h(v_h^2) \to \bar{v}^2$ strongly in $L^p(\Omega)$ for every $p \in [1, +\infty)$. Therefore, applying [24, Theorem 7.5] and passing to the limit as $h \to 0$ in (5.6), we get

$$\begin{aligned}\int_\Omega (\bar{v}^2 + \eta)|\nabla \bar{u}|^2 \,\mathrm{d}x &\le \liminf_{h\to 0} \int_\Omega \big(\mathrm{P}_h(v_h^2) + \eta\big)|\nabla u_h|^2 \,\mathrm{d}x \\ &\le \limsup_{h\to 0} \int_\Omega \big(\mathrm{P}_h(v_h^2) + \eta\big)|\nabla \psi_h|^2 \,\mathrm{d}x \\ &\le \int_\Omega \big(\bar{v}^2 + \eta\big)\big|\nabla \psi\big|^2 \,\mathrm{d}x \,.\end{aligned} \tag{5.7}$$

By density, we have that the chain of inequalities (5.7) holds for every $\psi \in \mathcal{A}(\bar{w})$. Moreover, it is easy to see that (5.7) is equivalent to (5.4).

Specifying (5.7) for $\psi = \bar{u}$, we get that

$$\lim_{h\to 0} \int_\Omega \big(\mathrm{P}_h(v_h^2) + \eta\big)|\nabla u_h|^2 \,\mathrm{d}x = \int_\Omega (\bar{v}^2 + \eta)|\nabla \bar{u}|^2 \,\mathrm{d}x \,,$$

which implies the strong convergence of u_h to $\bar{u}$ in $H^1(\Omega)$.

We now prove (5.5). Let us first consider a competitor $v \in H^1(\Omega) \cap L^\infty(\Omega)$, $v \le \bar{v}$ in Ω. Let $\varphi^k \in C^\infty(\overline{\Omega})$ be such that $\varphi^k \le 0$ in Ω and $\varphi^k \to v - \bar{v}$ in $H^1(\Omega)$ as $k \to +\infty$. Let us set $v_h^k := \mathrm{P}_h(\varphi^k) + v_h$. Then, $v_h^k \le v_h$ for every $h > 0$ and every $k \in \mathbb{N}$, $v_h^k \in \mathcal{F}_h$, and $v_h^k \rightharpoonup \bar{v} + \varphi^k$ weakly in $H^1(\Omega)$ as $h \to 0$. By the quadratic structure of $\mathcal{J}_h$, by (5.3), and by the definition of v_h^k,

$$\begin{aligned}&\frac{1}{2}\int_\Omega \big(\mathrm{P}_h(v_h^2) + \eta\big)|\nabla u_h|^2 \,\mathrm{d}x + \frac{1}{2}\int_\Omega \mathrm{P}_h\big((1 - v_h)^2\big)\,\mathrm{d}x \\ &\quad\le \frac{1}{2}\int_\Omega \big(\mathrm{P}_h((v_h^k)^2) + \eta\big)|\nabla u_h|^2 \,\mathrm{d}x + \frac{1}{2}\int_\Omega \mathrm{P}_h\big((1 - v_h^k)^2\big)\,\mathrm{d}x \\ &\qquad + \frac{1}{2}\int_\Omega \big|\nabla \mathrm{P}_h(\varphi^k)\big|^2 \,\mathrm{d}x + \int_\Omega \nabla \mathrm{P}_h(\varphi^k) \cdot \nabla v_h \,\mathrm{d}x \,.\end{aligned} \tag{5.8}$$

Since $u_h \to \bar{u}$ and $\mathrm{P}_h(\varphi^k) \to \varphi^k$ strongly in $H^1(\Omega)$, $\mathrm{P}_h(v_h^2) \to \bar{v}^2$ and $\mathrm{P}_h((v_h^k)^2) \to (\varphi^k + \bar{v})^2$ strongly in $L^p(\Omega)$ for every $p \in [1, +\infty)$ (see Lemma 5.1), and $v_h \rightharpoonup \bar{v}$ weakly in $H^1(\Omega)$, passing to the limit as $h \to 0$ in (5.8) we deduce that

$$\begin{aligned}&\frac{1}{2}\int_\Omega (\bar{v}^2 + \eta)|\nabla \bar{u}|^2 \,\mathrm{d}x + \frac{1}{2}\int_\Omega (1 - \bar{v})^2 \,\mathrm{d}x \\ &\quad\le \frac{1}{2}\int_\Omega \big((\varphi^k + \bar{v})^2 + \eta\big)|\nabla \bar{u}|^2 \,\mathrm{d}x + \frac{1}{2}\int_\Omega \big(1 - (\varphi^k + \bar{v})\big)^2 \,\mathrm{d}x \\ &\qquad + \frac{1}{2}\int_\Omega |\nabla \varphi^k|^2 \,\mathrm{d}x + \int_\Omega \nabla \varphi^k \cdot \nabla \bar{v} \,\mathrm{d}x \,.\end{aligned} \tag{5.9}$$

If we let $k \to +\infty$ in (5.9), recalling that $\varphi^k \to v - \bar{v}$ in $H^1(\Omega)$, we obtain

$$\begin{aligned} &\frac{1}{2}\int_\Omega (\bar{v}^2+\eta)|\nabla \bar{u}|^2 \,\mathrm{d}x + \frac{1}{2}\int_\Omega (1-\bar{v})^2 \,\mathrm{d}x \\ &\qquad \le \frac{1}{2}\int_\Omega (v^2+\eta)|\nabla \bar{u}|^2 \,\mathrm{d}x + \frac{1}{2}\int_\Omega (1-v)^2 \,\mathrm{d}x \\ &\qquad\quad + \frac{1}{2}\int_\Omega \big|\nabla (v-\bar{v})\big|^2 \,\mathrm{d}x + \int_\Omega \nabla(v-\bar{v})\cdot\nabla\bar{v}\,\mathrm{d}x\,. \end{aligned} \tag{5.10}$$

Rearranging the last two terms in the right-hand side of (5.10), we get the stability condition (5.5) for $v \in H^1(\Omega) \cap L^\infty(\Omega)$ with $v \le \bar{v}$. By a truncation argument, we get the same conclusion for $v \in H^1(\Omega)$ with $v \le \bar{v}$. □

We are now ready to prove Theorem 2.4.

Proof of Theorem 2.4. As already mentioned in Remark 2.5, in order to prove the existence of a quasi-static evolution in the sense of Definition 2.3 we show that any sequence of finite-dimensional quasi-static evolutions $(u_h, v_h)\colon [0,T] \to \mathcal{F}_h$ converges, up to a subsequence, to a quasi-static evolution as the mesh parameter h tends to 0.

For every $h > 0$, we need first to find the right sequence of finite-dimensional quasi-static evolutions (u_h, v_h) starting from a suitable initial datum $u_{0,h}, v_{0,h} \in \mathcal{F}_h$ and with a suitable boundary Dirichlet condition $w_h\colon [0,T] \to \mathcal{F}_h$.

As mentioned in Section 2, there exists a sequence $w_h \in W^{1,2}([0,T]; H^1(\Omega))$ such that $w_h \in W^{1,2}([0,T]; \mathcal{F}_h)$ and $w_h \to w$ in $W^{1,2}([0,T]; H^1(\Omega))$ as $h \to 0$ (see [37]). In particular, the last convergence implies that $w_h(t) \to w(t)$ in $H^1(\Omega)$ for every $t \in [0,T]$ and $\dot{w}_h(t) \to \dot{w}(t)$ in $H^1(\Omega)$ for a.e. $t \in [0,T]$. Again by [37], we can also find two sequences $u_{0,h}, v_{0,h} \in \mathcal{F}_h$ such that $u_{0,h} \in \mathcal{A}_h(w_h(0))$, $0 \le v_{0,h} \le 1$ in Ω, and, as $h \to 0$, $u_{0,h} \to u_0$ and $v_{0,h} \to v_0$ in $H^1(\Omega)$.

By Corollary 2.10, for every $h > 0$ there exists a finite-dimensional quasi-static evolution $(u_h, v_h)\colon [0,T] \to \mathcal{F}_h \times \mathcal{F}_h$ with $u_h(0) = u_{0,h}$, $v_h(0) = v_{0,h}$, and $u_h(t) \in \mathcal{A}(w_h(t))$ for every $t \in [0,T]$.

In view of (2.24)–(2.26) and of the construction of $u_{0,h}, v_{0,h}$, and w_h, we have that

$$\sup_{\substack{h>0\\ t\in[0,T]}} \|u_h(t)\|_{H^1} < +\infty \qquad \text{and} \qquad \sup_{\substack{h>0\\ t\in[0,T]}} \|v_h(t)\|_{H^1} < +\infty\,. \tag{5.11}$$

Since the sequence $v_h\colon [0,T] \to H^1(\Omega)$ is such that (5.11) holds, $t \mapsto v_h(t)$ is non-increasing, and, for every $t \in [0,T]$, $v_h(t)$ takes values in $[0,1]$, applying a generalized version of Helly's Selection Theorem (see [23, Theorem 2.3], we find a non-increasing function $v\colon [0,T] \to H^1(\Omega)$ such that, along a suitable subsequence $h_k \to 0$, for every $t \in [0,T]$ $v_{h_k}(t)$ converges to $v(t)$ weakly in $H^1(\Omega)$ and strongly in $L^p(\Omega)$ for every $p \in [1,+\infty)$. In particular, $v(0) = v_0$ and $0 \le v(t) \le 1$ in Ω for every $t \in [0,T]$, hence condition (1) of Definition 2.3 is satisfied.

In view of (5.11), for every $t \in [0,T]$ we have that, up to a subsequence (possibly dependent on t), $u_{h_k}(t) \rightharpoonup u(t)$ weakly in $H^1(\Omega)$ for some $u(t) \in \mathcal{A}(w(t))$. By Lemma 5.2, we deduce that the pair $(u(t), v(t))$ satisfies the stability conditions (2.9) and (2.10), and $u_{h_k}(t) \to u(t)$ strongly in $H^1(\Omega)$. Moreover, by Lemma 5.2 and by uniqueness of solution of the minimum problem

$$\min\left\{\mathcal{J}(u, v(t)) : \, u \in H^1(\Omega),\, u \in \mathcal{A}(w(t))\right\},$$

we have that the whole sequence $u_{h_k}(t)$ converges to $u(t)$ strongly in $H^1(\Omega)$ for every $t \in [0,T]$.

Finally, in order to prove the energy inequality (2.11), we need to pass to the limit in the finite-dimensional energy inequality (2.26) as $h_k \to 0$. Since $u_{0,h} \to u_0$

and $v_{0,h} \to v_0$ and Lemma 5.1 holds, we have that $\mathcal{J}_{h_k}(u_{0,h_k}, v_{0,h_k}) \to \mathcal{J}(u_0, v_0)$. Again by Lemma 5.1, $\mathrm{P}_{h_k}(v_{h_k}^2(t)) \to v(t)$ in $L^p(\Omega)$ for every $p \in [1, +\infty)$ and every $t \in [0, T]$. Finally, by construction, $w_h \to w$ in $W^{1,2}([0, T]; H^1(\Omega))$. Hence, passing to the limit in (2.26) as $h_k \to 0$ and applying the dominated convergence theorem, we get (2.11), and this concludes the proof of the theorem. □

6. Numerical Experiments

In this section we illustrate numerically the previous findings simulating brittle fracture propagation. We remark that the purpose of this section is not to challenge the efficiency of the numerical methods, but rather to show the consistency of the previously discussed theory. We keep an extensive numerical analysis for future work.

For the following numerical experiments we make use of the experiences gained from previous calculations in [5, 6, 8, 11, 19], where we also find some of the used examples used herein. Below we use the original Ambrosio-Tortorelli functional $\mathcal{J}_\varepsilon$ defined in (1.1), whose discretized version we still denote by $\mathcal{J}_h$. We recall it here for the reader's convenience: for all $u, v \in \mathcal{F}_h$

$$\mathcal{J}_h(u, v) := \frac{1}{2}\int_\Omega \big(\mathrm{P}_h(v^2) + \eta_\varepsilon\big)|\nabla u|^2 \,\mathrm{d}x + \int_\Omega \Big(\kappa\varepsilon|\nabla v|^2 + \frac{\kappa}{4\varepsilon}\mathrm{P}_h\big((1-v)^2\big)\Big)\,\mathrm{d}x\,. \tag{6.1}$$

The following implementations are made with `Freefem++` and are performed on a MacBook Pro, 2.6 GHz Intel Core i7, 8 GB 1600 MHz DDR3.

For all the examples in this section we fix the basic domain $\Omega := (0, 1) \times (0, 1)$ and we choose the time step $\tau := 0.01$. Hence, with an initial time $t_0 := 0$ we set $t_i := i\tau$. By v_i and u_i we denote the phase and the displacement fields at time t_i, respectively. We present three simulations for different boundary data w and initial cracks Γ, imposed by an initial phase-field of the form

$$v_0(x) := 1 - \exp\Big(-\frac{\mathrm{dist}(x, \Gamma)}{2\varepsilon}\Big)\,. \tag{6.2}$$

The specific choice made in (6.2) comes from the fact that v_0 is a recovery sequence for the Γ-limit of $\mathcal{J}_\varepsilon$ (see, e.g. [3, 15]). We stress once again the physical interpretation of the phase-field: Where it is close to zero the crack appears. On the contrary, where it takes the value one the elastic body is perfectly sound. As it follows from the proof of Γ-convergence of the functional $\mathcal{J}_\varepsilon$, the fracture theoretically gains a thickness of order ε. However, in our simulations we plot only the phase-field variable without giving a precise description of the crack set.

We notice that initializing the pre-crack by an initial phase-field is a crucial difference compared to the implementations in the existing literature, where the initial crack is imposed by a notch in the domain. Moreover, in the functional $\mathcal{J}_h$ we keep the projection operator P_h, which is dropped in the cited works. However, with these techniques we stay closer to the theoretic framework, which is more suitable for our purpose.

We repeat the basic algorithm that we want to implement from (4.1)–(4.2) dropping the indices k and h and adding a stopping criterion as described at the end of Section 4. Given $i \geq 1$ and the phase-field v_{i-1} at time t_{i-1}, we set $v_{i,0} := v_{i-1}$, and we define inductively on $j \geq 1$

$$u_{i,j} := \arg\min\,\{\mathcal{J}_h(u, v_{i,j-1}) : u \in \mathcal{A}_h(w_h(t_i))\}\,, \tag{6.3}$$

$$v_{i,j} := \arg\min\,\{\mathcal{J}_h(u_{i,j}, v) : v \in \mathcal{F}_h,\ v \leq v_{i,j-1}\}\,. \tag{6.4}$$

Due to the convergence of both sequences as $j \to +\infty$, we stop the loop when the phase-field does not show any significant changes. More precisely, we fix a threshold

$0 < Tol_v \ll 1$, and we perform the alternate minimization (6.3)-(6.4) until we find $\bar{\jmath} \in \mathbb{N}$ such that $\|v_{i,\bar{\jmath}} - v_{i,\bar{\jmath}-1}\|_\infty \leq Tol_v$ and set $(u_i, v_i) := (u_{i,\bar{\jmath}}, v_{i,\bar{\jmath}})$.

The minimum in (6.3) is simply obtained by solving the linear Euler-Lagrange equation. However, the numerical treatment of the irresistibility condition $v \leq v_{i,j-1}$ in (6.4) is an issue to be discussed. In many papers (see, for instance, [6, 8, 12, 18]) this constraint has been replaced by a Dirichlet boundary condition, forcing v to be zero where $v_{i,j-1}$ is below a certain threshold. The method turned out to be numerically very efficient. However, up to our knowledge there is a lack of a rigorous theoretical proof of convergence of the scheme in the framework of quasi-static evolution problems.

In order to be as close as possible to the theoretical results discussed in this paper, in our numerical simulations we do not exploit the algorithm proposed in [8, 12], but we rather perform the constrained minimization by a projected Newton method as in [19]. We precisely use the algorithm presented in [30, Section 5.5.2] (see also [10]). Due to the quadratic structure of the functional $\mathcal{J}_h(u, \cdot)$ this procedure is not too expensive.

On most parts of the domain the phase-field will be nearly constant. Only close to the crack it is expected to be very steep. To get an appropriate interpolation error the mesh has to be very fine in the neighborhood of the crack, while it can be coarse elsewhere. Thus, we use an adaptive triangulation refining the mesh where it is necessary. For our purposes, we regularly adapt the mesh in the iteration procedure exploiting the standard routine implemented in `Freefem++`, which uses the standard anisotropic interpolation error estimator.

We notice that in the theoretical part of this paper the mesh $\mathcal{T}_h$ is not allowed to vary. Working with a constant mesh throughout the whole algorithm would require it to be extremely fine on the entire domain in order not to influence the propagation of the crack path. With an adaptive procedure we can omit this difficulty and we can save a lot of computational effort. Moreover, we mention that such an approach has been already investigated and validated in [6, 18].

Concerning the choice of the mesh, we have theoretically considered two other restrictions that we do not take into account in the numerical simulations. Firstly, we require (2.17). This condition is only needed to ensure that the phase-field is non-negative (see Proposition 3.1). Nevertheless, even not making any restriction on the mesh triangles, the non-negativity of the phase-field turns out to be a posteriori fulfilled. Secondly, in (2.14) a certain isotropy of the triangles is required in order to guarantee the usual a priori interpolation estimate. However, looking at the final mesh, we observe that the assertion is fulfilled simply because the mesh consists of finitely many triangles.

The basic numerical scheme we use is described in Algorithm 1 and all the appearing numerical parameters are summarized in Table 1. As an output, at each time step we visualize the number of alternating minimizations as well as the total number of steps of the projected Newton method necessary to compute the minimizers in line 9 of Algorithm 1. Moreover, we plot the crack length as a function of time, approximating it by $\frac{1}{4\varepsilon}\int_\Omega (1 - v_i)^2 \, \mathrm{d}x + \varepsilon \int_\Omega |\nabla v_i|^2 \, \mathrm{d}x$.

In Algorithm 1 we also ensure a mesh adaption after at most 10 minimizations. This reduces the number of minimizations compared to a procedure where the mesh is adapted only when the change of the phase-fields is small enough. The "relative change of nodes" rel_{mesh} refers to the quotient of the number of added nodes and the number of nodes of the old mesh. In this way the difference between two consecutive meshes is quantified and the alternating procedure continues until rel_{mesh} goes below a certain threshold Tol_{mesh}.

Algorithm 1

```
for i = 1 to k do
    v_i ← v_{i-1}
    do
        cnt ← 0
        do
            cnt ← cnt + 1
            v_old ← v_i
            u_i ← arg min{ J_h(u, v_i) | u ∈ A_h(w(t_i)) }
            v_i ← arg min{ J_h(u_i, v) | v ∈ F_h, v ≤ v_old }
        while ||v_i − v_old||_∞ > Tol_v AND cnt < 10
        Perform the mesh adaption
        rel_mesh ← "relative change of nodes"
    while rel_mesh > Tol_mesh
end for
```

Table 1. Numerical Parameters

ε	η_ε	κ	Tol_v	Tol_{mesh}
$2 \cdot 10^{-3}$	10^{-5}	0.5	$2 \cdot 10^{-3}$	$2 \cdot 10^{-3}$

First Example. Our first example starts from a fully symmetrical setting, where we impose an initial crack orthogonal to and in the middle of the left boundary of the domain Ω described by the set

$$\Gamma_1 := \left[0, \tfrac{1}{4}\right] \times \left\{\tfrac{1}{2}\right\}.$$

The crack is initialized by choosing the phase-field v_0 in (6.2) for $\Gamma = \Gamma_1$. We also use the symmetric boundary condition

$$w_1(t) = \begin{cases} t & \text{on } \{0\} \times \left[0, \tfrac{1}{2}\right), \\ -t & \text{on } \{0\} \times \left(\tfrac{1}{2}, 1\right]. \end{cases}$$

The measured values are visualized in Figure 1. The crack propagates until the domain is fully broken at time 0.97. In Figure 2 and Figure 3 one can observe the corresponding phase-field at different time steps and the generated mesh, respectively.

For this simulation there were in total 9 874 alternations and 96 104 Newton steps computed, which means that 105 978 linear systems have been solved. The final mesh has 71 629 nodes.

The presented setting is quite standard and our result seems consistent compared to the ones in the literature. Note that a quasi-static evolution does not need to be continuous. Indeed, at the last time step before the crack completes it jumps instantaneously. Due to the large change in the phase-field the algorithm needs many iterations to complete, as it can be noticed from the peaks at time $t = 0.97$ in Figure 1.

One might realize that the phase-field spreads where it exits the domain. The reason for this phenomenon is that the irreversibility condition is imposed by using the intermediate step $v_{i,j-1}$ as an upper bound, which does not refer to a real physical state. However, the use of another upper bound, such as v_{i-1}, would break the consistency with the theory and, in particular, the energy-dissipation equality (2.33) would not be guaranteed anymore. This effect can be observed in all the examples.

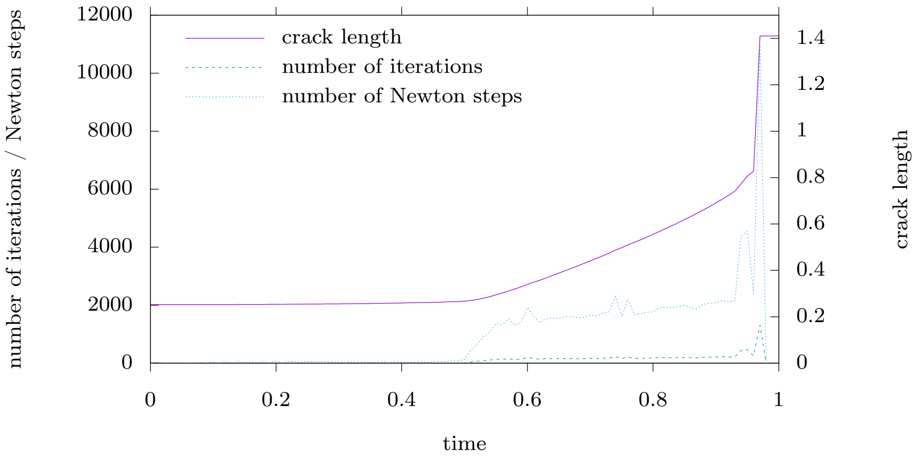
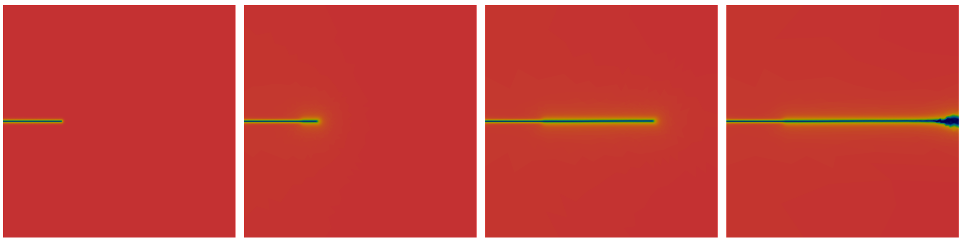
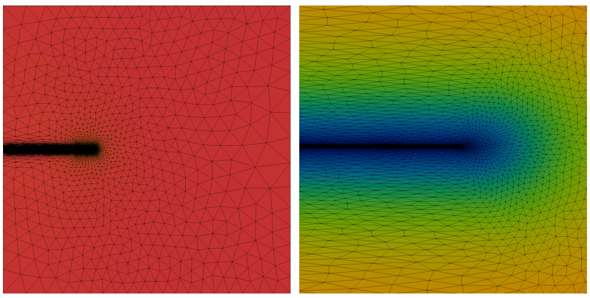


FIGURE 1. Numerical output data of the first example with initial crack Γ_1 and boundary condition w_1.



(A) $t = 0$ (B) $t = 0.6$ (C) $t = 0.96$ (D) $t = 0.97$

FIGURE 2. Phase-field of first example at various times.



(A) Complete mesh (B) Mesh at the crack tip

FIGURE 3. Mesh at time $t = 0.6$ used for the first example.

Second Example. In our second example we break the symmetry by turning the initial crack into a set not orthogonal to the left boundary and shifted out of the middle point. Namely, we use the initial phase-field v_0 from (6.2) with Γ replaced by

$$\Gamma_2 := \left\{ (x, y) \in \mathbb{R}^2 \mid x \in \left[0, \tfrac{1}{4}\right], y = \tfrac{1}{2}x + \tfrac{1}{4} \right\}.$$

The boundary condition stays the same in the sense that it is t below the crack and $-t$ above it. Thus, we have

$$w_2(t) = \begin{cases} t & \text{on } \{0\} \times \left[0, \frac{1}{4}\right), \\ -t & \text{on } \{0\} \times \left(\frac{1}{4}, 1\right]. \end{cases}$$

With this settings, the specimen completely breaks at time $t = 1.05$, where the crack crosses the bottom border as it can be seen in Figure 5. Figure 4 shows the corresponding numerical measurements. In this case we can observe a number of peaks appearing before the final jump of the fracture. The phenomena might be explained, as in the previous example, by the fact that the phase-field is actually experiencing jumps of different size at each time step, as it can be noticed by the oscillations in the graph of the crack length.

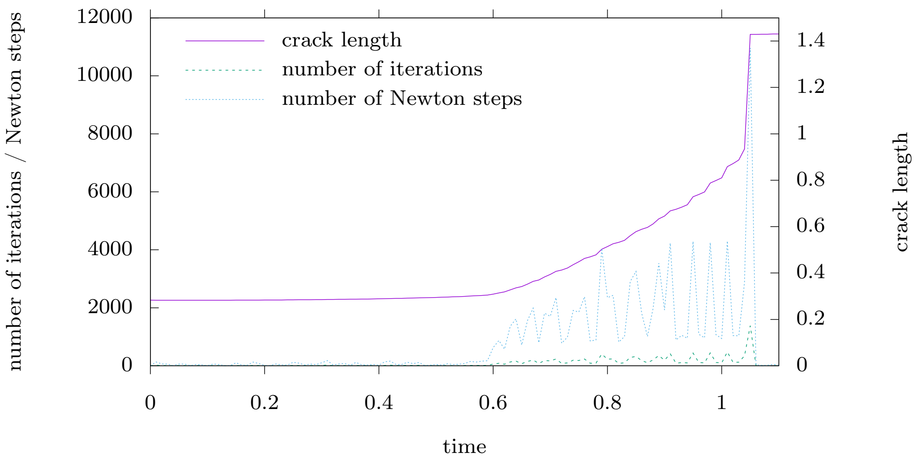
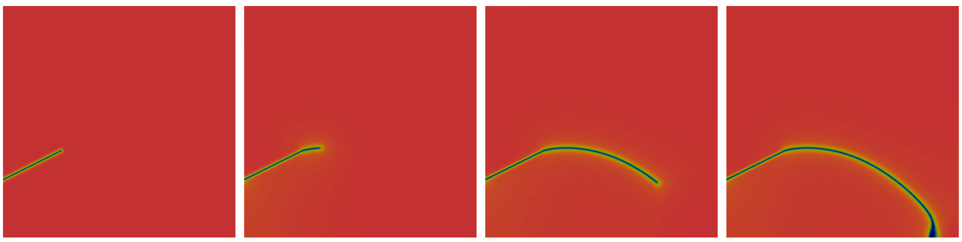


FIGURE 4. Numerical output data of the second example with initial crack Γ_2 and boundary condition w_2.

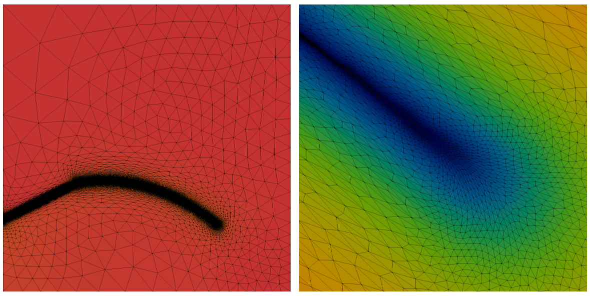


(A) $t = 0$ (B) $t = 0.7$ (C) $t = 1.04$ (D) $t = 1.05$

FIGURE 5. Phase-field of second example at various times.

For this example we do not know any comparable test case in the literature we could refer to. Since the crack tip is not in the middle of the domain and the boundary condition is not balanced, the shown crack path seems reasonable.

The total number of alternating minimizations until the crack completes is 10 222, and 96 528 Newton steps were performed. Thus, the simulation solved 106 750 linear systems. The mesh at the time of crack completion is composed of 108 020 nodes.



(A) Complete mesh (B) Mesh at the crack tip

FIGURE 6. Mesh at time $t = 1.04$ used for the first example.

Third Example. For the last example we again use Γ_1 as a pre-crack in (6.2) and w_1 as the driving boundary condition. Unlike the first example, we cut out a hole of the domain with center $(0.8, 0.25)$ and radius 0.1. This setting makes the crack deviate from the middle line into the hole, as it can be seen in Figure 8. The last part of the crack from the border of the hole to the right border of the domain appears instantaneously at time 1.11.

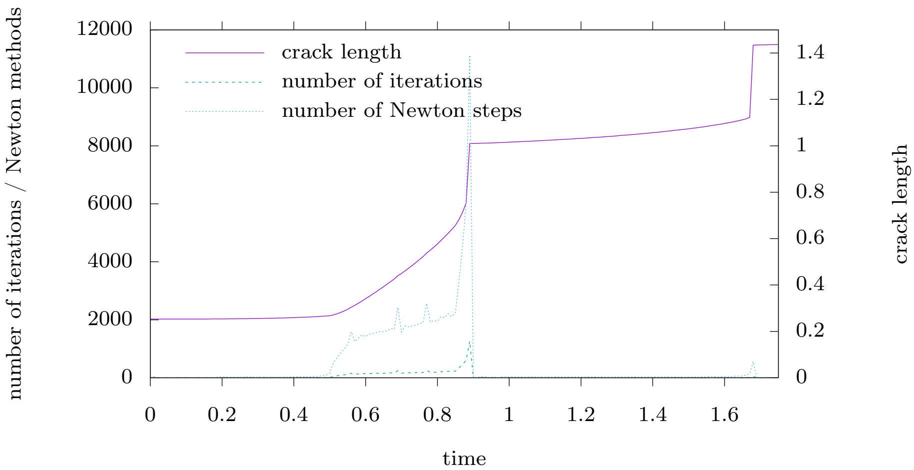
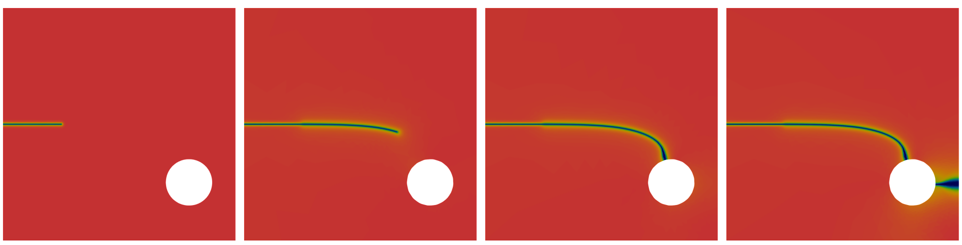


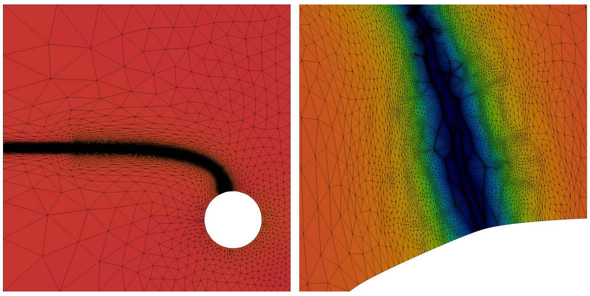
FIGURE 7. Numerical output data of the third example with initial crack Γ_1 and boundary condition w_1 and the hole $B_{0.1}(0.8, 0.25)$ cut out of the domain.



(A) $t = 0$ (B) $t = 0.88$ (C) $t = 0.89$ (D) $t = 1.68$

FIGURE 8. Phase-field of third example at various times.

One can find a similar example in [6, 18]. There the hole is equidistant from the bottom and the right boundary, and the crack completes downwards. However, from an energetically point of view there is no clearly preferred direction due to the same resulting crack length. In order to have a predictable situation we moved the hole such that it is closer to the right border than to the lower one. For this reason the obtained crack path is the expected one.



(A) Complete mesh (B) Mesh entering the hole

FIGURE 9. Mesh at time $t = 0.89$ used for the first example.

As in the two previous examples, in Figure 7 we notice the presence of two peaks in the number of iterations, coupled with a two stage behavior of the corresponding crack length. This phenomenon is related to the fact that the phase-field experiences this time two jumps: one when the fracture enters the hole and one when it exits the domain. In between, the phase-field is not significantly changing.

Having a look at the meshes in Figure 9, the diffusion of the phase-field, where it leaves the hole, goes along with an irregular mesh. This observation indicates large changes in the curvature of the phase-field where these diffusions appear.

The computation required in total 8 803 alternations with 87 602 Newton steps until the crack path was completed. It took therefore 96 405 linear systems to be solved. The final mesh consists of 55 050 nodes.

APPENDIX A

Lemma A.1. *Let $h > 0$. For every $M > 0$, there exists a positive constant $C = C(M)$ such that for every $u, v \in \mathcal{F}_h$ with $\|\nabla u\|_\infty, \|v\|_\infty \le M$, for every $\varphi \in \mathcal{A}_h$, and every $\psi \in \mathcal{F}_h$ we have*

$$\frac{1}{C}\|\varphi\|_{H^1} \le \|\varphi\|_{h,v} \le C\|\varphi\|_{H^1} \qquad \frac{1}{C}\|\psi\|_{H^1} \le \|\psi\|_{h,u} \le C\|\psi\|_{H^1}\,.$$

Proof. The proof follows directly from the definition (2.27)-(2.28) of the two norms. □

Lemma A.2. *Let $h > 0$. Let $w_j, v_j \in \mathcal{F}_h$ and let $u_j \in \mathcal{A}_h(w_j)$. Assume that there exist $w, v \in \mathcal{F}_h$ and $u \in \mathcal{A}_h(w)$ such that $w_j \to w$, $v_j \to v$, and $u_j \to u$ in $\mathcal{F}_h$ as $j \to +\infty$. Then*

$$|\partial_u \mathcal{J}_h|(u, v) = \lim_j |\partial_u \mathcal{J}_h|(u_j, v_j) \qquad |\partial_v \mathcal{J}_h|(u, v) = \lim_j |\partial_v \mathcal{J}_h|(u_j, v_j)\,. \tag{A.1}$$

Proof. Fix $\varphi \in \mathcal{A}_h(0)$ with $\|\varphi\|_{h,v} \le 1$. Since $\|\varphi\|_{h,v_j} \to \|\varphi\|_{h,v}$ as $j \to +\infty$, we consider the following function:

$$\varphi_j := \begin{cases} \varphi & \text{if } \|\varphi\|_{h,v} < 1\,, \\ \dfrac{\varphi}{\|\varphi\|_{h,v_j}} & \text{if } \|\varphi\|_{h,v} = 1\,. \end{cases}$$

Clearly $\varphi_j \to \varphi$ in $\mathcal{F}_h$ as $j \to +\infty$. Moreover, by definition (2.29) of the slope w.r.t. the displacement u, we have that

$$\liminf_j |\partial_u \mathcal{J}_h|(u_j, v_j) \geq -\lim_j \partial_u \mathcal{J}_h(u_j, v_j)[\varphi_j] = -\partial_u \mathcal{J}_h(u, v)[\varphi] .$$

Hence, we deduce that $|\partial_u \mathcal{J}_h|(u, v) \leq \liminf_j |\partial_u \mathcal{J}_h|(u_j, v_j)$.

For the opposite inequality, let $\varphi_j \in \mathcal{A}_h(0)$ be such that

$$|\partial_u \mathcal{J}_h|(u_j, v_j) = -\partial_u \mathcal{J}_h(u_j, v_j)[\varphi_j] .$$

Up to a subsequence, we may assume that $\varphi_j \to \varphi \in \mathcal{A}_h(0)$. Hence, $\|\varphi_j\|_{h,v_j} \to \|\varphi\|_{h,v} \leq 1$ and

$$\limsup_j |\partial_u \mathcal{J}_h|(u_j, v_j) = -\lim_j \partial_u \mathcal{J}_h(u_j, v_j)[\varphi_j] = -\partial_u \mathcal{J}_h(u, v)[\varphi] \leq |\partial_u \mathcal{J}_h|(u, v) ,$$

which conclude the proof of the first equality in (A.1). In a similar way we can prove the second. □

Proof of Proposition 3.1. It is enough to prove the statement for $j = 1$. In order to show the existence of a minimizer for (3.2) we want to apply the direct method of the calculus of variations. Since $\mathcal{J}_h(u_1, \cdot)$ is continuous with respect to the convergence in $\mathcal{F}_h$, we only need to show that a minimizing sequence $z_k \in \mathcal{F}_h$ for (3.2) admits a limit, at least up to a subsequence. Since

$$\sup_{k\in\mathbb{N}} \|\nabla z_k\|_2^2 \leq \sup_{k\in\mathbb{N}} 2\mathcal{J}_h(u_1, z_k) < +\infty ,$$

by construction of the function space $\mathcal{F}_h$ (2.15) and of the triangulation $\mathcal{T}_h$, we easily deduce that

$$\sup_k \|\nabla z_k\|_\infty < +\infty , \tag{A.2}$$

so that the functions z_k are uniformly Lipschitz in Ω. Moreover, since by hypothesis $v_0 \geq 0$, it is not restrictive to assume that there exists at least one vertex x_l, $l \in \{1, \dots, N_h\}$, such that $z_k(x_l) \geq 0$. Indeed, if $z_k < 0$ in Ω, it is readily seen that $\mathcal{J}_h(u_1, 0) \leq \mathcal{J}_h(u_1, z_k)$. Therefore, thanks to (A.2) and to the inequality $z_k \leq v_0$ in Ω, we also deduce that the sequence z_k is uniformly bounded in $W^{1,\infty}(\Omega)$. Thus, z_k converges, up to a subsequence, to some v_1 in $\mathcal{F}_h$, and this concludes the proof of existence. The uniqueness of solution follows by the strict convexity of the functional $\mathcal{J}_h(u_1, \cdot)$.

We now prove the second part of the statement, i.e., that $0 \leq v_1 \leq 1$. For the sake of contradiction, let us first assume that $v_1 \not\geq 0$. Using the notation described in Section 2, let $x_l \in \Delta_h$ be such that $v_1(x_l) \leq v_1(x_m)$ for every $m = 1, \dots, N_h$. In particular, we have $v_1(x_l) < 0$. Let $\xi_l \geq 0$ be the l-th element of the basis of $\mathcal{F}_h$ defined by (2.16). Being $v_0 \geq 0$ in Ω and $v_1(x_l) < 0$, for every $\varepsilon \in \mathbb{R}$ with $|\varepsilon|$ small enough we have that $v_1 + \varepsilon\xi_l \leq v_0$ in Ω, and, by the minimality of v_1, $\mathcal{J}_h(u_1, v_1) \leq \mathcal{J}_h(u_1, v_1 + \varepsilon\xi_l)$. By the quadratic structure of $\mathcal{J}_h(u_1, \cdot)$, from the previous inequality and the arbitrariness of ε we deduce that

$$\int_\Omega \mathrm{P}_h(v_1\xi_l)|\nabla u_1|^2 \,\mathrm{d}x + \int_\Omega \nabla v_1 \cdot \nabla \xi_l \mathrm{d}x - \int_\Omega \mathrm{P}_h\big((1 - v_1)\xi_l\big) \,\mathrm{d}x = 0 . \tag{A.3}$$

Since $v_1(x_l) < 0$ and (2.16) holds, we have that

$$\mathrm{P}_h(v_1\xi_l) \leq 0 \quad \text{and} \quad \mathrm{P}_h\big((1 - v_1)\xi_l\big) \geq 0 \qquad \text{in } \Omega. \tag{A.4}$$

Hence, from (A.3) and (A.4) we get

$$\int_\Omega \nabla v_1 \cdot \nabla \xi_l \,\mathrm{d}x = -\int_\Omega \mathrm{P}_h(v_1\xi_l)|\nabla u_1|^2 \,\mathrm{d}x + \int_\Omega \mathrm{P}_h\big((1 - v_1)\xi_l\big) \,\mathrm{d}x \geq 0 . \tag{A.5}$$

On the other hand, we can write v_1 as a linear combination of the elements of the basis $\{\xi_m\}_{m=1}^{N_h}$ of $\mathcal{F}_h$, namely, $v_1 = \sum_{m=1}^{N_h} v_1(x_m)\xi_m$. Then, by direct computation,

$$
\begin{aligned}
\int_\Omega \nabla v_1 \cdot \nabla \xi_l \,\mathrm{d}x &= \sum_{m=1}^{N_h} v_1(x_m) \int_\Omega \nabla \xi_m \cdot \nabla \xi_l \,\mathrm{d}x \\
&= v_1(x_l) \sum_{m=1}^{N_h} \int_\Omega \nabla \xi_m \cdot \nabla \xi_l \,\mathrm{d}x + \sum_{m=1}^{N_h} \big(v_1(x_m) - v_1(x_l)\big) \int_\Omega \nabla \xi_m \cdot \nabla \xi_l \,\mathrm{d}x \\
&= \sum_{m=1}^{N_h} \big(v_1(x_m) - v_1(x_l)\big) \int_\Omega \nabla \xi_m \cdot \nabla \xi_l \,\mathrm{d}x \leq 0 \,,
\end{aligned} \tag{A.6}
$$

where, in the last equality, we have used (2.17), the particular choice of the vertex x_l, and the fact that

$$\sum_{m=1}^{N_h} \int_\Omega \nabla \xi_m \cdot \nabla \xi_l \,\mathrm{d}x = 0 \,.$$

Therefore, combining (A.5) and (A.6) we get a contradiction, and thus $v_1 \geq 0$.

In order to show that $v_1 \leq 1$, we can argue again by contradiction, assuming that there exists a vertex x_l, $l \in \{1, \dots, N_h\}$ such that $v_1(x_l) \geq v_1(x_m)$ for every $m = 1, \dots, N_h$ and $v_1(x_l) > 1$. As before, being ξ_l the l-th element of the basis of $\mathcal{F}_h$, for $\varepsilon \in \mathbb{R}$ with $|\varepsilon|$ small enough there holds $v_1 + \varepsilon\xi_l \geq v_0$, such that we can repeat (A.3)–(A.6) with opposite inequality signs, concluding that $0 \leq v_1 \leq 1$ in Ω. This concludes the proof of the proposition. □

Proof of Lemma 5.1. Let $M > 0$, $g \in C^2(\mathbb{R})$, the mesh parameter $h > 0$, and the triangulation $\mathcal{T}_h$ be fixed, and let us consider a function $v \in \mathcal{F}_h$ such that $\|v\|_\infty \leq M$.

Given an element K of the triangulation $\mathcal{T}_h$, we first prove (5.1) on K. Without loss of generality, we may assume that the origin is a vertex of K and

$$K = \big\{x = (x_1, x_2) \in \Omega : x_1 \in [0, h],\, x_2 \in [0, x_1]\big\} \,.$$

The general case follows by an affine transformation.

Recalling that if a function $v \in \mathcal{F}_h$, then it is affine on every element of $\mathcal{T}_h$, for every $x \in K$ we have

$$v(x) = v(0) + \nabla v|_K \cdot x \,.$$

Hence, by Taylor expansion, there exists some $\xi \in K$ such that

$$g(v(x)) = g(v(0)) + g'(v(0))\big(v(x) - v(0)\big) + \frac{1}{2} g''(v(\xi))\big(v(x) - v(0)\big)^2 \,. \tag{A.7}$$

In particular, the last term in (A.7) can be simply estimated by

$$|g''(v(\xi))|\big(v(x) - v(0)\big)^2 \leq |g''(v(\xi))|\big|\nabla v|_K\big|^2 |x|^2 \,. \tag{A.8}$$

Recalling the definition of P_h (2.19) and using the expansion (A.7) we can write

$$
\begin{aligned}
\mathrm{P}_h(g \circ v)(x) &= g(v(0)) + \frac{1}{h}\big(g(v(h,0)) - g(v(0))\big)x_1 \\
&\quad + \frac{1}{h}\big(g(v(0,h)) - g(v(0))\big)x_2 \\
&= g(v(0)) + \frac{1}{h} g'(v(0))\Big((v(h,0) - v(0))x_1 + (v(0,h) - v(0))x_2\Big) \\
&\quad + \frac{1}{2h} g''(v(\xi_1))\big|v(h,0) - v(0)\big|^2 x_1 \\
&\quad + \frac{1}{2h} g''(v(\xi_2))\big|v(0,h) - v(0)\big|^2 x_2
\end{aligned} \tag{A.9}
$$

for some $\xi_1, \xi_2 \in K$. Hence, in view of (A.7)-(A.9), for $x \in K$ there holds

$$\big|g\big(v(x)\big) - \mathrm{P}_h(g \circ v)(x)\big| \le Ch^2 \big|\nabla v|_K\big|^2$$

for some positive constant $C = C(g, M)$ depending only on $g \in C^2(\mathbb{R})$ and on M. Integrating the previous inequality, we end up with (5.1), and the proof is concluded. □

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