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MULTI-DIMENSIONAL SUM-UP ROUNDING FOR ELLIPTIC CONTROL SYSTEMS*

PAUL MANNS† AND CHRISTIAN KIRCHES†

Abstract. Partial outer convexification has been used to derive relaxations of Mixed-Integer Optimal Control Problems (MIOCPs) that are constrained by time-dependent differential equations. The family of Sum-Up Rounding (SUR) algorithms provides a means to approximate feasible points of these relaxations, i.e. $[0,1]$-valued control trajectories, with $\{0,1\}$-valued points. The approximants computed by a SUR algorithm converge in a weak sense when the coarseness of the rounding grid of the SUR algorithm is driven to zero, which in turn induces norm convergence of the corresponding sequence of state vectors. We show that this approximation property can be transferred to MIOCPs with integer control variables distributed in more than one dimension when carrying out an appropriate grid refinement strategy. We deduce a norm convergence result for the state vector of elliptic PDE systems and provide computational results illustrating the applicability of the theoretical framework.

Key words. Mixed-Integer PDE-Constrained Optimization, Approximation Theory

AMS subject classifications. 49M20, 90C59, 65L50, 49J20, 90C11

1. Introduction.

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. We consider *partial outer convexification* reformulations [17, 19] of Mixed-Integer Optimal Control Problems (MIOCPs) constrained by elliptic state equations. In more detail, we consider optimization problems of the form

$$\begin{aligned} &\inf_{y,\omega}\ J(y) \\ \text{(BC)}\qquad &\text{s.t. } Ay = \sum_{i=1}^{M} \omega_i f_i, \quad 1 = \sum_{i=1}^{M} \omega_i \text{ and } \omega \in \{0,1\}^M \text{ a.e. on } \Omega, \end{aligned}$$

where $Ay = \sum_{i=1}^{M} \omega_i f_i$ is an elliptic state equation. The distributed binary-valued variable vector $\omega : \Omega \to \{0,1\}^M$ acts as a *one-hot* or *Special Ordered Set of Type 1 (SOS1)* encoding of the (spatially distributed) activation of the available discrete functions (right-hand sides) $f_1, \ldots, f_M$. This means that for a.a. $s \in \Omega$, we have $\omega_i(s) = 1$ for exactly one $i \in \{1, \ldots, M\}$ and $\omega_j(s) = 0$ for $j \neq i$. The f_i may take an additional continuous control as an input variable, but we omit this as it does not affect the theory we present and we refer to the articles [19, 13] for further information. Relaxing the SOS1 property to convex combinations increases the set of feasible activations. The relaxed problem reads

$$\begin{aligned} &\min_{x,\alpha}\ J(x) \\ \text{(RC)}\qquad &\text{s.t. } Ax = \sum_{i=1}^{M} \alpha_i f_i, \quad 1 = \sum_{i=1}^{M} \alpha_i \text{ and } \alpha \in [0,1]^M \text{ a.e. on } \Omega, \end{aligned}$$

where $\alpha : \Omega \to [0,1]^M$ is a continuous relaxation of ω. We assume that the relaxed problem (RC) is well-posed and has a solution. From a function space point of view, SUR

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is an abstract algorithm to compute approximations of solutions of (RC) that are feasible for (BC) and does so by rounding on a suitable grid. The feasible point may be obtained in linear time w.r.t. the number of grid cells. In general, the solution of (RC) cannot be obtained exactly and only finite dimensional approximations (RC_h) can be solved. Thus, we aim to find a minimizing sequence for (BC) by executing Algorithm 1.1, which computes solutions of improved finite-dimensional approximations ($\text{RC}_h^{(n)}$) of (RC) and uses them as inputs to the SUR algorithm, which is executed on a sequence of refined rounding grids. Assumption 3.1 states the precise functional analytic setting for our considerations on (BC), (RC), the abstract SUR algorithm and Algorithm 1.1. The aim of the methodology is to approximate the state vector x by means of the state approximants $y^{(n)}$, which arise from solving the Boundary Value Problem (BVP) for the binary-valued controls $\omega^{(n)}$. Our analysis will show that this approximation behavior can be obtained even if no infimal sequence of (BC) converges in norm, i.e. (BC) does not admit a minimizing binary control.

Algorithm 1.1 MIOCP Approximation

Input: J continuous in x
Input: Initial rounding grid $S^{(0)}$
  for $n = 1, \ldots$ **do**
    $S^{(n)} \leftarrow$ refine $S^{(n-1)}$
    $x^{(n)}, \alpha^{(n)} \leftarrow$ solve ($\text{RC}_h^{(n)}$)
    $\omega^{(n)} \leftarrow \text{SUR}(\alpha^{(n)}, S^{(n)})$
    $y^{(n)} \leftarrow A^{-1} \sum_{i=1}^{M} \omega_i^{(n)} f_i$
  end for

SUR is well-understood for time-dependent problems, i.e. $\alpha, \omega \in L^\infty((0,T), \mathbb{R}^M)$. We refer to the results by Sager et al. [17, 21, 18, 19, 6, 11] for ODE and DAE constraints, and Hante, Sager [9, 8] and the authors [13] for semilinear evolution equation constraints. In these settings, the SUR algorithm and its variants are applied to problems where the discrete variables are distributed in one dimension. Hahn and Sager [7] have transferred some ideas of *partial outer convexification* and SUR to elliptic PDEs and while revising this article, Yu and Anitescu [27] published a multi-dimensional variant of SUR for application to integral operators from optimum experimental design problems. We do not, know, however, of any rigorous analysis of a multi-dimensional variant of SUR from the function space point of view and the consequences on the approximation relationship between (BC) and (RC). Our work closes this gap.

Contribution. We generalize the SUR algorithm, which computes roundings based on intervals discretizing $[0, T]$ for $T > 0$, to dissections of multi-dimensional domains (see Definition 4.1). We show that the approximation properties from the one-dimensional setting translate to the multi-dimensional one. As the SUR algorithm is formulated from a function space point of view, we deduce that a sequence of roundings $(\omega^{(n)})_n$ computed on suitably refined grids approximates relaxed controls α in L^∞ equipped with the weak-* topology, an approximation property we cannot obtain for finite-dimensional control spaces where the weak topology coincides with the norm topology. This leverages the applicability of compactness properties from PDE theory to ensure the existence of a sequence of state vectors $y(\omega^{(n)})$ feasible for (BC) that converges to $x(\alpha)$ that minimizes (RC) in the norm topology. The continuity of J with respect to the state vector yields a minimizing sequence for (BC) even if no minimizing control function exists. This yields convergence of Algorithm 1.1 under an additional regularity assumption on the sequence of rounding grids, which has similarities to the assumptions in [27]. We have published a preliminary step in these results in the short proceedings article [14].

Furthermore, we provide computational experiments that demonstrate the theoretical results. We demonstrate both the behavior and the practical limits of Algorithm 1.1. We also test the method out of its intended scope in a staged control reconstruction problem.

Structure of the remainder. In Section 2, we introduce the one-dimensional SUR algorithm and summarize its properties. Section 3 outlines the PDE setting of this work and the techniques that establish convergence of the sequence of state vectors in the norm topology once weak-* convergence of the rounding approximation has been established. In Section 4, we introduce the multi-dimensional SUR algorithm and prove weak-* convergence for SUR when applied to the refined grids. Section 4 closes with a convergence proof of Algorithm 1.1 under suitable assumptions. We summarize the approximation relationship between (BC) and (RC) in Section 5. Section 6 illustrates the theoretical results computationally. Finally, we offer a conclusion in Section 7.

Notation. We denote the usual Lebesgue measure by the symbol λ. In cases of possible ambiguity, it is denoted by $\lambda_{\mathbb{R}^d}$. The Borel σ-algebra of a set $A \subset \mathbb{R}^d$ is denoted by $\mathcal{B}(A)$. The characteristic function of a set A is denoted by χ_A. The topological dual of a Banach space E is denoted by E^*. Convergence in the norm topology is indicated by $\to$, convergence in the weak topology by $\rightharpoonup$, and convergence in the weak-* topology by $\rightharpoonup^*$. For continuous (compact) embeddings from one Banach space into another, we use the symbol $\hookrightarrow$ ($\hookrightarrow^c$). $H_0^k(\Omega)$ is the space of all square-integrable functions over Ω that vanish on the boundary and whose k-th derivative is square-integrable, see [16]. We introduce the notation below for the feasible sets of (BC) and (RC) to simplify later statements:

$$\mathcal{F}_{BC} \coloneqq \left\{(y,\omega) \in V \times L^\infty(\Omega,\mathbb{R}^M) : (y,\omega) \text{ feasible for (BC)}\right\},$$
$$\mathcal{F}_{RC} \coloneqq \left\{(x,\alpha) \in V \times L^\infty(\Omega,\mathbb{R}^M) : (x,\alpha) \text{ feasible for (RC)}\right\},$$

where V denotes our state space, which will be detailed in Assumption 3.1.

2. What is Sum-Up Rounding (SUR)?

We introduce names for the functions α and ω that appear in the constraints of (RC) and (BC).

DEFINITION 2.1 (Binary and relaxed control). *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. A measurable function $\omega : \Omega \to \{0,1\}^M$ that satisfies $\sum_{i=1}^M \omega_i = 1$ a.e. in Ω is called* **binary control**. *A measurable function $\alpha : \Omega \to [0,1]^M$ that satisfies $\sum_{i=1}^M \alpha_i = 1$ a.e. in Ω is called* **relaxed control**.

Next, we state the SUR algorithm.

DEFINITION 2.2 (Sum-Up Rounding Algorithm, [17, 21, 19]). *Let $0 = t_0 < \ldots < t_N = T$ be a discretization of the interval $\bar{\Omega} = [0,T]$ with maximum discretization width $\Delta t \coloneqq \max_{i\in\{0,N-1\}} t_{i+1} - t_i$. For a relaxed control α, we define a binary-valued piecewise-constant function $\omega(\alpha) : [0,T] \to \{0,1\}^M$ iteratively for $i = 0,\ldots,N-1$ by*

$$\omega(\alpha)_j(t)|_{[t_i,t_{i+1})} \coloneqq \begin{cases} 1 : j = \underset{k\in\{1,\ldots,M\}}{\arg\max} \displaystyle\int_0^{t_{i+1}} \alpha_k(t)\,\mathrm{d}t - \int_0^{t_i} \omega(\alpha)_k(t)\,\mathrm{d}t, \\ 0 : \qquad\qquad\qquad\qquad\qquad\qquad \textit{otherwise} \end{cases}$$

and $\omega(\alpha)_j(t_N) \coloneqq \omega(\alpha)_j(t_{N-1})$ for $j \in \{1,\ldots,M\}$. If a tie arises with respect to the maximizing index k, one of the maximizing indices is chosen arbitrarily. In our implementation, we pick the smallest applicable index.

The rationale behind SUR can be described as follows. The algorithm proceeds forward with the index $i = 0,\ldots,N-1$ that identifies the current time interval on which the rounding is performed. The index $j \in \{1,\ldots,M\}$ identifies the discrete value under consideration. First, the entry of ω corresponding to the highest weighted mean $\int_0^{t_1} \alpha$ is set to one on the interval $[t_0,t_1)$. All other entries of ω are set to zero on that interval. The algorithm proceeds iteratively: for the i-th time interval index, it determines the integrated difference between α and ω up to time point t_i, the so-called *integrated control deviation*, which is denoted by $\Phi(t_i)$ in the remainder. To this quantity, it adds the weighted mean of the relaxed control over time interval i: $\int_{t_i}^{t_{i+1}} \alpha$. This sum is called *sum-up rounding gap*, and is denoted γ in the literature [12]. Then, the entry of ω to be set to one on interval $[t_i, t_{i+1}]$ is determined by choosing the one with maximum sum-up rounding gap. Again, all other entries of ω are set to zero on that interval. Now, the integrated control deviation until t_{i+1} can be computed and the algorithm loops with $i \leftarrow i + 1$. Clearly, SUR has a runtime complexity of $\mathcal{O}(N)$.

We define the notion of *vanishing integrality gap* to describe the type of approximation of feasible points of (RC) by feasible points of (BC) constructed by rounding as follows.

Definition 2.3 (Vanishing integrality gap). *Let $(\phi^{(n)})_n \subset L^\infty((0,T), \mathbb{R}^M)$ be a bounded sequence such that the sequence of the antiderivatives $\Phi^{(n)}(t) := \int_0^t \phi^{(n)}$ satisfies the convergence property*

$$\Phi^{(n)} \to 0 \text{ in } L^\infty((0,T), \mathbb{R}^M) \text{ (and in } C([0,T], \mathbb{R}^M)).$$

Then, we call $(\phi^{(n)})_n$ a sequence of **vanishing integrality gap**.

The following result is due to Sager and shows consistency of SUR, i.e. that the so-called control deviation $\phi^{(n)} := \alpha - \omega^{(n)}$ is of vanishing integrality gap if the grid coarseness tends to zero.

Proposition 2.4 (Vanishing Integrality Gap for SUR, [19]). *Let $\Omega = (0,T)$. There exists $C > 0$ such that for all relaxed controls α and all $\omega^{(n)}$, computed with SUR from α at a maximum discretization width $\Delta^{(n)}$ for $n \in \mathbb{N}$, we have*

1. *$\omega^{(n)}|_{(0,T)}$ is a binary control,*
2. *the sequence of control deviations $\phi^{(n)} := \alpha - \omega^{(n)}$ fulfills*

$$\sup_{t \in [0,T]} \left\| \int_0^t \phi^{(n)}(s)\,\mathrm{d}s \right\|_\infty \le C\Delta^{(n)}$$

In particular, $(\phi^{(n)})_n$ is of vanishing integrality gap if $\Delta^{(n)} \to 0$. ∎

From Lemma 2.1 in [13], we can deduce the following.

Proposition 2.5. *Let α be a relaxed control on $\Omega = (0,T)$. Let $\omega^{(n)}$ be computed with SUR from α and let $\Delta^{(n)} \to 0$. Then, $\phi^{(n)} \rightharpoonup^* 0$ in $L^\infty((0,T), \mathbb{R}^M)$.* ∎

This result is generalized in Section 4 to multi-dimensional domains and is key to obtaining norm convergence of the corresponding sequence of solutions $(y^{(n)})_n$ of the state equation in Section 3.

3. State vector convergence for elliptic systems. This section establishes convergence of (weak) solutions of the state equation of (BC) to the (weak) solution of the state equation of (RC) in the norm topology if the binary controls approach the relaxed ones in $L^\infty(\Omega)$ endowed with the weak-* topology (i.e. "in a weak sense"). This is shown in Theorem 3.2. We make the following assumption on our PDE setting.

Assumption 3.1. *V is a Hilbert space such that the so-called* Gelfand triple *$V \hookrightarrow^c L^2(\Omega) \cong L^2(\Omega)^* \hookrightarrow^c V^*$ holds with continuous, dense and compact embeddings and $A : V \to V^*$ is an isomorphism with bounded inverse, i.e. there exists $C > 0$ such that the estimate $\|y\|_V \le C\|f\|_{V^*}$ holds when y solves $Ay = f$ weakly for a given $f \in V^*$.*

This is a common setting for linear elliptic PDEs and operators, e.g. with $V = H_0^k(\Omega)$ for $k \geq 1$. For the most famous representative for this type of operator, the Dirichlet Laplacian, the derived results are illustrated numerically in Section 6.

THEOREM 3.2. *Let Assumption 3.1 hold and let $f_i \in L^2(\Omega)$ for $i \in \{1, \dots, M\}$. Let α be a relaxed control and let $(x, \alpha) \in \mathcal{F}_{RC}$. Let $(\omega^{(n)})_n \subset L^\infty(\Omega, \mathbb{R}^M)$ be a sequence of binary controls and $(y^{(n)})_n \subset V$ be such that $(y^{(n)}, \omega^{(n)})_n \subset \mathcal{F}_{BC}$. Let $(\phi^{(n)})_n$ with $\phi^{(n)} := \alpha - \omega^{(n)}$ satisfy*

$$\phi_i^{(n)} \rightharpoonup^* 0 \text{ in } L^\infty(\Omega).$$

for all $i \in \{1, \dots, M\}$. Then,

$$y^{(n)} \to x \text{ in } V.$$

Proof. We observe that $(\phi^{(n)})_n \subset L^\infty(\Omega, \mathbb{R}^M)$ and for all $v \in L^2(\Omega)$, we have $v f_i \in L^1(\Omega)$. The duality $(L^1(\Omega, \mathbb{R}^M))^* \cong L^\infty(\Omega, \mathbb{R}^M)$, see e.g. [5, Thm IV.1], implies $\int_\Omega \sum_{i=1}^M \phi_i^{(n)} \psi_i \,\mathrm{d}\lambda \to 0$ for all test functions $\psi \in L^1(\Omega, \mathbb{R}^M)$. Let $g \in L^2(\Omega)$. Then, we have $f_i g \in L^1(\Omega, \mathbb{R}^M)$ and we may choose $\psi_i := f_i g$, which implies $\int_\Omega \sum_{i=1}^M \phi_i^{(n)} f_i g \,\mathrm{d}\lambda \to 0$. As $\sum_{i=1}^M \phi_i^{(n)} f_i \in L^2(\Omega)$ by Hölder's inequality and $\|\phi^{(n)}\|_{L^\infty} \leq 1$, we obtain the weak convergence $\sum_{i=1}^M \phi_i^{(n)} f_i \rightharpoonup 0$ in $L^2(\Omega)$.

The compact embedding $L^2(\Omega) \hookrightarrow^c V^*$ in Assumption 3.1 implies $\sum_{i=1}^M \phi^{(n)} f_i \to 0$ in V^*. Employing the norm estimate in Assumption 3.1 we arrive at

$$\left\| x - y^{(n)} \right\|_V \leq C \left\| \sum_{i=1}^M \phi^{(n)} f_i \right\|_{V^*} \to 0,$$

which proves the claim. ∎

4. SUR on multi-dimensional domains. This section rephrases the SUR algorithm of Definition 2.2 and the convergence property of Proposition 2.5 for the multi-dimensional setting. Previous proofs of the approximation properties of the SUR algorithm have relied on the forward progression in time. We show that a spatial coherence property of the grid refinements can defined to transfer the approximation properties to multi-dimensional domains. Subsection 4.1 introduces the multi-dimensional SUR algorithm and an approximation property of relaxed controls, which are constant per cell on a fixed grid. Subsection 4.2 gives a sufficient condition on grid refinement strategies and the proofs that establish weak-* convergence of the sequence of binary controls that are computed on the refined rounding grids to a relaxed control. Subsection 4.3 uses these results to prove convergence of Algorithm 1.1.

4.1. Multi-dimensional SUR and vanishing integrality gap. We postulate the existence of a finite partition of Ω and compute a binary control ω from a relaxed control α using SUR.

DEFINITION 4.1 (SUR on multi-dimensional domains). *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. Let $\{S_1, \dots, S_N\} \subset \mathcal{B}(\Omega)$ be a finite partition of Ω indexed by $i \in \{1, \dots, N\}$. Let α be a relaxed control. For indices $j \in \{1, \dots, M\}$ of discrete controls, we define recursively*

$$\tilde{\omega}_{i,j} := \begin{cases} 1 : j = \underset{k \in \{1,\dots,M\}}{\arg\max} \displaystyle\int_{S_i} \alpha_k \,\mathrm{d}\lambda + \int_{\bigcup_{\ell=1}^{i-1} S_\ell} \alpha_k - \omega(\alpha)_k \,\mathrm{d}\lambda, \\ 0 : \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \textit{otherwise}, \end{cases}$$

$$\omega(\alpha)|_{S_i} :\equiv \tilde{\omega}_i.$$

If a tie arises with respect to the maximizing index k, one of the maximizing indices is chosen arbitrarily. In our implementation, we pick the smallest applicable index.

First, we transfer Proposition 2.4 to the multi-dimensional setting. By Definition 4.1, we have

$$\omega = \sum_{i=1}^{N} \tilde{\omega}_i \chi_{S_i}. \tag{4.1}$$

Analogously, we introduce piecewise-averaged versions of α,

$$\tilde{\alpha}_i \coloneqq \frac{1}{\lambda(S_i)} \int_{S_i} \alpha \,\mathrm{d}\lambda \quad \text{and} \quad \bar{\alpha} \coloneqq \sum_{i=1}^{N} \tilde{\alpha}_i \chi_{S_i} \tag{4.2}$$

and the control deviation ϕ,

$$\bar{\phi} \coloneqq \bar{\alpha} - \omega. \tag{4.3}$$

Applying Proposition 2.4 to the one-dimensional SUR algorithm, the multi-dimensional can be reduced to the one-dimensional setting.

COROLLARY 4.2 (Vanishing integrality gap for multi-dimensional SUR). *There exists $C > 0$ such that for all relaxed controls α and $\omega(\alpha)$ that are computed by the multi-dimensional variant of SUR of Definition 4.1, we obtain that*

1. *ω is a binary control and*
2. *for $\phi \coloneqq \alpha - \omega(\alpha)$ the following estimate holds:*

$$\max_{i \in \{1,\dots,N\}} \left\| \int_{\bigcup_{j=1}^{i} S_j} \phi \,\mathrm{d}\lambda \right\|_\infty \le C \cdot \max_{i \in \{1,\dots,N\}} \lambda(S_i).$$

Proof. We investigate ω produced by the SUR algorithm of Definition 4.1. We observe that by setting

$$t_i \coloneqq \lambda\left(\bigcup_{j=1}^{i} S_i \right)$$

for $i \in \{0, \dots, N\}$ and

$$\alpha^* \coloneqq \sum_{i=1}^{N} \tilde{\alpha}_i \chi_{[t_{i-1}, t_i)}, \; \omega^* \coloneqq \sum_{i=1}^{N} \tilde{\omega}_i \chi_{[t_{i-1}, t_i)}, \; \phi^* \coloneqq \alpha^* - \omega^*,$$

we obtain a curve $\alpha^* : [0, \lambda(\Omega)] \to [0,1]^M$ for which the application of SUR of Definition 2.2 would have produced the same result. This means that it would have produced the same sequence of piecewise constant function values on pieces with the same Lebesgue measure values as the multi-dimensional case, but in linear ordering along the time in ω^*. The equations (4.2) and (4.3) yield

$$\int_{S_i} \phi \,\mathrm{d}\lambda_{\mathbb{R}^d} = \int_{S_i} \bar{\phi} \,\mathrm{d}\lambda_{\mathbb{R}^d}$$

for $i \in \{1, \dots, N\}$. Now, splitting the integral of ϕ over $\mathbb{R}^d$ into a sum and recombining it into an integral of ϕ^* over $\mathbb{R}$ gives

$$\int_{\bigcup_{j=1}^{i} S_j} \phi \,\mathrm{d}\lambda_{\mathbb{R}^d} = \int_0^{t_i} \phi^* \,\mathrm{d}\lambda_{\mathbb{R}^1}.$$

Thus, we can apply Proposition 2.4 to obtain

$$\max_{i\in\{1,\dots,N\}} \left\| \int_{\bigcup_{j=1}^{i} S_j} \phi \,\mathrm{d}\lambda_{\mathbb{R}^d} \right\|_\infty \le \sup_{t\in[0,T]} \left\| \int_0^t \phi^* \,\mathrm{d}\lambda_{\mathbb{R}^1} \right\|_\infty \le C \cdot \max_{i\in\{1,\dots,N\}} \lambda(S_i). \qquad \square$$

The constant $C > 0$ can be improved by using different algorithms than SUR such as the integer optimization approach from [20] to minimize the integrated control deviation. These algorithms may be transferred to the multi-dimensional setting analogously.

4.2. Convergence for suitable grid refinements. Before stating the weak-* convergence of the binary controls to the relaxed control, we define a condition on sequences of grids, which will be sufficient to prove it.

DEFINITION 4.3 (Admissible sequences of refined rounding grids). *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. Then, we call a sequence $\left(\left\{S_1^{(n)},\dots,S_{N^{(n)}}^{(n)}\right\}\right)_n \subset 2^{\mathcal{B}(\Omega)}$ an* **order conserving domain dissection of** Ω *if*

1. $\left\{S_1^{(n)},\dots,S_{N^{(n)}}^{(n)}\right\}$ *is a finite partition of Ω for all $n \in \mathbb{N}$.*
2. $\max_{i\in\{1,\dots,N^{(n)}\}} \lambda(S_i^{(n)}) \to 0$.
3. *for all n and all $i \in \{1,\dots,N^{(n-1)}\}$, there exist $1 \le j < k \le N^{(n)}$ such that $\bigcup_{l=j}^{k} S_l^{(n)} = S_i^{(n-1)}$ and*
4. *the cells $S_j^{(n)}$ shrink regularly, that is there exists $C > 0$ such that for each $S_j^{(n)}$ there exists a Ball $B_j^{(n)}$ such that $S_j^{(n)} \subset B_j^{(n)}$ and $\lambda(S_j^{(n)}) \ge C\lambda(B_j^{(n)})$.*

The third property is particularly important for the proof below and means that the order of the grid cells is recursively preserved from grid iteration $n-1$ to grid iteration n. This is similar to the *two-level decomposition scheme* introduced in [27] for the case rectangular grid cells. We also note that the fourth property in Definition 4.3 bears similarities with finite element triangulations. It is similar to requiring a quasi-uniform mesh, which is refined with an isotropic strategy, see [2]. However, for our purpose, it is sufficient to restrict the eccentricity with a bound on the ratio between the measures of a cell and the circumscribed sphere without caring about the ratio of the diameter to the one of an inscribed sphere.

In fact, Definition 4.3 was obtained by studying the Hilbert curve, a so-called space-filling curve, which is a continuous and surjective mapping from $[0,1]$ to $[0,1]^2$. Space-filling curves can be defined as limits of approximating curves. The first three iterations of the Hilbert curve are displayed in Figure 1, a facsimile of the figure in Hilbert's article from 1891 [10]. The third property in Definition 4.3 can be observed in Figure 1. For example, the second square from iteration 1 is decomposed into squares 5-8 in iteration 2 and squares 17-32 in iteration 3, etc. In our proofs this property allows us to maintain a spatial coherence of the vanishing integrality gap, the error quantity, which can be controlled for the SUR algorithm by virtue of Corollary 4.2.

The following lemma provides an approximation argument that allows us to obtain the desired weak-* convergence. A preliminary version of this argument is published in the short proceedings article [14].

LEMMA 4.4. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. Let $\alpha \in L^\infty(\Omega,\mathbb{R}^M)$ be $[0,1]^M$-valued and let an order conserving domain dissection $\left\{S_1^{(n)},\dots,S_{N^{(n)}}^{(n)}\right\}$ of Ω be given. Let $(\phi^{(n)})_n$ be a sequence of $[-1,1]^M$-valued measurable functions. Let $C > 0$ be such*

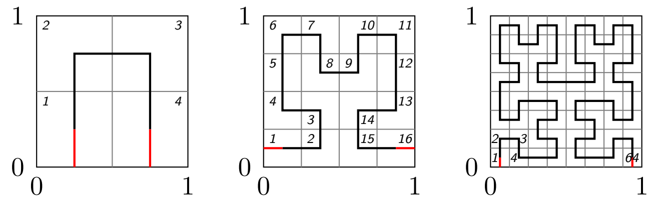

Fig. 1: Hilbert curve iterates H_1, H_2, H_3 (left to right) on $\bar{\Omega} = [0,1]^2$. The (additional) extension to the boundary is marked red (light gray in grayscale print). The induced discretization squares are circumscribed by the gray lines. Their ordering along the Hilbert curve iterates is indicated by the small numbers inside the cells.

that

$$\max_{i\in\{1,\dots,N^{(n)}\}} \left\| \int_{\bigcup_{j=1}^{i} S_j^{(n)}} \phi^{(n)} \,\mathrm{d}\lambda \right\|_\infty \le C \max_{i\in\{1,\dots,N\}} \lambda\left(S_i^{(n)}\right).$$

Let $f \in L^1(\Omega)$ and let $i \in \{1,\dots,M\}$. Then,

$$\int_\Omega \phi_i^{(n)} f \,\mathrm{d}\lambda \to 0.$$

Proof. In the remainder of the proof we abbreviate $\phi^{(n)} = \phi_i^{(n)}$ to avoid a bloated notation. We note that the products $\phi^{(n)} f$ are integrable for all $n \in \mathbb{N}$ because $\phi^{(n)} \in L^\infty(\Omega)$ for all $n \in \mathbb{N}$.

We have to show $\int_\Omega \phi_i^{(n)} f \,\mathrm{d}\lambda \to 0$. Since integrable functions can be written as the difference $f = f^+ - f^-$ with f^+ and f^- being positive integrable functions, it suffices to show the claim for functions $f \in L^1(\Omega)$ that are positive almost everywhere.

We use two approxmation steps. First, we approximate the function f^+ by simple functions. Second, we use the properties of Definition 4.3 to approximate the function by its average on the domain dissection of grid iteration n. This will allow us to apply Corollary 4.2, which then drives the integral to zero.

We recall that f^+ is the pointwise monotone limit of a sequence of simple functions, that is $0 \le f^{(1)}(x) \le f^{(2)}(x) \le \ldots \le f(x)$ for a.a. $x \in \Omega$ and $\lim_k \int_\Omega |f - f^{(k)}| = 0$. Let $\varepsilon > 0$. We pick $k \in \mathbb{N}$ such that $\int_\Omega |f - f^{(k)}| < \varepsilon/3$ and obtain that

$$\left|\int_\Omega f\phi^{(n)}\right| \le \left|\int_\Omega f^{(k)}\phi^{(n)}\right| + \left|\int_\Omega (f - f^{(k)})\phi^{(n)}\right| \le \left|\int_\Omega f^{(k)}\phi^{(n)}\right| + \frac{\varepsilon}{3}$$

for all $n \in \mathbb{N}$, where we have used the triangle inequality and that $\|\phi^{(n)}\|_{L^\infty} \le 1$.

Next, we set $g \coloneqq f^{(k)}$ and consider $\left|\int_\Omega g\phi^{(n)}\right|$. Again, $g \ge 0$ almost everywhere. For grid iteration $n \in \mathbb{N}$, we define the function $g^{(n)}$

$$g^{(n)}(x) \coloneqq \sum_{i=1}^{N^{(n)}} \chi_{S_i^{(n)}}(x) \frac{1}{\lambda(S_i^{(n)})} \int_{S_i^{(n)}} g \,\mathrm{d}\lambda \text{ for } x \in \Omega$$

The functions $g^{(n)}$ converge to g in a pointwise almost everywhere sense by virtue of Lebesgue's differentiation theorem, see [23, Chap. 3, Cor. 1.6 & 1.7], which may be applied because of the regular shrinkage assumption ensured by the fourth property of

Definition 4.3. Moreover, it holds that the $g^{(n)}$ pointwise almost everywhere are bounded by $\|g\|_{L^\infty}$, which is finite because g is a simple function. Thus, we apply Lebesgue's dominated convergence theorem to deduce $g^{(n)} \to g$ in $L^1(\Omega)$. Therefore, we may choose $n_0 \in \mathbb{N}$ such that

$$\|g - g^{(n_0)}\|_{L^1} < \frac{\varepsilon}{3}. \tag{4.4}$$

Let $n \in \mathbb{N}$. We use the triangle inequality and estimate

$$\left|\int_\Omega g\phi^{(n)}\,\mathrm{d}\lambda\right| \le \left|\int_\Omega g^{(n_0)}\phi^{(n)}\,\mathrm{d}\lambda\right| + \left|\int_\Omega (g - g^{(n_0)})\phi^{(n)}\,\mathrm{d}\lambda\right|.$$

For the first term it holds that

$$\left|\int_\Omega (g - g^{(n_0)})\phi^{(n)}\,\mathrm{d}\lambda\right| \le \left\|g - g^{(n_0)}\right\|_{L^1} \|\phi^{(n)}\|_{L^\infty} \overset{\|\phi^{(n)}\|_{L^\infty}\le 1}{\underset{(4.4)}{<}} \frac{\varepsilon}{3}.$$

Thus the proof is complete if we are able to show there exists $n_1 \in \mathbb{N}$ such that for all $n \ge n_1$ it holds that

$$\left|\int_\Omega g^{(n_0)}\phi^{(n)}\ \mathrm{d}\lambda\right| < \frac{\varepsilon}{3}.$$

For the remainder of the proof, we abbreviate $g_i \coloneqq \frac{1}{\lambda(S_i^{(n_0)})}\int_{S_i^{(n_0)}} g^{(n_0)}\,\mathrm{d}\lambda$. We can rewrite

$$\int_\Omega g^{(n_0)}\phi^{(n_0)}\,\mathrm{d}\lambda = \sum_{i=1}^{N^{(n_0)}} g_i \int_{S_i^{(n_0)}} \phi^{(n_0)}\,\mathrm{d}\lambda.$$

The third property of Definition 4.3 implies that the grid cell $S_i^{(n_0)}$ is decomposed into finitely many grid cells in iteration n_0+1. Since this property holds recursively, we deduce for all $n \ge n_0$ that

$$\int_\Omega g^{(n_0)}\phi^{(n)}\,\mathrm{d}\lambda = \sum_{i=1}^{N^{(n_0)}} g_i \int_{S_i^{(n_0)}} \phi^{(n)}\,\mathrm{d}\lambda. \tag{4.5}$$

Let $i \in \{1,\dots,N^{(n_0)}\}$ be fixed. The recursive decomposition property three of Definition 4.3 implies the following. For all $n \ge n_0$ there exist indices $a(i,n)$, $b(i,n) \in N^{(n)}$ such that $a(i,n)$ is the starting index and $b(i,n)$ the end index of the cells into which $S_i^{(n_0)}$ is decomposed in grid iteration $n \ge n_0$. Thus we have the decomposition $S_i^{(n_0)} = \bigcup_{j=a(i,n)}^{b(i,n)} S_j^{(n)}$. This means that This allows us to perform the following estimate

$$\begin{aligned}
\left|\int_{S_i^{(n_0)}} \phi^{(n)}\,\mathrm{d}\lambda\right| &= \left|\int_{\bigcup_{j=a(i,n)}^{b(i,n)} S_j^{(n)}} \phi^{(n)}\,\mathrm{d}\lambda\right| \\
&= \left|\int_{\bigcup_{j=1}^{b(i,n)} S_j^{(n)}} \phi^{(n)}\,\mathrm{d}\lambda - \int_{\bigcup_{j=1}^{a(i,n)-1} S_j^{(n)}} \phi^{(n)}\,\mathrm{d}\lambda\right| \\
&\le 2 \max_{k\in\{1,\dots,N^{(n)}\}} \left|\int_{\bigcup_{j=1}^{k} S_j^{(n)}} \phi^{(n)}\,\mathrm{d}\lambda\right|,
\end{aligned}$$

where the first equality follows by inserting the considerations above and the last inequality follows from the triangle inequality.

From the second property of Definition 4.3 and the prerequisites we deduce that

$$\left|\int_{S_i^{(n_0)}} \phi^{(n)}\,\mathrm{d}\lambda\right| < \frac{\varepsilon}{6\sum_{i=1}^{N^{(n_0)}} g_i}$$

for all $n \geq n_1$, where $n_1 \geq n_0$ is chosen such that

$$\max_{k\in\{1,\dots,N^{(n)}\}} \lambda\left(S_k^{(n)}\right) < \frac{\varepsilon}{6\max\{C,1\}\sum_{i=1}^{N^{(n_0)}} g_i}$$

holds for all $n \geq n_0$.

We insert the estimates we just obtained into (4.5), which gives

$$\left|\int_\Omega g^{(n_0)}\phi^{(n)}\,\mathrm{d}\lambda\right| < \frac{\varepsilon}{3}$$

for all $n \geq n_1$.

Thus for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ it holds that

$$\left|\int_\Omega f\phi^{(n)}\,\mathrm{d}\lambda\right| < \varepsilon,$$

which finishes the proof. ∎

Lemma 4.4 immediately establishes the desired weak-* and weak convergence properties, which we summarize in the following theorem.

THEOREM 4.5. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and let $\left(\left\{S_1^{(n)},\dots,S_{N^{(n)}}^{(n)}\right\}\right)_n \subset 2^{\mathcal{B}(\Omega)}$ be an admissible sequence of refined rounding grids of Ω. Let α be a relaxed control and for $n \in \mathbb{N}$, let $\omega^{(n)}$ be the binary control computed by the multi-dimensional SUR algorithm on the n-th rounding grid. Let $\phi^{(n)} := \alpha - \omega^{(n)}$ be the control deviation vector of the n-th grid. Then,*

$$\phi^{(n)} \rightharpoonup 0 \text{ in } L^p(\Omega,\mathbb{R}^M) \text{ for } 1 \leq p < \infty$$

and

$$\phi^{(n)} \rightharpoonup^* 0 \text{ in } L^p(\Omega,\mathbb{R}^M) \text{ for } 1 < p \leq \infty.$$

Proof. We consider the sequence of $(\omega^{(n)})_n$, where $\omega^{(n)} : \Omega \to \{0,1\}^M$ is generated by the multi-dimensional SUR algorithm on the n-th domain dissection along the subscript ordering. The function of the control deviation vector of the n-th grid is denoted by $\phi^{(n)} := \alpha - \omega^{(n)}$. It is $[-1,1]^M$-valued and satisfies the required estimate of Lemma 4.4 by virtue of Corollary 4.2.

We identify $\mathbb{R}^M \cong (\mathbb{R}^M)^*$ and use L^p-space duality for vector-valued function spaces, see e.g. [5, Thm IV.1], together with Lemma 4.4 and an ε/M-argument. ∎

This shows that the approximation properties hold for integral operators with L^1-kernels, which generalizes the results from [27] from Lipschitz continuous to L^1-functions. A spatial coherence property like the third property of Definition 4.3 is indeed necessary to obtain convergence in the weak* topology of $L^\infty(\Omega)$. We give a counterexample below that shows that one can choose a sequence of rounding grids such that weak* convergence does not follow.

EXAMPLE 4.6. *For $n \in \mathbb{N}$, we decompose the set $\Omega = [0, 1)$ and define the n-th rounding grid $\{S_1^{(n)}, \dots, S_{N^{(n)}}^{(n)}\}$ consisting of $N^{(n)} = 2^n$ intervals by setting*

$$S_{2i}^{(n)} := \left[(i-1)2^{-n}, i2^{-n}\right) \text{ and } S_{2i-1}^{(n)} := \left[0.5 + (i-1)2^{-n}, 0.5 + i2^{-n}\right)$$

for $i \in \{1, \dots, 2^{n-1}\}$. This implies that $S_k^{(n)} \subset [0, 0.5)$ if k is even and $S_k^{(n)} \subset [0.5, 1)$ if k is odd. The resulting sequence of rounding grids satisfies all properties from Definition 4.3 except the third.

Let $M = 2$ and consider the function $\alpha :\equiv \begin{pmatrix} 0.5 & 0.5 \end{pmatrix}^T$. Then the SUR algorithm applied to α on the n-th rounding yields a function ω that satisfies

$$\omega^{(n)}(x) = \begin{cases} \begin{pmatrix} 0 & 1 \end{pmatrix}^T & \text{if } x \in S_{2i}^{(n)} \text{ for some } i \in \{1, \dots, 2^{n-1}\}, \text{ and} \\ \begin{pmatrix} 1 & 0 \end{pmatrix}^T & \text{else, that is if } x \in S_{2i-1}^{(n)} \text{ for some } i \in \{1, \dots, 2^{n-1}\}. \end{cases} \tag{4.6}$$

Now, we consider the function $f = \chi_{[0,0.5)}$ and obtain

$$\int_\Omega f(\alpha - \omega^{(n)})_1 \, \mathrm{d}\lambda = \int_0^{0.5} (\alpha - \omega^{(n)})_1 \, \mathrm{d}\lambda = \int_{\bigcup_{i=1}^{2^{n-1}} S_{2i}^{(n)}} (\alpha - \omega^{(n)})_1 \, \mathrm{d}\lambda$$

because the cells $S_k^{(n)}$ for even k decompose $[0, 0.5)$ for all n. We insert (4.6) and obtain

$$\int_\Omega f(\alpha - \omega^{(n)})_1 \, \mathrm{d}\lambda = 0.5\lambda \left(\bigcup_{i=1}^{2^{n-1}} S_{2i}^{(n)} \right) = 0.5\lambda([0, 0.5)) = 0.25$$

for all $n \in \mathbb{N}$. Thus there exists $f \in L^1(\Omega)$ such that $\int_\Omega f(\alpha - \omega^{(n)}) \not\to 0$, and consequently, $\omega^{(n)} \not\rightharpoonup^* \alpha$ in $L^\infty(\Omega)$.

4.3. Algorithmic consequences. We prove our main result, the convergence of Algorithm 1.1 under Definition 4.3 with the results from Section 3 and Section 4 in the theorem below.

THEOREM 4.7 (Convergence of Algorithm 1.1). *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. Let Assumption 3.1 hold. Let $J : V \to \mathbb{R}$ be continuous. Let $f_1, \dots, f_M \in L^2(\Omega)$. Let the sequence $\left(S^{(n)}\right)_n$ produced by Algorithm 1.1 be an order conserving domain dissection. Then, for every norm-weak-*-accumulation point $(x^*, \alpha^*) \in \mathcal{F}_{RC}$ with approximating subsequences*

$$\alpha^{(n_k)} \rightharpoonup^* \alpha^* \text{ and } x^{(n_k)} \to x^*,$$

produced by Algorithm 1.1, the corresponding iterates $(y^{(n_k)}, \omega^{(n_k)}) \in \mathcal{F}_{BC}$ produced by Algorithm 1.1 satisfy

$$J(y^{(n_k)}) \to J(x^*).$$

Proof. To ease the notation, we denote the subsequence $(n_k)_k$ by $(n)_n$. It suffices to show $y^{(n)} \to x^*$ in V as J is continuous. This in turn follows from Theorem 3.2 if we can show $\omega^{(n)} \rightharpoonup^* \alpha^*$, i.e.

$$\int_\Omega (\alpha_i^* - \omega_i^{(n)})g = \int_\Omega (\alpha_i^* - \alpha_i^{(n)})g + \int_\Omega (\alpha_i^{(n)} - \omega_i^{(n)})g \to 0$$

for all $g \in L^1(\Omega)$ and $i \in \{1, \dots, M\}$.

The first term converges to zero by assumption. We observe that $\phi^{(n)} := \alpha^{(n)} - \omega^{(n)}$ satisfies the estimate required by Lemma 4.4 by virtue of Corollary 4.2 because $\omega^{(n)}$ is computed by SUR from the relaxed control function $\alpha^{(n)}$ on the n-th rounding grid. Consequently, the proof of Lemma 4.4 implies that for all $g \in L^1(\Omega)$ we have $\int_\Omega g\phi_i^{(n)} \to 0$ for all $i \in \{1, \dots, M\}$, which finishes the proof. ∎

5. Approximation relationship of (BC) and (RC). The previous sections show that the SUR algorithm gives an efficient means to compute binary-valued approximations of relaxed controls in weaker topologies. This is as good as we can expect and is not true in the norm topology if the relaxed control assumes fractional values on a set of positive measure. Regarding Theorem 4.7, we observe that if an accumulation point (α^*, x^*) minimizes (RC), we obtain a minimizing sequence for (RC) with binary controls. Thus, (BC) approximates (RC), which we summarize below.

THEOREM 5.1. *Let Assumption 3.1 hold. Let $J : V \to \mathbb{R}$ be continuous. Let $f_1, \ldots, f_M \in L^2(\Omega)$. Then, a sequence $(y^{(n)}, \omega^{(n)})_n \subset \mathcal{F}_{BC}$ exists such that*

$$\lim_{n\to\infty} J(y^{(n)}) \to \inf_{(y,\omega)\in\mathcal{F}_{BC}} J(y).$$

Furthermore,

$$\min_{(x,\alpha)\in\mathcal{F}_{RC}} J(x) = \inf_{(y,\omega)\in\mathcal{F}_{BC}} J(y).$$

Proof. By construction of (RC) and (BC), we have $\inf\{J(x) : (x,\alpha) \in \mathcal{F}_{RC}\} \leq \inf\{J(y) : (y,\omega) \in \mathcal{F}_{BC}\}$ as $\mathcal{F}_{BC} \subset \mathcal{F}_{RC}$. The set $\{\sum_{i=1}^M \alpha_i f_i : \alpha$ is a relaxed control$\}$ is convex, closed and bounded and consequently weakly compact in $L^2(\Omega)$ by virtue of the Banach-Alaoglu theorem. The map $\alpha \mapsto x(\alpha) = A^{-1}\sum_{i=1}^M \alpha_i f_i$ is continuous from the weak-* topology of $L^\infty(\Omega)$ to the norm topology of V with the same arguments as in Theorem 3.2. Thus the reduced objective map $\alpha \mapsto J(x(\alpha))$ is continuous from the weak-* topology of $L^\infty(\Omega)$ to $\mathbb{R}$. Thus, there exists a minimizer $(x^*, \alpha^*) \in \mathcal{F}_{RC}$ by virtue of the Weierstrass extreme value theorem for topological vector spaces, see [15].

The application of SUR to α^* on a sequence of uniformly refined uniform rounding grids (or any admissible sequence of rounding grids) yields existence of binary controls $\omega^{(n)}$ such that $\omega^{(n)} \rightharpoonup^* \alpha^*$ by virtue of Theorem 4.5. The arguments above imply that the corresponding state vectors $y^{(n)} = A^{-1}\sum_{i=1}^M f_i \omega_i^{(n)}$ satisfy $y^{(n)} \to x^*$ in V and thus,

$$\inf_{(y,\omega)\in\mathcal{F}_{BC}} J(y) \leq J(y^{(n)}) \to J(x^*) = \min_{(x,\alpha)\in\mathcal{F}_{RC}} J(x). \qquad \square$$

The result is constructive as applying SUR on sequences of rounding grids that are order conserving domain dissections yields sequences with these characteristics. Consider $f_1, \ldots, f_M \in \mathbb{R}$ and assume that (RC) is solved by means of an auxiliary variable $v := \sum_{i=1}^M \alpha_i f_i$ subject to box-constraints of the form $v \in [f_L, f_U]$ with $f_L = \min_i f_i$, $f_U = \max_i f_i$. Section 6 discloses that it may be more realistic to assume that the objective functional has the structure

$$J_1(x,\alpha) = J(x) + \frac{\gamma}{2}\left\|\sum_{i=1}^M \alpha_i f_i\right\|_{L^2}^2$$

or similar to ensure that (RC) is well-posed. If we keep the other assumptions the same and apply a similar reasoning, we obtain the following result, which introduces a suboptimality, but may be of interest in practice.

COROLLARY 5.2. *There exists a sequence $(y^{(n)}, \omega^{(n)})_n \subset \mathcal{F}_{BC}$ such that*

$$\min_{(x,\alpha)\in\mathcal{F}_{RC}} J_1(x,\alpha) \leq \lim_{n\to\infty} J_1(y^{(n)}, \omega^{(n)}) \leq \inf_{(y,\omega)\in\mathcal{F}_{BC}} J(y) + \frac{\gamma}{2}\|\max\{|f_L|, |f_U|\}\|_{L^2}^2.$$

Proof. Pointwise factorization, $0 \leq \alpha \leq 1$ a.e. and $\sum_{i=1}^{M} \alpha_i = 1$ a.e. give

$$\frac{\gamma}{2} \left\| \sum_{i=1}^{M} \alpha_i f_i \right\|_{L^2}^2 \leq \frac{\gamma}{2} \| \max\{|f_L|, |f_U|\} \|_{L^2}^2 =: K. \tag{5.1}$$

Let $(x^*, \alpha^*) \in \arg\min\{J_1(x, \alpha) : (x, \alpha) \in \mathcal{F}_{RC}\}$. Then,

$$J_1(x^*, \alpha^*) = J(x^*) + \frac{\gamma}{2} \left\| \sum_{i=1}^{M} \alpha_i^* f_i \right\|_{L^2}^2 \leq \inf_{(x,\alpha) \in \mathcal{F}_{RC}} J(x) + K.$$

If we assume the converse, (5.1) implies the contradictory inequality $J(x^*) > \inf\{J(x) : (x, \alpha) \in \mathcal{F}_{RC}\}$. The claim follows from Theorem 5.1, in particular from the existence of $(y^{(n)}, \omega^{(n)})_n$ such that $J(y^{(n)}) \to J(x^*)$, and (5.1) applied to the $\omega^{(n)}$. ∎

The box constraints $v \in [f_L, f_U]$ imply the existence of a (usually non-unique) feasible α. Thus, solving the relaxation for v is a consistent reduction of the problem. Furthermore, by rounding, the box constraints still hold as $\sum_{i=1}^{M} \omega_i f_i \in [f_L, f_U]$ for binary controls ω. Thus, the corollary states that we can approximate the infimum of (BC) up to a suboptimality in $\mathcal{O}(\gamma)$. Choosing γ small enough allows to control the limiting suboptimality a priori. However, γ may also be fixed a priori in practice. Corollary 5.2 can be generalized for arbitrary L^∞-functions f_i, i.e. we may solve for v in the convex hull of the f_i.

6. Numerical experiments. We illustrate our results computationally. All meshes and PDE solutions have been implemented in FEniCS [1]. As mentioned above, we consider the Dirichlet Laplacian, which satisfies our assumptions, see Example 6.1 below.

EXAMPLE 6.1. *We consider the Dirichlet Laplacian on the unit square $\bar{\Omega} = [0, 1]^2$, i.e. the constraint $-\Delta x = \sum_{i=1}^{M} \alpha_i f_i$, $x|_{\partial\Omega} = 0$ for relaxed controls α in (RC). In the interest of completeness, we note that the embeddings $H_0^1(\Omega) \hookrightarrow^c L^2(\Omega) \hookrightarrow^c H^{-1}(\Omega)$ are continuous, compact and dense, see [16, Thm 7.29], and that the Lax-Milgram theorem, see [16, Thm 9.14], yields the existence of a bounded inverse $A^{-1} : H^{-1}(\Omega) \to H_0^1(\Omega)$. Thus, Assumption 3.1 is satisfied.*

First, we demonstrate the approximation properties of SUR. Next, we use Algorithm 1.1 to approximately solve a tracking-type problem that is constrained by the Dirichlet Laplacian. Finally, we test the methodology outside of the intended scope in a control reconstruction problem.

6.1. Approximation properties of the SUR algorithm. We demonstrate Theorems 3.2 and 4.5 by computing the SUR approximation for eight uniformly refined square grids, where the side lengths of the cells are halved in each refinement, i.e. the number of grid cells quadruples from iteration to the next. The SUR approximation is computed along the orderings induced by the Hilbert curve approximants. A grayscale image of David Hilbert is used as input (relaxed control) for the SUR algorithm, see Figure 2 for the weak-* approximation with the Hilbert curve induced ordering of the cells. The resulting approximation errors for solutions of the state equation in Example 6.1, i.e. Theorem 3.2, are illustrated in Figure 3.

The weak-* convergence of $\omega^{(n)}$, i.e. Theorem 4.5, can be perceived visually in Figure 2 and the output of SUR resembles a dithering technique from computer graphics to display grayscale images with coarsely quantized gray colors such as the Floyd-Steinberg algorithm [24] or the digital half-toning algorithm from [25], which is very similar to SUR and also executed along a space-filling curve.

$\omega_{\text{HC}}^{(1)}$ $\omega_{\text{HC}}^{(3)}$ $\omega_{\text{HC}}^{(5)}$ $\omega_{\text{HC}}^{(7)}$ $\omega_{\text{HC}}^{(9)}$

Fig. 2: Weak-* approximants computed with SUR for a grayscale image of David Hilbert along the order defined by the 1st, 3rd, 5th, 7th and 9th Hilbert curve approximant.

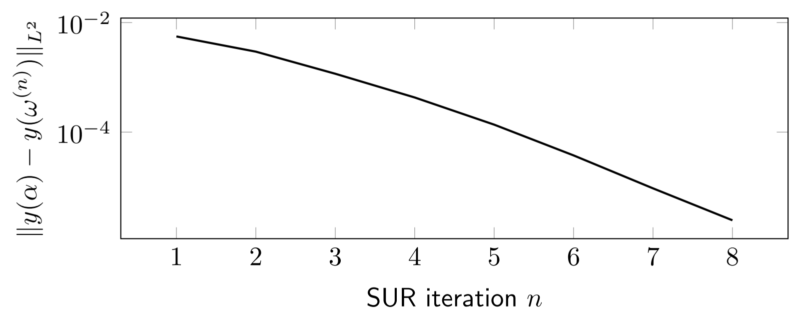

Fig. 3: State approximation error for SUR for uniformly refined grids and along the Hilbert curve approximant-induced orderings.

All results in the subsequent sections have been computed by executing the SUR algorithm along the cell ordering induced by Hilbert curve approximants.

6.2. Approximating the solution of an MIOCP with Algorithm 1.1. We consider the following problem

$$\text{(P)} \qquad \begin{aligned} &\min_{y,f} \frac{1}{2}\|y - y_d\|_{L^2}^2 + \frac{\gamma}{2}\|f\|_{L^2}^2 \\ &\text{s.t. } -\Delta y = f, \quad y|_{\partial\Omega} = 0, \quad f \in \{f_1, \ldots, f_M\} \subset \mathbb{R} \text{ a.e. on } \Omega \end{aligned}$$

with $f_1 < \ldots < f_M$. (P) is similar to problem (1.1) considered by Clason and Kunisch in [4] where they introduce the notions of *multi-bang controls* and *generalized multi-bang principle* for controls f satisfying the discrete-value constraint almost everywhere. Compared to (1.1) in [4], (P) lacks the term $\beta \int_\Omega \prod_{i=1}^M |f - f_i|_0$ with $|t|_0 = 1 - \delta_{t0}$ (using the real-valued Kronecker delta) that promotes $\{f_1, \ldots, f_M\}$-valued solutions. Furthermore, the box constraint $f_1 \le f \le f_M$ has been replaced by $f \in \{f_1, \ldots, f_M\}$. This is not a coincidence because, in an informal way, we can regard (P) as a limit problem of (1.1) in [4] for the homotopy arising from increasing their parameter β penalizing non-discreteness.

Reformulation and relaxation. We consider the following relaxed *partial outer convexification* of (P).

$$\text{(P RC1)} \qquad \begin{aligned} &\min_{x,f} \frac{1}{2}\|x - y_d\|_{L^2}^2 + \frac{\gamma}{2}\|f\|_{L^2}^2 \\ &\text{s.t. } -\Delta x = \sum_{i=1}^M \alpha_i f_i, \quad x|_{\partial\Omega} = 0, \quad \alpha \in [0,1]^M \text{ and } \sum_{i=1}^M \alpha_i = 1 \text{ a.e. on } \Omega. \end{aligned}$$

Of course, we reduce solving (P RC1) to solving

$$\text{(P RC2)} \qquad \begin{aligned} &\min_{x,f} \frac{1}{2}\|x - y_d\|^2_{L^2} + \frac{\gamma}{2}\|f\|^2_{L^2} \\ &\text{s.t. } -\Delta x = f, \quad x|_{\partial\Omega} = 0, \quad f \in [f_1, f_M] \text{ a.e. on } \Omega \end{aligned}$$

and compute α from f afterwards. Note that (P RC1) is ill-posed as the representation of f with convex combinations of the f_i is not unique and thus, the particular outcome of SUR and Algorithm 1.1 may depend on the chosen representation. The convergence results hold independently of the representation, but different $\alpha^{(n)}$ are computed and approximated by the $\omega^{(n)}$ in the weak-* sense. We have chosen the most natural representation from our point of view. Specifically, we represent a value $f(s)$ for $s \in \Omega$ as the convex combination of its two neighboring points in $\{f_1, \dots, f_M\}$. This means, we choose f_i and f_{i+1} such that $f_i \le f(s) \le f_{i+1}$ and compute $\alpha_i(s)$, $\alpha_{i+1}(s) = 1 - \alpha_i(s)$ such that $\alpha_i(s)f_i + (1 - \alpha_i(s))f_{i+1} = f(s)$. $\alpha_j(s) := 0$ for all $j \notin \{i, i+1\}$. Of course, a convex combination of neighboring points always exists due to Caratheodory's theorem.

It is well-known that L^1-regularized problems tend to produce large areas where the control is exactly zero. Thus, if there exists $f^* \in \{f_1, \dots, f_M\}$ that can be assumed to dominate the resulting control on large areas, it may be beneficial to solve an L^1-regularized problem with regularizer $\|f - f^*\|_{L^1}$ as a relaxed problem. Therefore, we include the following problem into our computational experiments:

$$\text{(P RC3)} \qquad \begin{aligned} &\min_{x,f} \frac{1}{2}\|x - y_d\|^2_{L^2} + \frac{\gamma}{2}\|f\|^2_{L^2} + \eta\|f\|_{L^1} \\ &\text{s.t. } -\Delta x = f, \quad x|_{\partial\Omega} = 0, \quad f \in [f_1, f_M] \text{ a.e. on } \Omega \end{aligned}$$

If $\gamma > 0$, the L^2-term improves the regularity of the solution without having to smooth the L^1-term. Elliptic control problems of the type of (P RC3) have been analyzed in [22] and [26] and we compute the solutions of the discretizations of (P RC3) with the *active set method* presented in [22]. We choose $\gamma \ll \eta$ to obtain a dominating effect of the L^1-regularization over the L^2-regularization.

Application of Algorithm 1.1. As the objective depends on α in (P RC2), we have a slight deviation from the setting in (BC) and (RC) and can only expect norm-convergence in the tracking type summand of the objective. Clearly, $\omega^{(n)} \rightharpoonup^* \alpha$ implies $v^{(n)} \rightharpoonup v$ with

$$v^{(n)} := \sum_{i=1}^{M} \omega_i^{(n)} f_i \text{ and } v = \sum_{i=1}^{M} \alpha_i f_i,$$

but the norm $\|\cdot\|_{L^2}$ is weakly lower semicontinuous and we obtain

$$\liminf_{n\to\infty} \frac{\gamma}{2}\left\|v^{(n)}\right\|^2_{L^2} \ge \frac{\gamma}{2}\left\|v\right\|^2_{L^2}$$

and equality holds if and only if $v^{(n)} \to v$ which cannot be assumed for the considered problems. Hence, we expect convergence of the tracking type summand in the progression of Algorithm 1.1 and convergence of the L^2-regularization to a suboptimal value.

We have taken y_d and the control quantization into $f_1 = -2, \dots, f_5 = 2$ from [4] to use their code for plausibility checks of our results. We solve (P RC2) approximately (with $\gamma = 10^{-3}$) and (P RC3) approximately (with $\gamma = 10^{-5}$ and $\eta = 5 \cdot 10^{-4}$) on refined triangular grid with first order Lagrange finite elements. For the right hand sides, we use a piecewise-constant discontinuous Galerkin discretization on square cells, which consist of two triangles each. The SUR algorithm is executed on the these square cells. We computed 9 iterations of Algorithm 1.1. The relative errors of the tracking term (J_t), the regularization term (J_r) and the state vector produced by SUR to the solution of ($\mathrm{RC}_h^{(9)}$) as well as the state vector difference along the iterates are given in Table 1 for (P RC2) and 2 for (P RC3).

Table 1: Self-convergence of the tracking term, the suboptimality gap in the regularizer, and the state vector iterates against the solution of the finest approximation of (P RC2) as well as convergence of the difference between the relaxed state vector and corresponding SUR approximation for the parameter $\gamma = 10^{-3}$.

It.	$\frac{\lvert J_t(\omega^{(n)})-J_t(\alpha^{(9)})\rvert}{J_t(\alpha^{(9)})}$	$\frac{\lvert J_r(\omega^{(n)})-J_r(\alpha^{(9)})\rvert}{J_r(\alpha^{(9)})}$	$\frac{\lVert y^{(n)}-x^{(9)}\rVert_{L^2}}{\lVert x^{(9)}\rVert_{L^2}}$	$\lVert y^{(n)} - x^{(n)}\rVert_{L^2}$
1	2.396×10^{-1}	4.231×10^{-1}	7.350×10^{-1}	7.366×10^{-3}
2	3.277×10^{-1}	9.867×10^{-2}	5.013×10^{-1}	8.496×10^{-3}
3	8.537×10^{-3}	2.752×10^{-2}	2.044×10^{-1}	3.881×10^{-3}
4	9.384×10^{-3}	6.808×10^{-2}	4.916×10^{-2}	7.960×10^{-4}
5	3.807×10^{-3}	5.963×10^{-2}	1.747×10^{-2}	3.888×10^{-4}
6	1.021×10^{-3}	6.174×10^{-2}	3.852×10^{-3}	8.069×10^{-5}
7	2.254×10^{-4}	6.146×10^{-2}	9.843×10^{-4}	2.233×10^{-5}
8	6.627×10^{-5}	6.142×10^{-2}	3.053×10^{-4}	7.945×10^{-6}
9	7.381×10^{-6}	6.142×10^{-2}	3.472×10^{-5}	1.214×10^{-6}

Table 2: Self-convergence of the tracking term, the suboptimality gap in the regularizer, and the state vector iterates against the solution of the finest approximation of (P RC3) as well as convergence of the difference between the relaxed state vector and corresponding SUR approximation for the parameters $\gamma = 10^{-5}$ and $\eta = 5 \cdot 10^{-4}$.

It.	$\frac{\lvert J_t(\omega^{(n)})-J_t(\alpha^{(9)})\rvert}{J_t(\alpha^{(9)})}$	$\frac{\lvert J_r(\omega^{(n)})-J_r(\alpha^{(9)})\rvert}{J_t(\alpha^{(9)})}$	$\frac{\lVert y^{(n)}-x^{(9)}\rVert_{L^2}}{\lVert x^{(9)}\rVert_{L^2}}$	$\lVert y^{(n)} - x^{(n)}\rVert_{L^2}$
1	2.287×10^{-1}	5.981×10^{-1}	7.580×10^{-1}	7.366×10^{-3}
2	3.333×10^{-1}	1.963×10^{-1}	5.030×10^{-1}	6.179×10^{-3}
3	1.793×10^{-2}	2.049×10^{-2}	3.112×10^{-1}	9.469×10^{-3}
4	9.653×10^{-3}	1.091×10^{-2}	7.762×10^{-2}	1.549×10^{-3}
5	2.654×10^{-3}	1.441×10^{-3}	1.372×10^{-2}	2.181×10^{-4}
6	6.477×10^{-4}	2.443×10^{-3}	3.326×10^{-3}	5.095×10^{-5}
7	1.728×10^{-4}	5.189×10^{-4}	8.461×10^{-4}	2.011×10^{-5}
8	5.953×10^{-5}	1.904×10^{-4}	3.318×10^{-4}	5.539×10^{-6}
9	5.382×10^{-6}	4.918×10^{-5}	3.012×10^{-5}	1.110×10^{-6}

The difference between the tracking type terms converges to zero in both cases while the difference between the regularizing terms converges to a suboptimal value in the case of (P RC2) due to the weak lower semicontinuity of $\|\cdot\|_{L^2}$ and the fact that $(v(\omega^{(n)}))_n$ does not converge in norm. In the case of (P RC3), the same happens, but the suboptimality is significantly smaller because the $v(\omega^{(n)})$ approximate the $v(\alpha^{(n)})$ closely in norm for fine grids. This strengthens our argument to employ L^1-regularization terms when possible. The relaxed solutions $v(\alpha^{(n)})$, their SUR approximants $v(\omega^{(n)})$ and the corresponding state vectors produced by Algorithm 1.1 are plotted in Figure 4 for the L^2-case and in

Figure 5 for the L^1-case. Due to their similarity to the L^2-case, the state vectors are omitted in the L^1-case. The better approximation of the right hand sides in the norm topology in the L^1 case is clearly visible when comparing the two figures.

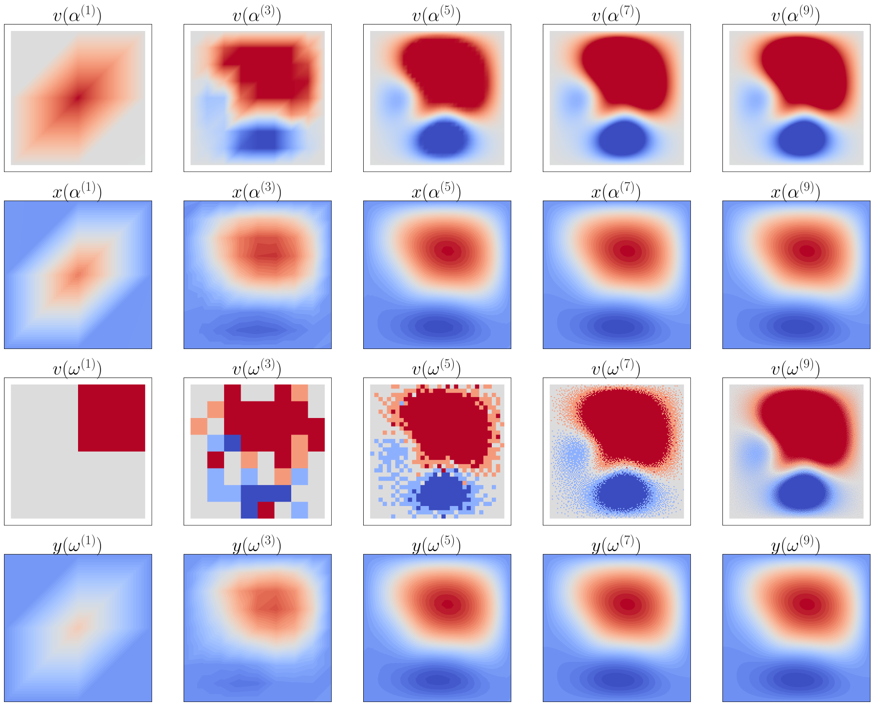

Fig. 4: Visualization of the (weak) convergence of $(v(\alpha^{(n)}))_n$, $(x(\alpha^{(n)}))_n$, $(v(\omega^{(n)}))_n$ and $(y(\omega^{(n)}))_n$ for (P RC2).

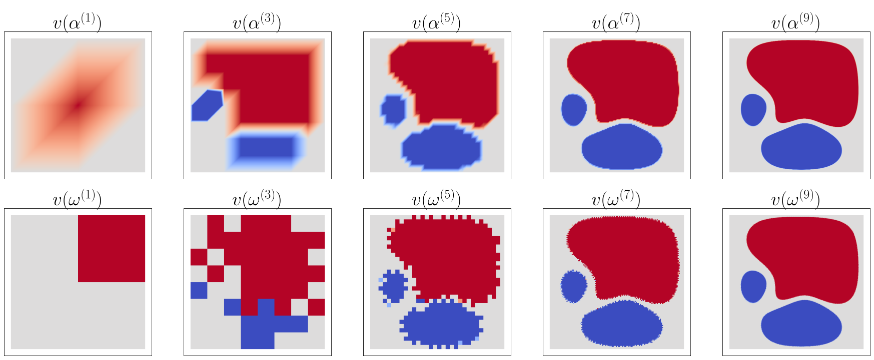

Fig. 5: Visualization of the (weak) convergence of $(v(\alpha^{(n)}))_n$ and $(v(\omega^{(n)}))_n$ for (P RC3).

6.3. Employing SUR for control reconstruction. We have shown that the multi-dimensional SUR algorithm is able to produce discrete-valued control trajectories such that a given state vector can be approximated arbitrarily well. Optimality of the approximated state vector holds if SUR is embedded into Algorithm 1.1. However, as we have only weak-* convergence in control space, a good approximation in control space in norm can only be expected if large parts of the relaxed control are already discrete-valued. We give it a try for control reconstruction and stage the following reconstruction problem to assess it. We pre-define a true binary control $\omega^* : [0,1] \to \{0,1\}$. Then, we solve the BVP

$$(-\nu\Delta + I)y_d = \omega^*, \quad y_d|_{\partial\Omega} = 0$$

to get a corresponding state y_d. Then, we employ the projected subgradient method to solve for the first order optimality conditions of a discretization of

$$\begin{aligned} &\min_{x,f} \frac{1}{2}\|x - y_d\|_{L^2}^2 \\ &\text{s.t. } (-\nu\Delta + I)x = \alpha, \quad x|_{\partial\Omega} = 0, \quad \alpha \in [0,1] \text{ a.e. on } \Omega, \end{aligned}$$

which yields a control α. The original pair (y_d, ω^*) is a (non-unique) minimizer with objective value zero. The optimization yields a blurred version α of the original control ω^*. We apply the SUR algorithm to compute $\tilde{\omega}$ from α and compare it to ω^*. We use a binary image of David Hilbert as original control ω^*. The operator $(-\nu\Delta + I)$ has a blurring effect, which can be controlled using the parameter ν. We have set $\nu = 10^{-4}$ and $\nu = 10^{-3}$ for our experiment. For the resulting relative L^2-error, we obtain $\|\omega^* - \tilde{\omega}\|_{L^2}/\|\omega^*\|_{L^2} = 1.6232 \times 10^{-1}$ for $\nu = 10^{-4}$ and $\|\omega^* - \tilde{\omega}\|_{L^2}/\|\omega^*\|_{L^2} = 2.8460 \times 10^{-1}$ for $\nu = 10^{-3}$. The controls ω^*, α and $\tilde{\omega}$ are visualized in Figure 6.

Fig. 6: Original binary-valued control ω^* (left), blurred reconstruction α (center) and binary-valued reconstruction $\tilde{\omega}$ (right) for $\nu = 10^{-4}$ (top) and $\nu = 10^{-3}$ (bottom).

The choice of which binary control is rounded to one by the SUR algorithm on a grid cell only depends on the average of the relaxed control on the current cell and the decisions for the previous grid cells. In particular, desirable features like edge detection or preservation cannot be expected as there is no optimality of the rounding with respect to any (semi-)norm like the Total Generalized Variation that is known to favor edge preservation, see [3]. This can be observed by closely inspecting the images in the bottom row, where the higher blurring was chosen.

7. Conclusion. We have addressed mixed-integer optimal control of elliptic PDEs. Theorem 5.1 shows that the infimal value of such problems may be approximated arbitrarily well by applying the SUR algorithm to a solution of a relaxation on a sufficiently fine rounding grid. The result is constructive and Theorem 4.7 shows that the approximations can be obtained on a computer by applying SUR with the input of a sufficiently fine approximation of the relaxed solution on a sufficiently fine rounding grid. An a priori estimate for the state vector convergence holds for piecewise constant relaxed controls under an ellipticity assumption on the differential operator.

If the relaxed control problem is regularized as in Section 6 to compute solutions more easily, the infimal value lies in the interval between the minimum of the regularized relaxed problem and the same value minus the upper bound of the regularizer. This interval can be controlled by the value of the penalty parameter in the regularizer. Regarding Theorems 4.5 and 4.7 we emphasize that Algorithm 1.1 and SUR are not restricted to Partial Differential Equation (PDE) settings but work for compact solution operators of dynamical systems in general.

Our approximation arguments have been known for MIOCPs with integer variables distributed in one dimension, i.e. the time domain, and are now available for integer variables distributed in more than one dimension for appropriate grid refinement strategies. We have applied the arguments to an elliptic PDE system and presented computational validations in an optimal control setting, which also showed the limitations of the approach mentioned in Section 5. The results in Subsection 6.3 indicate the difficulties arising when applying the method to recover a binary-valued control instead of approximating a desired state variable.

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