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A LINEAR BOUND ON THE INTEGRALITY GAP FOR SUM-UP ROUNDING IN THE PRESENCE OF VANISHING CONSTRAINTS

PAUL MANNS, CHRISTIAN KIRCHES, AND FELIX LENDERS

Abstract. In this article we consider the integrality gap between a relaxed continuous control trajectory and an integer feasible one that is constructively obtained in linear runtime by a family of sum-up-rounding algorithms. Such algorithms are typically invoked when solving Mixed-Integer Optimal Control Problems (MIOCPs) and serve to construct integer feasible approximations from optimal solutions of a particular relaxation. We give a constructive proof of a bound on the integrality gap in the presence of additional constraints on the discrete control. Our bound is linear in both the time discretization granularity and the number of discrete choices of the control. This result completes recent work on the approximation of feasible points of the relaxed problem by points of the combinatorial problem.

1. Introduction

Mixed-Integer Optimal Control Problems (MIOCPs) are a powerful tool to model many real-world problems, see e.g. [11] for a library of MIOCP problems. Interest in this problem class dates back to the 1980s, e.g. [2], and a recent survey of mathematical approaches and algorithms for solving MIOCPs may be found in [15]. Following the direct approach to optimal control when solving MIOCPs leads to mixed-integer nonlinear optimization problems (MINLPs). These fall into the class of NP-hard problems [5]. Relaxations also often turn out to be nonconvex due to the nonlinearity of the differential equation constraint. The comparative study [13] showed that MINLP approaches to MIOCP are generally not computationally attractive at the moment. Different authors have proposed to use optimal-control based branch&bound methods [6], or variable time transformation methods [4, 7]. Indirect approaches that make use of so-called hybrid maximum principles have been proposed by, e.g. [16] but are challenging to apply in practice due to their immediate applicability to only a selected and usually small problem class. A convexification and relaxation approach to MIOCP based on a density result in the space of measurable controls was proposed by [10], with follow-up work reported in, e.g., [8, 9, 12, 14]. A related approach was recently proposed by [17, 18].

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This work is concerned with this convexification and relaxation approach to MIOCP. We consider the following general class of MIOCPs, cf. [8,9],

$$\text{(MIOCP)}\qquad \begin{aligned} \min_{x,u,v}\ & \phi(x(1)) \\ \text{s.t.}\quad & \dot{x}(t) = f(x(t),u(t),v(t)) && \text{a.e. } t\in[0,1] \\ & x(0) = x_0 \\ & v(t) \in V && \text{a.e. } t\in[0,1] \\ & 0 \le c(x(t),u(t),v(t)) && \text{a.e. } t\in[0,1] \end{aligned}$$

where $x \in W^{1,\infty}([0,1],\mathbb{R}^{n_x})$, $u \in L^\infty([0,1],\mathbb{R}^{n_u})$, $v \in L^\infty([0,1],\mathbb{R}^{n_v})$, and where we assume a finite and discrete set $V = \{v_1,\dots,v_{|V|}\} \subset \mathbb{R}^{n_v}$, domains $D_x \subset \mathbb{R}^{n_x}$, $D_u \subset \mathbb{R}^{n_u}$, $V \subset D_v \subset \mathbb{R}^{n_v}$, $\phi \in C^1(D_x,\mathbb{R})$ and functions $f : D_x \times D_u \times D_v \to \mathbb{R}^{n_x}$ and $d : D_x \times D_u \times D_v \to \mathbb{R}^{n_d}$ being continuous in the first two arguments.

In work by Berkovitz [1], Cesari [3], Sager [10, 12, 14], and the authors [8], the equivalence of various classes of MIOCPs to their so-called *partially outer convexified* counterpart problems has been established. The *partial outer convexification* of (MIOCP) is given by

$$\text{(BC}_\delta\text{)}\qquad \begin{aligned} \phi_{\mathrm{BC}_\delta} := \min_{x,u,\omega}\ & \phi(x(1)) \\ \text{s.t.}\quad & \dot{x}(t) = \textstyle\sum_{i=1}^{|V|} \omega_i(t) f(x(t),u(t),v_i) && \text{a.e. } t\in[0,1] \\ & x(0) = x_0 \\ & \omega(t) \in \{0,1\}^{|V|} && \text{a.e. } t\in[0,1] \\ & 1 = \textstyle\sum_{i=1}^{|V|} \omega_i(t) && \text{a.e. } t\in[0,1] \\ & -\delta \le \omega_i(t) c(x(t),u(t),v_i),\ 1 \le i \le |V| && \text{a.e. } t\in[0,1] \end{aligned}$$

A counterexample in [3] shows that relaxing the mixed state-control constraint in (BC$_\delta$) is necessary to avoid certain degenerate situations. A relaxation naturally arises from weakening the SOS-1 property of ω to convex combinations.

$$\text{(RC}_\delta\text{)}\qquad \begin{aligned} \phi_{\mathrm{RC}_\delta} := \min_{x,u,\alpha}\ & \phi(x(1)) \\ \text{s.t.}\quad & \dot{x}(t) = \textstyle\sum_{i=1}^{|V|} \alpha_i(t) f(x(t),u(t),v_i) && \text{a.e. } t\in[0,1] \\ & x(0) = x_0 \\ & \alpha(t) \in [0,1]^{|V|} && \text{a.e. } t\in[0,1] \\ & 1 = \textstyle\sum_{i=1}^{|V|} \alpha_i(t) && \text{a.e. } t\in[0,1] \\ & -\delta \le \alpha_i(t) c(x(t),u(t),v_i),\ 1 \le i \le |V| && \text{a.e. } t\in[0,1] \end{aligned}$$

One is then interested in the relation of relaxed optimal solutions $(x(u^*,\alpha^*), u^*, \alpha^*)$ of (RC$_\delta$) to optimal solutions $(x(u^*,\omega^*), u^*, \omega^*)$ of (BC$_\delta$). For $u \in L^\infty([0,1],\mathbb{R}^{n_u})$, $v \in L^\infty([0,1],\mathbb{R}^{n_v})$, one has the follow very strong result: For all $\varepsilon > 0$ there is $\delta > 0$ such that

$$\phi_{\mathrm{RC}_\delta}(x^*(1;\ u^*,\alpha^*)) \le \phi_{\mathrm{BC}_\delta}(x^*(1;\ u^*,\omega^*)) < \phi_{\mathrm{RC}_\delta}(x^*(1;\ u^*,\alpha^*)) + \varepsilon$$

and $\delta \to 0$ if $\varepsilon \to 0$. The result is due to [10] in absence of the mixed state-control constraint c, and due to [8,9] including this constraint.

After discretization in time, the situation is more interesting for the second inequality. For u, v in the finite-dimensional space of piecewise constant functions on a partition of $[0,1]$ into $N \ge 1$ intervals with maximum length $\bar{\Delta}$, one has

$$\phi_{\mathrm{BC}_\delta}(x(1;\ u^*,\omega^*)) \le \phi_{\mathrm{RC}_\delta}(x(1;\ u^*,\alpha^*)) + \varepsilon \exp(L)$$

with L being the Lipschitz constant of f and under the prerequisite that said control space contains an admissible control ω^* that satisfies

$$\text{(P)} \qquad \sup_{t\in[0,1]} \left\| \int_0^t \alpha(s) - \omega(s)\,\mathrm{d}s \right\|_\infty \leq C\bar{\Delta}^E =: \varepsilon \text{ for all } t \in [0,1].$$

A related question was first given consideration by Veliov in [19, 20], who investigated the minimum Hausdorff distance between reachable sets. His result implies $E = \frac{1}{2}$, and $C = 1$ for $|V| = 2$. Sager in [10, 14] introduced the sum-up rounding technique for constructing an integer admissible piecewise constant control ω^*. In [12], he improved Veliov's result to $E = 1$ and obtained $C = |V|$ for $|V| > 2$, and established the application the MIOCP approximation problem. In [8], we showed $C \in O(\log |V|)$.

All results mentioned hold only in absence of the mixed state-control constraint c. In [8, 9], we built upon [12, 14] and showed that feasible points of (BC$_\delta$) may still be approximated arbitrarily well by feasible points of (RC$_\delta$) even if the nonlinear mixed state-control constraint c on the integer controls are present. By way of a counterexample, we showed that, after discretization, the constrained approximation problem is more difficult than the unconstrained one as a logarithmic bound does not exist and SUR-SOS-VC does not maintain feasibility. Addressing this issue, we proposed the modified sum-up-rounding algorithm SUR-SOS-VC.

1.1. Contributions.

We show that the modified sum-up-rounding algorithm SUR-SOS-VC satisfies the prerequisite (P) and constructs an integer feasible control ω with $E = 1$ and $C = \lfloor |V|/2 \rfloor$. This bound has not been reported before and is asymptotically as tight as possible.

1.2. Structure of the Article.

The remainder of this article is structured as follows. In Section 2, we introduce notation to be used throughout the article. Section 3 introduces fundamental definitions, states the new approximation result to be proved, and makes some observations and notes on the structure of the ensuing proof. Afterwards, Section 4 carries out the proof under the assumption that a certain sequence exists and can be constructed. A proof of the existence of the required sequence is the content of Section 5, where we also provide two algorithms for constructing the sequence. We close with some concluding remarks in Section 6.

2. Notation

The canonical unit vectors in $\mathbb{R}^n$ are denoted by e_i, $i \in \{1, \ldots, n\}$. For any vector $x \in \mathbb{R}^n$, we define its positive and negative part $x^+ := \max\{\mathbf{0}_{\mathbb{R}^n}, x\}$ and $x^- := -\min\{\mathbf{0}_{\mathbb{R}^n}, x\}$. We will make use of the Iverson bracket which generalizes the Kronecker delta as follows.

$$[P] := \begin{cases} 1 & : \quad \text{P is true} \\ 0 & : \quad \text{P is false} \end{cases}$$

The characteristic function of a set A is denoted $\chi_A(x) := [x \in A]$.

We denote the control dimension of the discrete variables in (BC$_\delta$) and (RC$_\delta$) as $M := |V|$. We consider temporal discretizations $0 = t_0 < \ldots < t_N = 1$ of the unit interval and $0 = t_0 < t_1 < \ldots$ of the positive half axis. We define the interval lengths $h_n := t_n - t_{n-1}$ for $n \geq 1$. Their supremum is denoted by $\bar{\Delta} := \sup_n h_n$

and their infimum by $\underline{\Delta} := \inf_n h_n$. The average values of the binary control and the relaxed control α on the n-th interval are denoted by $\omega_n \in \mathbb{R}^M$ and $\alpha_n \in \mathbb{R}^M$:

$$\omega_n := \frac{1}{h_n}\int_{t_{n-1}}^{t_n} \omega(t)\,\mathrm{d}t, \tag{2.1}$$

$$\alpha_n := \frac{1}{h_n}\int_{t_{n-1}}^{t_n} \alpha(t)\,\mathrm{d}t. \tag{2.2}$$

We define the control deviation between the relaxed and the binary control until the n-th interval (inclusively):

$$\phi_n := \int_0^{t_n} \alpha(t) - \omega(t)\,\mathrm{d}t = \sum_{k=1}^{n}(\alpha_k - \omega_k)h_k \in \mathbb{R}^M. \tag{2.3}$$

Many statements we make will make use of the order of the entries in the positive and negative part of ϕ_n for some grid point $n \in \mathbb{N}$. The following introduces the notation we require for this. For a given trajectory of control deviations $\phi = (\phi_n)_{n\in\mathbb{N}} \subset \mathbb{R}^M$, I_n^- denotes set of indices of non-positive entries of ϕ_n for some grid point $n \in \mathbb{N}$ and I_n^+ the set of indices of non-negative entries. Regarding their entries, $i_k^{n,+}$ denotes the index of the k-th biggest entry of ϕ_n^+ and $i_k^{n,-}$ denotes the index of the k-th biggest entry of ϕ_n^- which is the index of the k-th smallest index of ϕ_n. Furthermore, we introduce $\psi_{n,k}^+$ as the vector that contains the biggest k entries of ϕ_n^+ at their positions and zeros elsewhere and $\psi_{n,k}^-$ analogously. The following definition formalizes these descriptions.

Definition 2.1 (Encoding of the order within $\phi_n^\pm$)**.** *For $n \in \mathbb{N}$, we define sets of indices $I_n^- := \{i \in \{1,\dots,M\} : \phi_{n,i} \le 0\}$ and $I_n^+ := \{i \in \{1,\dots,M\} : \phi_{n,i} \ge 0\}$. The indices in $I_n^\pm$ are named as follows. $I_n^\pm = \{i_1^{n,\pm},\dots,i_{|I_n^\pm|}^{n,\pm}\}$ such that*

$$\left|\phi_{n,i_k^{n,\pm}}\right| \ge \left|\phi_{n,i_{k+1}^{n,\pm}}\right|$$

for all $k \in \{1,\dots,|I_n^\pm|-1\}$. For $k \in \{1,\dots,|I_n^\pm|\}$, we define

$$\psi_{n,k}^\pm := \sum_{j=1}^{k} e_{i_j^{n,\pm}}\phi_{n,i_j^{n,\pm}}$$

For subsets $J \subset I_0^\pm$, we define the indices $j_1^n,\dots,j_{|J|}^n$ for $n \in \mathbb{N}$ such that

$$\phi_{n,j_k^n}^\pm \ge \phi_{n,j_{k+1}^n}^\pm$$

for $k < |J|$ and $J = \{j_1^n,\dots,j_{|J|}^n\}$. Furthermore, we define

$$\psi_{n,k}^J := \sum_{m=1}^{k} e_{j_m^n}\phi_{j_m^n}^\pm.$$

For later convergence limits $\bar{\phi}^-$, we define $\bar{j}_k$ by

$$\bar{\phi}_{\bar{j}_k}^- \ge \bar{\phi}_{\bar{j}_{k+1}}^-.$$

for $k < |J|$, $J \subset I_0^-$.

3. Statement of the result

Before, we can state our main assertion, we require some preparatory definitions.

Definition 3.1 (SOS-Sum-Up Rounding Algorithms). *Let $0 = t_0 < \ldots < t_N = 1$ be a grid that discretizes the interval $[0, 1]$. We call an algorithm a SOS-Sum-Up Rounding Algorithm when it is defined recursively by*

$$\begin{aligned}
\omega(t) &:= \sum_{i=0}^{N-1} \chi_{(t_i,t_{i+1})}(t)\omega_i,\\
\omega_{i,j} &:= \left[j = \arg\max_{k\in F_i} \int_{t_i}^{t_{i+1}} \alpha_k(t)\, dt - \phi_{i,k} \right] \ , \ j \in \{1, \ldots, M\},\\
\emptyset &\neq F_i \subset \{1, \ldots, M\}.
\end{aligned}$$

In case of non-uniqueness of the maximizing index, $\arg\max$ *has to select one.*

Clearly, this algorithm is in $\mathcal{O}(N)$. We can define the original rounding scheme introduced by Sager [10] in this way.

Definition 3.2 (SUR-SOS). *The Standard SOS-Sum-Up Rounding Algorithm is defined as the SOS-Sum-Up Rounding Algorithm with*

$$F_i = \{1, \ldots, M\}. \tag{SUR-SOS}$$

It respects the required bound for problems without the combinatorial constraint $0 \leq c(x(t), u(t), v(t))$ [8, 12]. Partially outer convexification of an MIOCPs exhibiting such combinatorial constraint leads to a vanishing constraint in (BC_δ) which may experience arbitrarily large violations if treated with (SUR-SOS). In [8] we have introduced the following SOS-Sum-Up Rounding Algorithm to maintain vanishing constraint feasibility when rounding.

Definition 3.3 (SUR-SOS for Vanishing Constraints). *The Vanishing-Constraint SOS-Sum-Up Rounding Algorithm is defined as the SOS-Sum-Up Rounding Algorithm with*

$$F_i := \left\{ k \in \{1, \ldots, M\} : \int_{t_i}^{t_{i+1}} \alpha_k(t)\, dt > 0 \right\}. \tag{SUR-SOS-VC}$$

Remark 3.4. *Algorithm 5.1 will specify the choice of* $\arg\max$ *for* (SUR-SOS-VC) *in the case of non-uniqueness.*

We now define a property of (SUR-SOS-VC) that is sufficient to prove the existence of ε-feasible points for (BC_δ) from solutions of (RC_δ), cf. [8, 9].

Definition 3.5 (Vanishing-Constraint convergent algorithm). *Consider an algorithm that takes the following inputs*

(1) *a function $\alpha \in L^\infty([0, 1], \mathbb{R}^M)$ with $\sum_{j=1}^M \alpha_j(t) = 1$ for a.e. $t \in [0, 1]$ and*
(2) *a discretization grid $0 = t_0 < \ldots < t_N = 1$*

and produces the following output

(3) *a function $\omega \in L^\infty([0, 1], \mathbb{R}^M)$ such that $\sum_{j=1}^M \omega_j(t) = 1$ for a.e. $t \in [0, 1]$.*

We call such an algorithm Vanishing-Constraint convergent if

$$\int_{t_i}^{t_{i+1}} \alpha_j(t)\, dt = 0 \Rightarrow \omega_j(t) = 0 \text{ for a.e. } t \in (t_i, t_{i+1}) \ , \ i \in \{1, \ldots, N-1\}$$

and there exists a constant $C > 0$ such that

$$\sup_{t\in[0,1]} \left\| \int_0^t \alpha(s) - \omega(s)\, ds \right\|_\infty \leq C\bar{\Delta}. \tag{3.1}$$

Equipped with these definitions, we are able to state our main assertion that we prove in the following proposition.

Proposition 3.6 ((SUR-SOS-VC) is Vanishing-Constraint convergent)**.** *Let $\alpha \in L^\infty([0,1], \mathbb{R}^M)$ with $0 \leq \alpha(t) \leq 1$ and $\sum_{j=1}^M \alpha_j(t) = 1$ for a.e. $t \in [0,1]$ and $0 = t_0 < \ldots < t_N = 1$ be given. Then, (SUR-SOS-VC) is Vanishing-Constraint convergent with $C := \left\lfloor \frac{M}{2} \right\rfloor$.*

Remark 3.7.
(1) *Numerical results suggest that the sharpest choice for C is probably $\frac{1}{2}(M-1)$. It is not possible to get sharper than $\frac{1}{2}(M-1)$ as is demonstrated by an example in [8] reaching this bound.*
(2) *The important statement to prove is (3.1), the rest follows immediately from the definition of (SUR-SOS-VC).*

4. (SUR-SOS-VC) is Vanishing-Constraint convergent

This section is going to establish Proposition 3.6. We show the result for a sequence $(t_n)_{n\in\mathbb{N}_0}$ with $0 = t_0$, $t_{n-1} < t_n$, $h_n = t_n - t_{n-1} \in [\underline{\Delta}, \bar{\Delta}]$ with $\underline{\Delta} > 0$ that discretizes $\mathbb{R}_0^+$, consider $\alpha \in L^\infty(\mathbb{R}_0^+, \mathbb{R}^M)$ with $0 \leq \alpha(t) \leq 1$ and $\sum_{j=1}^M \alpha_j(t) = 1$ for a.e. $t \in \mathbb{R}_0^+$. This approach is valid because if the desired bound holds for the supremum over $\mathbb{R}_0^+$, it also holds for the supremum over the interval $[0,1]$. We will restrict to the negative part of $(\phi_n)_{n\in\mathbb{N}}$ for most of the reasoning and deduce the same result to the positive part because a completely analogous proof can be carried out. It suffices to work with the discrete trajectories $(\alpha_n)_{n\in\mathbb{N}}$, $(\omega_n)_{n\in\mathbb{N}}$ and $(\phi_n)_{n\in\mathbb{N}}$ because the norm is around the integral. Thus, we infer (3.1) by showing

$$\sup_{n\in\mathbb{N},\alpha\in A} \|\phi_n(\alpha)\|_\infty \leq C\bar{\Delta} \tag{4.1}$$

with the set of discretized control trajectories of the relaxed binary variables

$$A := \left\{ (\alpha_n)_{n\in\mathbb{N}} \subseteq [0,1] : \sum_{i=1}^M \alpha_{n,i} = 1 \text{ for all } n \in \mathbb{N} \right\}$$

and $\phi_n(\alpha) = \sum_{k=1}^n (\alpha_k - \omega_k) h_k$ for ω_k being computed recursively from $(\alpha_k)_{k\leq n}$ and $(\omega_k)_{k<n}$ with (SUR-SOS-VC). Due to this functional dependence, we abbreviate the notation and write ϕ_n instead of $\phi_n(\alpha)$ as well as $\sup_{n\in\mathbb{N}} \|\phi_n\|_\infty$ instead of $\sup_{n\in\mathbb{N},\alpha\in A} \|\phi_n(\alpha)\|_\infty$ in the remainder. After some preliminary assumptions and lemmata, we present the norm estimate (4.1) for a class of SOS-Sum-Up Rounding Algorithms with $C = M + 1$ in Theorem 4.8. Afterwards, we sharpen the estimate to $C = \left\lfloor \frac{M}{2} \right\rfloor$ for (SUR-SOS-VC) with the help of a proposition guaranteeing the possibility to choose the relaxed control variables $(\alpha_n)_n$ such that (SUR-SOS-VC) yields a certain structure in the resulting control deviations $(\phi_n)_n$.

Some of our results work for more SOS-Sum-Up Rounding Algorithms than (SUR-SOS-VC). Therefore, we introduce an assumption which is in particular satisfied by (SUR-SOS-VC), but is slightly more general, e.g. it also holds for (SUR-SOS).

Assumption 4.1 (Admissible indices for rounding)**.** *For all $n \in \mathbb{N}$*

$$\{1 \leq j \leq M : \alpha_{n,j} > 0\} \subset F_n$$

4.1. **Elementary properties of SOS-Sum-Up Rounding Algorithms.** Definition 3.1 and the prerequisites of Proposition 3.6 imply the following well-known properties of $(\omega_n)_{n\in\mathbb{N}}$ and $(\phi_n)_{n\in\mathbb{N}}$ produced by SOS-Sum-Up Rounding Algorithms for given $(\alpha_n)_{n\in\mathbb{N}}$ and $(h_n)_{n\in\mathbb{N}}$.

Lemma 4.2. *For all $n \in \mathbb{N}$, the following four properties are satisfied*

1. $\displaystyle\sum_{j=1}^{M} \phi^{+}_{n,j} = \sum_{j=1}^{M} \phi^{-}_{n,j}$,
2. $\alpha_{n,j} \geq 0$ *for all* $j \in \{1, \ldots, M\}$,
3. $\displaystyle\sum_{j=1}^{M} \alpha_{n,j} = 1$,
4. $\phi_{n,i} \geq 0$ *for at least one* $i \in \{1, \ldots, M\}$.

Proof. Follows from the definition of α and ω and the prerequisite of Proposition 3.6, cf. [8,10]. □

Lemma 4.3. *Let $n \in \mathbb{N}$, $\phi_{n,i} < \phi_{n-1,i}$ and i be selected by a SOS-Sum-Up Rounding Algorithm. Then, for all $j \in F_n \backslash \{i\}$*

$$\phi_{n,j} - \phi_{n,i} \leq h_n.$$

Furthermore, if $F_n = \{i\}$ and Assumption 4.1 holds, we get

$$\phi_n = \phi_{n-1}$$

Proof. To prove the first assertion, let the converse be true. Then,

$$\begin{aligned}
\phi_{n-1,j} + \alpha_{n,j} h_n &= \phi_{n,j} \\
&> \phi_{n,i} + h_n \\
&= \phi_{n-1,i} + \alpha_{n,i} h_n - h_n + h_n = \phi_{n-1,i} + \alpha_{n,i} h_n
\end{aligned}$$

which is a contradiction to i being selected as rounding index. The second assertion follows from Lemma 4.2 and Assumption 4.1. □

The following Lemma states an insight in the behavior of $(\|\phi_n\|_1)_{n\in\mathbb{N}}$ for $(\phi_n)_{n\in\mathbb{N}}$ produced by (SUR-SOS-VC) and will play an important role in the construction algorithms to obtain the desired result.

Lemma 4.4. *Let α be given and ω be generated by a SOS-Sum-Up Rounding Algorithm satisfying Assumption 4.1. Then, for all $n \in \mathbb{N}$, there holds:*

1. *If* $h_n \geq \max_{i \in F_n} \phi_{n-1,i} + \alpha_{n,i} h_n$ *then* $\|\phi_n\|_1 = \min \left\{ \|\phi'\|_1 \;\middle|\; \substack{\phi' = \phi_{n-1} + \alpha_n h_n - w h_n \\ w \in \{0,1\}^M, \|w\|_1 = 1 \\ w_i = 1 \Rightarrow i \in F_n} \right\}$
2. *If* $h_n \leq \max_{i \in F_n} \phi_{n-1,i} + \alpha_{n,i} h_n$ *then* $\|\phi_n\|_1 \leq \|\phi_{n-1}\|_1$
3. *If* $0 \geq \max_{i \in F_n} \phi_{n-1,i} + \alpha_{n,i} h_n$ *then* $\|\phi_n\|_1 = \|\phi_{n-1}\|_1$

Proof. To get rid of unused indices, we omit the temporal discretization index: $\phi := \phi_{n-1}, \alpha := \alpha_n, h := h_n,\ F := F_n$. Now, the Sum-Up Rounding Algorithm selects $i := \arg\max_{j \in F} \phi_j + \alpha_j h$.

(1) We need to prove $\|\phi'\|_1 \geq \|\phi_n\|_1$ for any admissible ϕ', ω. Let $\phi' = \phi + \alpha h - \omega h$ be admissible and $\omega_j = 1$ with $j \in F$. Then

$$\begin{aligned}\|\phi'\|_1 - \|\phi_n\|_1 &= \sum_{k=1}^{M}(|\phi_k + \alpha_k h - [j=k]h| - |\phi_k + \alpha_k h - [i=k]h|)\\ &= |\phi_j + \alpha_j h - h| - |\phi_j + \alpha_j h| + |\phi_i + \alpha_i h| - |\phi_i + \alpha_i h - h|.\end{aligned}$$

Let $j \in F$. It suffices to show

$$|\phi_j + \alpha_j h - h| + |\phi_i + \alpha_i h| \geq |\phi_j + \alpha_j h| + |\phi_i + \alpha_i h - h| \tag{4.2}$$

We know $\max_{j\in F} \phi_j + \alpha_j h - h \leq 0$ and thus, (4.2) is equivalent to

$$\begin{aligned}-\phi_j - \alpha_j h + h + |\phi_i + \alpha_i h| &\geq |\phi_j + \alpha_j h| - \phi_i - \alpha_i h + h\\ \Leftrightarrow |\phi_i + \alpha_i h| + \phi_i + \alpha_i h &\geq |\phi_j + \alpha_j h| + \phi_j + \alpha_j h\end{aligned}$$

First, we consider the case $\phi_i + \alpha_i h \geq 0$. If also $\phi_j + \alpha_j h \geq 0$, we obtain

$$2(\phi_i + \alpha_i h) \geq 2(\phi_j + \alpha_j h)$$

which is true due to the choice of i. If $\phi_j + \alpha_j h < 0$, we obtain

$$2(\phi_i + \alpha_i h) \geq -(\phi_j + \alpha_j h) + \phi_j + \alpha_j h = 0$$

which is true for $\phi_i + \alpha_i h \geq 0$. On the other hand if $\phi_i + \alpha_i h < 0$, then also $\phi_j + \alpha_j h < 0$ and (4.2) is equivalent to $0 \geq 0$ which is true.

(2) The claim is established as follows

$$\begin{aligned}\|\phi_n\|_1 = \sum_{j\in F} |\phi_j + \alpha_j h - \omega_j h| &= |\phi_i| + \alpha_i h - h + \sum_{j\in F, j\neq i} |\phi_j + \alpha_j h|\\ &\leq |\phi_i| + \alpha_i h - h + \sum_{j\in F, j\neq i} |\phi_j| + \alpha_j h\\ &= |\phi_i| + \alpha_i h - h + h - \alpha_i h + \sum_{j\in F, j\neq i} |\phi_j|\\ &= \sum_{j\in F} |\phi_j| = \|\phi_{n-1}\|_1\end{aligned}$$

where we have made use of $\phi_i + \alpha_i h - h \geq 0$, thus $|\phi_i + \alpha_i h - h| = \phi_i + \alpha_i h - h$ and as $\alpha_i h - h \leq 0$ also $\phi_i \geq 0$, so $|\phi_i| = \phi_i$ and Lemma 4.2 (3).

(3) From the prerequisite, we have $\phi_j + \alpha_j h \leq 0$ and consequently $\phi_j \leq 0$ for all $j \in F$ and so we get

$$\begin{aligned}\|\phi_n\|_1 = \sum_{j\in F} |\phi_j + \alpha_j h - \omega_j h| &= -\sum_{j\in F} \phi_j + \alpha_j h - \omega_j h\\ &= -\left(\sum_{j\in F} \phi_j + \sum_{j\in F} \alpha_j h - \sum_{j\in F} \omega_j h\right)\\ &= -\left(\sum_{j\in F} \phi_j + h - h\right)\\ &= \sum_{j\in F} |\phi| = \|\phi_{n-1}\|_1.\end{aligned}$$

□

Remark 4.5. (1) *For* (SUR-SOS)*, we have* $F_n = \{1, \ldots, M\}$ *and Lemma 4.2 implies that the third case in Lemma 4.4 cannot occur.*

(2) *In the case* $h_n \leq \max_{i \in F_n} \phi_{k,i} + \alpha_{k,i} h_k$*,* ω *is not necessarily chosen optimal w.r.t* $\|\cdot\|_1$ *among the admissible choices.*

(3) *According to Lemma 4.2, any increase* $\|\phi_n\|_1 - \|\phi_{n-1}\|_1$ *is equivalent to increases of half size in* ϕ_n^- *and* ϕ_n^+*:* $\|\phi_n^+\|_1 - \|\phi_{n-1}^+\|_1 = \|\phi_n^-\|_1 - \|\phi_{n-1}^-\|_1$*. Because* $\|\phi_n\|_1 \leq \|\phi_{n-1}\|_1$ *for the cases 2 and 3 of Lemma 4.4, there has to exist an index* i *such that* $0 > \phi_{n,i} > -h_n \geq -\bar{\Delta}$ *in that case.*

4.2. **Deriving the bound under Assumption 4.1.** Before starting to prove the bound, we supply the two following preparatory lemmata. The first gives a lower bound on entries of $\phi_n^\pm$ from bounds on $\|\psi_{n,k}^J\|_1$ for some $J \subset I_n^\pm$.

Lemma 4.6. *Let* $\phi_n \in \mathbb{R}^n$ *be a control deviation,* $k_0 < |J|$*,* $J \subset I_n^\pm$ *and the following bounds hold*

$$\|\psi_{n,k_0}^J\|_1 \leq \sum_{i=1}^{k_0} \xi - (i-1)\bar{\Delta} \tag{4.3}$$

with $\xi \geq k_0 \bar{\Delta}$ *and* $\zeta \geq 0$

$$\|\psi_{n,k_0+1}^J\|_1 > \sum_{i=1}^{k_0+1} \xi - (i-1)\bar{\Delta} - \zeta$$

Then,

$$|\phi_{n,j_k^n}| > \xi - k_0\bar{\Delta} - \zeta \underset{\text{if } \xi \geq (k_0+1)\bar{\Delta}, \zeta = 0}{\geq} \bar{\Delta}$$

for $1 \leq k \leq k_0 + 1$*.*

Proof. We assume the converse: $|\phi_{n,j_{k_0+1}^n}| \leq \xi - k_0\bar{\Delta} - \zeta$. Then,

$$\begin{aligned}
\|\psi_{n,k_0}^J\|_1 &> \sum_{i=1}^{k_0+1} \xi - (i-1)\bar{\Delta} - \zeta - |\phi_{n,j_{k_0+1}^n}| \\
&= \sum_{i=1}^{k_0} \xi - (i-1)\bar{\Delta} + \xi - k_0\bar{\Delta} - \zeta - |\phi_{n,j_{k_0+1}^n}| \\
&\geq \sum_{i=1}^{k_0} \xi - (i-1)\bar{\Delta}
\end{aligned}$$

which contradicts (4.3) and the claim holds for $k_0 + 1$ and also for $1 \leq k \leq k_0 + 1$ due to the order encoded by j_k^n. □

The second preparatory lemma states two very basic but repeatedly used inequalities on the behavior of the sum of the biggest elements from n to $n+1$.

Lemma 4.7. *Let* j *be the rounding index for the* n*-th interval generated by a SOS-Sum-Up Rounding Algorithm satisfying Assumption 4.1 and* $j \in J \subset \{1, \ldots, M\}$*.*

Let $\phi_{n+1,m}^+ > 0$ *for all* $m \in J$*. Then,*

$$\|\psi_{n,|J|}^+\|_1 \geq \sum_{m \in J} \phi_{n+1,m}^+$$

Let $\phi^-_{n+1,m} > 0$ for all $m \in J$. Then,

$$\|\psi^-_{n,|J|}\|_1 \geq \sum_{m \in J} \phi^-_{n+1,m} - \bar{\Delta}$$

Proof. The claim follows from the recursive update formula of the rounding algorithm, in particular the properties of α_n established in Lemma 4.2 and $\phi^\pm_{n+1,m} > 0$ for all $m \in J$ ensuring that everything is well-defined. For the first inequality, consider

$$\begin{aligned}\|\psi^+_{n,|J|}\|_1 &\geq \sum_{m \in J\setminus\{j\}} \phi_{n,m} + \phi_{n,j} \\ &= \sum_{m \in J\setminus\{j\}} \left(\phi^+_{n+1,m} - \alpha_{n+1,m}h_{n+1}\right) + \phi^+_{n+1,j} - \alpha_{n+1,j}h_{n+1} + h_{n+1} \\ &\geq \sum_{m \in J}^{k} \phi^+_{n+1,m}\end{aligned}$$

The derivation of the second inequality is very similar, but the sign in front of α and ω switches. This gives the additional summand $-\bar{\Delta}$ in the estimate. To understand the first inequality, we make two notes. First, let $\phi_{n,m} > 0$ and $\phi_{n+1,m} < 0$ for $m \in J$. Then, $m = j$ follows immediately as only one entry can decrease in one timestep and consequently $\phi_{n,m} \leq 0$ for $m \in J\setminus\{j\}$. Second, $\phi_{n,j} \leq 0$ implies $\phi^-_{n,j} = -\phi_{n,j}$ and in the other case, the inequality holds true as a positive value is subtracted.

$$\begin{aligned}\|\psi^-_{n,|J|}\|_1 &\geq \sum_{m \in J\setminus\{j\}} \phi^-_{n,m} - \phi_{n,j} \\ &= \sum_{m \in J\setminus\{j\}} \left(\phi^-_{n+1,m} + \alpha_{n+1,m}h_{n+1}\right) - \left(\phi_{n+1,j} - \alpha_{n+1,j}h_{n+1} + h_{n+1}\right) \\ &= \sum_{m \in J\setminus\{j\}} \left(\phi^-_{n+1,m} + \alpha_{n+1,m}h_{n+1}\right) + \phi^-_{n+1,j} + \alpha_{n+1,j}h_{n+1} - h_{n+1} \\ &\geq \sum_{m \in J}^{k} \phi^-_{n+1,m} - h_{n+1} \geq \sum_{m \in J}^{k} \phi^-_{n+1,m} - \bar{\Delta}.\end{aligned}$$

□

The following theorem establishes the desired bound with $C = M + 1$ for all SOS-Sum-Up Rounding Algorithms satisfying Assumption 4.1. The proof makes use of an inductive argument, and has already been presented in [8]. In this article, we sharpen the bound in the sequel to the proof.

Theorem 4.8. *For SOS-Sum-Up Rounding Algorithms satisfying Assumption 4.1, we obtain*

$$\sup_{n \in \mathbb{N}} \|\phi^\pm_n\|_\infty \leq (M+1)\bar{\Delta}$$

Proof. For the reasoning below, we define the first grid point at which the biggest k entries of $\phi^\pm_n$ sum up to a value greater than $\sum_{i=1}^{k}(C - (i-1))\bar{\Delta}$ as $n^\pm_k$ for a fixed $C > 0$:

$$n_k^{\pm} := \min \left\{ n \in \mathbb{N} : \|\psi_{n,k}^{\pm}\|_1 > \sum_{i=1}^{k} (C - (i-1))\bar{\Delta} \right\}$$

and $n_k^{\pm} := \infty$ if it does not exist.

We start with the positive part. Throughout the reasoning for the positive part, we abbreviate $i_j^n := i_j^{n,+}$ and $n_k := n_k^+$. We show the even tighter bound $M\bar{\Delta}$ for the positive part by contradiction. Thus, we set $C := M$ and assume $\sup_{n\in\mathbb{N}} \|\phi_n^+\|_\infty > M\bar{\Delta}$. We deduce that $n_1 \in \mathbb{N}$ as defined above exists such that

$$\phi^+_{n_1, i_1^{n_1}} > M\bar{\Delta}.$$

Assume we knew

$$n_k < \infty \text{ and } n_k < n_{k-1} \text{ for } 2 \le k \tag{*}$$

for $1 \le k \le M$. From (*), we infer that $n_M < \infty$ and

$$\|\psi^+_{n_M,M}\|_1 > \sum_{i=1}^{M} (M - (i-1))\bar{\Delta}$$

and $n_M < n_{M-1}$ yielding

$$\|\psi^+_{n_M,M-1}\|_1 \le \sum_{i=1}^{M-1} (M - (i-1))\bar{\Delta}.$$

Lemma 4.6 implies

$$\phi^+_{n_M,j} > 0$$

for all $j \in \{1, \ldots, M\}$ which contradicts Lemma 4.2.

Now, we establish (*) inductively. Note that n_1 is the first grid point where $\phi^+_{n,i} > M\bar{\Delta}$ happens for some i. Thus, we know $\phi^+_{n_1,i_1^{n_1}} > \phi^+_{n_1-1,j}$ for all $j \in \{1, \ldots, M\}$. As $\phi^+_{n_1,i_1^{n_1}}$ was increased, it cannot have been the rounding index. We denote the rounding index as j and get $\phi^+_{n_1,j} > (M-1)\bar{\Delta}$ from Lemma 4.3 which implies

$$\phi^+_{n_1,i_2^{n_1}} > (M-1)\bar{\Delta} \quad \text{and} \quad \|\psi^+_{n_1,2}\|_1 > M\bar{\Delta} + (M-1)\bar{\Delta}$$

from which we immediately deduce $n_2 \le n_1$. For the grid point $n_1 - 1$, we obtain with the help of Lemma 4.7

$$\|\psi^+_{n_1-1,2}\|_1 \ge \phi^+_{n_1,i_1^{n_1}} + \phi^+_{n_1,j} > M\bar{\Delta} + (M-1)\bar{\Delta}.$$

This means $n_2 < n_1$. We proceed for $k \le M-1$ and assume inductively that (*) holds for k. Again, we apply Lemma 4.6 and arrive at $\phi^+_{n_k,m} > (M-(k-1))\bar{\Delta}$ for $m \in \{i_1^{n_k}, \ldots, i_k^{n_k}\}$. Again, we denote the rounding index from $n_k - 1$ to n_k as j. We know $j \notin \{i_1^{n_k}, \ldots, i_k^{n_k}\}$ because $\|\psi^+_{n_k,k}\|_1$ was increased. Analogously to the case n_1, we obtain $\phi^+_{n_k,j} > (M-k)\bar{\Delta}$ giving the existence of n_{k+1} and $n_{k+1} \le n_k$. To see $n_{k+1} < n_k$ and close the induction, we use Lemma 4.7 and obtain

$$\|\psi^+_{n_k-1,n_{k+1}}\|_1 \ge \sum_{m=1}^{k} \phi^+_{n_k,i_m^{n_k}} + \phi^+_{n_k,j} > \sum_{i=1}^{k+1} (M-(i-1))\bar{\Delta}.$$

This proves (*).

Now, we execute a similar argument for the negative part. Throughout the reasoning for the negative part, we abbreviate $i^n_j := i^{n,-}_j$ and $n_k := n^-_k$. We prove the claim by contradiction. Thus we assume $\sup_{n\in\mathbb{N}} \|\phi^-_n\|_\infty > (M+1)\bar{\Delta}$. We deduce that there exists $n_1 \in \mathbb{N}$ such that

$$\phi^-_{n_1,i_1^{n_1}} > C\bar{\Delta}.$$

for some $C \geq M+1$. Assume we knew:

(**) For all $2 \leq k \leq M$ at least one of the following two cases holds:

(1) $n_k < \infty$, and $n_k < n_{k-1}$

(2) $n_k < \infty$, and $n_{k+1} < \infty$, and $n_k \leq n_{k-1}$, $n_{k+1} < n_{k-1}$

Then, for $k = M$, we infer that the case (1) has to be present as only M entries exist. Hence, n_M exists and $n_M < n_{M-1}$. As for the positive part, Lemma 4.6 implies

$$\phi^+_{n_M,j} > 0$$

for all $j \in \{1,\dots,M\}$ which contradicts Lemma 4.2.

Now, we prove (**) inductively and start with the base case. By definition of n_1, we know $\phi^-_{n_1,i_1^{n_1}} > \phi^-_{n_1-1,j}$ for all $j \in \{1,\dots,M\}$ and $i_1^{n_1}$ has to be the rounding index from grid point $n_1 - 1$ to n_1 because at most one entry of the negative part of the control deviation can be increased in one step. From $\phi^-_{n_1,i_1^{n_1}} > \phi^-_{n_1-1,i_1^{n_1}}$ we obtain $\alpha_{n_1-1,i_1^{n_1}} < 1$ and by virtue of Lemma 4.3, we infer that there exists $j \neq i_1^{n_1}$, $j \in F_{n_1-1}$ such that $\phi^-_{n_1,j} \geq (C-1)\bar{\Delta}$. Hence

$$\|\psi^-_{n_1,2}\|_1 > C\bar{\Delta} + (C-1)\bar{\Delta}$$

and $n_2 \leq n_1$. As the rounding index was $i_1^{n_1}$, we infer $\|\psi^-_{n_1-1,2}\|_1 > C\bar{\Delta} + (C-1)\bar{\Delta}$ by virtue of Assumption 4.1 if $F_{n_1-1} = \{i_1^{n_1}, j\}$ and we get $n_2 < n_1$. If $\{i_1^{n_1}, j\} \subsetneq F_{n_1-1}$, there exists $m \in F_{n_1-1}\backslash\{i_1^{n_1}, j\}$ with $\phi^-_{n_1,m} \geq (C-1)\bar{\Delta}$ and by virtue of Lemma 4.7, we obtain

$$\begin{aligned}\|\psi^-_{n_1-1,3}\|_1 &\geq \phi^-_{n_1,i_1^{n_1}} + \phi^-_{n_1,j} + \phi^-_{n_1,m} - \bar{\Delta}\\ &> C\bar{\Delta} + 2(C-1)\bar{\Delta} - \bar{\Delta} > C\bar{\Delta} + (C-1)\bar{\Delta} + (C-2)\bar{\Delta}\end{aligned}$$

which proves the claim for n_1. Now, assume that (**) holds for $k \leq M-1$. Assuming the case (1), i.e. $n_k < n_{k-1}$, Lemma 4.6 yields

$$\phi^-_{n_k,i_k^{n_k}} > (C-(k-1))\bar{\Delta}.$$

Analogously to previous reasoning, we infer from the definition of n_k that the rounding index is in $\{i_1^{n_k},\dots,i_k^{n_k}\}$ and $\phi^-_{n_k,j} \geq (C-k)\bar{\Delta}$ from Lemma 4.3 and Assumption 4.1 for $j \notin \{i_1^{n_k},\dots,i_k^{n_k}\}$, $j \in F_{n_k-1}$. Hence, n_{k+1} exists with $n_{k+1} \leq n_k$. If $F_{n_k-1} \subset \{i_1^{n_k},\dots,i_k^{n_k}, j\}$, we deduce $n_{k+1} < n_k$ from Lemma 4.7. If on the other hand there exists $m \in F_{n_k-1}\backslash\{i_1^{n_k},\dots,i_k^{n_k}, j\}$, then an analogous reasoning to the base case above yields $n_{k+2} < n_k$. Now, we assume case (2) is present. If $\phi^-_{n_{k+1},i_{k+1}^{n_{k+1}}} \leq (C-k)\bar{\Delta}$, we infer

$$\|\psi^-_{n_{k+1},k}\|_1 > \sum_{i=1}^{k+1}(C-(i-1))\bar{\Delta} - \phi^-_{n_{k+1},i_{k+1}^{n_{k+1}}} \geq \sum_{i=1}^{k}(C-(i-1))\bar{\Delta}$$

which implies $n_k \le n_{k+1}$ and with the help of induction hypothesis $n_k < n_{k-1}$. Hence, case (1) is present and the reasoning has already been handled. So, we can restrict us to

$$\phi^-_{n_{k+1}, i_k^{n_{k+1}}} > (C-k)\bar{\Delta} \text{ and } \phi^-_{n_{k+1}, i_{k+1}^{n_{k+1}}} > (C-k)\bar{\Delta}$$

and

$$\phi^-_{n_{k+1}, i_k^{n_{k+1}}} + \phi^-_{n_{k+1}, i_{k+1}^{n_{k+1}}} > (C-(k-1))\bar{\Delta} + (C-k)\bar{\Delta}$$

which follows from the induction hypothesis $n_{k+1} < n_{k-1}$ similar to the argument in Lemma 4.6. The rounding index has to be in $\{i_1^{n_{k+1}}, \dots, i_{k+1}^{n_{k+1}}\}$ by definition of n_{k+1}. From Lemma 4.3 and Assumption 4.1, we infer $\phi^-_{n_{k+1}, j} \ge (C-(k+1))\bar{\Delta}$ for some $j \notin \{i_1^{n_{k+1}}, \dots, i_{k+1}^{n_{k+1}}\}$, $j \in F_{n_{k+1}-1}$. Hence, n_{k+2} exists with $n_{k+2} \le n_{k+1}$. If $F_{n_{k+1}-1} \subset \{i_1^{n_{k+1}}, \dots, i_{k+1}^{n_{k+1}}, j\}$, we deduce $n_{k+2} < n_{k+1}$ from Lemma 4.7. Alternatively, if there exists $m \in F_{n_{k+1}-1} \backslash \{i_1^{n_{k+1}}, \dots, i_{k+1}^{n_{k+1}}, j\}$, Lemma 4.3 and 4.7 establish

$$\begin{aligned} \|\psi^-_{n_{k+1}-1, k+3}\|_1 &\ge \sum_{l=1}^{k+1} \phi^-_{n_{k+1}, i_l^{n_{k+1}}} + \phi^-_{n_{k+1}, j} + \phi^-_{n_{k+1}, m} - \bar{\Delta} \\ &> \sum_{i=1}^{k+1} (C-(i-1))\bar{\Delta} + 2(C-(k+1))\bar{\Delta} - \bar{\Delta} \\ &= \sum_{i=1}^{k+3} (C-(i-1))\bar{\Delta} \end{aligned}$$

which yields $n_{k+3} < n_{k+1}$ and closes the induction proving (**). □

4.3. **Assumptions.** In order to sharpen the bound, some technical constructions are necessary. The following assumption is necessary to enable them.

Assumption 4.9 (Recurrence of $\bar{\Delta}$ in $(h_n)_{n\in\mathbb{N}}$)**.** *The sequence of interval lengths $(h_n)_{n\in\mathbb{N}}$ has a subsequence $(h_{n_k})_{k\in\mathbb{N}}$ with $h_{n_k} \equiv \bar{\Delta}$.*

Remark 4.10. *Note that Assumption 4.9 is not restrictive towards proving Proposition 3.6 as the elements of $(h_{n_k})_{k\in\mathbb{N}}$ can occur at later grid points than $t_N = 1$.*

Our proof relies on the existence of a finite subsequence $(\alpha_n)_{n_1 \le n \le n_2}$ of $(\alpha_n)_{n\in\mathbb{N}}$ that can be constructed for every ϕ_{n_1} (which does not depend on $(\alpha_n)_{n>n_1}$) such that the application of (SUR-SOS-VC) implies that the negative part of ϕ_{n_2} is ε-*stairs-shaped* which is defined below in Definition 4.11 and $\|\phi_{n_1}\|_1 = \|\phi_{n_1+1}\|_1 = \dots = \|\phi_{n_2}\|_1$.

Definition 4.11 (ε-stairs-shape)**.** *Let $n \in \mathbb{N}$ and $\varepsilon > 0$. We call ϕ_n ε-stairs-shaped in a subset $J := \{j_1, \dots, j_{|J|}\} \subset I_n^{\pm}$ of the indices of the negative / positive part if there exists $1 \le m \le |J|$ such that for all $1 \le i \le m$*

$$\left|\phi^{\pm}_{n,j_i} - \phi^{\pm}_{n,j_{i+1}}\right| \in B_\varepsilon(\bar{\Delta}) \tag{4.4}$$

and for all $m+1 \le i \le |J|$

$$\left|\phi^{\pm}_{n,j_i}\right| \in [0, \bar{\Delta}). \tag{4.5}$$

The existence of such a sequence is not clear a priori and well be proven constructively in Section 5. We state its existence in the following proposition which is sufficient for the proof of (4.1) to work.

Proposition 4.12. *Let $0 < \varepsilon < \frac{\bar{\Delta}}{2}$, Assumption 4.9 hold and consider* (SUR-SOS-VC) *as SOS-Sum-Up Rounding Algorithm, $(\alpha_n)_{n\in\mathbb{N}} \in A$ be given and $n_1 \in \mathbb{N}$ with $\phi_{n_1} = \phi_{n_1}(\alpha)$. Then, there exists $(\beta_n)_{n\in\mathbb{N}} \in A$ such that $\phi_{n_1} = \phi_{n_1}(\beta)$, $\beta_k = \alpha_k$ for $k \le n_1$ and there exists $n_2 \ge n_1$ such that $\phi_{n_2}(\beta)$ is ε-stairs-shaped in $J \subset I^{\pm}_{n_2} = I^{\pm}_{n_1}$. Furthermore, $\|\phi_{n_1}(\beta)\|_1 = \ldots = \|\phi_{n_2}(\beta)\|_1$ and $\phi_{n_1,j}(\beta) = \ldots = \phi_{n_2,j}(\beta)$ for $j \notin J$.*

4.4. **Sharpening the bound for** (SUR-SOS-VC)**.** As mentioned before, the reasoning that leads to the bound $C = \left\lfloor \frac{M}{2} \right\rfloor \bar{\Delta}$ is done for ϕ^- only, as it can be carried out for ϕ^+ completely analogously. We abbreviate $i^k_j := i^{k,-}_j$ where no ambiguity is present. Furthermore, we assume (SUR-SOS-VC) to be the SOS-Sum-Up Rounding Algorithm in this section.

Definition 4.13. *We define the quantity we want to drive to zero*

$$\psi_M := \sup_{n\in\mathbb{N}} \|\psi^-_{n,1}\|_1 = \sup_{n\in\mathbb{N}} \|\psi^-_{n,1}\|_\infty = \sup_{n\in\mathbb{N}} \|\phi^-_n\|_\infty.$$

Thanks to having Theorem 4.8 already at hand, the suprema are bounded and ψ_M is well-defined. In order to formulate claims on suprema of parts of ϕ, we need to introduce the following set of iteration indices.

Definition 4.14. *For $k \in \{1,\ldots,M\}$, we define the set of iteration indices for which at least k strictly negative entries of ϕ exist*

$$N_k := \{n \in \mathbb{N} : k \le |I^-_n| \text{ and } \phi_{n,i^n_k} < 0\}.$$

With an assumption on N_k, we bind the supremum on the sum of the k biggest entries of ϕ^- from above.

Lemma 4.15. *Let Assumption 4.9 hold, $K = \left\lfloor \frac{\psi_M}{\bar{\Delta}} \right\rfloor$ and $N_k \neq \emptyset$ for all $k \in \{1,\ldots,K\}$. Then,*

$$\sup_{n\in N_k} \|\psi^-_{n,k}\|_1 \le \sum_{i=1}^{k} \psi_M - (i-1)\bar{\Delta}$$

Proof. By definition, the assertion holds true for $k = 1$. We proceed inductively and assume the claim holds for $k \in \mathbb{N}$. We can restrict to the case $k+1 \le K$ and close by contradiction. Suppose the claim holds not true for $k+1$, i.e.

$$\sup_{n\in N_{k+1}} \|\psi^-_{n,k+1}\|_1 > \sum_{i=1}^{k+1} \psi_M - (i-1)\bar{\Delta}\ . \tag{4.6}$$

We set

$$d := \sup_{n\in N_{k+1}} \|\psi^-_{n,k+1}\|_1 - \sum_{i=1}^{k+1} \psi_M - (i-1)\bar{\Delta}$$

and infer that there exist $n_1 \in \mathbb{N}$, $0 < \varepsilon < d$ and $(\alpha_n)_{n\in\mathbb{N}}$ such that

$$\|\psi^-_{n_1,k+1}\|_1 = \sum_{i=1}^{k+1} \psi_M - (i-1)\bar{\Delta} + (d - \varepsilon) \tag{4.7}$$

We remind ourselves that ϕ_{n_1} depends only on $(\alpha_n)_{n\leq n_1}$ and the same must also hold for any $(\beta_n)_{n\in\mathbb{N}}$ with $(\alpha_n)_{n\leq n_1} = (\beta_n)_{n\leq n_1}$ and in particular for the sequence $(\beta_n)_{n\in\mathbb{N}}$ provided by Proposition 4.12. For $\phi_n = \phi_n(\beta)$ we find $n_2 := n_2(\delta) \geq n_1$ such that $\|\phi_{n_1}\|_1 = \ldots = \|\phi_{n_2}\|_1$ and ϕ_{n_2} being δ-stairs-shaped in $J := \{i_1^{n_1}, \ldots, i_{k+1}^{n_1}\}$ comprising the largest $k+1$ entries of ϕ_{n_1} for all $\delta > 0$. We apply Lemma 4.6 to (4.6) and the induction hypothesis and obtain $\phi_{n,j} > \bar{\Delta}$ for $j \in J$ and $n_1 < n < n_2$ which allows us to restrict to the case (4.4). For $\delta \to 0$, we get the limit behavior

$$\begin{aligned}
\phi^-_{n_2,j_1^{n_2}} &\to \psi_M + \frac{d-\varepsilon}{k+1}\\
&\cdots\\
\phi^-_{n_2,j_{k+1}^{n_2}} &\to \psi_M - k\bar{\Delta} + \frac{d-\varepsilon}{k+1}
\end{aligned}$$

We obtain $\phi^-_{n_2,j_1^{n_2}} > \psi_M$ for some δ small enough which conflicts with ψ_M being the supremum. Hence, the assumption (4.6) was wrong and we conclude the proof. □

The upper bound on the same supremum is also proven inductively. The induction step gives much insight and we pay an extra lemma for it.

Lemma 4.16. *Let $0 < \varepsilon < \bar{\Delta}$, Assumption 4.9 be satisfied, $k+1 \leq K$, $K = \left\lfloor \frac{\psi_M}{\bar{\Delta}} \right\rfloor$, $N_k \neq \emptyset$ and define*

$$n_k^\varepsilon := \min\{n \in N_k : \sup_{l\in N_k} \|\psi^-_{l,k}\|_1 - \|\psi^-_{n,k}\|_1 < \varepsilon\}.$$

Assume

$$\|\psi^-_{n_k^\varepsilon,k}\|_1 > \sum_{i=1}^{k} \psi_M - (i-1)\bar{\Delta} - \varepsilon. \tag{4.8}$$

Then, $n_k^\varepsilon \in N_{k+1} \neq \emptyset$ and the following inequalities hold

$$\begin{aligned}
\|\psi^-_{n_{k+1}^{2\varepsilon},k+1}\|_1 &> \sum_{i=1}^{k+1} \psi_M - (i-1)\bar{\Delta} - 2\varepsilon,\\
\|\psi^-_{n_k^\varepsilon,k+1}\|_1 &> \sum_{i=1}^{k+1} \psi_M - (i-1)\bar{\Delta} - 2\varepsilon,\\
\phi^-_{n_k^\varepsilon,i_k^{n_k^\varepsilon}} &\geq \psi_M - (k-1)\bar{\Delta} - \varepsilon,\\
n_{k+1}^{2\varepsilon} &\leq n_k^\varepsilon.
\end{aligned} \tag{4.9}$$

Proof. Lemma 4.15 gives

$$\sup_{l\in\mathbb{N}} \|\psi^-_{l,k-1}\|_1 \leq \sum_{i=1}^{k-1} \psi_M - (i-1)\bar{\Delta}.$$

Using this together with (4.8), Lemma 4.6 establishes

$$\phi^-_{n_k^\varepsilon,i_k^{n_k^\varepsilon}} > \psi_M - (k-1)\bar{\Delta} - \varepsilon \underset{k+1\leq K}{\geq} 2\bar{\Delta} - \varepsilon > \bar{\Delta}. \tag{4.10}$$

By definition of n_k^ε we know that one entry of $\phi^-_{n_k^\varepsilon}$ was increased compared to $\phi^-_{n_k^\varepsilon-1}$ and by virtue of Lemma 4.3, there exists $j \in F_{n_k-1}$, $j \notin \left\{ i_1^{n_k^\varepsilon}, \dots, i_k^{n_k^\varepsilon} \right\}$ with

$$0 > \phi_{n_k^\varepsilon, i_k^{n_k^\varepsilon}} - \phi_{n_k^\varepsilon, j} \geq -\bar{\Delta}. \tag{4.11}$$

Plugging in our observation (4.10) into (4.11) gives

$$\phi^-_{n_k^\varepsilon, j} \geq \phi^-_{n_k^\varepsilon, i_k^{n_k^\varepsilon}} - \bar{\Delta} > \psi_M - k\bar{\Delta} - \varepsilon \underset{k+1 \leq K}{\geq} \bar{\Delta} - \varepsilon > 0$$

which establishes $n_k^\varepsilon \in N_{k+1}$. This yields

$$\begin{aligned}
\|\psi^-_{n_k^\varepsilon, k+1}\|_1 &\geq \|\psi^-_{n_k^\varepsilon, k}\|_1 + \phi^-_{n_k^\varepsilon, j} \\
&> \sum_{i=1}^{k} \psi_M - (i-1)\bar{\Delta} - \varepsilon + \psi_M - k\bar{\Delta} - \varepsilon \\
&\geq \sum_{i=1}^{k+1} \psi_M - (i-1)\bar{\Delta} - 2\varepsilon.
\end{aligned}$$

Consequently, we get

$$\|\psi^-_{n_{k+1}^{2\varepsilon}, k+1}\|_1 \geq \sum_{i=1}^{k+1} \psi_M - (i-1)\bar{\Delta} - 2\varepsilon \text{ and } n_{k+1}^{2\varepsilon} \leq n_k^\varepsilon. \tag{4.12}$$

□

Induction with Lemma 4.16 yields the upper bound.

Lemma 4.17. *Let $0 < \varepsilon < \bar{\Delta}$, Assumption 4.9 be satisfied, $k \in \{1, \dots, K\}$, $K = \left\lfloor \frac{\psi_M}{\bar{\Delta}} \right\rfloor$, $\varepsilon_k := \frac{\varepsilon}{2^{K-k}}$. Then, $N_k \neq \emptyset$*

$$\sup_{n \in N_k} \|\psi^-_{n,k}\|_1 > \sum_{i=1}^{k} \psi_M - (i-1)\bar{\Delta} - \varepsilon_k. \tag{4.13}$$

Proof. By definition, the claim holds true for $k = 1$. Assume the claim holds true for some k. We proceed with induction to show the assertion for $k+1$ if $k+1 \leq K$. We can apply Lemma 4.16, (4.9) with $\varepsilon = \varepsilon_k$. We obtain $N_{k+1} \neq \emptyset$ and

$$\sup_{n \in N_{k+1}} \|\psi^-_{n,k+1}\|_1 \geq \|\psi^-_{n_{k+1}^{\varepsilon_{k+1}}, k+1}\|_1 > \sum_{i=1}^{k+1} \psi_M - (i-1)\bar{\Delta} - \varepsilon_{k+1}. \tag{4.14}$$

□

We summarize our insights from Lemma 4.15, 4.16, 4.17 in the following theorem.

Theorem 4.18. *Let Assumption 4.9 be satisfied, $k \in \{1, \dots, K\}$, $K = \left\lfloor \frac{\psi_M}{\bar{\Delta}} \right\rfloor$. Then, $N_k \neq \emptyset$ and*

$$\sup_{n \in N_k} \|\psi^-_{n,k}\|_1 = \sum_{i=1}^{k} \psi_M - (i-1)\bar{\Delta}. \tag{4.15}$$

Due to (SUR-SOS-VC) (only entries $\phi_{n,k}, k \in F_n$ can undergo modifications), we can perform all constructions to prove the suprema for $(\phi_n^-)_{n\in\mathbb{N}}$ completely analogously for $(\phi_n^+)_{n\in\mathbb{N}}$ without touching $(\phi_n^-)_{n\in\mathbb{N}}$. This yields the following theorem.

Theorem 4.19. *We define ψ_M^+ and ψ_M^- analogously to ψ_M*

$$\psi_M^{\pm} := \sup_{n\in\mathbb{N}} \|\psi_{n,1}^{\pm}\|_1 = \sup_{n\in\mathbb{N}} \|\psi_{n,1}^{\pm}\|_\infty$$

and N_k^+ and N_k^- analogously to N_k

$$N_k^{\pm} := \{n \in \mathbb{N} : k \le |I_n^{\pm}| \text{ and } \phi^{\pm}_{n,i_k^{n,\pm}} > 0\}.$$

Let Assumption 4.9 be satisfied, $k^\pm \in \{1,\dots,K^\pm\}$, $K^\pm = \left\lfloor \frac{\psi_M^\pm}{\bar{\Delta}} \right\rfloor$. Then, $N^{\pm}_{k^\pm} \neq \emptyset$ and

$$\sup_{n\in N_k^\pm} \|\psi_{n,k}^{\pm}\|_1 = \sum_{i=1}^{k} \psi_M^{\pm} - (i-1)\bar{\Delta}.$$

Theorem 4.18 implies the following corollary which also establishes (3.1), (4.1) and Proposition 3.6 with $C := \left\lfloor \frac{M}{2} \right\rfloor \bar{\Delta}$.

Corollary 4.20. *Let Assumption 4.9 be satisfied. Then,*

$$\psi_M^{\pm} \le \left\lfloor \frac{M}{2} \right\rfloor \bar{\Delta}$$

Proof. We use the notation $K^\pm = \left\lfloor \frac{\psi_M^\pm}{\bar{\Delta}} \right\rfloor$. We assume the converse

$$\psi_M^{+} > \left(\left\lfloor \frac{M}{2} \right\rfloor + c \right) \bar{\Delta}, \tag{4.16}$$

for some $c > 0$ which is equivalent to $K^+ \ge \left\lfloor \frac{M}{2} \right\rfloor$. We observe from Theorem 4.19 that for all $\zeta > 0$ and especially $\zeta < c\bar{\Delta}$ there exists a minimal $n_\zeta \in \mathbb{N}$ such that

$$\|\psi^{+}_{n_\zeta,\lfloor \frac{M}{2} \rfloor}\|_1 > \sum_{i=1}^{\lfloor \frac{M}{2} \rfloor} \psi_M^{+} - (i-1)\bar{\Delta} - \zeta$$

and

$$\|\psi^{+}_{n_\zeta,\lfloor \frac{M}{2} \rfloor}\|_1 \le \sum_{i=1}^{\lfloor \frac{M}{2} \rfloor} \psi_M^{+} - (i-1)\bar{\Delta}.$$

From Lemma 4.6, we infer

$$\phi^{+}_{n_\zeta, i^{n_\zeta,+}_{\lfloor \frac{M}{2} \rfloor}} = \bar{\Delta} + c\bar{\Delta} - \zeta \underset{\zeta < c\bar{\Delta}}{>} \bar{\Delta}.$$

Because $n_\zeta \in \mathbb{N}$ is minimal, we deduce that there is $j \notin \{i_1^{n_\zeta,+}, \dots, i^{n_\zeta,+}_{\lfloor \frac{M}{2} \rfloor}\}$ such that

$$\phi^{+}_{n_\zeta,j} > 0,$$

i.e. $N^{+}_{\lfloor \frac{M}{2} \rfloor + 1} \neq \emptyset$. Then, $|I_n^-| \le \left\lfloor \frac{M}{2} \right\rfloor$ for all $n \in N^{+}_{\lfloor \frac{M}{2} \rfloor + 1}$. We assume $K^- \ge |I^-_{n_\zeta}|$. From Lemma 4.2 we deduce

$$\left\| \psi^{+}_{n_\zeta, \lfloor \frac{M}{2} \rfloor + 1} \right\|_1 \le \left\| \psi^{-}_{n_\zeta, |I^-_{n_\zeta}|} \right\|_1 \le \sum_{i=1}^{|I^-_{n_\zeta}|} \psi_M^{-} - (i-1)\bar{\Delta}$$

which implies $\psi^+_M < \psi^-_M$. Now assume that $K^- < |I^-_{n_\varsigma}| \leq \left\lfloor \frac{M}{2} \right\rfloor \leq K^+$. Then, an ε-stairs-shaped reorganization of $\phi^-_{n_\varsigma}$ exists by virtue of Proposition 4.12 which also has to abide the suprema from Theorem 4.19 for all $\varepsilon > 0$. Thus we get

$$\sum_{i=1}^{\left\lfloor \frac{M}{2} \right\rfloor} \psi^+_M - (i-1)\bar{\Delta} \leq \left\| \psi^-_{n_\varsigma, |I^-_{n_\varsigma}|} \right\|_1 \leq \sum_{i=1}^{K^-} \psi^-_M - (i-1)\bar{\Delta} + (|I^-_{n_\varsigma}| - K^-)\bar{\Delta}$$

which implies

$$K^-\psi^+_M + \sum_{i=K^-+1}^{\left\lfloor \frac{M}{2} \right\rfloor} \psi^+_M - (i-1)\bar{\Delta} \leq K^-\psi^-_M + (|I^-_{n_\varsigma}| - K^-)\bar{\Delta}$$

We know $\psi^+_M - (i-1)\bar{\Delta} > \bar{\Delta}$ for $i \in \{K^-, \ldots, |I^-_{n_\varsigma}|, \ldots, \left\lfloor \frac{M}{2} \right\rfloor\}$ and infer $\psi^+_M < \psi^-_M$ which contradicts $K^- < |I^-_{n_\varsigma}|$. So this case is impossible and the other one gives

$$\left\lfloor \frac{M}{2} \right\rfloor \bar{\Delta} \leq \psi^+_M < \psi^-_M.$$

The analogous reasoning for ψ^-_M implies $\psi^+_M > \psi^-_M$ which is a contradiction. Hence, the assumption (4.16) was false. □

We can transfer the result to all SOS-Sum-Up Rounding Algorithms satisfying Assumption 4.1.

Corollary 4.21 (The integrality gap under Assumption 4.1)**.** *Consider a SOS-Sum-Up Rounding Algorithm satisfying Assumption 4.1. Let $\alpha \in L^\infty([0,1], \mathbb{R}^M)$ with $0 \leq \alpha(t) \leq 1$ and $\sum_{j=1}^M \alpha_j(t) = 1$ for a.e. $t \in [0,1]$ and $0 = t_0 < \ldots < t_N = 1$ be given. Then, (4.1) holds with $C := \left\lfloor \frac{M}{2} \right\rfloor$.*

Proof. Assumption 4.1 states

$$\{1 \leq j \leq M : \alpha_{n,j} > 0\} \subset F_n$$

$\{1 \leq j \leq M : \alpha_{n,j} > 0\}$ is exactly the set of admissible rounding indices indices for (SUR-SOS-VC). In every step, the arg max is taken over the set of indices F_n which leads to a sequence $(\phi_n)_{n\in\mathbb{N}}$ which $\|\phi_n\|_\infty$ is at least as small as if $F_n = \{1 \leq j \leq M : \alpha_{n,j} > 0\}$ for all $n \in \mathbb{N}$. Hence, the bound from Proposition 3.6 / Corollary 4.20 transfers to all SOS-Sum-Up Rounding Algorithms satisfying Assumption 4.1. □

5. Construction Algorithms

This section establishes Proposition 4.12 constructively. We present several algorithms which generate $(\alpha_n)_{n_1 \leq n \leq n_2}$ for some starting control deviation ϕ_{n_1} such that the successive application of (SUR-SOS-VC) leads to $n_2 \geq n_1$ where $\phi^-_{n_2}$ is ε-stairs-shaped. Furthermore, the algorithm leaves $\|\cdot\|_1$ invariant, i.e. $\|\phi^-_{n_1}\|_1 = \ldots = \|\phi^-_{n_2}\|_1$. These make up the two important ingredients exploited in the proof of Lemma 4.15. To establish Proposition 4.12, we use a bottom-up approach. First, we analyze the properties of Algorithm 5.1. Then, we use it as a building block in Algorithm 5.2 which starts from a subset of the indices in a well-defined shape and establishes the ε-shaped-property for a subset containing one more index. We finish by showing that Algorithm 5.3 produces a sequence satisfying the requirements of Proposition 4.12 in an divide-and-conquer manner by executing Algorithm 5.2 repeatedly on increasing subsets of indices.

Algorithm 5.1 Generation of $(\alpha_n)_{n\geq n_1}$ to achieve $\lim_n \phi_{n,i} = \phi_{n,j} - \bar{\Delta}$

Require: Interval index n_1
Require: Control indices $i, j \in \{1, \ldots, M\}$,
Require: Initial value ϕ_{n_1} satisfying $\phi_{n_1,i} \leq \phi_{n_1,j} \leq 0$, $\phi_{n_1,i} \leq -\bar{\Delta}$
Require: $0 < \varepsilon < \frac{1}{2}$, $\quad (\varepsilon \ll \frac{1}{2})$
Require: $(h_n)_{n\geq 0} \subseteq [\underline{\Delta}, \bar{\Delta}]$

$n \leftarrow n_1$
for $n \geq n_1$ **do**

$$t \leftarrow \begin{cases} 1, & \text{A}: \quad \phi_{n,i} = \phi_{n,j} - \bar{\Delta}, \\ 1-\varepsilon, & \text{B}: \quad \phi_{n,i} - \phi_{n,j} \leq -3\bar{\Delta}, \\ \min\left\{1-\varepsilon, \frac{\phi_{n,j}-\phi_{n,i}-\bar{\Delta}}{2h_{n+1}}\right\}, & \text{C}: \quad -3\bar{\Delta} < \phi_{n,i} - \phi_{n,j} < -\bar{\Delta}, \\ 1, & \text{D}: \quad -\bar{\Delta} < \phi_{n,i} - \phi_{n,j} \leq -h_{n+1}, \\ \frac{\phi_{n,j}-\phi_{n,i}+h_{n+1}}{2h_{n+1}}, & \text{E}: \quad -h_{n+1} < \phi_{n,i} - \phi_{n,j}. \end{cases}$$

$\alpha_{n+1} \leftarrow te_i + (1-t)e_j$
$\phi_{n+1} \leftarrow$ apply (SUR-SOS-VC) $(\phi_n, \alpha_{n+1}h_{n+1}$, round i on parity)
end for

Algorithm 5.1 is used to construct a sequence $(\alpha_n)_{n\geq n_1}$ from an initial control deviation vector ϕ_{n_1} such that we achieve $\lim_n \phi_{n,i} = \phi_{n,j} - \bar{\Delta}$ once indices i, j are chosen with $\phi_{n_1,i} \leq \phi_{n_1,j} \leq 0$ and $\phi_{n_1,i} \leq -\bar{\Delta}$. Moreover, n_2 exists with and $(\phi_n)_{n\geq n_2}$ is ultimately constant under Assumption 4.9.

Lemma 5.1 (Asymptotics, $\|\cdot\|_1$-preservation of $(\phi_n)_{n\geq k}$ generated by Alg. 5.1)**.** *Let $\phi_{n_1} \in \mathbb{R}^M$ with $\phi_{n_1,i} \leq \phi_{k,j} \leq 0$, $\phi_{k,i} \leq -\bar{\Delta}$ and $\underline{\Delta} > 0$.*

Then Alg. 5.1 is well-defined: $\phi_{n,i} - \phi_{n,j} \leq 0$, $\phi_{n,i} \leq -\bar{\Delta}$, $\phi_{n,j} \leq 0$ for all $n \geq n_1$.

It produces iterates ϕ_n, α_n which satisfy $-\bar{\Delta} \leq \phi_{n,i} - \phi_{n,j} \leq 0$ for $n \geq n_\infty := n_1 + \left\lceil \frac{\phi_{n_1,j}-\phi_{n_1,i}-\bar{\Delta}}{2(1-\varepsilon)\underline{\Delta}} \right\rceil$ and $\phi_{n,i} - \phi_{n,j}$ approaches $-\bar{\Delta}$ in the following sense:

1. *If $\phi_{n_1,i} - \phi_{n_1,j} < -\bar{\Delta}$ there is $n'_\infty \leq n_\infty$ such that $\phi_{n,i} - \phi_{n,j} = -\bar{\Delta}$ for $n \geq n'_\infty$ and $(\phi_{n,i} - \phi_{n,j})_{n\geq n_1}$ increases monotonically.*
2. *If $-\bar{\Delta} < \phi_{n_1,i} - \phi_{n_1,j}$ and $\bar{\Delta}$ is an accumulation point of $(h_n)_{n\geq 0}$, the limiting behavior $\lim_n \phi_{n,i} - \phi_{n,j} = -\bar{\Delta}$ holds. If in addition $\bar{\Delta} \in (h_n)_{n\geq n_1}$, then there is $n''_\infty \geq k$ such that $\phi_{k,i} - \phi_{k,j} = -\bar{\Delta}$ for $n \geq n''_\infty$. The sequence $(\phi_{n,i} - \phi_{n,j})_{n\geq n_1}$ decreases monotonically.*
3. *For $n \geq k$ the iterates satisfy $\phi_{n,i} + \phi_{n,j} = \phi_{n_1,i} + \phi_{n_1,j}$ and $\|\phi_n\|_1 = \|\phi_k\|_1$.*
4. *For the limit $\bar{\phi}$, we get*

$$\bar{\phi}_l = \begin{cases} \frac{\phi_{k,j}+\phi_{k,i}+\bar{\Delta}}{2}, & l = j, \\ \frac{\phi_{k,j}+\phi_{k,i}-\bar{\Delta}}{2}, & l = i \\ \phi_{k,l}, & l \notin \{i,j\} \end{cases}.$$

Proof. We first analyze the effect of (SUR-SOS-VC) in the cases A, B, C, D, E in the algorithm.

If case A occurs for $n_0 \geq n_1$, $\alpha_{n_0+1,i} = 1$ and Lemma 4.3 implies $(\phi_n)_{n\geq n_0} \equiv \phi_{n_0}$.

In the cases B and C, $\phi_{n,j} > \phi_{n,i} + \bar{\Delta} \geq \phi_{n,i} + h_{n+1}\alpha_{n+1,i}$. Thus for every choice of $0 \leq t \leq 1$, the rounding decision yields $\omega_{n+1,j} = 1$. On the other hand

$\omega_{n+1,i} = 1$ if and only if

$$0 \le \phi_{n,i} + h_{n+1}\alpha_{n+1,i} - \phi_{n,j} - h_{n+1}\alpha_{n+1,j} = \phi_{n,i} - \phi_{n,j} + (2t-1)h_{n+1}$$

which is equivalent to

$$t \ge \tfrac{\phi_{n,j} - \phi_{n,i} + h_{n+1}}{2h_{n+1}}.$$

Thus in cases D and E the rounding decision yields $\omega_{n+1,i} = 1$. The new control difference can then be readily computed:

(5.1)

$$\begin{aligned}\phi_{n+1,i} - \phi_{n+1,j} &= \phi_{n,i} - \phi_{n,j} + th_{n+1} - (1-t)h_{n+1} - (\omega_{n+1,i} - \omega_{n+1,j})h_{n+1} \\ &= \begin{cases} \phi_{n,i} - \phi_{n,j} = -\bar{\Delta}, & \text{case A} \\ \phi_{n,i} - \phi_{n,j} + t2h_{n+1}, & \text{cases B and C}, \\ \phi_{n,i} - \phi_{n,j} - (1-t)2h_{n+1}, & \text{cases D and E.} \end{cases}\end{aligned}$$

To prove that the algorithm is well-defined, first note that $0 \le t \le 1$ in all cases by construction. Second we have to prove the upper bounds $\phi_{n,i} - \phi_{n,j} \le 0$, $\phi_{n,i} \le -\bar{\Delta}$, $\phi_{n,j} \le 0$ for all $n \ge n_1$, as t is undefined otherwise and $\|\phi_n\|_1$ would not necessarily be left invariant.

This is true by assumption for $n = n_1$, so we have to show that it stays true for $n+1$. By (5.1) the difference increases only in cases B and C and then by $t2h_{n+1}$. Thus $\phi_{n+1,i} - \phi_{n+1,j} \le 0$ if and only if $t \le -\frac{\phi_{n,i} - \phi_{n,j}}{2h_{n+1}}$. By construction, the choice of t satisfies this requirement. In the cases A-D, $\phi_{n+1,i} \le -\bar{\Delta}$, $\phi_{n+1,j} \le 0$ is established immediately. In the case E, the rounding decision yields

$$\begin{aligned}&\phi_{n+1,i} \le \phi_{n,i} \le -\bar{\Delta} \\ &\phi_{n+1,j} = \phi_{n,j} + (1-t)h_{n+1} \le \phi_{n,i} + th_{n+1} \underset{t \le 1, h_{n+1} \le \bar{\Delta}}{\le} 0.\end{aligned}$$

Next, we prove that $-\bar{\Delta} \le \phi_{n+1,i} - \phi_{n+1,j} \le -h_{n+1}$ if $-\bar{\Delta} \le \phi_{n,i} - \phi_{n,j}$ (cases A, D and E). There is nothing to prove in cases A and D since $t = 1$. In case E $t = \frac{\phi_{n,j} - \phi_{n,i} + h_{n+1}}{2h_{n+1}}$ and $\phi_{n+1,i} - \phi_{n+1,j}$ is reduced by $(1-t)2h_{n+1} = h_{n+1} + \phi_{n,i} - \phi_{n,j}$ so that

$$\phi_{n+1,i} - \phi_{n+1,j} = -h_{n+1} \ge -\bar{\Delta}. \tag{5.2}$$

We can deduce the assertion $-\bar{\Delta} \le \phi_{n,i} - \phi_{n,j} \le 0$ for $n \ge n_\infty$ if already $-\bar{\Delta} \le \phi_{n_1,i} - \phi_{n_1,j} \le 0$. In the case $\phi_{n_1,i} - \phi_{n_1,j} < -\bar{\Delta}$ we have to prove that the algorithm generates controls such that $-\bar{\Delta} \le \phi_{n_\infty,i} - \phi_{n_\infty,j}$. To this end, we first note that $\phi_{n,i} - \phi_{n,j}$ is strict monotonically increasing as long as $\phi_{n,i} - \phi_{n,j} < -\bar{\Delta}$ (cases B and C) and the increase is $t2h_{n+1} > 0$. By choice of t it is ensured $\phi_{n+1,j} - \phi_{n+1,i} = \min\{-\bar{\Delta}, \phi_{n,j} - \phi_{n,i} + (1-\varepsilon)2h_{n+1}\}$ where the choice $t = \frac{\phi_{n,j} - \phi_{n,i} - \bar{\Delta}}{2h_{n+1}}$ is selected at most once and leads to $\phi_{n+1,j} - \phi_{n+1,i} = -\bar{\Delta}$. Otherwise $t = 1 - \varepsilon$ and the increase satisfy $t2h_{n+1} \ge 2(1-\varepsilon)\underline{\Delta}$ proving the assertion as well as the first statement.

Now, we prove the second statement. Then, either case D or E occurs, with $\phi_{n+1,i} - \phi_{n+1,j}$ unaltered in case D and decreased in case E. As has been already shown in (5.2), if case E occurs $\phi_{n+1,i} - \phi_{n+1,j} = -h_{n+1}$. Thus $(\phi_{n,i} - \phi_{n,j})_{n \ge n_1}$ is monotonically decreasing and bounded from below by $-\bar{\Delta}$. If $\bar{\Delta}$ is an accumulation point of $(h_n)_{n \ge k}$, $\phi_{n,j} - \phi_{n,i} > -h_{n+1}$ infinitely often for $n \ge n_1$ as $\phi_{n,j} - \phi_{n,i} \ge -\bar{\Delta}$ if $h_{n+1} < \bar{\Delta}$, proving $\lim_n \phi_{n,j} - \phi_{n,i} = \liminf_n -h_n = -\bar{\Delta}$. If furthermore there

Algorithm 5.2 Inner loop of ϕ^- reordering

Require: $\phi_{n_1} \in \mathbb{R}^M$, $J_0 \subset I^-_{n_1}$, $l = |J_0| - 1$, $x > 0$, $\varepsilon_1 > 0$, $\varepsilon_1 \geq \varepsilon_0 \geq 0$
Require: $\phi^-_{n_1, j_1^{n_1}} > \phi^-_{n_1, j_2^{n_1}} > \ldots > \phi^-_{n_1, j_l^{n_1}} \geq \phi^-_{n_1, j_{l+1}^{n_1}} \geq 0$
Require: $\begin{cases} & \text{(a)} \quad \phi^-_{n_1, j_k^{n_1}} > (l - (k-1))\bar{\Delta} + x & \text{for} \quad k \leq l \\ \text{or} & \text{(b1)} \quad \phi^-_{n_1, j_k^{n_1}} > (l - (k-1))\bar{\Delta} - \frac{x}{2(l+1)} & \text{for} \quad k \leq l \\ & \text{(b2)} \quad \phi^-_{n_1, j_{l+1}^{n_1}} \geq x & \end{cases}$

$n \leftarrow 0, m \leftarrow n_1$
$\varphi_0 \leftarrow \phi_{n_1}$
$\tilde{\phi}^-_1 \leftarrow \frac{\sum_{k=1}^{l+1} \phi^-_{0, j_k^0} + \frac{l(l+1)}{2}\bar{\Delta}}{l+1}$
$\tilde{\phi}^-_k \leftarrow \tilde{\phi}^-_1 - (k-1)\bar{\Delta}\ ,\ 2 \leq k \leq l+1$
while $\exists k \in \{1, \ldots, l+1\} : |\varphi^-_{n, j_k^n} - \tilde{\phi}^-_k| \geq \min\left\{\frac{\varepsilon_1}{2}, \varepsilon_0\right\}$ **do**
    $\eta_0 \leftarrow \varphi_n$
    $a, b \leftarrow j_1^n, j_1^n$
    for $k \in \{1, \ldots, l\}$ **do**
        $a, b \leftarrow \arg\min\left\{\eta_{k-1,b}, \varphi_{n, j_{k+1}^n}\right\}, \arg\max\left\{\eta_{k-1,b}, \varphi_{n, j_{k+1}^n}\right\}$
        $n_2, (\alpha_m, \phi_m), \ldots, (\alpha_{m+n_2}, \phi_{m+n_2}) \leftarrow$ Algorithm 5.1 $(\phi_{n_1} = \eta_{k-1}, a, b)$
        $m \leftarrow m + n_2$
        $\eta_k \leftarrow \phi_m$
    end for
    $\varphi_{n+1} \leftarrow \eta_l$
    $n \leftarrow n + 1$
end while
return ϕ_n

is $\ell \geq n_1$ such that $h_\ell = \bar{\Delta}$, then $\phi_{\ell+1,i} - \phi_{\ell+1,j} = -\bar{\Delta}$ and thus $\phi_{n,i} - \phi_{n,j} = -\bar{\Delta}$ for $n \geq n''_\infty = \ell + 1$.

The third assertion follows from Lemma 4.4. The fourth follows from the third. □

Remark 5.2. (1) *Algorithm 5.1 stipulates a choice of* $\arg\max$ *for rounding in the case of non-uniqueness. If we were unwilling make this choice, the same reasoning works when i denotes the index of the smaller of the two entries and j the index of the other one instead.*

(2) *Note that* (SUR-SOS-VC) *cannot be replaced by* (SUR-SOS) *in Algorithm 5.1 because the absence of the restriction on F_n prevents us from preventing the modification of entries other than ϕ^-_j and ϕ^-_i.*

(3) *Assumption 4.9 implies the prerequisite $\bar{\Delta} \in (h_n)_{n \geq n_1}$ for Lemma 5.1 (2). Hence, we can stop Algorithm 5.1 after* $\max\{n_\infty, n''_\infty\}$ *steps with an iterate exhibiting $\phi_{n,i} - \phi_{n,j} = -\bar{\Delta}$.*

We move on to Algorithm 5.2 which starts from an iterate ϕ_{n_1} which fulfills two either the requirement case (a) or (b) ensuring that for every execution of Algorithm 5.1, the prerequisites of Lemma 5.1 hold. It constructs an iterate ϕ_{n_2} in which $l+1$ entries of the negative part are ε_1-stairs-shaped. The following Lemma establishes well-definedness and the desired convergence of the iterates under Assumption 4.9.

Lemma 5.3 (Convergence, well-definedness of Alg. 5.2)**.** *Let Assumption 4.9 hold and* ε_1, x, ε_0, $\phi^-_{n_1}$, $l \le |I^-_{n_1}| - 1$ *such that the input requirements of Algorithm 5.2 are fulfilled. Then, Algorithm 5.2 produces iterates* $(\alpha_n, \phi_n(\alpha))_{n > n_1}$ *satisfying* $I^-_n = I^-_{n-1}$ *and* $\|\phi^-_n\|_1 = \|\phi^-_{n-1}\|_1$ *for* $n > n_1$*.*

The iterates exhibit the convergence $\phi^-_n \to \bar{\phi}^-$ *with:*

$$\bar{\phi}_j = \begin{cases} -\frac{\sum_{k=1}^{l+1} \phi^-_{n_1, j^{n_1}_k} + \frac{l(l+1)}{2}\bar{\Delta}}{l+1}, & j = \bar{j}_1, \\ -\left(\bar{\phi}^-_{\bar{j}_1} - (k-1)\bar{\Delta}\right), & j = \bar{j}_k,\ 2 \le k \le l+1, \\ \phi_{j,n_1}, & j \notin J_0. \end{cases}$$

Let $\varepsilon_0 > 0$*. Then, Algorithm 5.2 stops after a finitely many steps with an iterate* $\phi^-_{n_2}$ *satisfying*

$$\phi^-_{n_2, j^{n_2}_k} \in B_{\min\left\{\varepsilon_0, \frac{\varepsilon_1}{2}\right\}}(\bar{\phi}^-_{\bar{j}_k}) \text{ and } \phi^-_{n_2, j^{n_2}_k} - \phi^-_{n_2, j^{n_2}_{k+1}} \in B_{\varepsilon_1}(\bar{\Delta})$$

for $1 \le k \le l+1$*.*

Proof. We start indexing with 0 instead of n_1 to enhance readability. It suffices to analyze the behavior of $\phi^-_{n,j}$ for $j \in J_0$ and $n \in \mathbb{N}$ because all other entries of ϕ^-_n are never touched during the execution of Algorithm 5.2. We will consider the **for**-loop as a whole and therefore work with φ_n to which the control deviation vector ϕ is assigned at the end of the n-th **for**-loop is assigned.

We will work with the differences between the k-th and $k+1$-th biggest entry of ϕ^- among the of entries in J_0 at the end of iteration n. We denote this quantity by $d_{n,k} := \varphi^-_{n,j^n_k} - \varphi^-_{n,j^n_{k+1}} \ge 0$. First, we derive update formulas to describe one cycle of the **for**-loop (line 8). For $k \in \{1, \dots, l\}$, Lemma 5.1 implies that Algorithm 5.1 touches the entries j^{n-1}_k and j^{n-1}_{k+1} to produce η_k from η_{k-1}. To avoid cumbersome notation, we write $\eta^-_{k,j}$ for the j-th biggest entry of η^-_k and obtain that the k-th iteration sets the k-th biggest entry of $\eta^-_k, \dots, \eta^-_l$ and consequently also of φ^-_{n+1}. The first iteration of the **for**-loop yields

$$\begin{aligned} \eta^-_{1,1} &= \frac{\varphi^-_{n,j^n_1} + \varphi^-_{n,j^n_2} + \bar{\Delta}}{2} = \eta^-_{2,1} = \dots = \eta^-_{l,1} = \varphi^-_{n+1,j^{n+1}_1} \\ \eta^-_{1,2} &= \frac{\varphi^-_{n,j^n_1} + \varphi^-_{n,j^n_2} - \bar{\Delta}}{2} \end{aligned} \tag{5.3}$$

and inductively, we obtain for the subsequent iterations inside the **for**-loop

$$\begin{aligned} \eta^-_{k-1,k-1} \ge \eta^-_{k,k} &= \frac{\varphi^-_{n,j^n_{k+1}} + \eta^-_{k-1,k} + \bar{\Delta}}{2} = \eta^-_{k+1,k} = \dots = \eta^-_{l,k} = \varphi^-_{n+1,j^{n+1}_k} \\ \eta^-_{k,k+1} &= \frac{\varphi^-_{n,j^n_{k+1}} + \eta^-_{k-1,k} - \bar{\Delta}}{2} \end{aligned} \tag{5.4}$$

and for the smallest entry

$$\eta^-_{l,l+1} = \frac{\varphi^-_{n,j^n_{l+1}} + \eta^-_{l-1,l} - \bar{\Delta}}{2} = \varphi^-_{n+1,j^{n+1}_{l+1}}.$$

By virtue of Lemma A.1, we obtain for $k \in \{1, \dots, l\}$

$$\varphi^-_{n+1,j_k^{n+1}} = \frac{\varphi^-_{n,j^n_{k+1}} + \varphi^-_{n+1,j^{n+1}_{k-1}}}{2} = \frac{\bar{\Delta} + \varphi^-_{n,j_1^n} + \varphi^-_{n,j_2^n} + \sum_{i=3}^{k+1} 2^{i-2}\varphi^-_{n,j_i^n}}{2^k} \tag{5.5}$$

and

$$\varphi^-_{n+1,j^{n+1}_{l+1}} = \frac{\bar{\Delta}(1-2^l) + \varphi^-_{n,j_1^n} + \varphi^-_{n,j_2^n} + \sum_{i=3}^{l+1} 2^{i-2}\varphi^-_{n,j_i^n}}{2^l}.$$

We obtain $d_{n+1,l} = \bar{\Delta}$ and by virtue of Lemma A.2 for $k \in \{1, \dots, l-1\}$

$$d_{n+1,k} = \frac{\bar{\Delta} + \sum_{j=1}^{k+1} 2^{j-1} d_{n,j}}{2^{k+1}}.$$

Algorithm 5.1 always touches the two considered entries only. Thus, $j_1^{n+1}, \dots, j_{l+1}^{n+1}$ is just a permutation of $j_1^n, \dots, j_{l+1}^n$ encoding the order of φ^- in J_0 during Algorithm 5.2 .

Well-definedness: For this part of the proof, we use the following abbreviating notation: $\varphi_{n,i} := \varphi_{n,j_i^n}$, $\phi_{m,i} := \phi_{m,j_i^n}$ for $n \in \mathbb{N}$ and $i \in \{1, \dots, l+1\}$. The iterates are well-defined if $\varphi^-_{n,i} \geq 0$, $\phi^-_{m+n_2,i} \geq 0$ in every iteration. This is in particular true if $\varphi^-_{n,l+1} \geq 0$ in every iteration. We state the observations on the update (line 14) from above in matrix form.

$$\begin{pmatrix} \bar{\Delta} \\ \varphi^-_{n+1,1} \\ \varphi^-_{n+1,2} \\ \varphi^-_{n+1,3} \\ \vdots \\ \varphi^-_{n+1,l} \\ \varphi^-_{n+1,l+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & \dots & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & 0 & \dots & 0 \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \frac{1}{2^l} & \frac{1}{2^l} & \frac{1}{2^l} & \frac{1}{2^{l-1}} & \frac{1}{2^{l-2}} & \dots & \frac{1}{2} \\ -\frac{2^l-1}{2^l} & \frac{1}{2^l} & \frac{1}{2^l} & \frac{1}{2^{l-1}} & \frac{1}{2^{l-2}} & \dots & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \bar{\Delta} \\ \varphi^-_{n,1} \\ \varphi^-_{n,2} \\ \varphi^-_{n,3} \\ \vdots \\ \varphi^-_{n,l} \\ \varphi^-_{n,l+1} \end{pmatrix}$$

Recall the prerequisite for case (a)

$$\varphi^-_{0,i} > (l-(i-1))\bar{\Delta} + x \quad \text{and} \quad \varphi^-_{0,l+1} \geq 0$$

for $i \in \{1, \dots, l\}$. By virtue of Lemma A.4 and Lemma A.3 used as induction step for induction over n, we obtain

$$\varphi^-_{n,i} > (l+1-i)\bar{\Delta} + \frac{x}{2} \geq 0$$

for all $i \in \{1, \dots, l+1\}$ and $n \geq 1$ which proves the well-definedness of all executions of Algorithm 5.1 in the iteration in case (a). Recall the prerequisites for case (b)

$$\varphi^-_{0,i} > (l-(i-1))\bar{\Delta} - \frac{x}{2(l+1)} \text{ for } i \leq l \text{ and } \varphi^-_{0,l+1} \geq x$$

and apply the reasoning from Lemma A.5 l times for the first l iterations yielding

$$\varphi^-_{k,l+1} > \tilde{\xi}$$

for $1 \leq k \leq l$ with

$$\tilde{\xi} = \prod_{j=1}^{l} \frac{2(l-(j-1))+1}{2(l+1-(j-1))} x > 0$$

which proves in particular well-definedness of the executions of Algorithm 5.1 for $k \leq l$ and for the l-th step we obtain

$$\varphi^{-}_{l,i} > (l-(i-1))\bar{\Delta} + \tilde{\xi}$$

for all $i \in \{1, \dots, l+1\}$. Now, we can apply Lemma A.3 which establishes the well-definedness for all subsequent iterations.

Convergence: To establish convergence, a view on the distances between subsequent elements of ϕ^{-}_{n} is beneficial. For the change from d_n to d_{n+1}, we get

$$d_{n+1,i} = \frac{\bar{\Delta} + \sum_{j=1}^{i+1} d_{n,j} 2^{j-1}}{2^{i+1}}$$

for $i \in \{1, \dots, l-1\}$. We obtain

$$d_{n+1,i} = \frac{1}{2} d_{n+1,i-1} + \frac{1}{2} d_{n,i+1}$$

with the boundary conditions

$$d_{n+1,1} = \frac{d_{n,1} + \bar{\Delta}}{4} + \frac{1}{2} d_{n,2}, \quad d_{n+1,l} = \bar{\Delta} \text{ for } n \geq 1$$

because the last distance after every outer loop iteration is just the result of Algorithm 5.1, i.e. $\bar{\Delta}$. We arrive at a linear update after one **for**-cycle for d_{n+1} (line 14):

$$\begin{pmatrix} \varphi^{-}_{n+1,j_1^{n+1}} \\ d_{n+1,1} \\ d_{n+1,2} \\ \vdots \\ d_{n+1,l-1} \\ d_{n+1,l} \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{2} & 0 & 0 & \dots & \frac{1}{2} \\ 0 & \frac{1}{4} & \frac{1}{2} & 0 & \dots & \frac{1}{4} \\ 0 & \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & \dots & \frac{1}{8} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \frac{1}{2^l} & \frac{1}{2^{l-1}} & \frac{1}{2^{l-2}} & \dots & \frac{1}{2} + \frac{1}{2^l} \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} \varphi^{-}_{n,j_1^{n}} \\ d_{n,1} \\ d_{n,2} \\ \vdots \\ d_{n,l-1} \\ d_{n,l} \end{pmatrix}$$

For the remainder, we denote the update matrix T. The first line and column of T imply that e_1 is an eigenvector to the eigenvalue 1. The Gershgorin circle theorem binds all other eigenvalues by 1. The minor from second column and row on is a row-stochastic matrix implying that $\begin{pmatrix} 0 & 1 & \dots & 1 \end{pmatrix}^T$ is an eigenvector to the eigenvalue 1. Due to the last line the $d_{n,l}$ is left invariant and thus the first line yields for every eigenvector v to the eigenvalue 1: $v_2 = v_{l+1}$. Inductively, we obtain $v_2 = v_3 = v_4 = \dots = v_{l+1}$. Hence, the geometric multiplicity of the eigenvalue 1 is 2 and the corresponding eigenspace is

$$\operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \right\}.$$

Analogously to the convergence of the von-Mises-iteration, we get convergence to an element in the eigenspace to the eigenvalue 1. A close look on the last row of T reveals $d_{n,l} = \bar{\Delta}$ for $n \geq 1$, yielding

$$\begin{pmatrix} d_{n,1} \\ \vdots \\ d_{n,l} \end{pmatrix} \underset{n\to\infty}{\to} \begin{pmatrix} d_{1,l} \\ \vdots \\ d_{1,l} \end{pmatrix} = \begin{pmatrix} \bar{\Delta} \\ \vdots \\ \bar{\Delta} \end{pmatrix}$$

The convergence of φ^-_{n,j_1^n} follows. As Algorithm 5.1 leaves $\|\varphi^-_n\|_1$ invariant, we can compute the convergence limit of φ^-_{n,j_1^n}:

$$\bar{\phi}^-_{\bar{j}_1} = \varphi^-_{0,j_1^n} + \frac{\frac{l(l+1)}{2}\bar{\Delta} - \sum_{k=1}^{l}(l+1-k)d_{0,k}}{l+1} = \frac{\sum_{i=1}^{l+1}\varphi^-_{0,j_i^n} + \frac{l(l+1)}{2}\bar{\Delta}}{l+1} \tag{5.6}$$

From the construction, it is clear that $j_1^0 = j_1^1 = \ldots = \bar{j}_1$. When $\varphi^-_{n,j_k^n} \geq \varphi^-_{n,j_{k+1}^n} + \frac{\bar{\Delta}}{2}$ for all $k \in \{1, \ldots, l\}$, inductive reasoning yields $\eta^-_{k-1,k} > \varphi^-_{n,j_{k+1}^n}$ which implies $j_k^n = j_k^{n+1}$ for all $k \in \{1, \ldots, l+1\}$ and consequently, there exists some $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, the order of the entries $\varphi^-_{n,j}$, $j \in J_0$, by value does not change anymore. Consequently, also ϕ^- converges in the same indices to the same limit.

Termination: An $\varepsilon_1/2$-argument infers the termination at the claimed iterate from the convergence and the termination criterion in line 5. □

We introduce Algorithm 5.3 which produces an ε-stairs-shape iterate from arbitrary inputs by executing Algorithm 5.2 on subsets of I^- in a divide-and-conquer approach: Algorithm 5.2 starts with the biggest $i-1$ entries being ε-stairs-shaped and is executed on i entries of ϕ^- until they are ε-stairs-shaped. This procedure is repeated until their $\|\cdot\|_1$-norm cannot be increased any further by replacing the lowest entry under consideration with a bigger one not considered so far and the next iteration of the **for**-loop can start from the biggest i elements being ε-stairs-shaped.

Before every execution of Algorithm 5.2, Algorithm 5.3 takes care that the inputs to Algorithm 5.2 are well-defined, i.e. either input requirement (a) or (b) is present. This ensures that within Algorithm 5.2, Algorithm 5.1 is never executed on two entries with both values strictly lower than $\bar{\Delta}$. This in turn ensures that the input requirements of all executions of Algorithm 5.1 are well-defined as we have established in Lemma 5.3. Lemma 5.4 formalizes and proves this behavior.

Lemma 5.4 (Termination of Alg. 5.3 with ε-stairs-shaped iterate ϕ_n)**.** *Let Assumption 4.9 hold, $\phi_0 \in \mathbb{R}^M$ and $\varepsilon > 0$ and L, $J_0 \subset I_0^-$ as required by Algorithm 5.3.*

Then, Algorithm 5.3 produces iterates α_n, ϕ_n which are well-defined, i.e. $I_n^- = I_{n-1}^-$, and $\|\cdot\|_1$-preserving, i.e. $\|\phi_n^-\|_1 = \|\phi_{n-1}^-\|_1$.

After a finite number of intervals n, it terminates with an iterate α_n such that ϕ_n is ε-stairs-shaped in $\{j_1^n, \ldots, j_L^n\} = J_0 \subset I_0^-$.

Proof. Calls of Algorithm 5.2 only modify elements of $J \subset J_0 \subset I_0^-$ in their calls. Consequently, only elements of J_0 are touched if all calls to Algorithm 5.2 are well-defined. The well-definedness of the iterates follows from the well-definedness of Algorithm 5.2, i.e. applicability of Lemma 5.3. Algorithm 5.3 uses φ_n which gets assigned the value of ϕ at the n-th execution of Algorithm 5.2. Thus, it suffices to

Algorithm 5.3 Outer loop of ϕ^- reordering

Require: $\phi_{n_1} \in \mathbb{R}^M$, $\frac{\bar{\Delta}}{2} > \varepsilon_1 > 0$, $J_0 \subset I^-_{n_1}$, $L = |J_0|$,
Require: $\phi^-_{n_1, j^{n_1}_1} \geq \dots \geq \phi^-_{n_1, j^{n_1}_L} \geq 0$

$n \leftarrow 0, m \leftarrow n_1$
$\varphi_0 \leftarrow \phi_{n_1}$
$\varepsilon_0 \leftarrow \varepsilon_1$
for $k \in \{1, \dots, L-1\}$ **do**
  if $\varphi^-_{n,j^n_{k+1}} \geq \bar{\Delta}$ or $(|\varphi^-_{n,j^n_k} - \varphi^-_{n,j^n_{k+1}} - \bar{\Delta}| \geq \varepsilon_1$ and $\varphi^-_{n,j^n_k} \geq \bar{\Delta})$ **then**
    while $\exists i \in \{1, \dots, k\} : |\varphi^-_{n,j^n_i} - \varphi^-_{n,j^n_{i+1}} - \bar{\Delta}| \geq \varepsilon_1$ **do**
      $J \leftarrow \{j^n_1, \dots, j^n_{k+1}\}$ if $k + 2 \leq L$ else $z = \varepsilon_1$
      $\bar{\phi}^-_{n,1} \leftarrow \varphi^-_{n,1} + \frac{\frac{k(k+1)}{2}\bar{\Delta} - \sum_{i=1}^{k}(k+1-i)d_{0,i}}{k+1}$
      $\bar{\phi}^-_{n,i} \leftarrow \bar{\phi}^-_{n,1} - (i-1)\bar{\Delta}$, $2 \leq i \leq k+1$
      if $\bar{\phi}^-_{n,k+1} - \bar{\Delta} \neq 0$ **then**
        $\varepsilon_0 \leftarrow \min\left\{\varepsilon_0, \frac{\left|\bar{\phi}^-_{n,j^n_{k+1}} - \bar{\Delta}\right|}{2}\right\}$
      else if $k + 2 \leq L$ and $\varphi^-_{n,j^n_{k+2}} > 0$ **then**
        $\varepsilon_0 \leftarrow \min\left\{\varepsilon_0, \frac{\varphi^-_{n,j^n_{k+2}}}{2(k+2)}\right\}$
      end if
      if $k + 2 \leq L$ and $\bar{\phi}^-_{n,k+1} \neq \varphi^-_{n,j^n_{k+2}}$ **then**
        $\varepsilon_0 \leftarrow \min\left\{\varepsilon_0, \frac{\left|\bar{\phi}^-_{n,j^n_{k+1}} - \varphi^-_{n,j^n_{k+2}}\right|}{2}\right\}$
      end if
      $n_2, (\alpha_m, \phi_m), \dots, (\alpha_{m+n_2}, \phi_{m+n_2}) \leftarrow$ Algorithm 5.2 $(\phi_m,\ \varepsilon_1,\ \varepsilon_0,\ J)$
      $\varphi_{n+1} \leftarrow \phi_{m+n_2}$
      $m \leftarrow m + n_2$
      $n \leftarrow n + 1$
    end while
  else
    break
  end if
end for
return ϕ_n

do the reasoning for φ as their values coincide at every step in Algorithm 5.3, but φ provides the easier indexing.

Correct result on termination: Assume the Algorithm 5.3 terminates. Then, either first **if**-procedure branched into **else** or the **for**-loop terminates after the iteration $k = L - 1$ is done. In the first case, the while loop terminated such that for $m = k - 1$, property (4.4) is satisfied with $\varepsilon = \varepsilon_1$ because $\varepsilon_0 \leq \varepsilon_1$ holds by construction. If $\phi^-_{n,j^n_k} < \bar{\Delta}$, then $m = k - 1$ also satisfies the condition (4.5) and φ_n is ε_1-stairs-shaped in $J_0 = \{j^n_1, \dots, j^n_L\}$. If on the other hand $\varphi^-_{n,j^n_k} \geq \bar{\Delta}$, the unsatisfied **if**-condition also implies $|\varphi^-_{n,j^n_k} - \varphi^-_{n,j^n_{k+1}}| \in B_{\varepsilon_1}(\bar{\Delta})$ and $\varphi^-_{n,j^n_{k+1}} < \bar{\Delta}$ implying that φ_n is ε-stairs-shaped in $J_n = \{j^n_1, \ldots, j^n_L\}$ with k and ε_1.

In the second case, the **while**-loop is the last statement inside one **for**-loop iteration. As the **for**-loop terminated, the **while**-loop terminated and φ_n is ε_1-stairs-shaped in $J_0 = \{j^n_1, \ldots, j^n_L\}$ with $m = L$.

Termination and well-definedness: We show that the execution of the first **if**-branch terminates for the iterate $k \le L-1$ if it terminated for $k-1$. For $k = 1$, Algorithm 5.2 executes Algorithm 5.1 once and $\left|\varphi^-_{1,j^0_1} - \varphi^-_{1,j^0_2}\right| = \bar{\Delta}$ after the first iteration of the **while**-loop. If exists $j \neq j^0_1 = j^1_1$ with $\varphi^-_{1,j^2_1} > \varphi^-_{1,j^1_1} - \bar{\Delta} + \varepsilon_1$, then the while loop cycles again and φ^-_{1,j^2_1} is increased by at least $0.5\varepsilon_1$. This can happen for at most $L-2$ times as the lower of the two elements taking part in the operation is decreased and cannot be selected in the next iteration anymore because either the termination criterion is satisfied or there is another bigger element that is selected. After the the first iteration, the **if**-condition leads to termination. If the first condition of the **if**-condition is true, case (a) is satisfied for the next execution of Algorithm 5.2. This is also the case if the second condition is true and $\varphi^-_{n,j^n_2} > \bar{\Delta}$. If $\varphi^-_{n,j^n_2} = \bar{\Delta}$, $|\phi^-_{n,j^n_2} - \phi^-_{n,j^n_2} - \bar{\Delta}| \ge \varepsilon_1$ implies that case (b) holds with the choice $x = \varepsilon_1$.

So, we inductively assume that for the first execution of the **while**-loop in a **for**-iteration, case (a) or (b) of Algorithm 5.2 is satisfied and proceed inductively.

We observe that the decrease of ε_0 in line 11 ensures that if the convergence limit of the lowest entry under consideration by Algorithm 5.2 is strictly greater or lower than $\bar{\Delta}$ so is the resulting iterate $\varphi_{n+1,j^k_{n+1}}$ Similarly the decrease of ε_0 in line 16 ensures that if the convergence limit of the lowest entry under consideration by Algorithm 5.2 is strictly greater or lower than the value of the next biggest entry, so is the resulting iterate.

Assume $\bar{\phi}^-_{n,k+1} > \bar{\Delta}$ and the **while**-loop terminates after its current iteration. Either the **if**-condition leads to termination or for the next execution of Algorithm 5.2 case (a) of the prerequisites is present and its execution is well-defined. If line 16 was not executed, then $\varphi^-_{n+1,j^{k+1}_{n+1}} \in B_{\frac{\varepsilon_1}{2}}(\bar{\phi}^-_{n,k+1})$ and the **while**-loop terminates. So we consider that line 16 is executed and the **while**-loop does not terminate. Clearly, case (a) is also satisfied for the next execution of Algorithm 5.2 and an increase of the $k+1$-th biggest entry happens in this execution because the lowest entry was exchanged by a bigger one. But as ε_0 is only sharpened, the entry from the previous iteration cannot be taken into account any more because the $k+1$-th entry is strictly greater than it and can only increase. This can only happen $L-k$ times and the **while**-loop terminates after at most $L-k$ iterations with case (a) satisfied for the next execution of the **while**-loop.

Assume $\bar{\phi}^-_{n,k+1} = \bar{\Delta}$. The same reasoning holds true as for the case $\bar{\phi}^-_{n,k+1} > \bar{\Delta}$ with the exception that in the last iteration of the **while**-loop, case (b) is ensured for the next execution of Algorithm 5.2 by line 13 with $x = \varepsilon_0$ if the $k+2$-th entry exists and is nonzero. If it does not exist, the **for**-loop has iterated to its end. If it is zero, then the **if**-condition is violated in the next **for**-iteration. In both cases, the algorithm terminates.

Assume $\bar{\phi}^-_{n,k+1} < \bar{\Delta}$. The same reasoning holds true as for the case $\bar{\phi}^-_{n,k+1} > \bar{\Delta}$ with the exception that for the termination iteration of the **while**-loop. But when

Algorithm 5.4 Reordering of negative part of ϕ_k

Require: ϕ_{n_1}, $\varepsilon > 0$,
```
n ← 0
φ_0 ← φ_{n_1}
while ∃j ∈ {1, ..., |I_n^-| − 1} : |φ^-_{n,i_j^{n,-}} − φ^-_{n,i_{j+1}^{n,-}} − Δ̄| ≥ ε ∧ φ_{n,i_{j+1}^{n,-}} ≤ −Δ̄ do
    for j ∈ {1, ..., |I_n^-| − 1} do
        if φ^-_{n,i_j^{n,-}} > Δ̄ or φ^-_{n,i_{j+1}^{n,-}} > Δ̄ then
            n_2, (α_m, φ_m), ..., (α_{m+n_2}, φ_{m+n_2}) ← Algorithm 5.1 (φ_m, i_j^{m,-}, i_{j+1}^{m,-})
            φ_{n+1} ← φ_{m+n_2}
            m ← m + n_2
            n ← n + 1
        end if
    end for
end while
```

the **while**-loop terminates, the **for**-loop also iterated to its end or the **if**-condition is violated and the Algorithm terminates.

Thus, in every **for**-iteration, the **while**-loop terminates after at most $L - k$ iterations and Algorithm 5.3 terminates with

$$n \leq \frac{L(L-1)}{2}$$

and all executions of Algorithm 5.2 being well-defined. □

Corollary 5.5. *Proposition 4.12 holds.*

For practical purposes, we also state the much simpler Algorithm 5.4 which has shown to do the job of Algorithm 5.3 numerically. However, we haven't found a convergence and termination proof so far and leave this assertion as a conjecture.

Conjecture 5.6 (Convergence, Termination of Alg. 5.4)**.** *Let $\varepsilon > 0$ and Assumption 4.9 hold. Then, Algorithm 5.4 produces takes an control deviation ϕ_k and produces a finite number of iterates ϕ_n, α_n, $n \in \{k+1, \ldots, N_\varepsilon\}$ such that $\|\phi_n\|_1 = \|\phi_{n-1}\|_1$, $I_n^- = I_{n-1}^-$ and $\phi^-_{N_\varepsilon}$ is ε-stairs-shaped in $I^-_{N_\varepsilon}$.*

Although we were unable to prove a convergence result and have to leave a conjecture on its convergence, we like to note that Algorithm 5.4 is the right choice if one wants to implement a reorganization of ϕ_n^- as needed for the proof of Proposition 3.6 because Algorithm 5.3 potentially drives ε_0 to exactly zero on a computer with finite precision. Figure 1 shows its behavior exemplary for 10 nonzero entries of the negative part of the initial control deviation ϕ_{n_1}.

6. Concluding Remarks

We have shown the supremum of the integrated rounding gap (4.1) between a relaxed control trajectory feasible for (RC_δ) and its rounded correspondent produced by the SOS-Sum-Up Rounding Algorithm (SUR-SOS-VC) is linearly bounded by the interval length of the temporal discretization grid. We proved the result with the constant $C = M + 1$ straightforwardly in Theorem 4.8 and sharpened it to $C = \left\lfloor \frac{M}{2} \right\rfloor$ in Corollary 4.20. As an example reaching this bound is given in [8],

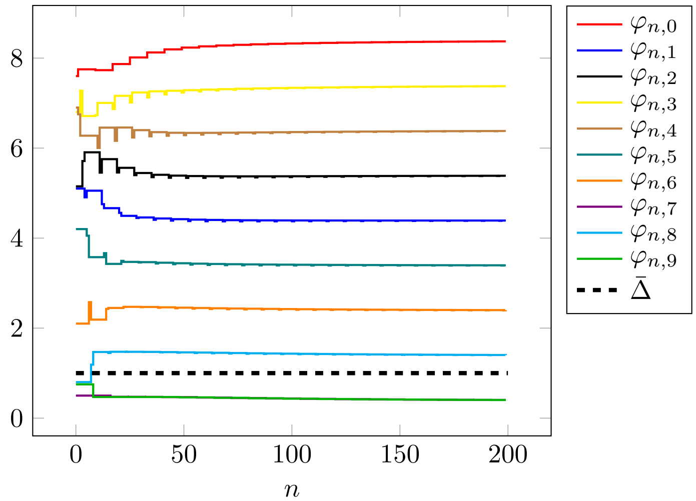

FIGURE 1. Trajectory of the nonzero entries of φ^- produced by the algorithms of Algorithm 5.4 for some exemplary φ_0^-.

the bound is as tight as possible for M being odd and asymptotically as tight as possible tight for M being even. For the proof of the sharper bound, the existence of a certain sequence was required. In particular, the sequence was employed in the proof of Lemma 4.15. Its construction was carried out in Section 5 proving Proposition 4.12 which makes up the constructive nature of our argument. Finally, we concluded in Corollary 4.21 that the derived bound holds for a family of SOS-Sum-Up Rounding Algorithms as (SUR-SOS-VC) shows their worst-case behavior.

Appendix A. Formulas in the Proof of Lemma 5.3

Lemma A.1. *In Algorithm 5.2, we have for $k \in \{1, \dots, l\}$*

$$\varphi^-_{n+1,j_k^{n+1}} = \frac{\bar{\Delta} + \varphi^-_{n,j_1^n} + \varphi^-_{n,j_2^n} + \sum_{i=3}^{k+1} 2^{i-2} \varphi^-_{n,j_i^n}}{2^k}$$

Proof. The base case $k = 1$ follows from (5.3). We close inductively with

$$\begin{aligned}
\varphi^-_{n+1,j^{n+1}_k} &\underset{(5.4)}{=} \frac{\varphi^-_{n,j^n_{k+1}} + \eta^-_{k-1,k} + \bar{\Delta}}{2} \\
&= \frac{\varphi^-_{n,j^n_{k+1}} + \eta^-_{k-1,k-1}}{2} \\
&= \frac{\varphi^-_{n,j^n_{k+1}} + \varphi^-_{n+1,j^{n+1}_{k-1}}}{2} \\
&\underset{\text{IH}}{=} \frac{\varphi^-_{n,j^n_{k+1}} + \frac{\bar{\Delta} + \varphi^-_{n,j^n_1} + \varphi^-_{n,j^n_2} + \sum_{i=3}^{k} 2^{i-2}\varphi^-_{n,j^n_i}}{2^{k-1}}}{2} \\
&= \frac{\bar{\Delta} + \varphi^-_{n,j^n_1} + \varphi^-_{n,j^n_2} + \sum_{i=3}^{k} 2^{i-2}\varphi^-_{n,j^n_i} + 2^{k-1}\varphi^-_{n,j^n_{k+1}}}{2^k} \\
&= \frac{\bar{\Delta} + \varphi^-_{n,j^n_1} + \varphi^-_{n,j^n_2} + \sum_{i=3}^{k+1} 2^{i-2}\varphi^-_{n,j^n_i}}{2^k}
\end{aligned}$$

□

Lemma A.2. *In Algorithm 5.2, we have for $k \in \{2, \dots, l\}$*

$$d_{n+1,k-1} = \frac{\bar{\Delta} + \sum_{j=1}^{k} 2^{j-1} d_{n,j}}{2^k}$$

Proof. With the help of Lemma A.1, we obtain

$$\begin{aligned}
d_{n+1,k-1} &= \varphi^-_{n+1,j^{n+1}_{k-1}} - \varphi^-_{n+1,j^{n+1}_k} \\
&\underset{(5.5)}{=} \varphi^-_{n+1,j^{n+1}_{k-1}} - \frac{\varphi^-_{n+1,j^{n+1}_{k-1}} + \varphi^-_{n,j^n_{k+1}}}{2} \\
&= \frac{\bar{\Delta} + \varphi^-_{n,j^n_1} + \varphi^-_{n,j^n_2} + \sum_{i=3}^{k} 2^{i-2}\varphi^-_{n,j^n_i} - 2^{k-1}\varphi^-_{n,j^n_{k+1}}}{2^k} \\
&= \frac{\bar{\Delta} + (\varphi^-_{n,j^n_1} - \varphi^-_{n,j^n_{k+1}}) + (\varphi^-_{n,j^n_2} - \varphi^-_{n,j^n_{k+1}}) + \sum_{i=3}^{k} 2^{i-2}(\varphi^-_{n,j^n_i} - \varphi^-_{n,j^n_{k+1}})}{2^k} \\
&= \frac{\bar{\Delta} + \sum_{j=1}^{k} d_{n,j} + \sum_{j=2}^{k} d_{n,j} + \sum_{j=3}^{k} 2^{j-2}(\sum_{k=j}^{i} d_{n,k})}{2^i} \\
&= \frac{\bar{\Delta} + \sum_{j=1}^{k} 2^{j-1} d_{n,j}}{2^k}.
\end{aligned}$$

□

Lemma A.3. *Let $\xi > 0$. Assume we have $\varphi^-_{n,j^n_i} > (l - (i-1))\bar{\Delta} + \xi$ for $i \in \{1, \dots, l+1\}$ in Algorithm 5.2 Then,*

$$\varphi^-_{n+1,j^{n+1}_i} > (l - (i-1))\bar{\Delta} + \xi$$

for $i \in \{1, \dots, l+1\}$.

Proof. For the base case, Lemma A.1 gives

$$\varphi^-_{n+1,j^{n+1}_1} = \frac{\varphi^-_{n,j^n_1} + \varphi^-_{n,j^n_2} + \bar{\Delta}}{2} > \frac{l\bar{\Delta} + \xi + (l-1)\bar{\Delta} + \xi + \bar{\Delta}}{2} = l\bar{\Delta} + \xi.$$

We proceed inductively for $i \le l$ and obtain

$$\varphi^-_{n+1,j_i^{n+1}} \underset{(5.5)}{=} \frac{\varphi^-_{n+1,j_{i-1}^{n+1}} + \varphi^-_{n,j_{i+1}^n}}{2}$$
$$\underset{\text{IH}}{>} \frac{(l-i+2)\bar{\Delta} + \xi + (l-i)\bar{\Delta} + \xi}{2} = (l-i+1)\bar{\Delta} + \xi.$$

and for the $l+1$-th entry

$$\varphi^-_{n+1,j_{l+1}^{n+1}} = \varphi^-_{n+1,j_l^{n+1}} - \bar{\Delta} > \xi.$$

□

Lemma A.4. *Let $\xi > 0$. Assume we have $\varphi^-_{n,j_i^n} > (l-(i-1))\bar{\Delta}+\xi$ for $i \in \{1,\dots,l\}$ and $\varphi^-_{n,j_{l+1}^n} \ge 0$ in Algorithm 5.2. Then,*

$$\varphi^-_{n+1,j_i^{n+1}} > (l-(i-1))\bar{\Delta} + \frac{\xi}{2}$$

for $i \in \{1,\dots,l+1\}$.

Proof. For the base case, Lemma A.1 gives

$$\phi^-_{n+1,j_1^{n+1}} = \frac{\varphi^-_{n,j_1^n} + \varphi^-_{n,j_2^n} + \bar{\Delta}}{2} > \frac{l\bar{\Delta} + \xi + (l-1)\bar{\Delta} + \xi + \bar{\Delta}}{2} = l\bar{\Delta} + \xi.$$

We proceed inductively for $i < l$ and obtain

$$\varphi^-_{n+1,j_i^{n+1}} \underset{(5.5)}{=} \frac{\varphi^-_{n+1,j_{i-1}^{n+1}} + \varphi^-_{n,j_{i+1}^n}}{2}$$
$$\underset{\text{IH}}{>} \frac{(l-i+2)\bar{\Delta} + \xi + (l-i)\bar{\Delta} + \xi}{2} = (l-i+1)\bar{\Delta} + \xi.$$

For the l-th entry, we get

$$\varphi^-_{n+1,j_l^{n+1}} = \frac{\varphi^-_{n+1,i_{l-1}^{n+1}} + \varphi^-_{n,j_l^n}}{2} > \frac{2\bar{\Delta} + \xi}{2} = \bar{\Delta} + \frac{\xi}{2}$$

and for the $l+1$-th

$$\varphi^-_{n+1,j_{l+1}^{n+1}} = \varphi^-_{n+1,j_l^{n+1}} - \bar{\Delta} > \frac{\xi}{2}.$$

□

Lemma A.5. *Let $\xi > 0$. Assume we have $\varphi^-_{n,j_i^n} > (l-(i-1))\bar{\Delta} - \frac{\xi}{2(l+1)}$ for $i \in \{1,\dots,l\}$ and $\varphi^-_{n,j_{l+1}^n} \ge \xi$ in Algorithm 5.2. Then,*

$$\varphi^-_{n+1,j_i^{n+1}} > (l-(i-1))\bar{\Delta} - \frac{\tilde{\xi}}{2l}$$

for $i \in \{1,\dots,l-1\}$ and

$$\varphi^-_{n+1,j_l^{n+1}} > \bar{\Delta} + \tilde{\xi}$$

and

$$\varphi^-_{n+1,j_l^{n+1}} > \bar{\Delta} + \tilde{\xi} \quad \textit{and} \quad \varphi^-_{n+1,j_{l+1}^{n+1}} > \tilde{\xi}$$

with

$$\tilde{\xi} := \frac{(2l+1)\xi}{2(l+1)}$$

Proof. For the base case, Lemma A.1 gives

$$\varphi^-_{n+1,j_1^{n+1}} = \frac{\varphi^-_{n,j_1^n} + \varphi^-_{n,j_2^n} + \bar{\Delta}}{2} > \frac{l\bar{\Delta} - \frac{\xi}{2(l+1)} + (l-1)\bar{\Delta} - \frac{\xi}{2(l+1)} + \bar{\Delta}}{2} = l\bar{\Delta} - \frac{\xi}{2(l+1)}.$$

We proceed inductively for $i < l$ and obtain

$$\begin{aligned}\varphi^-_{n+1,j_i^{n+1}} &\underset{(5.5)}{=} \frac{\varphi^-_{n+1,j_{i-1}^{n+1}} + \varphi^-_{n,j_{i+1}^n}}{2}\\ &\underset{\text{IH}}{>} \frac{(l-i+2)\bar{\Delta} - \frac{\xi}{2(l+1)} + (l-i)\bar{\Delta} - \frac{\xi}{2(l+1)}}{2} = (l-i+1)\bar{\Delta} - \frac{\xi}{2(l+1)}.\end{aligned}$$

For the l-th entry, we get

$$\varphi^-_{n+1,j_l^{n+1}} = \frac{\varphi^-_{n+1,i_{l-1}^{n+1}} + \varphi^-_{n,j_l^n}}{2} > \frac{2\bar{\Delta} - \frac{\xi}{2(l+1)} + \xi}{2} = \bar{\Delta} + \frac{(2l+1)\xi}{2(l+1)}$$

and for the $l+1$-th

$$\varphi^-_{n+1,j_{l+1}^{n+1}} = \varphi^-_{n+1,j_l^{n+1}} - \bar{\Delta} > \frac{(2l+1)\xi}{2(l+1)}.$$

We observe

$$-\frac{\xi}{2(l+1)} = -\frac{\frac{2l\xi}{2(l+1)}}{2l} > -\frac{\frac{(2l+1)\xi}{2(l+1)}}{2l} = -\frac{\tilde{\xi}}{2l}$$

which closes the argument. ☐

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