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SPP1962 at Weierstrass Institute for Applied Analysis and Stochastics (WIAS)
Leibniz Institute in the Forschungsverbund Berlin e.V.
Mohrenstraße 39, 10117 Berlin, Germany
E-Mail: spp1962@wias-berlin.de

World Wide Web: http://spp1962.wias-berlin.de/

Variational source condition for Ill-posed backward nonlinear Maxwell's equations

De-Han Chen* Irwin Yousept †

Abstract

This paper analyzes the Tikhonov regularization for ill-posed backward nonlinear Maxwell's equations. We propose a variational source condition (VSC), leading to power-type convergence rates for the Tikhonov regularization. By means of the complex interpolation theory, the proposed VSC is verified under a piecewise H^s-type Sobolev regularity assumption on the exact initial data. The second part of the paper is focused on the sensitivity analysis for the nonlinear forward Maxwell system and the first-order optimality conditions for the Tikhonov regularization. As a final result, we prove a strong convergence result for the corresponding adjoint state with power-type convergence rates based on VSC.

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1 Introduction

We examine the mathematical analysis for an electromagnetic inverse problem governed by evolutionary Maxwell's equations. The inverse problem is to recover (unknown) electromagnetic fields at the past time ($t = 0$) by measurement at the present time ($t = T > 0$). As the governing forward problem, we focus on nonlinear hyperbolic Maxwell's equations, where the nonlinearity arises from the modelling of nonlinear material properties as encountered in various electromagnetic materials such as ferromagnetic

*School of Mathematics & Statistics, Central China Normal University. Wuhan 430079, China, and Universität Duisburg-Essen, Fakultät für Mathematik, Thea-Leymann-Str. 9, D-45127 Essen, Germany, (chen.dehan@uni-due.de.). The work of DC was supported by the Alexander von Humboldt foundation through a postdoctoral research fellowship and hosted by Universität Duisburg-Essen.

†Universität Duisburg-Essen, Fakultät für Mathematik, Thea-Leymann-Str. 9, D-45127 Essen, Germany, (irwin.yousept@uni-due.de.). This work of IY was supported by the German Research Foundation Priority Programm DFG SPP 1962 "Non-smooth and Complementarity-based Distributed Parameter Systems: Simulation and Hierarchical Optimization", Project YO 159/2-1.

materials or superconductors (cf. [18, 32, 34]). To be more precise, we consider the following nonlinear hyperbolic Maxwell system:

$$\begin{cases} \varepsilon\partial_t \mathbf{E}(x,t)) - \nabla\times\mathbf{H} = \mathbf{F}_1(t,x,\mathbf{E}(x,t),\mathbf{H}(x,t)) & \text{in } \Omega\times(0,T), \\ \mu\partial_t \mathbf{H}(x,t) + \nabla\times\mathbf{E} = \mathbf{F}_2(t,x,\mathbf{E}(x,t),\mathbf{H}(x,t)) & \text{in } \Omega\times(0,T), \\ \mathbf{E}(x,t)\times\mathbf{n} = 0 & \text{on } \partial\Omega\times(0,T). \end{cases} \tag{1.1}$$

In the setting of (1.1), $\Omega\subset\mathbb{R}^3$ denotes a bounded Lipschitz domain and $\mathbf{F}_1, \mathbf{F}_2 : [0,T]\times\Omega\times\mathbb{R}^3\times\mathbb{R}^3\to\mathbb{R}^3$ are (given) nonlinear functions. Furthermore, $\mathbf{E} : \Omega\times[0,T]\to\mathbb{R}^3$ denotes the electric field, $\mathbf{H} : \Omega\times[0,T]\to\mathbb{R}^3$ the magnetic field, $\epsilon : \Omega\to\mathbb{R}^{3\times3}$ the electric permittivity and $\mu : \Omega\to\mathbb{R}^{3\times3}$ the magnetic permeability. The precise mathematical assumptions for all the data involved in (1.1) will be specified in Section 2. Let us underline that, in the Maxwell forward system (1.1), the material parameters ϵ and μ are assumed to be known data. We refer to [11, 20] for ill-posed identification problems of the electric permittivity and the magnetic permeability in the linear counterpart to (1.1).

Given data for the electromagnetic fields at the final time $(\mathbf{e}^\dagger, \mathbf{h}^\dagger)$, our goal is to recover the initial value $(\mathbf{E}(\cdot,0), \mathbf{H}(\cdot,0)) =: (\mathbf{u}^\dagger, \mathbf{v}^\dagger)$ in the space

$$\mathbf{Y} := \{(\mathbf{u},\mathbf{v}) \in \mathbf{H_0}(\mathbf{curl})\times\mathbf{H}(\mathbf{curl}) \mid \varepsilon\mathbf{u}\in\mathbf{H}(\mathrm{div}),\ \ \mu\mathbf{v}\in\mathbf{H_0}(\mathrm{div})\},$$

where the imposed divergence conditions in $\mathbf{Y}$ are motivated by the physical Gauss law for magnetic and electric fields. Note that, since ϵ and μ are only of class $L^\infty(\Omega)^{3\times3}$, the Hilbert space $\mathbf{Y}$ is not embedded to any Sobolev space $\mathbf{H}^s(\Omega)\times\mathbf{H}^s(\Omega)$ for $s>0$. However, by the Maxwell compactness embedding theory [28, 22], the embedding $\mathbf{Y}\hookrightarrow\mathbf{L}^2(\Omega)\times\mathbf{L}^2(\Omega)$ is compact. Our analysis will benefit from this compactness result. The considered inverse problem is ill-posed in the following sense: If we replace the exact terminal data $(\mathbf{e}^\dagger, \mathbf{h}^\dagger)$ by a noisy pattern $(\mathbf{e}^\delta, \mathbf{h}^\delta)$ satisfying

$$\|(\mathbf{e}^\delta, \mathbf{h}^\delta) - (\mathbf{e}^\dagger, \mathbf{h}^\dagger)\|_{\mathbf{L}^2_\epsilon(\Omega)\times\mathbf{L}^2_\mu(\Omega)} \le \delta,$$

where the parameter $\delta>0$ is used to represent a noise level in the data, then the initial value of the (mild) solution to (1.1) may not belong to the space $\mathbf{Y}$. Even if it belongs to $\mathbf{Y}$ and δ is small, the solution may still be far from the exact initial value $(\mathbf{u}^\dagger, \mathbf{v}^\dagger)\in\mathbf{Y}$.

To deal with the ill-posedness, we shall consider the Tikhonov regularization technique by solving a least-squares nonlinear minimization problem. Our main goal is to examine the convergence rate of the regularized solution under an appropriate choice of the noise level δ and the Tikhonov regularization parameter. To obtain the convergence rate, one usually requires an additional smoothness assumption on the true solution, well-known as the so-called *source condition.* In general, (classical) source conditions could be restrictive since they require the Fréchet differentiability of the forward operator and further properties on the adjoint of the Fréchet-derivative (cf. [17, 19]). Our present work shall focus on the so-called *variational source condition* (VSC). To the best of our knowledge, the concept of VSC was introduced independently by Flemming [7] and Grasmair [9]. In contrast to the classical source condition, VSC does not require any differentiability assumption on the forward operator. More importantly, under appropriate parameter choice rules (see [16]), convergence rates can be deduced from VSC in a straightforward manner.

In literature, there is only a small number of contributions towards verification of VSC for inverse problems. In particular, little is known concerning verification of VSC for inverse problems governed by partial differential equations. For abstract linear operators and ℓ^p penalties with respect to certain bases, we refer to [1, 4, 5] and references therein. Recently, Hohage and Weilding [13, 14] shown that VSC is valid for the Tikhonov regularization of inverse scattering problems. In particular, based on VSC, they obtained convergence with logarithmic-type rates for the corresponding regularized solutions. For more details between VSC and classical source conditions, we refer the reader to [13, 14, 15]. See also [5] for recent results on VSC for elastic-net regularizations.

In this work, we shall propose VSC for the Tikhonov regularization of the ill-posed backward nonlinear Maxwell's equations (1.1). The main novelty of our contribution includes the verification of VSC under piecewise H^s-type Sobolev regularity assumptions on the exact initial value by means of the interpolation theory. This piecewise regularity assumption is related to the physical material structure of the medium Ω consisting of different homogeneous materials (see **(A4)** in page 17). Since our techniques are different from those proposed in [13, 14, 15], we believe that our results may help enrich the works on VSC for inverse problems. In addition to the verification of VSC, we also examine the sensitivity analysis of the associated Maxwell forward operator and establish its Gâteaux-differentiability property. In particular, this result allows us to develop adjoint techniques and derive first-order optimality conditions for the associated Tikhonov regularization problem. Then, based on VSC, we obtain convergence rates for the corresponding adjoint state. To our best knowledge, these results have not been obtained in the literature of PDE-constrained optimization.

The outline of the paper is as follows. In Section 2, we provide the mathematical formulation for the ill-posed backward nonlinear Maxwell's equations (1.1) and the associated Tikhohov regularization. Section 3 is concerned with the well-posedneess of Tikhohov regularization and its first-order sensitivity analysis. In section 4, we establish the validity of VSC for the considered inverse problem and derive convergence rates for regularised solutions and adjoint states.

2 Preliminaries and mathematical formulation

Throughout this paper, for a given Hilbert space H, we denote by $(\cdot, \cdot)_H$ the standard inner product and by $\|\cdot\|_H$ the standard norm of H. If H is continuously embedded into a normed vector space V, then we write $V \hookrightarrow H$. The notation $\mathcal{L}(X, Y)$ stands for the space of all bounded linear operator from a normed space X into another normed space Y endowed with the standard operator norm $\|\cdot\|_{\mathcal{L}(X,Y)}$. If $Y = X$, then we use the abbreviation $\mathcal{L}(X)$. For an interval $J \subset \mathbb{R}$, $1 \leq p \leq \infty$, and a normed space X, let $L^p(J; X)$ denote the classical L^p-Bochner space. Moreover, $C([a, b]; X)$ denotes the Banach space of all continuous function from $[a, b]$ to X. A bold typeface is used to indicate a three-dimensional vector function or a Hilbert space of three-dimensional vector functions. In our analysis, we mainly deal with the following Hilbert spaces:

$$\begin{aligned}
\mathbf{H}(\mathbf{curl}) &:= \{\mathbf{q} \in \mathbf{L^2}(\Omega); \mathbf{curl}\,\mathbf{q} \in \mathbf{L^2}(\Omega)\}, \\
\mathbf{H_0}(\mathbf{curl}) &:= \{\mathbf{q} \in \mathbf{H}(\mathbf{curl}); \mathbf{q} \times \mathbf{n} = 0 \text{ on } \partial\Omega\}, \\
\mathbf{H}(\mathrm{div}) &: = \{\mathbf{q} \in \mathbf{L^2}(\Omega); \mathrm{div}\,\mathbf{q} \in L^2(\Omega)\}, \\
\mathbf{H_0}(\mathrm{div}) &: = \{\mathbf{q} \in \mathbf{H}(\mathrm{div}); \mathbf{q} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\},
\end{aligned}$$

where the **curl**- and div-operators as well as the normal and tangential traces are understood in the sense of distributions (cf. [8]).

For a symmetric and uniformly positive definite function $\alpha \in L^\infty(\Omega)^{3\times 3}$, the notation $\mathbf{L}^2_\alpha(\Omega)$ stands for the α-weighted $\mathbf{L^2}(\Omega)$-space with the weighted scalar product $(\alpha\cdot, \cdot)_{\mathbf{L^2}(\Omega)}$. Then, we define the weighted Hilbert space $\mathbf{X} := \mathbf{L}^2_\varepsilon(\Omega) \times \mathbf{L}^2_\mu(\Omega)$, equipped with the weighted scalar product

$$((\mathbf{u}, \mathbf{v}), (\hat{\mathbf{u}}, \hat{\mathbf{v}}))_{\mathbf{X}} := (\varepsilon \mathbf{u}, \hat{\mathbf{u}})_{\mathbf{L^2}(\Omega)} + (\mu \mathbf{v}, \hat{\mathbf{v}})_{\mathbf{L^2}(\Omega)} \quad \forall\, (\mathbf{u}, \mathbf{v}),\, (\hat{\mathbf{u}}, \hat{\mathbf{v}}) \in \mathbf{X}.$$

Let us now introduce the (unbounded) Maxwell operator

$$\mathcal{A} : D(\mathcal{A}) \subset \mathbf{X} \to \mathbf{X}, \quad \mathcal{A} := -\begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix}^{-1} \begin{pmatrix} 0 & -\mathbf{curl} \\ \mathbf{curl} & 0 \end{pmatrix},$$

whose domain is given by $D(\mathcal{A}) := \mathbf{H_0}(\mathbf{curl}) \times \mathbf{H}(\mathbf{curl})$.

Throughout this paper, we assume that the following standing assumptions hold:

(A0) Let the electric permittivity and the magetic permeability $\varepsilon, \mu : \Omega \to \mathbb{R}^{3\times 3}$ be of class $L^\infty(\Omega)^{3\times 3}$, symmetric and uniformly positive definite in the sense that there exit positive real numbers $\underline{\epsilon}$ and $\underline{\mu}$ such that

$$\xi^T \varepsilon(x) \xi \geq \underline{\epsilon} |\xi|^2 \quad \text{and} \quad \xi^T \mu(x) \xi \geq \underline{\mu} |\xi|^2 \quad \text{for a.e. } x \in \Omega \text{ and all } \xi \in \mathbb{R}^3.$$

(A1) For every $t \in [0, T]$, the operator $\mathbf{F}(t, \cdot) : \mathbf{X} \to \mathbf{X}$ given by

$$(\mathbf{u}, \mathbf{v}) \to (\mu^{-1} \mathbf{F}_1(t, x, \mathbf{u}, \mathbf{v}), \varepsilon^{-1} \mathbf{F}_2(t, x, \mathbf{u}, \mathbf{v}))$$

is well-defined, and the mapping $\mathbf{F} : [0, T] \times \mathbf{X} \to \mathbf{X}$ is globally Lipschitz-continuous with the Lipschitz constant $L > 0$, i.e.,

$$\|\mathbf{F}(t_1, (\mathbf{u}_1, \mathbf{v}_1)) - \mathbf{F}(t_2, (\mathbf{u}_2, \mathbf{v}_2))\|_{\mathbf{X}} \leq L(|t_1 - t_2| + \|(\mathbf{u}_1, \mathbf{v}_1) - (\mathbf{u}_2, \mathbf{v}_2)\|_{\mathbf{X}})$$

for all $t_1, t_2 \in [0, T]$ and $(\mathbf{u}_1, \mathbf{v}_1), (\mathbf{u}_2, \mathbf{v}_2) \in \mathbf{X}$.

Applying the Maxwell operator $\mathcal{A}$, the nonlinear hyperbolic Maxwell system (1.1) can be reformulate as the following abstract Cauchy problem:

$$\begin{cases} \left(\frac{d}{dt} - \mathcal{A}\right)(\mathbf{E}, \mathbf{H})(t) = \mathbf{F}(t, \mathbf{E}(t), \mathbf{H}(t)), \quad t \in (0, T] \\ (\mathbf{E}, \mathbf{H})(0) = (\mathbf{u}, \mathbf{v}). \end{cases} \tag{2.1}$$

Obviously, $\mathcal{A} : D(\mathcal{A}) \subset \mathbf{X} \to \mathbf{X}$ is a densely defined, and closed operator. Moreover, due to the choice of the weighted Hilbert space X, the operator $\mathcal{A}$ is skew-adjoint, i.e., $\mathcal{A}^* = -\mathcal{A}$ with $D(\mathcal{A}^*) = D(\mathcal{A})$. Thus, by virtue of Stone's theorem [27, Theorem 10.8, p.41], $\mathcal{A}$ generates a strongly continuous group $\{\mathbb{T}\}_{t\in\mathbb{R}}$ of unitary operators on $\mathbf{X}$.

Definition 2.1. *Let* $(\mathbf{u}, \mathbf{v}) \in \mathbf{X}$*. A continuous function* $(\mathbf{E}, \mathbf{H}) \in C([0, T]; \mathbf{X})$ *is called a mild solution of (2.1) associated with* $(\mathbf{u}, \mathbf{v})$*, if and only if*

$$(\mathbf{E}, \mathbf{H})(t) = \mathbb{T}_t(\mathbf{u}, \mathbf{v}) + \int_0^t \mathbb{T}_{t-s}\mathbf{F}(s, \mathbf{E}(s), \mathbf{H}(s))ds, \quad \forall\, t \in [0, T].$$

Thanks to the Lipschitz property **(A1)**, a classical result [27] implies that the Cauchy problem (2.1) admits for every $(\mathbf{u}, \mathbf{v}) \in \mathbf{X}$ a unique mild solution $(\mathbf{E}, \mathbf{H}) \in C([0, T]; \mathbf{X})$. We denote the mild solution operator associated with (2.1) by

$$G : \mathbf{X} \to C([0, T]; \mathbf{X}), \quad (\mathbf{u}, \mathbf{v}) \mapsto (\mathbf{E}, \mathbf{H}),$$

which assigns to every initial data $(\mathbf{u}, \mathbf{v}) \in \mathbf{X}$ the unique mild solution $(\mathbf{E}, \mathbf{H}) \in C([0, T]; \mathbf{X})$ of (2.1). Then, we introduce the operator

$$S : \mathbf{X} \to \mathbf{X}, \quad S(\mathbf{u}, \mathbf{v}) := G(\mathbf{u}, \mathbf{v})(T).$$

Let us now state the regularity assumption on the initial data we consider throughout this paper:

(**A2**) Suppose that the exact true initial value satisfies $(\mathbf{u}^\dagger, \mathbf{v}^\dagger) \in \mathbf{Y}$.

In all what follows, we set

$$(\mathbf{e}^\dagger, \mathbf{h}^\dagger) := S(\mathbf{u}^\dagger, \mathbf{v}^\dagger).$$

As pointed out in the introduction, we aim at recovering the initial value of the mild solution for (2.1) from a given measurement at final time with noise data, which we denote by $(\mathbf{e}^\delta, \mathbf{h}^\delta) \in \mathbf{X}$ satisfying

$$\|(\mathbf{e}^\delta, \mathbf{h}^\delta) - (\mathbf{e}^\dagger, \mathbf{h}^\dagger)\|_{\mathbf{X}} \leq \delta,$$

with $\delta > 0$ representing the noise level in the data. Then, the inverse problem reads as follows: Find $(\mathbf{u}, \mathbf{v}) \in \mathbf{Y}$ such that $S(\mathbf{u}, \mathbf{v}) = (\mathbf{e}^\delta, \mathbf{h}^\delta)$. Our goal is to investigate the corresponding Tikhonov regularization method:

$$\mathcal{J}_\delta^\kappa(\mathbf{u}, \mathbf{v}) := \frac{1}{2}\|S(\mathbf{u}, \mathbf{v}) - (\mathbf{e}^\delta, \mathbf{h}^\delta)\|_{\mathbf{X}}^2 + \frac{\kappa}{2}\|(\mathbf{u}, \mathbf{v})\|_{\mathbf{Y}}^2 \to \min, \quad \text{subject to } (\mathbf{u}, \mathbf{v}) \in \mathbf{Y}. \tag{2.2}$$

We close this section by providing a classical result on the energy balance equality for every strongly continuous group of unitary operators on $\mathbf{X}$.

Lemma 2.2. *Let $\{\mathbb{S}_t\}_{t\in\mathbb{R}}$ be a strongly continuous group of unitary operators on $\mathbf{X}$. Furthermore, suppose that $(\mathbf{e},\mathbf{h}) \in C([0,T];\mathbf{X})$, $(\mathbf{e}_0,\mathbf{h_0}) \in \mathbf{X}$ and $(\mathbf{f_1},\mathbf{f_2}) \in L^1((0,T);\mathbf{X})$ satisfy*

$$(\mathbf{e},\mathbf{h})(t) = \mathbb{S}_t(\mathbf{e}_0,\mathbf{h_0}) + \int_0^t \mathbb{S}_{t-s}(\mathbf{f_1},\mathbf{f_2})(s)ds, \quad \forall\, t \in [0,T].$$

Then, the energy balance equality

$$\|(\mathbf{e},\mathbf{h})(t)\|_{\mathbf{X}}^2 = \|(\mathbf{e}_0,\mathbf{h_0})\|_{\mathbf{X}}^2 + 2\int_0^t ((\mathbf{f_1},\mathbf{f_2})(s),(\mathbf{e},\mathbf{h})(s))_{\mathbf{X}} ds \tag{2.3}$$

holds for all $t \in [0,T]$.

As investigated in [33], the energy balance equality is an important tool for the mathematical analysis of the optimal control of nonlinear evolutionary Maxwell's equations. In our case, the energy balance equality is important for the case, where $\mathbf{F}$ is monotone (see Proposition 3.4).

3 Tikhonov regularization and its sensitivity analysis

In the following, we recall some results from the Tikhonov regularization theory (cf. [2, 12, 7, 9]). Then, in Section 3.2, we establish the well-posedness for the Tikhonov regularization (2.2) and derive some important properties of the regularised solution. The final part of this section is devoted to the analysis of the adjoint state associated with the Tikhonov regularization (2.2) and its convergence behavior.

3.1 Tikhonov regularization

Let W and Z be Hilbert spaces and $F : W \to Z$ be an operator with an unbounded inverse F^{-1}. Given $z \in Z$, we look for a solution $w \in W$ of the following operator equation:

$$F(w) \,=\, z. \tag{3.1}$$

This operator equation is ill-posed in the sense that a solution possibly does not exist, if the exact data $z = F(w^\dagger)$ comes with small noisy, namely only the noise data z^δ of y available satisfying

$$\|F(w^\dagger) - z^\delta\|_Z \le \delta, \tag{3.2}$$

for some small noisy level $\delta > 0$. Even if a solution exists, then it could be crucially far away from the original one $w^\dagger$. To obtain a stable approximation of the original solution of (3.1), Tikhonov proposed the use of a regularized solution w_κ^δ defined by the minimizer of the following minimization problem:

$$T_\gamma^\delta(w) := \frac{1}{2}\|F(w) - z^\delta\|_Z^2 + \kappa\,\|w\|_W^2 \to \min, \quad \text{subject to} \quad w \in W. \tag{3.3}$$

Under some natural assumptions on F, the minimization problem (3.3) admits for every $\kappa > 0$ and $z^\delta \in Z$ a solution $w^\delta_\kappa \in W$. We call this solution the (Tikhonov-)regularized solution of the ill-posed operator equation (3.1). This regularized solution has the advantage that it is stable with respect to small perturbation in the data z^δ [12, Theoem 3.2] (cf. also [24, 25]).

Proposition 3.1 ([12, 25]). *If $F : W \to Z$ is sequentially weakly continuous, then for every $z^\delta \in Z$ and $\kappa > 0$, there exists a minimizer of (3.3). Furthermore, the minimizer of (3.3) is stable with respect to a small perturbation. More precisely, if $\{z_n\}_{n=1}^\infty \subset Z$ converges strongly towards z^δ, then the sequence $\{w_n\}_{n=1}^\infty$ of minimizers to (3.3) with z^δ replaced by z_n admits a subsequence that converges to a minimizer of (3.3) in the strong topology of W.*

In addition, if $\{\delta_k\}_{k=1}^\infty$ is a sequence of positive real numbers converging monotonically to zero, and the a priori parameter choice $\kappa(\delta_k)$ is chosen such that

$$\kappa(\delta_k) \to 0 \text{ and } \frac{\delta_k^2}{\kappa(\delta_k)} \to 0 \text{ as } k \to \infty,$$

then the sequence $\{w^{\delta_k}_{\kappa(\delta_k)}\}_{k=1}^\infty$ of regularised solutions to (3.3) admits a subsequence that converges strongly in W to a norm-minimising solution w' satisfying

$$\|w'\|_X^2 = \min_{w \in W}\{\|w\|_W^2 \,:\, F(w) = z\}.$$

If the norm-minimising solution is unique, then $w^{\delta_k}_{\kappa(\delta_k)} \to w'$ strongly in W as $k \to \infty$.

In general, the convergence rate of w^δ_κ may be arbitrary slow. To achieve an explicit convergence rate of the regularized solution w^δ_κ, one needs to choose an appropriate parameter $\kappa = \kappa(\delta, z^\delta)$ and impose an additional conditions on the true solution $w^\dagger$, which can also be considered as a certain smoothness of $w^\dagger$ with respect to the forward operator F. In the literature, this condition is called *source condition.* In particular, the *variational source condition* of the form

$$\frac{\beta}{2}\|w^\dagger - w\|_W^2 \le \frac{1}{2}\|w\|_W^2 - \frac{1}{2}\|w^\dagger\|_W^2 + \Psi(\|F(w^\dagger) - F(w)\|_Z) \quad \forall w \in W \tag{3.4}$$

became popular for the description of the solution smoothness. In the setting of (3.4), $\beta \in (0, 1]$ is a fixed constant, and Ψ is a concave index function. Notice that a function Ψ is called an index function if and only if $\Psi : (0, \infty) \to (0, \infty)$ is continuous and strictly increasing and satisfies $\lim_{t\to 0} \Psi(t) = 0$. As already established in [7, 9], the condition (3.4) implies the following convergence rate:

$$\|w^\delta_{\kappa(\delta)} - w^\dagger\|_W = \mathcal{O}(\Psi(\delta)) \quad \text{as} \quad \delta \to 0 \tag{3.5}$$

In other words, the index function can determine the convergence rate, if the regularization parameter is chosen appropriately. In particular, we have following result:

Proposition 3.2 ([16, Theorem 1]). *Let the regularization parameter is chosen a priori as* $\kappa = \kappa(\delta) = \frac{\delta^2}{\Psi(\delta)}$. *Then, under the variational source condition (3.4), the convergence property (3.5) holds for the regularized solution* $w^\delta_{\kappa(\delta)}$.

In this paper we only apply a priori rules for choosing the regularization parameter when minimizing T^δ_γ, because the discussion of a posteriori rules such as variants of the discrepancy principle or Lepskiĭ principle, which also depend on z^δ, is beyond the scope of this paper. We refer the reader e.g. to [5, 16] for a posteriori parameter choice rules under variational source conditions.

3.2 Well-posedness of regularised solutions

We begin by recalling some standard results from the semigroup theory.

Lemma 3.3. *Let assumptions* (**A0**) – (**A1**) *hold. Then, the mild solution operator* $G : \mathbf{X} \to C([0,T];\mathbf{X})$ *satisfies*

$$\|G(\mathbf{u}_1, \mathbf{v}_1)(t) - G(\mathbf{u}_2, \mathbf{v}_2)(t)\|_{\mathbf{X}} \lesssim e^{Lt} \|(\mathbf{u}_1, \mathbf{v}_1) - (\mathbf{u}_2, \mathbf{v}_2)\|_{\mathbf{X}} \tag{3.6}$$

for all $(\mathbf{u}_i, \mathbf{v}_i) \in \mathbf{X}$, $i = 1, 2$. *In addition, if* $(\mathbf{u}, \mathbf{v}) \in \mathbf{Y}$, *then* $G(\mathbf{u}, \mathbf{v}) \in C([0,T]; D(\mathcal{A})) \cap C^1([0,T];\mathbf{X})$ *is the classical solution of the Cauchy problem (2.1).*

Proof. The standard argument in [27, pp. 184] implies that G is well-defined and the estimate (3.6) holds. On the other hand, the argument in [27, pp. 189] ensures that the mild solution $(\mathbf{E}, \mathbf{H})$ is Lipschitz continuous. Thus, the function $\mathbf{f}(t) := \mathbf{F}(t, \mathbf{E}(t), \mathbf{H}(t))$, $t \in (0,T)$ is also Lipschitz continuous with respect to $t \in [0,T]$ and hence in $C^{0,1}([0,T];\mathbf{X})$. The reflexivity of $\mathbf{X}$ implies that $\mathbf{f} \in W^{1,\infty}((0,T);\mathbf{X})$. By the regularity property $(\mathbf{u}, \mathbf{v}) \in \mathbf{Y}$, we apply [6, Corollary 7.6, p. 440] to deduce that the solution $(\mathbf{E}, \mathbf{H}) \in C([0,T]; D(\mathcal{A})) \cap C^1([0,T];\mathbf{X})$ is the classical solution. □

Setting $t = T$ into (3.6), we immediately obtain the Lipschitz continuity of the operator $S : \mathbf{X} \to \mathbf{X}$:

$$\|S(\mathbf{u}_1, \mathbf{v}_1) - S(\mathbf{u}_2, \mathbf{v}_2)\|_{\mathbf{X}} \lesssim e^{LT} \|(\mathbf{u}_1, \mathbf{v}_1) - (\mathbf{u}_2, \mathbf{v}_2)\|_{\mathbf{X}} \quad \forall (\mathbf{u}_1, \mathbf{v}_1), (\mathbf{u}_2, \mathbf{v}_2) \in \mathbf{X}. \tag{3.7}$$

In following lemma, we establish another estimate result for the operator $S : \mathbf{X} \to \mathbf{X}$. This estimate is significant for our analysis related to the variational source condition.

Proposition 3.4. *Under the assumptions of Lemma 3.3, the forward operator* S *is sequentially weak-to-strong continuous from* $\mathbf{Y}$ *to* $\mathbf{X}$. *More importantly, there exits a constant* $C_S > 0$ *such that*

$$\|(\mathbf{u}_1, \mathbf{v}_1) - (\mathbf{u}_2, \mathbf{v}_2)\|_{\mathbf{X}} \le C_S \|S(\mathbf{u}_1, \mathbf{v}_1) - S(\mathbf{u}_2, \mathbf{v}_2)\|_{\mathbf{X}} \quad \forall (\mathbf{u}_1, \mathbf{u}_2), (\mathbf{v}_1, \mathbf{v}_2) \in \mathbf{X}. \tag{3.8}$$

If $\mathbf{F}$, *in addition, satisfies the following monotonicity condition:*

$$(\mathbf{F}(t, (\mathbf{u}_1, \mathbf{v}_1)) - \mathbf{F}(t, (\mathbf{u}_2, \mathbf{v}_2)), (\mathbf{u}_1, \mathbf{v}_1) - (\mathbf{u}_2, \mathbf{v}_2))_{\mathbf{X}} \ge 0 \quad \forall t \in [0,T], \tag{3.9}$$

for all $(\mathbf{u}_i, \mathbf{v}_i) \in \mathbf{X}$, $(i = 1, 2)$, *then the estimate* (3.8) *holds with* $C_S = 1$.

Proof. Let $\{(\mathbf{u}_n, \mathbf{v}_n)\}_{n=1}^{\infty} \subset \mathbf{Y}$ be a sequence, converging weakly to $(\mathbf{u}, \mathbf{v})$ in $\mathbf{Y}$. The compact embedding $\mathbf{Y} \hookrightarrow \mathbf{X}$ (see [28, 22]) implies that $(\mathbf{u}_n, \mathbf{v}_n) \to (\mathbf{u}, \mathbf{v})$ strongly in $\mathbf{X}$ as $n \to \infty$. Then (3.7) implies that the strong convergence $S(\mathbf{u}_n, \mathbf{v}_n) \to S(\mathbf{u}, \mathbf{v})$ in $\mathbf{X}$. In conclusion, $S : \mathbf{X} \supset \mathbf{Y} \to \mathbf{X}$ is sequentially weak-to-strong continuous.

Let us now prove the main estimate (3.8). For all $0 \leq s \leq T$, we obtain by the definition of mild solution $(\mathbf{E}, \mathbf{H})$ at $t = s$ and $t = T$ that

$$(\mathbf{E}, \mathbf{H})(T) = \mathbb{T}_T(\mathbf{E}, \mathbf{H})(0) + \int_0^T \mathbb{T}_{T-\tau}\mathbf{F}(\tau, (\mathbf{E}, \mathbf{H})(\tau))d\tau, \tag{3.10}$$

and

$$(\mathbf{E}, \mathbf{H})(s) = \mathbb{T}_s(\mathbf{E}, \mathbf{H})(0) + \int_0^s \mathbb{T}_{s-\tau}\mathbf{F}(\tau, (\mathbf{E}, \mathbf{H})(\tau))d\tau. \tag{3.11}$$

Applying $\mathbb{T}_{T-s}$ to both sides of (3.11) and taking advantage of the group property of the strongly continuous group $\{\mathbb{T}_t\}_{t\in\mathbb{R}}$,

$$\mathbb{T}_{T-s}(\mathbf{E}, \mathbf{H})(s) = \mathbb{T}_T(\mathbf{E}, \mathbf{H})(0) + \int_0^s \mathbb{T}_{T-\tau}\mathbf{F}(\tau, (\mathbf{E}, \mathbf{H})(\tau))d\tau.$$

In view of this and (3.10), we obtain that

$$(\mathbf{E}, \mathbf{H})(T) = \mathbb{T}_{T-s}(\mathbf{E}, \mathbf{H})(s) + \int_s^T \mathbb{T}_{T-\tau}\mathbf{F}(\tau, (\mathbf{E}, \mathbf{H})(\tau))d\tau. \tag{3.12}$$

Then, applying $\mathbb{T}_{s-T}$ to the above equality yields that

$$(\mathbf{E}, \mathbf{H})(s) = \mathbb{T}_{s-T}(\mathbf{E}, \mathbf{H})(T) - \int_s^T \mathbb{T}_{s-\tau}\mathbf{F}(\tau, (\mathbf{E}, \mathbf{H})(\tau))d\tau.$$

Therefore, for two different solution $\mathbf{U}_1 := G(\mathbf{u}_1, \mathbf{v}_1)$ and $\mathbf{U}_2 := G(\mathbf{u}_2, \mathbf{v}_2)$, it holds that

$$(\mathbf{U}_1 - \mathbf{U}_2)(s) = \mathbb{T}_{s-T}(\mathbf{U}_1 - \mathbf{U}_2)(T) - \int_s^T \mathbb{T}_{s-\tau}\left(\mathbf{F}(\tau, \mathbf{U}_1(\tau)) - \mathbf{F}(\tau, \mathbf{U}_2(\tau))\right)d\tau.$$

As the group $\{\mathbb{T}_t\}_{t\in\mathbb{R}}$ is unitary and $\mathbf{F}$ is globally Lipschitz-continuous with the Lipschitz constant $L > 0$, we obtain from the above identity that

$$\|(\mathbf{U}_1 - \mathbf{U}_2)(s)\|_{\mathbf{X}} \leq \|(\mathbf{U}_1 - \mathbf{U}_2)(T)\|_{\mathbf{X}} + \int_s^T L\|(\mathbf{U}_1 - \mathbf{U}_2)(\tau)\|_{\mathbf{X}}d\tau \quad \forall\, s \in [0, T],$$

which implies that

$$\|(\mathbf{U}_1 - \mathbf{U}_2)(T - s)\|_{\mathbf{X}} \leq \|(\mathbf{U}_1 - \mathbf{U}_2)(T)\|_{\mathbf{X}} + \int_{T-s}^T L\|(\mathbf{U}_1 - \mathbf{U}_2)(\tau)\|_{\mathbf{X}}d\tau \quad \forall\, s \in [0, T].$$

Setting $f(s) := \|(\mathbf{U}_1-\mathbf{U}_2)(T-s)\|_{\mathbf{X}}$ for all $s \in [0,T]$, we deduce from the above inequality by changing of variables that

$$\begin{aligned} f(s) &\le \|(\mathbf{U}_1-\mathbf{U}_2)(T)\|_{\mathbf{X}} + \int_0^s L\|(\mathbf{U}_1-\mathbf{U}_2)(T-w)\|_{\mathbf{X}} dw \\ &\le \|(\mathbf{U}_1-\mathbf{U}_2)(T)\|_{\mathbf{X}} + \int_0^s Lf(w)dw \quad \forall\, s \in [0,T]. \end{aligned}$$

The classical Gronwall's inequality implies $f(s) \le \exp(Ls)\|(\mathbf{U}_1-\mathbf{U}_2)(T)\|$ for all $s \in [0,T]$. Taking $s = T$, we get

$$\|(\mathbf{u}_1,\mathbf{v}_1)-(\mathbf{u}_2,\mathbf{v}_2)\|_{\mathbf{X}} \le \exp(LT)\|(\mathbf{U}_1-\mathbf{U}_2)(T)\|_{\mathbf{X}} = \exp(LT)\|S(\mathbf{u}_1,\mathbf{v}_1)-S(\mathbf{u}_2,\mathbf{v}_2)\|_{\mathbf{X}}.$$

Now suppose that $\mathbf{F}$ additionally satisfies (3.9). Then, the energy balanced equality implies that

$$\begin{aligned} \|(\mathbf{U}_1-\mathbf{U}_2)(T)\|^2_{\mathbf{X}} =& \|(\mathbf{U}_1-\mathbf{U}_2)(0)\|^2_{\mathbf{X}} \\ &+ 2\int_0^T (\mathbf{F}(t,\mathbf{U}_1(t))-\mathbf{F}(t,\mathbf{U}_2(t)),\mathbf{U}_1(t)-\mathbf{U}_2(t))_{\mathbf{X}} dt, \end{aligned}$$

which due to (3.9) yields that $\|(\mathbf{u}_1,\mathbf{v}_1)-(\mathbf{u}_2,\mathbf{v}_2)\|_{\mathbf{X}} \le \|S(\mathbf{u}_1,\mathbf{v}_1)-S(\mathbf{u}_2,\mathbf{v}_2)\|_{\mathbf{X}}$. □

Propositions 3.1 and 3.4 lead to the following result:

Theorem 3.5. *Assume that the hypothesises* $(\mathbf{A0})-(\mathbf{A2})$ *hold.*

(1) For each $\kappa > 0$ *and* $(\mathbf{E}^\delta,\mathbf{H}^\delta) \in \mathbf{X}$*, there exits a minimiser* $(\mathbf{u}^\delta_\kappa,\mathbf{v}^\delta_\kappa) \in \mathbf{Y}$ *of (2.2).*

(2) The minimizer of (2.2) is stable with respect to the perturbation in the sense of Proposition 3.2.

(3) Let $\{\delta_n\}_{n=1}^\infty \subset \mathbb{R}^+$ *be a sequence converging monotonically to zero and* $(\mathbf{e}^{\delta_n},\mathbf{h}^{\delta_n})$ *satisfy*

$$\|(\mathbf{e}^{\delta_n},\mathbf{h}^{\delta_n})-(\mathbf{e}^\dagger,\mathbf{h}^\dagger)\|_{\mathbf{X}} \le \delta_n.$$

Moreover, we assume that the regularization parameter κ_n *fulfils*

$$\kappa_n \to 0 \text{ and } \frac{\delta_n^2}{\kappa_n} \to 0 \quad \text{as } n \to \infty. \tag{3.13}$$

If $(\mathbf{u}_n,\mathbf{v}_n)$ *is the minimiser of (2.2) with* $(\mathbf{e}^\delta,\mathbf{h}^\delta)$ *and* κ *replaced by* $(\mathbf{e}^{\delta_n},\mathbf{h}^{\delta_n})$ *and* κ_n*, respectively, then the sequence* $\{(\mathbf{u}_n,\mathbf{v}_n)\}_{n=1}^\infty$ *converges strongly in* $\mathbf{Y}$ *to the exact solution* $(\mathbf{u}^\dagger,\mathbf{v}^\dagger)$ *as* $n\to\infty$.

3.3 Optimal system for (2.2) and convergence

Our first-order analysis relies on the following assumptions for $\mathbf{F} : [0, T] \times \mathbf{X} \to \mathbf{X}$:

(A3a) For each $t \in [0, T]$, $\mathbf{F}(t, \cdot) := (\mathbf{F}_1(t, \cdot), \mathbf{F}_2(t, \cdot)) : \mathbf{X} \to \mathbf{X}$ is Gâteaux-differentiable.

(A3b) The Gâteaux derivative is assumed to satisfy the following property: If

$$t_n \to t \text{ in } [0, T] \quad \text{and} \quad (\mathbf{u}_n, \mathbf{v}_n) \to (\mathbf{u}, \mathbf{v}) \text{ strongly in } \mathbf{X},$$

then for every $(\mathbf{w}, \mathbf{y}) \in \mathbf{X}$ it holds that

$$\lim_{n\to\infty} \|\mathbf{F}'(t_n, \mathbf{u}_n, \mathbf{v}_n)(\mathbf{w}, \mathbf{y}) - \mathbf{F}'(t, \mathbf{u}, \mathbf{v})(\mathbf{w}, \mathbf{y})\|_{\mathbf{X}} = 0.$$

(A3c) The Gâteaux derivative maps every bounded set in $[0, T] \times \mathbf{X}$ into a bounded set in $\mathcal{L}(\mathbf{X})$.

Lemma 3.6. *Let* $(\mathbf{A0}) - (\mathbf{A1})$ *and* $(\mathbf{A3})$ *be satisfied. Then the operator* $S : \mathbf{X} \to \mathbf{X}$ *is weakly directional differentiable, and the weak directional derivative of* S *at* $(\mathbf{u}, \mathbf{v}) \in \mathbf{X}$ *in the direction* $(\delta\mathbf{u}, \delta\mathbf{v}) \in \mathbf{X}$ *is given* $S'(\mathbf{u}, \mathbf{v})(\delta\mathbf{u}, \delta\mathbf{v}) = (\delta\mathbf{E}(T), \delta\mathbf{H}(T))$*, where* $(\delta\mathbf{E}, \delta\mathbf{H}) \in C([0, T]; \mathbf{X})$ *satisfies the following integral equation*

$$(\delta\mathbf{E}, \delta\mathbf{H})(t) = \mathbb{T}_t(\delta\mathbf{u}, \delta\mathbf{v}) + \int_0^t \mathbb{T}_{t-s}\mathbf{F}'(t, \mathbf{E}(s), \mathbf{H}(s))(\delta\mathbf{E}(s), \delta\mathbf{H}(s))ds \quad \forall\, t \in [0, T]. \tag{3.14}$$

Proof. Let $(\mathbf{u}, \mathbf{v})$, $(\delta\mathbf{u}, \delta\mathbf{v}) \in \mathbf{X}$ and $(\mathbf{E}, \mathbf{H}) = G(\mathbf{u}, \mathbf{v})$. Further, for every $\tau \in \mathbb{R}^+$, we write $(\mathbf{E}_\tau, \mathbf{H}_\tau) = G(\mathbf{u} + \tau\delta\mathbf{u}, \mathbf{v} + \tau\delta\mathbf{v})$. Thus, according to (3.6), we have

$$\left(\frac{\mathbf{E}_\tau - \mathbf{E}}{\tau}, \frac{\mathbf{H}_\tau - \mathbf{H}}{\tau}\right)(t) = \mathbb{T}_t(\delta\mathbf{u}, \delta\mathbf{v}) + \int_0^t \mathbb{T}_{t-s} I_\tau(s)ds, \tag{3.15}$$

where

$$I_\tau(s) := \frac{\mathbf{F}(s, \mathbf{E}_\tau(s), \mathbf{H}_\tau(s)) - \mathbf{F}(s, \mathbf{E}(s), \mathbf{H}(s))}{\tau}. \tag{3.16}$$

Lemma 3.3 implies that $\{(\frac{\mathbf{E}_\tau - \mathbf{E}}{\tau}, \frac{\mathbf{H}_\tau - \mathbf{H}}{\tau})\}_{\tau>0}$ is bounded in $L^\infty(0, T; X)$ and hence it has a weak star converging subsequence, which we still denoted by $\{(\frac{\mathbf{E}_\tau - \mathbf{E}}{\tau}, \frac{\mathbf{H}_\tau - \mathbf{H}}{\tau})\}_{\tau>0}$, such that

$$(\frac{\mathbf{E}_\tau - \mathbf{E}}{\tau}, \frac{\mathbf{H}_\tau - \mathbf{H}}{\tau}) \rightharpoonup (\delta\mathbf{E}, \delta\mathbf{H}) \text{ weakly star in } L^\infty((0, T), \mathbf{X}) \text{ as } \tau \to 0^+, \tag{3.17}$$

for some $(\delta\mathbf{E}, \delta\mathbf{H}) \in L^\infty((0, T), \mathbf{X})$.

For the sake of brevity, we write $\mathbf{U}_\tau = (\mathbf{E}_\tau, \mathbf{H}_\tau)$. Let $\mathbf{w} \in L^1((0, T); \mathbf{X})$ be arbitrarily fixed. The mean-value theorem in the integral form implies that for almost all $s \in (0, T)$,

$$\begin{aligned}&(I_\tau(s), \mathbf{w}(s))_{\mathbf{X}} \\ =&(\mathbf{F}'(s, \mathbf{U}(s))\frac{\mathbf{U}_\tau(s) - \mathbf{U}(s)}{\tau}, \mathbf{w}(s))_{\mathbf{X}} + (\mathbf{G}_\tau(s)\frac{\mathbf{U}_\tau(s) - \mathbf{U}(s)}{\tau}, \mathbf{w}(s))_{\mathbf{X}},\end{aligned} \tag{3.18}$$

where

$$\mathbf{G}_\tau(s)\boldsymbol{x} := \int_0^1 \mathbf{F}'(s, (\mathbf{U} + \theta(\mathbf{U}_\tau - \mathbf{U}))(s))\boldsymbol{x} - \mathbf{F}'(s, \mathbf{U}(s))\boldsymbol{x} d\theta \quad \forall \boldsymbol{x} \in \mathbf{X}.$$

Thanks to **(A3b)**, it holds that $\mathbf{G}_\tau(s) \in \mathcal{L}(\mathbf{X})$ for almost all $s \in (0,T)$ and $\mathbf{G}_\tau(\cdot)\boldsymbol{x} \in \mathcal{C}([0,T];\mathbf{X})$ for all $\boldsymbol{x} \in \mathbf{X}$. Therefore, it follows that $\mathbf{G}_\tau(\cdot)^*\mathbf{v}(\cdot)$ belongs to $L^1((0,T);\mathbf{X})$. Since for almost every $s \in (0,T)$, $\mathbf{G}_\tau(s)^*\mathbf{w}(s) \to 0$ as $\tau \to 0$, and according to **(A3c)** $\mathbf{G}_\tau(s)^*\mathbf{w}(s)$ can be bounded by $M\|\mathbf{w}(s)\|$ for some constant $M > 0$, we can apply dominated convergence theorem to get that $\mathbf{G}_\tau^*\mathbf{w} \to 0$ strongly in $L^1((0,T);\mathbf{X})$. Then the weak star convergence (3.17) implies

$$\int_0^T (\mathbf{G}_\tau(s)\frac{\mathbf{U}_\tau(s) - \mathbf{U}(s)}{\tau}, \mathbf{w}(s))_{\mathbf{X}} ds = \int_0^T (\frac{\mathbf{U}_\tau(s) - \mathbf{U}(s)}{\tau}, \mathbf{G}_\tau^*(s)\mathbf{w}(s))_{\mathbf{X}} ds \to 0, \quad (3.19)$$

as $\tau \to 0$. In addition, (3.17) yields

$$\lim_{\tau\to 0}\int_0^T (\mathbf{F}'(s, \mathbf{U}(s))\frac{\mathbf{U}_\tau(s) - \mathbf{U}(s)}{\tau}, \mathbf{w}(s))_{\mathbf{X}} ds = \int_0^T (\mathbf{F}'(s, \mathbf{U}(s))(\delta\mathbf{E}(s), \delta\mathbf{H}(s)), \mathbf{w}(s))_{\mathbf{X}}\, ds. \quad (3.20)$$

Concluding from (3.18)–(3.20) and since $\mathbf{w} \in L^1((0,T);\mathbf{X})$ was chosen arbitrarily fixed, we obtain that

$$I_\tau \rightharpoonup \mathbf{F}'(\cdot, \mathbf{U}(\cdot))(\delta\mathbf{E}, \delta\mathbf{H}) \quad \text{weakly star in } L^\infty((0,T);\mathbf{X}). \quad (3.21)$$

On the other hand, it holds that

$$\int_0^T \int_0^t (\mathbb{T}_{t-s} I_\tau(s), \mathbf{w}(t))_{\mathbf{X}}\, ds dt = \int_0^T \left(I_\tau(s), \int_s^T \mathbb{T}_{t-s}^*\mathbf{w}(\mathbf{t})dt \right)_{\mathbf{X}} ds \quad \forall\, \mathbf{w} \in L^1((0,T);\mathbf{X}).$$

Since the mapping $s \mapsto \int_s^T \mathbb{T}_{t-s}^*\mathbf{w}(t)dt$ also belongs to $L^1(0,T;\mathbf{X})$, we obtain from (3.21) and (3.15) that the weak star limit $(\delta\mathbf{E}, \delta\mathbf{H})$ satisfies the integral equation (3.14). □

Corollary 3.7. *Under the assumptions of Lemma 3.6, the operator* $S'(\mathbf{u}, \mathbf{v}) : \mathbf{X} \to \mathbf{X}$ *is weakly-Gâteaux-differentiable.*

Proof. Let $(\mathbf{u}, \mathbf{v}) \in \mathbf{X}$ and $(\mathbf{E}, \mathbf{H}) = G(\mathbf{u}, \mathbf{v})$. In view of (3.14), the operator $S'(\mathbf{u}, \mathbf{v}) : \mathbf{X} \to \mathbf{X}$ is linear. Let us show the boundedness result. In fact, the energy balance equality implies that $(\delta\mathbf{E}, \delta\mathbf{H}) = G(\mathbf{u}, \mathbf{v})(\delta\mathbf{u}, \delta\mathbf{v})$ satisfies

$$\|(\delta\mathbf{E}(t), \delta\mathbf{H}(t))\|_{\mathbf{X}}^2 = \|(\delta\mathbf{u}, \delta\mathbf{v})\|_X^2 + 2\int_0^t (\mathbf{F}'(t, \mathbf{E}(s), \mathbf{H}(s))(\delta\mathbf{E}(s), \delta\mathbf{H}(s)), (\delta\mathbf{E}(s), \delta\mathbf{H}(s)))_{\mathbf{X}}\, ds \quad \forall\, t \in [0,T].$$

Then, according to **(A3c)**, we can find a constant $C > 0$ independent of $(\delta\mathbf{E}, \delta\mathbf{H})$ such that

$$\|(\delta\mathbf{E}(t), \delta\mathbf{H}(t))\|_{\mathbf{X}}^2 \le \|(\delta\mathbf{u}, \delta\mathbf{v})\|_X^2 + C\int_0^t \|(\delta\mathbf{E}(s), \delta\mathbf{H}(s))\|_{\mathbf{X}}^2 ds \quad \forall\, t \in [0,T].$$

From this inequality, the Gronwall lemma implies that $S'(\mathbf{u}, \mathbf{v}) : \mathbf{X} \to \mathbf{X}$ is bounded. □

From this result and a classical argument we obtain the following result:

Corollary 3.8. *Let the assumptions of Lemma 3.6 hold. Then, for every* $\kappa > 0$*, the functional* $\mathcal{J}_\delta^\kappa : \mathbf{Y} \to \mathbb{R}$ *is Gâteaux-differentiable with the Gâteaux derivative*

$$\mathcal{J}_\delta^{\kappa\prime}(\mathbf{u}, \mathbf{v})(\delta\mathbf{u}, \delta\mathbf{v}) = (S(\mathbf{u}, \mathbf{v}) - (\mathbf{E}^\delta, \mathbf{H}^\delta), S'(\mathbf{u}, \mathbf{v})(\delta\mathbf{u}, \delta\mathbf{v}))_{\mathbf{X}} + \kappa((\mathbf{u}, \mathbf{v}), (\delta\mathbf{u}, \delta\mathbf{v}))_{\mathbf{Y}}, \quad (3.22)$$

for all $(\mathbf{u}, \mathbf{v}), (\delta\mathbf{u}, \delta\mathbf{v}) \in \mathbf{Y}$.

Let us now establish an explicit formula for the adjoint operator associated with the Gâteaux-derivative $S'(\mathbf{u}, \mathbf{v}) : \mathbf{X} \to \mathbf{X}$.

Lemma 3.9. *Assume the assumptions of Lemma 3.6 hold. Let* $(\mathbf{u}, \mathbf{v}) \in \mathbf{X}$ *and* $(\mathbf{E}, \mathbf{H}) \in \mathcal{C}([0, T], \mathbf{X})$ *denote the mild solution associated with the initial value* $(\mathbf{u}, \mathbf{v})$*. Then, for every* $(\mathbf{w}, \mathbf{h}) \in \mathbf{X}$*, the adjoint operator* $S'(\mathbf{u}, \mathbf{v})^* : \mathbf{X} \to \mathbf{X}$ *satisfies*

$$S'(\mathbf{u}, \mathbf{v})^*(\mathbf{w}, \mathbf{z}) := (\mathbf{K}(0), \mathbf{Q}(0)),$$

where $(\mathbf{K}, \mathbf{Q}) \in C([0, T]; \mathbf{X})$ *is the unique solution of the following integral equation:*

$$(\mathbf{K}(t), \mathbf{Q}(t)) = \mathbb{T}_{t-T}(\mathbf{w}, \mathbf{h}) + \int_t^T \mathbb{T}_{t-s}\mathbf{F}'(s, \mathbf{E}(s), \mathbf{H}(s))^*(\mathbf{K}(s), \mathbf{Q}(s))ds \quad (3.23)$$

for all $t \in [0, T]$.

Proof. Let $B(t) := \mathbf{F}'(t, \mathbf{E}(t), \mathbf{H}(t))$, $t \in [0, T]$. From **(A3b)** together with $(\mathbf{E}, \mathbf{H}) \in C([0, T], \mathbf{X})$, it follows that

$$B(\cdot)\boldsymbol{x} \in C([0, T]; \mathbf{X}), \quad B(\cdot)\mathbf{w}(\cdot) \in L^1((0, T); \mathbf{X}) \quad \forall \boldsymbol{x} \in \mathbf{X}, \mathbf{w} \in L^1((0, T); \mathbf{X}). \quad (3.24)$$

Let us define the Yosida-approximation

$$B_n(t) := n(n - \mathcal{A})^{-1}B(t), \quad \forall n \in \mathbb{N}, \ \forall t \in [0, T]. \quad (3.25)$$

In view of (3.24), the Yosida-approximation satisfies

$$B_n(\cdot)\boldsymbol{x} \in C([0, T], D(\mathcal{A})), \quad B_n(\cdot)\mathbf{w}(\cdot) \in L^1((0, T); D(\mathcal{A})) \quad \forall \boldsymbol{x} \in \mathbf{X}, \mathbf{w} \in L^1((0, T); \mathbf{X})$$

For this reason, for every $(\hat{\mathbf{w}}, \hat{\mathbf{z}}) \in D(\mathcal{A})$, the integral equation

$$(\delta\mathbf{E}_n(t), \delta\mathbf{H}_n(t)) = \mathbb{T}_t(\hat{\mathbf{w}}, \hat{\mathbf{z}}) + \int_0^t \mathbb{T}_{t-s}B_n(s)(\delta\mathbf{E}_n(s), \delta\mathbf{H}_n(s))ds \quad \forall t \in [0, T] \quad (3.26)$$

admits a unique solution $(\delta\mathbf{E}_n, \delta\mathbf{H}_n) \in \mathcal{C}^1([0, T]; \mathbf{X}) \cap \mathcal{C}([0, T]; D(\mathcal{A}))$ satisfying

$$\begin{cases} \frac{d}{dt}(\delta\mathbf{E}_n, \delta\mathbf{H}_n)(t) = (\mathcal{A} + B_n(t))(\delta\mathbf{E}_n, \delta\mathbf{H}_n)(t) & \forall t \in [0, T] \\ (\delta\mathbf{E}_n(0), \delta\mathbf{H}(0)) = (\hat{\mathbf{w}}, \hat{\mathbf{z}}). \end{cases} \quad (3.27)$$

Similarly, for every $(\mathbf{w}, \mathbf{z}) \in D(\mathcal{A})$, the following integral equation

$$(\mathbf{K}_n(t), \mathbf{Q}_n(t)) = \mathbb{T}_{t-T}(\mathbf{w}, \mathbf{h}) + \int_t^T \mathbb{T}_{t-s} B_n(s)^* (\mathbf{K}_n(s), \mathbf{Q}_n(s)) ds \quad \forall\, t \in [0, T].$$

admits a unique solution $(\mathbf{K}_n, \mathbf{Q}_n) \in \mathcal{C}^1([0,T]; \mathbf{X}) \cap \mathcal{C}([0,T]; D(\mathcal{A}))$ satisfying

$$\begin{cases} -\frac{d}{dt}(\mathbf{K}_n, \mathbf{Q}_n)(t) = (-\mathcal{A} + B_n(t)^*)(\mathbf{K}_n, \mathbf{Q}_n)(t) \quad \forall t \in [0, T] \\ (\delta \mathbf{E}_n(T), \delta \mathbf{H}(T)) = (\mathbf{w}, \mathbf{z}). \end{cases} \tag{3.28}$$

Combining (3.27) and (3.28), we obtain that

$$\begin{aligned} &((\delta \mathbf{E}_n(T), \delta \mathbf{H}_n(T)), (\mathbf{K}_n(T), \mathbf{Q}_n(T))_{\mathbf{X}} - ((\delta \mathbf{E}_n(0), \delta \mathbf{H}_n(0)), (\mathbf{K}_n(0), \mathbf{Q}_n(0))_{\mathbf{X}} \\ =& \int_0^T \big(\frac{d}{dt}(\delta \mathbf{E}_n, \delta \mathbf{H}_n)(t), (\mathbf{K}_n(t), \mathbf{Q}_n(t)) \big)_{\mathbf{X}} + \big((\delta \mathbf{E}_n, \delta \mathbf{H}_n)(t), \frac{d}{dt}((\mathbf{K}_n(t), \mathbf{Q}_n(t)) \big)_{\mathbf{X}} dt \\ =& \int_0^T \big((\delta \mathbf{E}_n, \delta \mathbf{H}_n)(t), (\mathcal{A} + B_n(t))^* (\mathbf{K}_n(t), \mathbf{Q}_n(t)) + \frac{d}{dt}(\mathbf{K}_n(t), \mathbf{Q}_n(t)) \big)_{\mathbf{X}} dt = 0. \end{aligned} \tag{3.29}$$

A direct computation based on (3.26) yields

$$\begin{aligned} &\|(\delta \mathbf{E}_n(t), \delta \mathbf{H}_n(t)) - (\delta \mathbf{E}(t), \delta \mathbf{H}(t))\|_{\mathbf{X}} \\ =&\| \int_0^t \mathbb{T}_{t-s} B_n(s) (\delta \mathbf{E}_n(s), \delta \mathbf{H}_n(s)) - (\delta \mathbf{E}(s), \delta \mathbf{H}(s)) ds \|_{\mathbf{X}} \\ &+ \| \int_0^t \mathbb{T}_{t-s} (B_n(s) - B(s)) (\delta \mathbf{E}(s), \delta \mathbf{H}(s)) ds \|_{\mathbf{X}} \quad \forall\, t \in [0, T]. \end{aligned} \tag{3.30}$$

Since for each $t \in [0, T]$, $B_n(t) \to B(t)$ strongly in $\mathbf{X}$ and the set $\{(\delta \mathbf{E}, \delta \mathbf{H})(t) \mid t \in [0, T]\}$ is compact in $\mathbf{X}$, it follows that

$$\forall \varepsilon > 0 \, \exists \, N \in \mathbb{N} \; \forall n \geq N : \max_{s \in [0,T]} \|(B_n(s) - B(s))(\delta \mathbf{E}(s), \delta \mathbf{H}(s))\|_{\mathbf{X}} \leq \varepsilon. \tag{3.31}$$

On the other hand, by definition (3.25), there exits a constant $C_B > 0$, independent of n, such that

$$\sup_{t \in [0,T]} \|B_n(t)\|_{\mathcal{L}(\mathbf{X})} \leq C_B \quad \forall\, n \in \mathbb{N}. \tag{3.32}$$

Therefore, in view of (3.30)–(3.32), it holds that for $\varepsilon > 0$, there exits an $N \in \mathbb{N}$ such that

$$\begin{aligned} &\|(\delta \mathbf{E}_n(t), \delta \mathbf{H}_n(t)) - (\delta \mathbf{E}(t), \delta \mathbf{H}(t))\|_{\mathbf{X}} \\ \leq& C_B \int_0^t \|(\delta \mathbf{E}_n(s), \delta \mathbf{H}_n(s)) - (\delta \mathbf{E}(s), \delta \mathbf{H}(s))\|_{\mathbf{X}} + t\varepsilon \quad \forall\, t \in [0, T], \; \forall\, n \geq N. \end{aligned}$$

Hence the Gronwall's lemma implies that

$$\lim_{n \to \infty} (\delta \mathbf{E}_n(t), \delta \mathbf{H}_n(t)) = (\delta \mathbf{E}(t), \delta \mathbf{H}(t)) \quad \forall t \in [0, T]. \tag{3.33}$$

Similarly, one can show that

$$\lim_{n\to\infty}(\mathbf{K}_n(t), \mathbf{Q}_n(t)) = (\mathbf{K}(t), \mathbf{Q}(t)) \quad \forall t \in [0,T], \tag{3.34}$$

where $(\mathbf{K}, \mathbf{Q})$ is the solution of the integral solution (3.23). Combining (3.29), (3.33) and (3.34), we obtain that

$$(S'(\mathbf{u}, \mathbf{v})(\hat{\mathbf{w}}, \hat{\mathbf{z}}), (\mathbf{w}, \mathbf{z}))_{\mathbf{X}} = ((\hat{\mathbf{w}}, \hat{\mathbf{z}}), (\mathbf{K}(0), \mathbf{Q}(0)))_{\mathbf{X}}$$

for all $(\mathbf{w}, \mathbf{z}), (\hat{\mathbf{w}}, \hat{\mathbf{z}}) \in D(\mathcal{A})$. Then the density of $D(\mathcal{A})$ in $\mathbf{X}$ completes the proof. □

Theorem 3.10. *Assume that* $(\mathbf{A0})-(\mathbf{A1})$ *and* $(\mathbf{A3})$ *hold. Furthermore, let* $(\mathbf{u}^\delta_\kappa, \mathbf{v}^\delta_\kappa) \in \mathbf{Y}$ *be the minimiser of (2.2) and* $(\mathbf{E}^\delta_\kappa, \mathbf{H}^\delta_\kappa) \in C([0,T]; D(\mathcal{A})) \cap C^1([0,T]; \mathbf{X})$ *the corresponding solution of (2.1) associated with* $(\mathbf{u}^\delta_\kappa, \mathbf{v}^\delta_\kappa) \in \mathbf{Y}$*. Then, there exits a unique pair* $(\mathbf{K}^\delta_\kappa, \mathbf{Q}^\delta_\kappa) \in C([0,T]; \mathbf{X})$ *satisfying*

$$\begin{aligned}(\mathbf{K}^\delta_\kappa(t), \mathbf{Q}^\delta_\kappa(t)) =& \mathbb{T}_{t-T}(\mathbf{E}^\delta_\kappa(T) - \mathbf{e}^\delta, \mathbf{H}^\delta_\kappa(T) - \mathbf{h}^\delta) \\ &+ \int_t^T \mathbb{T}_{t-s}\mathbf{F}'(s, \mathbf{E}^\delta_\kappa(s), \mathbf{H}^\delta_\kappa(s))^*(\mathbf{K}^\delta_\kappa(s), \mathbf{Q}^\delta_\kappa(s))ds. \quad \forall t \in [0,T]\end{aligned} \tag{3.35}$$

$$(\kappa(\mathbf{u}^\delta_\kappa, \mathbf{v}^\delta_\kappa), (\delta\mathbf{u}, \delta\mathbf{v}))_{\mathbf{Y}} = -((\mathbf{K}^\delta_\kappa(0), \mathbf{Q}^\delta_\kappa(0)), (\delta\mathbf{u}, \delta\mathbf{v}))_{\mathbf{X}} \ \forall (\delta\mathbf{u}, \delta\mathbf{v}) \in \mathbf{Y}. \tag{3.36}$$

In the sequel, we call $(\mathbf{K}^\delta_\kappa, \mathbf{Q}^\delta_\kappa) \in C([0,T]; \mathbf{X})$ the adjoint state associated with $(\mathbf{u}^\delta_\kappa, \mathbf{v}^\delta_\kappa)$.

Proof. The necessary optimality condition for (2.2) reads as

$$\mathcal{J}^\kappa_\delta(\mathbf{u}^\delta_\kappa, \mathbf{v}^\delta_\kappa)(\delta\mathbf{u}, \delta\mathbf{v}) = 0, \quad \forall\, (\delta\mathbf{u}, \delta\mathbf{v}) \in \mathbf{Y},$$

which is according to (3.22) equivalent to

$$(S(\mathbf{u}^\delta_\kappa, \mathbf{v}^\delta_\kappa) - (\mathbf{E}^\delta, \mathbf{H}^\delta), S'(\mathbf{u}^\delta_\kappa, \mathbf{v}^\delta_\kappa)(\delta\mathbf{u}, \delta\mathbf{v}))_{\mathbf{X}} + (\kappa(\mathbf{u}^\delta_\kappa, \mathbf{v}^\delta_\kappa), (\delta\mathbf{u}, \delta\mathbf{v}))_{\mathbf{Y}} = 0, \quad \forall\, (\delta\mathbf{u}, \delta\mathbf{v}) \in \mathbf{Y}.$$

Thus, by Lemma 3.9, we obtain the desired result. □

Corollary 3.11. *Assume that hypothesises* $(\mathbf{A0})-(\mathbf{A3})$ *hold. Let* $\{\delta_n\}_{n=1}^\infty, \{\kappa_n\}_{n=1}^\infty \subset \mathbb{R}$, $\{(\mathbf{e}^{\delta_n}, \mathbf{h}^{\delta_n})\}_{n=1}^\infty \subset \mathbf{X}$*, and* $\{\mathbf{u}_n, \mathbf{v}_n\}_{n=1}^\infty \subset \mathbf{Y}$ *be sequences as defined in Theorem 3.5. Moreover, for every* $n \in \mathbb{N}$*, let* $(\mathbf{E}^{\delta_n}_{\kappa_n}, \mathbf{H}^{\delta_n}_{\kappa_n}) \in C([0,T]; D(\mathcal{A})) \cap C^1([0,T]; \mathbf{X})$ *denote the corresponding solution of (2.1) associated with* $(\mathbf{u}_n, \mathbf{v}_n) \in \mathbf{Y}$*, and* $(\mathbf{K}_n, \mathbf{Q}_n) \in C([0,T]; \mathbf{X})$ *denote the adjoint state satisfying* $(3.35)-(3.36)$ *with* $(\mathbf{E}^\delta_\kappa, \mathbf{H}^\delta_\kappa)$, $(\mathbf{e}^\delta, \mathbf{h}^\delta)$ *and* $(\mathbf{u}^\delta_\kappa, \mathbf{v}^\delta_\kappa)$ *replaced by* $(\mathbf{E}^{\delta_n}_{\kappa_n}, \mathbf{H}^{\delta_n}_{\kappa_n})$, $(\mathbf{e}^{\delta_n}, \mathbf{h}^{\delta_n})$ *and* $(\mathbf{u}_n, \mathbf{v}_n)$*, respectively. Then, the adjoint state* $(\mathbf{K}_n, \mathbf{Q}_n)$ *converges strongly in* $C([0,T]; \mathbf{X})$ *to zero as* $n \to \infty$*.*

Proof. From Theorem 3.5 it follows that $(\mathbf{u}_n, \mathbf{v}_n)$ converges strongly to $(\mathbf{u}^\dagger, \mathbf{v}^\dagger)$. The Lipschtiz continuity of operator $S : \mathbf{X} \to \mathbf{X}$ implies that $(\mathbf{E}^{\delta_n}_{\kappa_n}(T), \mathbf{H}^{\delta_n}_{\kappa_n}(T))$ converges strongly to $(\mathbf{e}^\dagger, \mathbf{h}^\dagger)$. As a consequence, we obtain that

$$\lim_{n\to\infty} \|(\mathbf{E}^{\delta_n}_{\kappa_n}(T) - \mathbf{e}^{\delta_n}, \mathbf{H}^{\delta_n}_{\kappa_n}(T) - \mathbf{h}^{\delta_n})\|_{\mathbf{X}} = 0. \tag{3.37}$$

On the other hand, since $\{(\mathbf{E}_n, \mathbf{H}_n)\}_{n=1}^{\infty}$ is a bounded set, we may apply the assumption (**A3**) to the equation (3.36) and obtain a constant $M > 0$ such that

$$\begin{aligned}
&\|(\mathbf{K}_n(t), \mathbf{Q}_n(t))\|_{\mathbf{X}} \\
\leq &\|(\mathbf{E}_{\kappa_n}^{\delta_n}(T) - \mathbf{e}^{\delta_n}, \mathbf{H}_{\kappa_n}^{\delta_n}(T) - \mathbf{h}^{\delta_n})\|_{\mathbf{X}} + M \int_t^T \|(\mathbf{K}_n(s), \mathbf{Q}_n(s))\|_{\mathbf{X}} ds \quad s \in [0, T].
\end{aligned}$$

Using a similar argument as in the proof of Proposition 3.4, we have

$$\|(\mathbf{K}_n(t), \mathbf{Q}_n(t))\|_{\mathbf{X}} \leq e^{M(T-t)} \|(\mathbf{E}_{\kappa_n}^{\delta_n}(T) - \mathbf{e}^{\delta_n}, \mathbf{H}_{\kappa_n}^{\delta_n}(T) - \mathbf{h}^{\delta_n})\|_{\mathbf{X}}. \tag{3.38}$$

Now, the assertion follows from (3.37) and (3.38). □

4 Convergence rates analysis under variational source conditions

It is worth mentioning that the convergence speed for the Tikhonov regularization (2.2) and the corresponding adjoint state (Theorem 3.5 and Corollary 3.11) can be arbitrarily slow. It is our goal of this section to study their convergence rates. Our analysis is mainly based on results from the variational regularization theory. From Section 3.1, we know that it suffices to verify that

$$\begin{aligned}
\frac{\beta}{2} \|(\mathbf{u}^\dagger - \mathbf{u}, \mathbf{v}^\dagger - \mathbf{v})\|_{\mathbf{Y}}^2 \leq &\frac{1}{2} \|(\mathbf{u}^\dagger, \mathbf{v}^\dagger)\|_{\mathbf{Y}}^2 - \frac{1}{2} \|(\mathbf{u}, \mathbf{v})\|_{\mathbf{Y}}^2 \\
&+ \Psi(\|S(\mathbf{u}^\dagger, \mathbf{v}^\dagger) - S(\mathbf{u}, \mathbf{v})\|_{\mathbf{X}}), \quad \forall (\mathbf{u}, \mathbf{v}) \in \mathbf{Y}
\end{aligned} \tag{4.1}$$

holds for some constant $0 < \beta \leq 1$ and some index function Ψ. To this end, we shall first establish some auxiliary results in Section 4.1 and recall some results on interpolation spaces. Then, the verification of (4.1) will be investigated in Section 4.2. Throughout this section, we make the following additional material assumption.

(**A4**) There exist C^2-domains $\Omega_j \subset \mathbb{R}^3$, $j = 1, \ldots, N$ such that

$$\Omega_i \cap \Omega_j = \emptyset \quad \forall i \neq j \quad \text{and} \quad \overline{\Omega} = \cup_{j=1}^N \overline{\Omega}_j,$$

and

$$\varepsilon_{|\Omega_j}, \mu_{|\Omega_j} \in C^2(\overline{\Omega_j})^{3\times3} \quad \forall j = 1, 2, \ldots, N. \tag{4.2}$$

Let us underline that the above material assumption means physically that Ω is a heterogeneous medium containing finitely many different homogeneous materials Ω_j.

Lemma 4.1. *Let (**A0**) and (**A4**) be satisfied. Then the space **Y** is dense in **X**.*

Proof. Let us consider the linear spaces

$$\mathcal{D} := \{\mathbf{u} \in C_0^\infty(\Omega)^3; \mathbf{u}\,|_{\Omega_i} \in C_0^\infty(\Omega_i^1)^3 \text{ for } i = 1, 2, \cdots, N\}. \tag{4.3}$$

Thanks to (4.2), it holds that

$$\mathcal{D} \subset \mathbf{H_0}(\mathbf{curl}) \cap \varepsilon^{-1}\mathbf{H}(\mathrm{div}), \quad \mathcal{D} \subset \mathbf{H}(\mathbf{curl}) \cap \mu^{-1}\mathbf{H_0}(\mathrm{div}),$$

from which it follows that $\mathcal{D} \times \mathcal{D} \subset \mathbf{Y}$. Moreover, by the construction, $\mathcal{D} \times \mathcal{D}$ is dense in $\mathbf{X}$. This completes the proof. □

4.1 An auxiliary result

In this subsection, we investigate the connection between the inner products of $\mathbf{X}$ and $\mathbf{Y}$, which will be characterized by an unbounded self-adjoint operator, which will play a key role in our analysis below. To take advantage of the spectral theorem of operators in complex Hilbert spaces and the complex interpolation theory, we need to consider the complexification of $\mathbf{X}$ and $\mathbf{Y}$. More precisely, let $\mathbf{X}_\mathbb{C}$ be a complex linear space consists of all complex-valued functions $(\mathbf{u}, \mathbf{v})$ with $(\mathrm{Re}\,\mathbf{u}, \mathrm{Re}\,\mathbf{v}), (\mathrm{Im}\,\mathbf{u}, \mathrm{Im}\,\mathbf{v}) \in \mathbf{X}$, equipped with inner product

$$\begin{aligned}((\mathbf{u}_1, \mathbf{v}_1), (\mathbf{u}_2, \mathbf{v}_2))_{\mathbf{X}_\mathbb{C}} =& (\mathrm{Re}(\mathbf{u}_1, \mathbf{v}_1), \mathrm{Re}(\mathbf{u}_2, \mathbf{v}_2))_{\mathbf{X}} + (\mathrm{Im}(\mathbf{u}_1, \mathbf{v}_1), \mathrm{Im}(\mathbf{u}_2, \mathbf{v}_2))_{\mathbf{X}} \\ &- i(\mathrm{Re}(\mathbf{u}_1, \mathbf{v}_1), \mathrm{Im}(\mathbf{u}_2, \mathbf{v}_2))_{\mathbf{X}} + i(\mathrm{Im}(\mathbf{u}_1, \mathbf{v}_1), \mathrm{Re}(\mathbf{u}_2, \mathbf{v}_2))_{\mathbf{X}}\end{aligned}$$

It is obvious that $((\mathbf{u}_1, \mathbf{v}_1), (\mathbf{u}_2, \mathbf{v}_2))_{\mathbf{X}_\mathbb{C}} = ((\mathbf{u}_1, \mathbf{v}_1), (\mathbf{u}_2, \mathbf{v}_2))_{\mathbf{X}}$ for all $(\mathbf{u}_1, \mathbf{v}_1), (\mathbf{u}_2, \mathbf{v}_2) \in \mathbf{X}$. Similarly, we define the complexification $\mathbf{Y}_\mathbb{C}$ of $\mathbf{Y}$.

Under the assumptions (**A0**) and (**A4**), according to Lemma 4.1 the embedding $\mathbf{Y}_\mathbb{C} \subset \mathbf{X}_\mathbb{C}$ is dense and continuous. Then there exits a (unique) extension of $\mathbf{X}_\mathbb{C}$, called *extrapolation space* $\mathbf{Y}_\mathbb{C}^*$, which is isometric to the dual space of $\mathbf{Y}_\mathbb{C}$, such that the triple $\mathbf{Y}_\mathbb{C} \subset \mathbf{X}_\mathbb{C} \subset \mathbf{Y}_\mathbb{C}^*$ satisfies the following conditions (see e.g. [30, Sect. 7, Chap. 1]).

(1) $\mathbf{Y}_\mathbb{C} \subset \mathbf{X}_\mathbb{C} \subset \mathbf{Y}_\mathbb{C}^*$ with dense and continuous embeddings.

(2) $\{\mathbf{Y}_\mathbb{C}^*, \mathbf{Y}_\mathbb{C}\}$ forms an adjoint pair with duality product $\langle \cdot, \cdot \rangle_{\mathbf{Y}_\mathbb{C}^*, \mathbf{Y}_\mathbb{C}}$.

(3) the duality product $\langle \cdot, \cdot \rangle_{\mathbf{Y}_\mathbb{C}^*, \mathbf{Y}_\mathbb{C}}$ satisfies

$$\langle u, f \rangle_{\mathbf{Y}_\mathbb{C}^*, \mathbf{Y}_\mathbb{C}} = (u, f)_{\mathbf{X}_\mathbb{C}} \quad \text{for all } f \in \mathbf{Y}_\mathbb{C},\ u \in \mathbf{X}_\mathbb{C}.$$

Since the inner-product $(\cdot, \cdot)_{\mathbf{Y}_\mathbb{C}}$ is a symmetric sesquilinear form over $\mathbf{Y}_\mathbb{C}$, the operator $\mathcal{B}_{\mathbf{Y}_\mathbb{C}} : \mathbf{Y}_\mathbb{C} \to \mathbf{Y}_\mathbb{C}^*$ defined by

$$\langle B_{\mathbf{Y}_\mathbb{C}}(\mathbf{u}_1, \mathbf{v}_1), (\mathbf{u}_2, \mathbf{v}_2) \rangle_{\mathbf{Y}_\mathbb{C}^* \times \mathbf{Y}_\mathbb{C}} := ((\mathbf{u}_1, \mathbf{v}_1), (\mathbf{u}_2, \mathbf{v}_2))_{\mathbf{Y}_\mathbb{C}} \quad \forall\, (\mathbf{u}_1, \mathbf{v}_1), (\mathbf{u}_2, \mathbf{v}_2) \in \mathbf{Y}_\mathbb{C}$$

is linear and bounded. In addition, if we define

$$\mathcal{B}(\mathbf{u},\mathbf{v}) := \mathcal{B}_{\mathbf{Y}_{\mathbb{C}}}(\mathbf{u},\mathbf{v}) \quad \forall\, (\mathbf{u},\mathbf{v}) \in D(\mathcal{B})$$

with the domain

$$D(\mathcal{B}) := \{(\mathbf{u},\mathbf{v}) \in \mathbf{Y}_{\mathbb{C}}; \mathcal{B}_{\mathbf{Y}}(\mathbf{u},\mathbf{v}) \in \mathbf{X}_{\mathbb{C}}\},$$

then $\mathcal{B} : D(\mathcal{B}) \subset \mathbf{X}_{\mathbb{C}} \to \mathbf{X}_{\mathbb{C}}$ is a densely defined and close operator (cf. [30, Theorem 1.25]) and it satisfies many other mathematical properties. Some of them are summarized in the following lemma:

Lemma 4.2 ([30, Theorem 2.34 and Corollary 2.4])**.** *Assume that* (**A0**) *and* (**A4**) *hold. Then, the unbounded operator* $\mathcal{B} : D(\mathcal{B}) \subset \mathbf{X}_{\mathbb{C}} \to \mathbf{X}_{\mathbb{C}}$ *is densely defined, closed, self-adjoint operator and m-accretive. Furthermore, it satisfies*

$$(\mathcal{B}(\mathbf{u}_1,\mathbf{v}_1),(\mathbf{u}_2,\mathbf{v}_2))_{\mathbf{X}_{\mathbb{C}}} = ((\mathbf{u}_1,\mathbf{v}_1),(\mathbf{u}_2,\mathbf{v}_2))_{\mathbf{Y}_{\mathbb{C}}} \quad \forall\, (\mathbf{u}_1,\mathbf{v}_1) \in D(\mathcal{B}), (\mathbf{u}_2,\mathbf{v}_2) \in \mathbf{Y}_{\mathbb{C}}. \tag{4.4}$$

In addition, $(\mathcal{B}, D(\mathcal{B}))$ *is maximal in the sense that if* $(\mathbf{u},\mathbf{v})$ *is an element in* $\mathbf{Y}_{\mathbb{C}}$ *satisfying*

$$((\mathbf{u}^*,\mathbf{v}^*),(\mathbf{e},\mathbf{h}))_{\mathbf{X}_{\mathbb{C}}} = ((\mathbf{u},\mathbf{v}),(\mathbf{e},\mathbf{h}))_{\mathbf{Y}_{\mathbb{C}}} \quad \forall (\mathbf{e},\mathbf{h}) \in \mathbf{Y}_{\mathbb{C}}$$

for some $(\mathbf{u}^*,\mathbf{v}^*) \in \mathbf{X}_{\mathbb{C}}$*, then* $(\mathbf{u},\mathbf{v}) \in D(\mathcal{B})$ *and* $\mathcal{B}(\mathbf{u},\mathbf{v}) = (\mathbf{u}^*,\mathbf{v}^*)$*.*

From (4.4) we obtain that

$$(\mathcal{B}(\mathbf{u},\mathbf{v}),(\mathbf{u},\mathbf{v}))_{\mathbf{X}_{\mathbb{C}}} = \|(\mathbf{u},\mathbf{v})\|^2_{\mathbf{Y}_{\mathbb{C}}} \geq \|(\mathbf{u},\mathbf{v})\|^2_{\mathbf{X}_{\mathbb{C}}} \quad \forall (\mathbf{u},\mathbf{v}) \in D(\mathcal{B}).$$

Then, in view of the compactness of the embedding $D(\mathcal{B}) \subset \mathbf{X}_{\mathbb{C}}$, we can infer that there exits a complete orthonormal basis $\{(\mathbf{a}_n,\mathbf{b}_n)\}_{n=1}^{\infty}$ in $\mathbf{X}_{\mathbb{C}}$ such that

$$(\mathcal{B}(\mathbf{u},\mathbf{v}),(\mathbf{u},\mathbf{v}))_{\mathbf{X}_{\mathbb{C}}} = \sum_{n=1}^{\infty} \lambda_n |((\mathbf{u},\mathbf{v}),(\mathbf{a}_n,\mathbf{b}_n))_{\mathbf{X}_{\mathbb{C}}}|^2 \quad \forall\, (\mathbf{u},\mathbf{v}) \in D(\mathcal{B}), \tag{4.5}$$

where $1 \leq \lambda_1 \leq \lambda_2 \leq \cdots$, $\lim_{n\to\infty} \lambda_n = +\infty$ and, for every $n \in \mathbb{N}$, $(\mathbf{a}_n,\mathbf{b}_n)$ is the eigenfunction for the eigenvalue of λ_n, i.e.,

$$\mathcal{B}(\mathbf{a}_n,\mathbf{b}_n) = \lambda_n(\mathbf{a}_n,\mathbf{b}_n) \quad \forall n \geq 1$$

In addition, for every $s \geq 0$, the fractional power $\mathcal{B}^s$ of $\mathcal{B}$ can be defined as

$$\mathcal{B}^s(\mathbf{u},\mathbf{v}) := \sum_{n=1}^{\infty} \lambda_n^s ((\mathbf{u},\mathbf{v}),(\mathbf{a}_n,\mathbf{b}_n))_{\mathbf{X}_{\mathbb{C}}} (\mathbf{a}_n,\mathbf{b}_n) \quad \forall\, (\mathbf{u},\mathbf{v}) \in D(\mathcal{B}^s), \tag{4.6}$$

where the domain $D(\mathcal{B}^s)$ is given by

$$D(\mathcal{B}^s) = \{(\mathbf{u},\mathbf{v}) \in \mathbf{X}_{\mathbb{C}} \mid \sum_{n=1}^{\infty} \lambda_n^{2s} |((\mathbf{u},\mathbf{v}),(\mathbf{a}_n,\mathbf{b}_n))_{\mathbf{X}_{\mathbb{C}}}|^2 < \infty\}. \tag{4.7}$$

Then, for each $s \geq 0$, $\mathcal{B}^s : D(\mathcal{B}^s) \subset \mathbf{X} \to \mathbf{X}$ is also self-adjoint and $D(\mathcal{B}^s)$ is a Banach space equipped with the norm

$$\|(\mathbf{u},\mathbf{v})\|_{D(\mathcal{B}^s)} := \|\mathcal{B}^s(\mathbf{u},\mathbf{v})\|_{\mathbf{X}_{\mathbb{C}}} \quad \forall (\mathbf{u},\mathbf{v}) \in D(\mathcal{B}^s), \tag{4.8}$$

which is also equivalent to the corresponding graph norm of $(\mathcal{B}^s, D(\mathcal{B}^s))$ (for more details, we refer to [23, 29]). Let us mention that

$$D(\mathcal{B}^{1/2}) = \mathbf{Y}_{\mathbb{C}} \tag{4.9}$$

holds with norm equivalence (see [30, Theorem 2.33]).

4.2 Verification of VSC and convergence rates

First of all, let us remark that the variational source condition (4.1) is equivalent to

$$\begin{aligned} &((\mathbf{u}^\dagger,\mathbf{v}^\dagger),(\mathbf{u}^\dagger-\mathbf{u},\mathbf{v}^\dagger-\mathbf{v}))_{\mathbf{Y}} \\ &\leq \frac{1-\beta}{2}\|(\mathbf{u}^\dagger,\mathbf{v}^\dagger)-(\mathbf{u},\mathbf{v})\|_{\mathbf{Y}}^2 + \Psi(\|S(\mathbf{u}^\dagger,\mathbf{v}^\dagger)-S(\mathbf{u},\mathbf{v})\|_{\mathbf{X}}), \quad \forall\, (\mathbf{u},\mathbf{v}) \in \mathbf{Y}. \end{aligned} \tag{4.10}$$

At this point, we shall recall that $\mathbf{Y}$ only contains real-valued functions. Our goal now is to verify (4.10) for some concave index function $\Psi : (0,\infty) \to (0,\infty)$ and some constant $\beta \in (0,1]$. The arguments used in the following theorem were partly inspired from [4, Lemma 5.1] and [15, Theorem 2.1].

Theorem 4.3. *Let* $(\mathbf{A0}) - (\mathbf{A2})$ *and* $(\mathbf{A4})$ *be satisfied, and let* $(\mathbf{u}^\dagger,\mathbf{v}^\dagger) \in D(\mathcal{B}^{1/2+s})$ *be a real-valued function with* $s \in (0,1/2]$. *Then, there exits a concave index function* $\Psi : (0,\infty) \to (0,\infty)$ *satisfying the variational source condition (4.10) with* $\beta = \frac{1}{2}$ *and*

$$\Psi(\delta) = O(\delta^{\frac{2s}{2s+1}}) \quad as \quad \delta \to 0^+. \tag{4.11}$$

Proof. We prove that (4.10) is satisfied for $\beta = 1/2$ and an appropriate index function $\Psi : (0,\infty) \to (0,\infty)$. To this end, $(\mathbf{u},\mathbf{v}) \in \mathbf{Y}$ and for every $\lambda \geq \lambda_1$ we introduce the following orthogonal projection:

$$P_\lambda(\mathbf{w},\mathbf{z}) = \sum_{\lambda_1 \leq \lambda_n \leq \lambda} ((\mathbf{w},\mathbf{z}),(\mathbf{a}_n,\mathbf{b}_n))_{\mathbf{X}_{\mathbb{C}}}(\mathbf{a}_n,\mathbf{b}_n) \quad \forall\, (\mathbf{w},\mathbf{z}) \in \mathbf{X}_{\mathbb{C}}.$$

We then infer that for all $\lambda \geq \lambda_1$,

$$\begin{aligned} &((\mathbf{u}^\dagger,\mathbf{v}^\dagger),(\mathbf{u}^\dagger-\mathbf{u},\mathbf{v}^\dagger-\mathbf{v}))_{\mathbf{Y}} \\ =&\mathrm{Re}\,((I-P_\lambda)(\mathbf{u}^\dagger,\mathbf{v}^\dagger),(\mathbf{u}^\dagger-\mathbf{u},\mathbf{v}^\dagger-\mathbf{v}))_{\mathbf{Y}_{\mathbb{C}}} + \mathrm{Re}\,((\mathbf{u}^\dagger,\mathbf{v}^\dagger),P_\lambda(\mathbf{u}^\dagger-\mathbf{u},\mathbf{v}^\dagger-\mathbf{v}))_{\mathbf{Y}_{\mathbb{C}}} \\ =&:\mathbf{I_1}+\mathbf{I_2}, \end{aligned}$$

since $P_\lambda : \mathbf{X}_{\mathbb{C}} \to \mathbf{X}_{\mathbb{C}}$ is self-adjoint.

On the one hand, the Cauchy-Schwarz inequality and Young's inequality yield

$$\begin{aligned}\mathbf{I_1} \leq & \|(\mathbf{u}^\dagger - \mathbf{u}, \mathbf{v}^\dagger - \mathbf{v}))\|_{\mathbf{Y}} \|(I - P_\lambda)(\mathbf{u}^\dagger, \mathbf{v}^\dagger)\|_{\mathbf{Y}_\mathbb{C}} \\ \leq & \frac{1}{4}\|(\mathbf{u}^\dagger - \mathbf{u}, \mathbf{v}^\dagger - \mathbf{v})\|_{\mathbf{Y}}^2 + 4\|(I - P_\lambda)(\mathbf{u}^\dagger, \mathbf{v}^\dagger)\|_{\mathbf{Y}_\mathbb{C}}^2.\end{aligned}$$

From (4.8) and (4.9), it follows that $\|(I - P_\lambda)(\mathbf{u}^\dagger, \mathbf{v}^\dagger)\|_{\mathbf{Y}_\mathbb{C}}^2 = \|\mathcal{B}^{1/2}((I - P_\lambda)(\mathbf{u}^\dagger, \mathbf{v}^\dagger))\|_{\mathbf{X}_\mathbb{C}}^2$. Then, the definition (4.6) of $\mathcal{B}^s$ implies

$$\begin{aligned}\|(I - P_\lambda)(\mathbf{u}^\dagger, \mathbf{v}^\dagger)\|_{\mathbf{Y}_\mathbb{C}}^2 \leq & C\sum_{\lambda_n > \lambda} \lambda_n |((\mathbf{u}, \mathbf{v}), (\mathbf{a}_n, \mathbf{b}_n))_{\mathbf{X}_\mathbb{C}}|^2 \leq C\sum_{\lambda_n > \lambda} \frac{\lambda_n^{1+2s}}{\lambda^{2s}} |((\mathbf{u}, \mathbf{v}), (\mathbf{a}_n, \mathbf{b}_n))_{\mathbf{X}_\mathbb{C}}|^2 \\ \leq & \frac{C}{\lambda^{2s}}\|\mathcal{B}^{1/2+s}(\mathbf{u}^\dagger, \mathbf{v}^\dagger)\|_{\mathbf{X}_\mathbb{C}}^2,\end{aligned}$$

for some constant $C > 0$. In conclusion, we obtain

$$\mathbf{I_1} \leq \frac{1}{4}\|(\mathbf{u}^\dagger - \mathbf{u}, \mathbf{v}^\dagger - \mathbf{v})\|_{\mathbf{Y}}^2 + \frac{4C}{\lambda^{2s}}\|\mathcal{B}^{1/2+s}(\mathbf{u}^\dagger, \mathbf{v}^\dagger)\|_{\mathbf{X}_\mathbb{C}}^2. \tag{4.12}$$

An interplay of Proposition 3.4 and definition of projection P_λ implies that

$$\begin{aligned}\mathbf{I_2} \leq & \sum_{\lambda_n \leq \lambda} \lambda_n |((\mathbf{u}^\dagger, \mathbf{v}^\dagger), (\mathbf{a}_n, \mathbf{b}_n))_{\mathbf{X}_\mathbb{C}}| \cdot |((\mathbf{u}^\dagger - \mathbf{u}, \mathbf{v}^\dagger - \mathbf{v}), (\mathbf{a}_n, \mathbf{b}_n))_{\mathbf{X}_\mathbb{C}}| \\ \leq & \lambda\|(\mathbf{u}^\dagger, \mathbf{v}^\dagger)\|_{\mathbf{X}_\mathbb{C}}\|(\mathbf{u}^\dagger, \mathbf{v}^\dagger) - (\mathbf{u}, \mathbf{v})\|_{\mathbf{X}_\mathbb{C}} \\ = & \lambda\|(\mathbf{u}^\dagger, \mathbf{v}^\dagger)\|_{\mathbf{X}}\|(\mathbf{u}^\dagger, \mathbf{v}^\dagger) - (\mathbf{u}, \mathbf{v})\|_{\mathbf{X}} \\ \leq & C_S\lambda\|(\mathbf{u}^\dagger, \mathbf{v}^\dagger)\|_{\mathbf{X}}\|S(\mathbf{u}^\dagger, \mathbf{v}^\dagger) - S(\mathbf{u}, \mathbf{v})\|_{\mathbf{X}} \\ \leq & C_S\lambda(\|(\mathbf{u}^\dagger, \mathbf{v}^\dagger)\|_{\mathbf{X}} + 1)\|S(\mathbf{u}^\dagger, \mathbf{v}^\dagger) - S(\mathbf{u}, \mathbf{v})\|_{\mathbf{X}}\end{aligned} \tag{4.13}$$

Combing (4.12) and (4.13) and noticing that $\lambda \geq \lambda_1$ is arbitrary, we have

$$\begin{aligned}& ((\mathbf{u}^\dagger, \mathbf{v}^\dagger), (\mathbf{u}^\dagger - \mathbf{u}, \mathbf{v}^\dagger - \mathbf{v}))_{\mathbf{Y}} \\ \leq & \frac{1}{4}\|(\mathbf{u}^\dagger - \mathbf{u}, \mathbf{v}^\dagger - \mathbf{v})\|_{\mathbf{Y}}^2 \\ & + \inf_{\lambda \geq \lambda_1}\left(\frac{4C}{\lambda^{2s}}\|\mathcal{B}^{1/2+s}(\mathbf{u}^\dagger, \mathbf{v}^\dagger)\|_{\mathbf{X}_\mathbb{C}}^2 + C_S\lambda(\|(\mathbf{u}^\dagger, \mathbf{v}^\dagger)\|_{\mathbf{X}} + 1)\|S(\mathbf{u}^\dagger, \mathbf{v}^\dagger) - S(\mathbf{u}, \mathbf{v})\|_{\mathbf{X}}\right)\end{aligned}$$

for all $(\mathbf{u}, \mathbf{v}) \in \mathbf{Y}$. Therefore, it remains to show that the function

$$\Psi_s : (0, \infty) \to (0, \infty), \quad \Psi_s(\delta) := \inf_{\lambda \geq \lambda_1}\left(\frac{A_s}{\lambda^{2s}} + B_s\lambda\delta\right) \tag{4.14}$$

is a concave index function, where we set constants $A_s := 4C\|\mathcal{B}^{1/2+s}(\mathbf{u}^\dagger, \mathbf{v}^\dagger)\|_{\mathbf{X}_\mathbb{C}}^2$ and $B_s := C_S(\|(\mathbf{u}^\dagger, \mathbf{v}^\dagger)\|_{\mathbf{X}} + 1)$. As Ψ_s is an infimum of concave functions, we have that $\Psi_s : (0, \infty) \to (0, \infty)$ is concave. In particular, a classical result yields that $\Psi_s : (0, \infty) \to (0, \infty)$ is continuous (cf. [35, Corollary 47.6]). Now, we prove the decay estimate (4.11), which also implies the continuity of Ψ_s at 0. For any $\delta \in (0, \frac{A_s}{B_s \lambda_1^{(2s+1)}}]$, we choose $\lambda = \sqrt[2s+1]{\frac{A_s}{B_s}}\delta^{-\frac{1}{(2s+1)}}$ and obtain that

$$\Psi_s(\delta) \le 2A_s^{\frac{1}{2s+1}} B_s^{\frac{2s}{2s+1}} \delta^{\frac{2s}{(2s+1)}}.$$

Finally, we verify now that Ψ_s is strictly increasing. To this end, let $\delta_1, \delta_2 > 0$ with $\delta_1 < \delta_2$. Since both $\lambda \to \infty$ implies that the right-hand side of (4.14) blows up, the infimum in the definition of $\Psi(\delta_2)$ can be attained at some $\lambda = \lambda^* < +\infty$. Thus it follows that

$$\Psi_s(\delta_1) \le \left(\frac{A_s}{(\lambda^*)^{2s}} + B_s \lambda^* \delta_1 \right) < \Psi_s(\delta_2). \tag{4.15}$$

This completes the proof. □

An interplay of Lemma 4.3 and Proposition 3.2 yields the following result:

Theorem 4.4. *Assume that* $(\mathbf{A0}) - (\mathbf{A4})$ *hold, and* $(\mathbf{u}^\dagger, \mathbf{v}^\dagger) \in D(\mathcal{B}^{1/2+s})$ *with* $1/2 \ge s > 0$, *and let* Ψ_s *be a concave index function defined as in* (4.14).

(a) Given any regularization parameter $\kappa > 0$, *let* $(\mathbf{u}_\kappa^\delta, \mathbf{v}_\kappa^\delta)$ *denote the regularised solution for Tikhonov-regularization problem (2.2). If the regularization parameter* $\kappa > 0$ *is chosen as* $\kappa = \kappa(\delta) := \frac{2\delta^2}{\Psi_s(\delta)}$, *then the convergence rate*

$$\|(\mathbf{u}_{\kappa(\delta)}^\delta, \mathbf{v}_{\kappa(\delta)}^\delta) - (\mathbf{u}^\dagger, \mathbf{v}^\dagger)\|_{\mathbf{Y}} = O(\delta^{\frac{2s}{2s+1}}) \quad \text{as } \delta \to 0^+ \tag{4.16}$$

holds.

(b) Assume further that ($\mathbf{A3}$) *holds and let* $(\mathbf{K}_{\kappa(\delta)}^\delta, \mathbf{Q}_{\kappa(\delta)}^\delta)$ *be the adjoint state satisfying (3.35)-(3.36) associated with* $(\mathbf{u}_{\kappa(\delta)}^\delta, \mathbf{v}_{\kappa(\delta)}^\delta)$. *Then, it holds that*

$$\|(\mathbf{K}_{\kappa(\delta)}^\delta, \mathbf{Q}_{\kappa(\delta)}^\delta)\|_{C([0,T];\mathbf{X})} = O(\delta^{\frac{2s}{2s+1}}) \quad \text{as } \delta \to 0^+$$

provided that the regularization parameter $\kappa > 0$ *is chosen as* $\kappa = \kappa(\delta) = \frac{2\delta^2}{\Psi_s(\delta^2)}$.

Proof. Assertion (a) is merely a direct consequence of the Proposition 3.2 and Lemma 4.3.

Let $(\mathbf{E}_{\kappa(\delta)}^\delta, \mathbf{H}_{\kappa(\delta)}^\delta) \in C([0,T];\mathbf{X})$ be the mild solution of (2.1) associated with $(\mathbf{u}_{\kappa(\delta)}^\delta, \mathbf{v}_{\kappa(\delta)}^\delta)$. To prove the second assertion, we first obtain from (4.16) and the Lipschtiz continuity of operator $S : \mathbf{X} \to \mathbf{X}$ that

$$\|(\mathbf{E}_{\kappa(\delta)}^\delta(T), \mathbf{H}_{\kappa(\delta)}^\delta(T)) - (\mathbf{e}^\dagger, \mathbf{h}^\dagger)\| = O(\delta^{\frac{2s}{2s+1}}) \quad \text{as } \delta \to 0^+,$$

which, together with (3.2), implies

$$\|(\mathbf{E}_{\kappa(\delta)}^\delta(T), \mathbf{H}_{\kappa(\delta)}^{\delta_n}(T)) - (\mathbf{e}^\delta, \mathbf{h}^\delta)\| = O(\delta^{\frac{2s}{2s+1}}) \quad \text{as } \delta \to 0^+,$$

Then, we use the argument as in the proof of Corollary 3.11 to complete the proof. □

4.3 Concrete realization of $D(\mathcal{B}^s)$ by Sobolev functions

Theorem 4.4 shows the explicit convergence results of regularised solution under the condition that the true initial value belongs to $D(\mathcal{B}^{s+\frac{1}{2}})$, $0 < s \leq \frac{1}{2}$. Our goal now is to present an explicit characterization of $D(\mathcal{B}^{s+\frac{1}{2}})$ in terms of Sobolev space. To this end, we shall utilize the complex interpolation theory.

If two normed complex Banach space X and Y are continuously embedded in a Hausdorff topological vector space, then for every $\theta \in (0,1)$, we can define the complex interpolation $[X,Y]_\theta$ between X and Y (cf. e.g. [21, 26] for details). From the classical theory on the complex interpolation space and [30, Theorem 2.34 and Corollary 2.4], we have the following result.

Lemma 4.5. *Assume that $\mathcal{B} : D(\mathcal{B}) \subset \mathbf{X}_\mathbb{C} \to \mathbf{X}_\mathbb{C}$ is the self-adjoint operator defined as in Lemma 4.2. Then, it holds that*

$$D(\mathcal{B}^s) := [\mathbf{X}_\mathbb{C}, D(\mathcal{B})]_s \quad \forall\, s \in [0,1],$$

and

$$D(\mathcal{B}^s) = [\mathbf{X}_\mathbb{C}, \mathbf{Y}_\mathbb{C}]_{2s} \quad \forall\, 0 \leq s \leq \frac{1}{2}$$

with norm equivalence.

To characterize sufficient space between $\mathbf{X}_\mathbb{C}$ and $D(\mathcal{B})$, we shall utilize fractional Sobolev-Lebesgue spaces of complex valued functions. To more be precise, for any $0 \leq s < \infty$, we define

$$H^s(\mathbb{R}^n;\mathbb{C}) := \{u \in \mathcal{S}(\mathbb{R}^n;\mathbb{C})' \mid \mathcal{F}^{-1}((1+|\xi|^2)^{s/2}\mathcal{F}(u)) \in L^2(\mathbb{R}^2;\mathbb{C})\},$$

where $\mathcal{F} : \mathcal{S}(\mathbb{R}^n;\mathbb{C})' \to \mathcal{S}(\mathbb{R}^n;\mathbb{C})'$ represents the Fourier transform and $\mathcal{S}(\mathbb{R}^n;\mathbb{C})'$ denotes the tempted distribution space. Let us point out that $H^s(\mathbb{R}^n;\mathbb{C})$ is a Hilbert space equipped with inner product

$$(u,v)_{H^s(\mathbb{R}^n;\mathbb{C})} := ((1+|\xi|^2)^{s/2}\mathcal{F}(u), ((1+|\xi|^2)^{s/2}\mathcal{F}(v))_{L^2(\mathbb{R}^n;\mathbb{C})} \quad \forall\, u,v \in H^s(\mathbb{R}^n;\mathbb{C}).$$

In addition, for a bounded domain $\Omega \subset \mathbb{R}^n$ with a Lipschitz boundary $\partial\Omega$, the space $H^s(\Omega;\mathbb{C})$ with a possibly non-integer exponent $s > 0$ is defined as the space of all complex-valued functions $u \in L^2(\Omega;\mathbb{C})$ with some $U \in H^s(\mathbb{R}^n;\mathbb{C})$ such that $U_{|\Omega} = u$, endowed with the norm

$$\|u\|_{H^s(\Omega;\mathbb{C})} := \inf_{\substack{U_{|\Omega}=u \\ U \in H^s(\mathbb{R}^n)}} \|U\|_{H^s(\mathbb{R}^n;\mathbb{C})}.$$

It is well-known that (see e.g. [31]) the above norm is equivalent to

$$\left(\|u\|^2_{L^2(\mathbb{R}^2;\mathbb{C})} + \sum_{|\alpha| \leq \lfloor s \rfloor} \int_{\Omega\times\Omega} \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{n+2s}} dxdy \right)^{\frac{1}{2}},$$

where $\lfloor s \rfloor$ is the largest integer less or equal to s

If s is an integer, then this space coincides with the classical Sobolev space. In particular, $H^0(\Omega;\mathbb{C}) = L^2(\Omega;\mathbb{C})$. For $s > \frac{1}{2}$, the trace operator $\gamma : H^s(\Omega;\mathbb{C}) \to H^{s-\frac{1}{2}}(\partial\Omega;\mathbb{C})$ is linear and bounded. Let $\mathring{H}^s(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ with respect to the norm of $H^s(\Omega)$. Then it is known that $H^s(\Omega) = \mathring{H}^s(\Omega)$ for all $0 \le s < 1/2$. If Ω is $C^{1,1}$, then

$$\mathring{H}^s(\Omega;\mathbb{C}) = \{u \in H^s(\Omega;\mathbb{C}) \mid \gamma(\frac{\partial^k u}{\partial \nu^k}) = 0 \quad \forall\, 0 \le k \le \lfloor s \rfloor\}, \quad \forall\, s \in \big(\frac{1}{2}, 2\big] \setminus \{\frac{3}{2}\}$$

(cf. e.g. [10]). It is worthy mentioning that the characterization

$$[L^2(\Omega;\mathbb{C}), \mathring{H}^2(\Omega;\mathbb{C})]_s = \begin{cases} \mathring{H}^{2s}(\Omega;\mathbb{C}) & 1/4 < s \le 1, \; s \ne \frac{3}{4} \\ H^{2s}(\Omega;\mathbb{C}) & 0 \le s < 1/4. \end{cases} \tag{4.17}$$

holds with norm equivalence (see [10] and [21]).

Let Ω satisfy the assumption **(A4)**. Then we introduce the following function spaces, which will be important in the sequel. We define Banach spaces

$$\mathcal{X}^s : = \{\mathbf{u} \in \mathbf{L}^2(\Omega;\mathbb{C}) \mid \mathbf{u}\,|_{\Omega_j} \in H^{2s}(\Omega_j;\mathbb{C})^3, \; j = 1, 2, \ldots N\}$$
$$\mathring{\mathcal{X}}^s : = \{\mathbf{u} \in \mathbf{L}^2(\Omega;\mathbb{C}) \mid \mathbf{u}\,|_{\Omega_j} \in \mathring{H}^{2s}(\Omega_j;\mathbb{C})^3 \; j = 1, 2, \ldots N\}$$

where the norms are give by $\|\mathbf{u}\|_{\mathcal{X}^s} = \Big(\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega;\mathbb{C})} + \sum_{j=1}^N \|\mathbf{u}\,|_{\Omega_j}\|^2_{H^{2s}(\Omega_j;\mathbb{C})^3}\Big)^{1/2}$. It follows from (4.17) that

$$[\mathbf{L}^2(\Omega;\mathbb{C}), \mathring{\mathcal{X}}^1]_s = \begin{cases} \mathring{\mathcal{X}}^s & 1/4 < s \le 1, \quad s \ne \frac{3}{4}, \\ \mathcal{X}^s & 0 \le s < 1/4. \end{cases} \tag{4.18}$$

Proposition 4.6. *Under the assumptions* **(A0)** *and* **(A4)***, then we have the following continuous embeddings:*

$$\mathring{\mathcal{X}}^s \times \mathring{\mathcal{X}}^s \subset D(\mathcal{B}^s) \;\text{ for } 1 \ge s > \frac{1}{4} \text{ and } s \ne \frac{3}{4},$$

and

$$\mathcal{X}^s \times \mathcal{X}^s \subset D(\mathcal{B}^s) \;\text{ for } 0 \le s < \frac{1}{4}.$$

Proof. Let us define the linear space

$$D(\mathcal{C}) := \mathring{\mathcal{X}}^1 \times \mathring{\mathcal{X}}^1,$$

which is a Banach space under the norm

$$\|(\mathbf{u}, \mathbf{v})\|_{D(\mathcal{C})} = (\|\mathbf{u}\|^2_{\mathcal{X}^1} + \|\mathbf{v}\|^2_{\mathcal{X}^1})^{1/2}. \tag{4.19}$$

Making use of this Banach space, we introduce an unbounded operator $C : D(\mathcal{C}) \subset \mathbf{X}_\mathbb{C} \to \mathbf{X}_\mathbb{C}$, defined by $\mathcal{C}(\mathbf{u}, \mathbf{v}) = (\mathcal{C}_1\mathbf{u}, \mathcal{C}_2\mathbf{v})$, where

$$\mathcal{C}_1\mathbf{u} = \varepsilon^{-1}\mathbf{u} + \varepsilon^{-1}\mathbf{curl}\, \times \mathbf{curl}\,\mathbf{u} - \nabla\mathrm{div}\,(\varepsilon\mathbf{u}),$$

and

$$\mathcal{C}_2\mathbf{v} = \mu^{-1}\mathbf{v} + \mu^{-1}\mathbf{curl}\ \times \mathbf{curl}\,\mathbf{v} - \nabla \mathrm{div}\,(\mu\mathbf{v}).$$

Let us underline that $\mathcal{C}(\mathbf{u},\mathbf{v}) \in \mathbf{X}_{\mathbb{C}}$ holds for all $(\mathbf{u},\mathbf{v}) \in D(\mathcal{C})$ since $\epsilon\mid_{\Omega_j}, \mu\mid_{\Omega_j} \in C^2$ and $\mathbf{u}\mid_{\Omega_j}, \mathbf{v}\mid_{\Omega_j} \in \mathring{H}^2(\Omega_j;\mathbb{C})^3$ for all $j = 1, 2, \ldots, N$. We recall that every $(\mathbf{u},\mathbf{v}) \in \mathbf{Y}_{\mathbb{C}}$, it holds

$$\mathrm{Re}(\mathbf{u},\mathbf{v}), \mathrm{Im}(\mathbf{u},\mathbf{v}) \in \mathbf{Y} := \{(\mathbf{u},\mathbf{v}) \in \mathbf{H_0}(\mathbf{curl}) \times \mathbf{H}(\mathbf{curl}) \mid \varepsilon\mathbf{u} \in \mathbf{H}(\mathrm{div}),\ \ \mu\mathbf{v} \in \mathbf{H_0}(\mathrm{div})\}$$

Therefore, every $(\mathbf{u},\mathbf{v}) \in \mathbf{Y}_{\mathbb{C}}$ satisfies

$$(\mathcal{C}(\mathbf{z}_1,\mathbf{z}_2),(\mathbf{u},\mathbf{v}))_{\mathbf{X}_{\mathbb{C}}} = ((\mathbf{z}_1,\mathbf{z}_2),(\mathbf{u},\mathbf{v}))_{\mathbf{Y}_{\mathbb{C}}} \quad \forall\,(\mathbf{z}_1,\mathbf{z}_2) \in \mathcal{D}_{\mathbb{C}} \times \mathcal{D}_{\mathbb{C}},$$

where

$$\mathcal{D}_{\mathbb{C}} := \{\mathbf{u} \in C_0^\infty(\Omega;\mathbb{C})^3; \mathbf{u}\mid_{\Omega_j} \in C_0^\infty(\Omega_j;\mathbb{C})^3 \text{ for } j = 1, 2, \cdots, N\}.$$

Then, because $D(\mathcal{C})$ is merely the closure of $\mathcal{D}_{\mathbb{C}} \times \mathcal{D}_{\mathbb{C}}$ under the norm (4.19), it holds that

$$(\mathcal{C}(\mathbf{u}_1,\mathbf{v}_1),(\mathbf{u}_2,\mathbf{v}_2))_{\mathbf{X}_{\mathbb{C}}} = ((\mathbf{u}_1,\mathbf{v}_1),(\mathbf{u}_2,\mathbf{v}_2))_{\mathbf{Y}_{\mathbb{C}}} \quad \forall(\mathbf{u}_1,\mathbf{v}_1) \in D(\mathcal{C}),\ (\mathbf{u}_2,\mathbf{v}_2) \in \mathbf{Y}_{\mathbb{C}}.$$

Therefore it follows from Lemma 4.2 that the operator $(\mathcal{C}, D(\mathcal{C}))$ is the restriction of $(\mathcal{B}, D(\mathcal{B}))$ to the domain $D(\mathcal{C})$. Thus, we have that

$$\|(\mathbf{u},\mathbf{v})\|_{D(\mathcal{B})} = \|\mathcal{B}(\mathbf{u},\mathbf{v})\|_{\mathbf{X}_{\mathbb{C}}} = \|\mathcal{C}(\mathbf{u},\mathbf{v})\|_{\mathbf{X}_{\mathbb{C}}} \lesssim \|(\mathbf{u},\mathbf{v})\|_{D(\mathcal{C})} \quad \forall(\mathbf{u},\mathbf{v}) \in D(\mathcal{C}),$$

which ensures that the embedding $D(\mathcal{C}) \subset D(\mathcal{B})$ is continuous. This implies

$$[\mathbf{X}, D(\mathcal{C})]_s \subset [\mathbf{X}, D(\mathcal{B})]_s = D(\mathcal{B}^s) \quad \forall\, s \in [0,1].$$

Now the assertion follows from the above inclusion and (4.18). □

As a consequence of Proposition 4.6 and Theorem 4.4, we can obtain the following result, which characterizes the a priori bound of $(\mathbf{u}^\dagger,\mathbf{v}^\dagger)$ in terms of Sobolev spaces instead of abstract spaces.

Corollary 4.7. *Assume that* $(\mathbf{A0}) - (\mathbf{A4})$ *hold, and* $(\mathbf{u}^\dagger,\mathbf{v}^\dagger)$ *is real-valued and satisfies*

$$(\mathbf{u}^\dagger,\mathbf{v}^\dagger) \in \mathring{\mathcal{X}}_1^{s+\frac{1}{2}} \times \mathring{\mathcal{X}}^{s+\frac{1}{2}}$$

with $1/2 \geq s > 0$ *and* $s \neq \frac{1}{4}$ *. Then all the statements of Theorem 4.4 are valid.*

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