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Differential Sensitivity Analysis of Variational Inequalities with Locally Lipschitz Continuous Solution Operators*

Constantin Christof[†] Gerd Wachsmuth[‡]

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This paper is concerned with the differential sensitivity analysis of variational inequalities in Banach spaces whose solution operators satisfy a generalized Lipschitz condition. We prove a sufficient criterion for the directional differentiability of the solution map that turns out to be also necessary for elliptic variational inequalities in Hilbert spaces (even in the presence of asymmetric bilinear forms, nonlinear operators and nonconvex functionals). In contrast to classical results, our method of proof does not rely on Attouch's theorem on the characterization of Mosco convergence but is fully elementary. Moreover, our technique allows us to also study those cases where the variational inequality at hand is not uniquely solvable and where directional differentiability can only be obtained w.r.t. the weak or the weak-$\star$ topology of the underlying space. As tangible examples, we consider a variational inequality arising in elastoplasticity, the projection onto prox-regular sets, and a bang-bang optimal control problem.

Keywords: Variational Inequalities, Sensitivity Analysis, Directional Differentiability, Bang-Bang, Optimal Control, Differential Stability, Second-Order Epi-Differentiability

MSC: 90C31, 49K40, 47J20

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1 Introduction

The aim of this paper is to study the differentiability properties of the solution operator to a parametrized variational inequality (VI) of the form

$$\bar{x} \in X, \qquad \langle A(p, \bar{x}), x - \bar{x}\rangle + j(x) - j(\bar{x}) \geq 0 \qquad \forall x \in X. \tag{1}$$

Here, X denotes a Banach space, A is an operator into the topological dual $X^\star$ of X, $j : X \to (-\infty, \infty]$ is a proper function (i.e., $j \not\equiv \infty$), and p is an element of some parameter space P (the argument of the solution map). For the precise assumptions on the quantities in (1), we refer to Section 2.

Note that VIs of the type (1) occur naturally as optimality conditions for minimization problems of the form

$$\text{Minimize} \quad J(p, x) + j(x) \qquad \text{w.r.t. } x \in X. \tag{2}$$

Indeed, if j is convex and $J(p, \cdot) : X \to \mathbb{R}$ is convex and Gâteaux differentiable, then (1) with $A(p, \cdot) = \partial_x J(p, \cdot)$ is a necessary and sufficient optimality condition for (2).

Differentiability results for special instances of the VI (1) or the minimization problem (2) can be found frequently in the literature. Especially the case where X is a Hilbert space and where (2) describes the metric projection onto a closed convex nonempty set K (i.e., where $P = X$, $J(p, x) = \frac{1}{2}\|x - p\|_X^2$, and $j = \delta_K : X \to \{0, \infty\}$ is the indicator function of K) has been studied extensively throughout the years in a wide variety of different settings. Exemplarily, we mention [Zarantonello, 1971; Mignot, 1976; Haraux, 1977; Fitzpatrick and Phelps, 1982; Rockafellar, 1990; Shapiro, 1994; Noll, 1995; Rockafellar and Wets, 1998; Levy, 1999; Shapiro, 2016]. Results that cover cases where j is not the indicator function of some set K may be found, e.g., in [Sokołowski, 1988; Sokołowski and Zolésio, 1992; Do, 1992; Borwein and Noll, 1994] and the more recent [De los Reyes and Meyer, 2016; Christof and Meyer, 2016; Adly and Bourdin, 2017; Hintermüller and Surowiec, 2017; Christof and G. Wachsmuth, 2017b].

The contributions that shed the most light on the mechanisms that underlie the sensitivity analysis of VIs of the form (1) are probably [Do, 1992; Borwein and Noll, 1994; Rockafellar and Wets, 1998] and [Adly and Bourdin, 2017]. In these works, it is shown that the differentiability properties of the solution operator to (1) are directly related to the so-called second-order epi-differentiability of the functional $j : X \to (-\infty, \infty]$, cf. Definition 2.9. More precisely, in [Rockafellar and Wets, 1998, Chapter 13G], [Do, 1992, Theorems 3.9 and 4.3] and [Borwein and Noll, 1994, Proposition 6.3] it is established that the directional differentiability of the solution operator to (2) in a point p is equivalent to the (strong) second-order epi-differentiability of the functional j in $\bar{x}$ and the proto-differentiability of the subdifferential ∂j in $\bar{x}$, respectively, provided X is a Hilbert space, $J(p, x) = \frac{1}{2}\|x - p\|_X^2$, and j is a convex and lower semicontinuous function. In [Adly and Bourdin, 2017], a similar (but only sufficient) criterion for the directional differentiability of the solution map is obtained for problems that are not only perturbed in the operator A but also in the functional j, see [Adly and Bourdin, 2017, Theorem 41].

What the approaches in [Do, 1992; Borwein and Noll, 1994; Rockafellar and Wets, 1998; Adly and Bourdin, 2017] have in common is that they rely heavily on rather involved concepts and theorems from set-valued and convex analysis and the theory of monotone operators, cf., e.g., the proof of [Do, 1992, Theorem 4.3]. In this paper, we will demonstrate that the majority of the results in [Do, 1992; Borwein and Noll, 1994; Rockafellar and Wets, 1998; Adly and Bourdin, 2017] can be reproduced and even extended using only elementary tools from functional analysis (the most complicated are the theorem of Banach-Alaoglu and Banach's fixed-point theorem). The main advantages and novel features of our approach are the following:

(i) We can establish that the second-order epi-differentiability of the functional j in $\bar{x}$ is sufficient for the directional differentiability of the solution operator to (1) without making use of involved instruments from convex and set-valued analysis, see Theorem 2.13 and the more tangible Corollary 3.1. We do not have to invoke, e.g., Attouch's theorem which is at the heart of the proofs in [Do, 1992]. We further emphasize that the proof of Theorem 2.13 is shorter than one page and its most complicated argument is the selection of a weak-$\star$ convergent subsequence.

(ii) Because of its simplicity, our analysis allows for various generalizations and extensions. In particular, it is also applicable when (1) is not uniquely solvable, when X is not a Hilbert space, when j is not convex, and when the directional differentiability is only obtainable in the weak or the weak-$\star$ topology of the underlying space, cf. the analysis in Section 2 and the examples in Section 5.

(iii) In the case of an elliptic variational inequality in a Hilbert space, our approach yields the equivalence of the (strong) second-order epi-differentiability of j in $\bar{x}$ and the directional differentiability of the solution operator to (1) even in the presence of nonlinear operators, asymmetric bilinear forms and nonconvex functionals (see Theorem 4.1). We are thus able to extend [Do, 1992, Theorem 4.3] and [Borwein and Noll, 1994, Proposition 6.3] (which require A to be given by $A(p, x) := x - p$ and j to be convex and lower semicontinuous, and which rely heavily on results for the classical Moreau-Yosida regularization) to cases where the VI at hand cannot be identified with a minimization problem of the form (2) and where the method of proof in [Do, 1992; Borwein and Noll, 1994] cannot be employed.

We hope that the self-containedness and conciseness of our approach make this paper in particular helpful for those readers who are interested in the sensitivity analysis of VIs of the form (1) but who are not familiar with, e.g., the concepts of graphical convergence and protodifferentiability.

We would like to point out that the ideas that our analysis is based on can also be used to obtain differentiability results for VIs that involve not only a parameter-dependent operator A but also a parameter-dependent functional j. In [Christof and Meyer, 2016], for example, our approach was used to study the directional differentiability of the solution map $L^\infty_+(\Omega) \times H^{-1}(\Omega) \to H^1_0(\Omega)$, $(c, p) \mapsto \bar{x}$, to a $H^1_0(\Omega)$-elliptic VI of the form

$$\bar{x} \in H^1_0(\Omega), \qquad \langle A(\bar{x}) - p, x - \bar{x}\rangle + \int_\Omega c\, k(x) \mathrm{d}\lambda - \int_\Omega c\, k(\bar{x}) \mathrm{d}\lambda \geq 0 \qquad \forall x \in H^1_0(\Omega),$$

where k is the Nemytskii operator of a piecewise-smooth convex real-valued function. We restrict our analysis to perturbations in the operator A since a unified description of the sensitivity analysis becomes rather involved when perturbations in the functional j are considered, and since somewhat peculiar effects occur when the functional j is manipulated. See, e.g., the results in [Christof and Meyer, 2016, Section 5] for some examples and [Adly and Bourdin, 2017] where the approach of [Do, 1992] is generalized to parameter-dependent functionals j.

Before we begin with our analysis, we give a short overview of the contents and the structure of this paper.

In Section 2, we study the differentiability properties of the solution operator to the VI (1) in an abstract setting. Here, we also motivate and introduce the notions of "weak-$\star$ second subderivative" (Definition 2.5) and "second-order epi-differentiability" (Definition 2.9) that are needed for our approach. The main results of Section 2, Theorem 2.7 and Theorem 2.13, yield that directional derivatives of the solution map to (1) are themselves solutions to suitably defined variational inequalities and that the second-order epi-differentiability of j is sufficient for the directional differentiability of the solution operator to (1).

In Section 3, we state a self-contained corollary of Theorem 2.13 that is more tangible than the results of Section 2. Section 3 further contains a criterion for second-order epi-differentiability that is of major importance not only for practical applications but also for the development of the theory.

In Section 4, we consider the special case that X is a Hilbert space and that A is strongly monotone. In this situation, the sufficient differentiability criterion proved in Section 2 is also necessary and, as a consequence, sharp.

In Section 5, we apply our results to three model problems. These examples are not covered by the classical theory and, thus, highlight the broad applicability of our results. The first problem is a variational inequality of the first kind with saddle-point structure that arises in elastoplasticity and has been studied, e.g., in [Herzog et al., 2013]. Second, we study the projection onto prox-regular sets. Finally, we apply our theorems to bang-bang optimal control problems in the measure space $\mathcal{M}(\Omega)$. The results that we obtain here underline that it makes sense to study the variational inequality (1) in a Banach space setting and that the generality of our approach is not only of theoretical interest but also of relevance in practice.

Lastly, in Section 6, we summarize our findings and make some concluding remarks.

2 Sensitivity Analysis in an Abstract Setting

As already mentioned in the introduction, the aim of this paper is to study variational inequalities of the form (1), i.e., problems of the type

$$\bar{x} \in X, \qquad \langle A(p, \bar{x}), x - \bar{x} \rangle + j(x) - j(\bar{x}) \geq 0 \qquad \forall x \in X.$$

Our standing assumptions on the quantities in (1) are as follows:

Assumption 2.1 (Functional Analytic Setting)**.**

- *X is the (topological) dual of a reflexive or separable Banach space Y.*
- *p is an element of a normed vector space P (the space of parameters).*
- *$j : X \to (-\infty, \infty]$ is a proper function (not necessarily convex).*
- *$A : P \times X \to Y$ is a mapping into the predual Y of X.*

Our main interest is in the differentiability properties of the (potentially set-valued) solution operator

$$S : P \rightrightarrows X, \qquad p \mapsto \{\bar{x} \in X \mid \bar{x} \text{ solves (1) with parameter } p\}.$$

To study the latter in the greatest possible generality, we avoid discussing the solvability of the problem (1) and simply state the minimal assumptions that the solutions to (1) have to satisfy for our sensitivity analysis to hold. Tangible examples (e.g., applications with elliptic variational inequalities in Hilbert spaces) will be addressed later on, cf. Sections 3 to 5.

Assumption 2.2 (Standing Assumptions for the Sensitivity Analysis)**.** *We are given two families $\{q_t\}_{0<t<t_0} \subset P$ and $\{\bar{x}_t\}_{0\le t<t_0} \subset X$, $t_0 > 0$, such that the following is satisfied:*

(i) It holds $q_t \to q$ in P for $t \searrow 0$ with some $q \in P$.

(ii) It holds $\bar{x}_t \in S(tq_t)$ for all $0 < t < t_0$, $\bar{x}_0 \in S(0)$, and there exists a constant $L > 0$ with

$$\|\bar{x}_t - \bar{x}_0\|_X \le Lt \quad \forall t \in [0, t_0). \tag{3}$$

(iii) There exist bounded linear operators $A_p \in \mathcal{L}(P, Y)$ and $A_x \in \mathcal{L}(X, Y)$ such that the difference quotients $y_t := (\bar{x}_t - \bar{x}_0)/t$, $0 < t < t_0$, satisfy

$$A(t\,q_t, \bar{x}_0 + t\,y_t) = A(0, \bar{x}_0) + t\,A_p q_t + t\,A_x y_t + r(t) \tag{4}$$

with a remainder $r : (0, t_0) \to Y$ such that $\|r(t)\|_Y / t \to 0$ for $t \searrow 0$.

Remark 2.3.

(i) Instead of $\bar{x}_t \in S(tq_t)$ for all $0 < t < t_0$ and $\bar{x}_0 \in S(0)$ we could also assume $\bar{x}_t \in S(p + tq_t)$ for all $0 < t < t_0$ and $\bar{x}_0 \in S(p)$ with some fixed $p \in P$. Since such a p can always be "hidden" by redefining A, we consider w.l.o.g. the case $p = 0$.

(ii) The Lipschitz condition in Assumption 2.2 (ii) is, e.g., satisfied in case of an elliptic variational inequality in a Hilbert space, cf. Section 4. If (1) can be identified with a minimization problem of the form (2), (3) can further be recovered from a quadratic growth condition for the solution $\bar{x}_0$ of the unperturbed problem, cf. Section 5.3.

(iii) Assumption 2.2 (iii) is, e.g., satisfied if A is Fréchet differentiable in $(0, \bar{x}_0)$.

We emphasize that we do not say anything about the uniqueness of solutions in the above. We just assume that a family $\{\bar{x}_t\}_{0\leq t<t_0}$ with the properties in Assumption 2.2 exists. In what follows, our aim will be to prove necessary and sufficient conditions for the weak-$\star$ and the strong convergence of the difference quotients y_t. Note that, if $q_t = q$ and if $S(tq)$ is a singleton for all $0 \leq t < t_0$, then the weak-$\star$ (respectively, strong) convergence of y_t to some y for $t \searrow 0$ is equivalent to the weak-$\star$ (respectively, strong) directional differentiability of the solution operator S in the point $p = 0$ in the direction q with directional derivative y. To study the behavior of the difference quotients $\{y_t\}_{0<t<t_0}$, we make the following observation:

Lemma 2.4. *The difference quotients y_t, $0 < t < t_0$, satisfy*

$$\begin{aligned}\left\langle A_p q_t + A_x y_t, z - y_t\right\rangle &+ \frac{1}{2}\left(\frac{j(\bar{x}_0 + t\,z) - j(\bar{x}_0) - t\,\langle a_0, z\rangle}{t^2/2}\right)\\ &- \frac{1}{2}\left(\frac{j(\bar{x}_0 + t\,y_t) - j(\bar{x}_0) - t\,\langle a_0, y_t\rangle}{t^2/2}\right) + \hat{r}(t)\,\|z - y_t\|_X \geq 0\end{aligned} \tag{5}$$

for all $z \in X$. Here, $a_0 := -A(0, \bar{x}_0)$ and $\hat{r}(t) := \|r(t)\|_Y/t$, so that $\hat{r}(t) = o(1)$ as $t \searrow 0$.

Proof. Since $\bar{x}_t = \bar{x}_0 + t y_t$ solves (1) with $p = t\,q_t$ and because of (4), it holds

$$\begin{aligned}0 &\leq \left\langle A(t\,q_t, \bar{x}_t), \bar{x}_0 + t\,z - \bar{x}_t\right\rangle + j(\bar{x}_0 + t\,z) - j(\bar{x}_t)\\ &= \left\langle -a_0 + t\,A_p q_t + t\,A_x y_t + r(t), t\,(z - y_t)\right\rangle + j(\bar{x}_0 + t\,z) - j(\bar{x}_0 + t\,y_t)\end{aligned}$$

for all $z \in X$, where $\|r(t)\|_Y/t \to 0$ as $t \searrow 0$. Dividing by t^2, rearranging terms and using that $j(\bar{x}_0) \in \mathbb{R}$, (5) follows immediately with $\hat{r}(t) = \|r(t)\|_Y/t$. ☐

Note that from $\bar{x}_0 \in S(0)$ and (1), we obtain that $\bar{x}_0$ and $a_0 := -A(0, \bar{x}_0)$ satisfy $\bar{x}_0 \in \operatorname{dom}(j) := \{x \in X \mid j(x) \in \mathbb{R}\}$ and $a_0 \in \partial j(\bar{x}_0)$, where

$$\partial j(x) := \{g \in Y \mid j(z) \geq j(x) + \langle g, z - x\rangle\ \forall z \in X\} \quad \forall x \in X.$$

This shows that the bracketed expressions in (5) may be interpreted as second-order difference quotients in which the (possibly nonexistent) derivative of j at $\bar{x}_0$ is replaced with an element of the subdifferential $\partial j(\bar{x}_0)$ (where we use the term subdifferential somewhat loosely here since j is not assumed to be convex). The structure of (5) motivates the following definition.

Definition 2.5 (Weak-$\star$ Second Subderivative)**.** *Let $x \in \operatorname{dom}(j)$ and $g \in Y$ be given. Then the (weak-$\star$) second subderivative $Q_j^{x,g} : X \to [-\infty, \infty]$ of j in x for g is defined by*

$$Q_j^{x,g}(z) := \inf\left\{\liminf_{n\to\infty} \frac{j(x + t_n\,z_n) - j(x) - t_n\langle g, z_n\rangle}{t_n^2/2}\ \middle|\ t_n \searrow 0, z_n \overset{\star}{\rightharpoonup} z\right\}.$$

The notion of second subderivatives goes (at least to the authors' best knowledge) back to Rockafellar who introduced the concept in finite dimensions in 1985, see [Rockafellar, 1985]. Since then second subderivatives have appeared frequently in the literature, although under different names. [Do, 1992] and [Noll, 1995], for example, use a construction analogous to that in Definition 2.5 in the Hilbert space setting and call the resulting functional second-order epi-derivative and second-order Mosco derivative, respectively. We remark that the epigraph of $Q_j^{x,g}$ can be identified with an appropriately defined Kuratowski limit of the epigraphs of the difference quotient functions appearing in (5), cf. [Do, 1992, Section 1].

The next lemma collects some basic properties of the functional $Q_j^{x,g}$.

Lemma 2.6. *Let $x \in \operatorname{dom}(j)$ and $g \in Y$ be arbitrary but fixed. Then, it holds $Q_j^{x,g}(\alpha\, z) = \alpha^2\, Q_j^{x,g}(z)$ for all $\alpha > 0$ and all $z \in X$ and $Q_j^{x,g}(0) \le 0$. Moreover, if $g \in \partial j(x)$, then $Q_j^{x,g}(z) \ge 0$ for all $z \in X$ and $Q_j^{x,g}(0) = 0$.*

Proof. The first formula follows from a simple scaling argument. In the case $g \in \partial j(x)$, the nonnegativity of $Q_j^{x,g}$ follows from the definition of ∂j. The formulas for $Q_j^{x,g}(0)$ follow from the choice $z_n = 0$. □

In the case that $g \in \partial j(x)$, Lemma 2.6 implies in particular that $Q_j^{x,g}$ is proper and that the domain of the second subderivative $Q_j^{x,g}$ is a pointed cone (where "pointed" means that the cone contains the origin). In what follows, we will call this cone the *reduced critical cone* $\mathcal{K}_j^{x,g}$, i.e.,

$$\mathcal{K}_j^{x,g} := \operatorname{dom}\big(Q_j^{x,g}\big) = \big\{z \in X \;\big|\; Q_j^{x,g}(z) < +\infty\big\}.$$

The motivation behind this naming convention will become clear in Lemma 2.11 (iii) and the examples in Section 5.

We are now in the position to prove that a limit point y of the difference quotients y_t has to be the solution of a certain VI that involves the second subderivative.

Theorem 2.7 (Necessary Condition for Limit Points of the Difference Quotients y_t)**.** *Suppose that there exists a $y \in X$ such that the difference quotients y_t satisfy $y_t \overset{\star}{\rightharpoonup} y$ in X and $A_x y_t \to A_x y$ in Y for $t \searrow 0$. Then, y satisfies*

$$\langle A_p q + A_x y, z - y\rangle + \frac{1}{2} Q_j^{\bar{x}_0,a_0}(z) - \frac{1}{2} Q_j^{\bar{x}_0,a_0}(y) \ge 0 \qquad \forall z \in X \tag{6}$$

and it holds

$$Q_j^{\bar{x}_0,a_0}(y) = -\langle A_p q + A_x y, y\rangle < +\infty \tag{7}$$

as well as

$$Q_j^{\bar{x}_0,a_0}(y) = \lim_{t \searrow 0} \frac{j(\bar{x}_0 + t\, y_t) - j(\bar{x}_0) - t\langle a_0, y_t\rangle}{t^2/2}. \tag{8}$$

Proof. We first prove (6). Let $z \in \mathcal{K}_j^{\bar{x}_0,a_0}$ and $\varepsilon > 0$ be given. Then, the definitions of $Q_j^{\bar{x}_0,a_0}(z)$ and $\mathcal{K}_j^{\bar{x}_0,a_0}$ yield that there exist sequences $z_n \overset{\star}{\rightharpoonup} z$ and $t_n \searrow 0$ with

$$Q_j^{\bar{x}_0,a_0}(z) \le \lim_{n\to\infty} \frac{j(\bar{x}_0 + t_n\, z_n) - j(\bar{x}_0) - t_n\langle g, z_n\rangle}{t_n^2/2} \le Q_j^{\bar{x}_0,a_0}(z) + \varepsilon < \infty.$$

From (5) with t_n and z_n, we infer

$$\left\langle A_p q_{t_n} + A_x y_{t_n}, z_n - y_{t_n} \right\rangle + \frac{1}{2}\,\frac{j(\bar{x}_0 + t_n\, z_n) - j(\bar{x}_0) - t_n\,\langle a_0, z_n\rangle}{t_n^2/2} \\ - \frac{1}{2}\,\frac{j(\bar{x}_0 + t_n\, y_{t_n}) - j(\bar{x}_0) - t_n\,\langle a_0, y_{t_n}\rangle}{t_n^2/2} + \hat{r}_n \geq 0$$

with $\hat{r}_n \to 0$ as $n \to \infty$. Passing to the limit $n \to \infty$ in the above, we find

$$0 \leq \left\langle A_p q + A_x y, z - y \right\rangle + \frac{1}{2}\, Q_j^{\bar{x}_0, a_0}(z) + \varepsilon - \liminf_{n\to\infty} \frac{1}{2}\,\frac{j(\bar{x}_0 + t_n\, y_{t_n}) - j(\bar{x}_0) - t_n\,\langle a_0, y_{t_n}\rangle}{t_n^2/2} \\ \leq \left\langle A_p q + A_x y, z - y \right\rangle + \frac{1}{2}\, Q_j^{\bar{x}_0, a_0}(z) + \varepsilon - \frac{1}{2}\, Q_j^{\bar{x}_0, a_0}(y).$$

Letting $\varepsilon \searrow 0$, (6) now follows immediately.

In order to prove (7), we first note that (6) with $z = 0 \in \mathcal{K}_j^{\bar{x}_0, a_0}$ yields $Q_j^{\bar{x}_0, a_0}(y) < +\infty$. Choosing $z = s\,y$ with arbitrary $s \geq 0$ in (6) and using the positive homogeneity of $Q_j^{\bar{x}_0, a_0}$, we further find that

$$(s-1)\left\langle A_p q + A_x y, y \right\rangle + \frac{s^2 - 1}{2}\, Q_j^{\bar{x}_0, a_0}(y) \geq 0 \qquad \forall s \geq 0.$$

Dividing this inequality by $s - 1$ and passing to the limits $s \searrow 1$ and $s \nearrow 1$, we obtain (7) as claimed.

It remains to check (8). To this end, we fix a sequence $\{t_n\} \subset \mathbb{R}^+$ with $t_n \searrow 0$ and consider for $s > 0$ the function

$$\Theta(s) := \limsup_{n\to\infty} \frac{j(\bar{x}_0 + s\,t_n\, y_{s\,t_n}) - j(\bar{x}_0) - s\,t_n\langle a_0, y_{s\,t_n}\rangle}{(s\,t_n)^2/2} \geq Q_j^{\bar{x}_0, a_0}(y).$$

For arbitrary $s_1, s_2 > 0$, (5) with $t = s_1\, t_n$, $z = \frac{s_2}{s_1}\, y_{s_2\, t_n}$ yields

$$\left\langle A_p q_{s_1\, t_n} + A_x y_{s_1\, t_n}, \frac{s_2}{s_1}\, y_{s_2\, t_n} - y_{s_1\, t_n} \right\rangle \\ + \frac{1}{2}\,\frac{j(\bar{x}_0 + s_2\, t_n\, y_{s_2\, t_n}) - j(\bar{x}_0) - s_2\, t_n\,\langle a_0, y_{s_2\, t_n}\rangle}{(s_1\, t_n)^2/2} \\ - \frac{1}{2}\,\frac{j(\bar{x}_0 + s_1\, t_n\, y_{s_1\, t_n}) - j(\bar{x}_0) - s_1\, t_n\,\langle a_0, y_{s_1\, t_n}\rangle}{(s_1\, t_n)^2/2} + \hat{r}_n \geq 0,$$

where $\hat{r}_n \to 0$ as $n \to \infty$. Passing to the limit in the above (with a suitable subsequence), using (7) and multiplying by 2, we find

$$2\left(1 - \frac{s_2}{s_1}\right) Q_j^{\bar{x}_0, a_0}(y) + \frac{s_2^2}{s_1^2}\,\Theta(s_2) - \Theta(s_1) \geq 0. \tag{9}$$

The same arguments with $s_2 = 0$ yield

$$2\, Q_j^{\bar{x}_0, a_0}(y) - \Theta(s_1) \geq 0 \quad \forall s_1 > 0. \tag{10}$$

We will now prove that

$$\Theta(s) \le \frac{m+1}{m}\, Q_j^{\bar{x}_0,a_0}(y) \qquad \forall s>0 \quad \forall m\in\mathbb{N}. \tag{11}$$

To obtain (11), we use induction over m. For $m=1$, (11) is equivalent to (10) so there is nothing to prove. For the induction step $m\mapsto m+1$, we choose $s_1=s$ and $s_2=\frac{m}{m+1}s$ in (9). This yields

$$\begin{aligned}\Theta(s) &\le 2\left(1-\frac{m}{m+1}\right) Q_j^{\bar{x}_0,a_0}(y) + \frac{m^2}{(m+1)^2}\,\Theta(s_2)\\ &\le 2\left(1-\frac{m}{m+1}\right) Q_j^{\bar{x}_0,a_0}(y) + \frac{m^2}{(m+1)^2}\,\frac{m+1}{m}\, Q_j^{\bar{x}_0,a_0}(y) \le \frac{m+2}{m+1}\, Q_j^{\bar{x}_0,a_0}(y),\end{aligned}$$

where in the second estimate we have used the induction hypothesis. Hence, (11) is valid and the induction is complete. Letting $m\to\infty$ in (11), we arrive at $\Theta(s)\le Q_j^{\bar{x}_0,a_0}(y)$ for all $s>0$. With $s=1$, we obtain in particular

$$Q_j^{\bar{x}_0,a_0}(y) \le \liminf_{n\to\infty} \frac{j(\bar{x}_0+t_n\,y_{t_n}) - j(\bar{x}_0) - t_n\langle a_0, y_{t_n}\rangle}{t_n^2/2} \le \Theta(1) \le Q_j^{\bar{x}_0,a_0}(y).$$

Since $\{t_n\}$ was arbitrary, (8) now follows immediately and the proof is complete. □

Remark 2.8.

(i) The assumptions $y_t \overset{\star}{\rightharpoonup} y$ in X and $A_x y_t \to A_x y$ in Y in Theorem 2.7 are satisfied in two interesting situations: Firstly, if y_t converges even strongly to y and secondly if $y_t \overset{\star}{\rightharpoonup} y$ and if A_x is weakly-$\star$ completely continuous. We will see in Sections 4 and 5 that both these cases appear in practice (the first one in the Hilbert space setting, the second one in case of our bang-bang example).

(ii) Assume for the moment that $q_t = q$, that $S(tq)$ is a singleton for all $0\le t<t_0$, and that S is (strongly) directionally differentiable in $p=0$ in the direction q with derivative y. Then, Theorem 2.7 implies that y has to be a solution to (6) and that the difference quotients y_t have to be a recovery sequence for the weak-$\star$ second subderivative $Q_j^{\bar{x}_0,a_0}(y)$, see (8). These necessary conditions for directional derivatives (that also apply when S is only directionally differentiable in some directions) have, at least to the authors' best knowledge, not been known before.

The above observation that the difference quotients y_t provide a recovery sequence motivates the following definition.

Definition 2.9 (Second-Order Epi-Differentiability). *Let $x\in\operatorname{dom}(j)$ and $g\in Y$ be given. The functional j is said to be weakly-$\star$ twice epi-differentiable (respectively, strictly twice epi-differentiable, respectively, strongly twice epi-differentiable) in x for g in a direction $z\in X$, if for all $\{t_n\}\subset\mathbb{R}^+$ with $t_n\searrow 0$ there exists a sequence z_n satisfying $z_n \overset{\star}{\rightharpoonup} z$ (respectively, $z_n \overset{\star}{\rightharpoonup} z$ and $\|z_n\|_X\to\|z\|_X$, respectively, $z_n\to z$) and*

$$Q_j^{x,g}(z) = \lim_{n\to\infty} \frac{j(x+t_n\,z_n) - j(x) - t_n\langle g, z_n\rangle}{t_n^2/2}. \tag{12}$$

The functional j is called weakly-$\star$/strictly/strongly twice epi-differentiable in x for g if it is weakly-$\star$/strictly/strongly twice epi-differentiable in x for g in all directions $z \in X$.

Remark 2.10.

(i) We emphasize that the prefixes "weakly-$\star$", "strictly" and "strongly" in Definition 2.9 refer to the mode of convergence of the recovery sequence. In all cases, the considered second subderivative is that in Definition 2.5.

(ii) If X is reflexive, then strong second-order epi-differentiability is equivalent to the Mosco epi-convergence of the sequence of second-order difference quotient functions appearing in (5), see, e.g., [Do, 1992, Section 2].

(iii) Note that j is weakly-$\star$/strictly/strongly twice epi-differentiable in an $x \in \operatorname{dom}(j)$ for a $g \in \partial j(x)$ if and only if j is weakly-$\star$/strictly/strongly twice epi-differentiable in x for g in all directions $z \in \mathcal{K}_j^{x,g}$. This follows from the fact that for all $z \in X \setminus \mathcal{K}_j^{x,g}$, recovery sequences can trivially be found (just choose, e.g., $z_n := z$).

If j is twice epi-differentiable in a point $x \in \operatorname{dom}(j)$ for some $g \in \partial j(x)$, then $Q_j^{x,g}$ enjoys additional properties as the following lemma shows.

Lemma 2.11. *Let $x \in \operatorname{dom}(j)$ and $g \in \partial j(x)$ be given.*

(i) If j is convex and weakly-$\star$ twice epi-differentiable in x for g, then $Q_j^{x,g}$ is convex.

(ii) If j is strictly twice epi-differentiable in x for g, then $Q_j^{x,g}$ is weakly-$\star$ sequentially lower semicontinuous.

(iii) If j is Hadamard directionally differentiable in x and strongly twice epi-differentiable in x for g, then it holds $j'(x; z) = \langle g, z\rangle$ for all $z \in \mathcal{K}_j^{x,g}$.

Proof. We first prove (i): Let $z, \hat{z} \in \mathcal{K}_j^{x,g}$ and $\lambda \in [0, 1]$ be given, and let $\{t_n\} \subset \mathbb{R}^+$ be an arbitrary but fixed sequence with $t_n \searrow 0$. Then, the definition of weak-$\star$ second-order epi-differentiability implies that there exist recovery sequences $z_n, \hat{z}_n$ with $z_n \overset{\star}{\rightharpoonup} z$ and $\hat{z}_n \overset{\star}{\rightharpoonup} \hat{z}$ such that (12) holds for z and $\hat{z}$, respectively. Using these sequences, the convexity of j and the definition of $Q_j^{x,g}\big(\lambda z + (1 - \lambda) \hat{z}\big)$, we may compute

$$
\begin{aligned}
&\lambda Q_j^{x,g}(z) + (1 - \lambda) Q_j^{x,g}(\hat{z}) \\
&\qquad = \lim_{n\to\infty} \frac{\lambda j(x + t_n z_n) + (1 - \lambda) j(x + t_n \hat{z}_n) - j(x) - t_n \langle g, \lambda z_n + (1 - \lambda) \hat{z}_n\rangle}{t_n^2/2} \\
&\qquad \geq \liminf_{n\to\infty} \frac{j\big(x + t_n (\lambda z_n + (1 - \lambda) \hat{z}_n)\big) - j(x) - t_n \langle g, \lambda z_n + (1 - \lambda) \hat{z}_n\rangle}{t_n^2/2} \\
&\qquad \geq Q_j^{x,g}\big(\lambda z + (1 - \lambda) \hat{z}\big).
\end{aligned}
$$

This establishes (i).

To obtain (ii), we consider an arbitrary but fixed $z \in X$ and a sequence z_k with $z_k \overset{\star}{\rightharpoonup} z$. We assume w.l.o.g. that $\liminf_{k\to\infty} Q_j^{x,g}(z_k) = \lim_{k\to\infty} Q_j^{x,g}(z_k) \in \mathbb{R}$ (if it holds $\liminf_{k\to\infty} Q_j^{x,g}(z_k) = \infty$, then the claim is vacuously true). Suppose for the time being that Y is separable with a countable dense subset $\{w_i\}_{i\in\mathbb{N}}$, let $\{t_n\} \subset \mathbb{R}^+$ be some sequence with $t_n \searrow 0$, and let $\{z_{k,n}\}_{n\in\mathbb{N}}$ be recovery sequences for the z_k as in the definition of the strict second-order epi-differentiability. Then, we may find a strictly increasing sequence $\{N_k\}$ such that

$$\sum_{i=1}^{k} |\langle w_i, z_{k,n} - z_k\rangle| + \big| \|z_{k,n}\|_X - \|z_k\|_X \big| + \Big| Q_j^{x,g}(z_k) - \frac{j(x + t_n z_{k,n}) - j(x) - t_n\langle g, z_{k,n}\rangle}{t_n^2/2} \Big| \leq \frac{1}{k} \tag{13}$$

holds for all $n \geq N_k$ and all $k \in \mathbb{N}$. Redefine $N_1 := 1$ and set $k_n := \sup\{k \in \mathbb{N} \mid n \geq N_k\}$ for all $n \in \mathbb{N}$. Then, it holds $n \geq N_{k_n}$ for all n by definition, $k_n \in \mathbb{N}$ for all n by the strict monotonicity of $\{N_k\}$ and $k_n \to \infty$ monotonously for $n \to \infty$. The latter implies in tandem with (13) that $\hat{z}_n := z_{k_n,n}$ satisfies $\langle w_i, \hat{z}_n - z_{k_n}\rangle \to 0$ for all $i \in \mathbb{N}$, $|\|\hat{z}_n\|_X - \|z_{k_n}\|_X| \to 0$ and

$$\Big| \frac{j(x + t_n \hat{z}_n) - j(x) - t_n\langle g, \hat{z}_n\rangle}{t_n^2/2} - Q_j^{x,g}(z_{k_n}) \Big| \to 0$$

as $n \to \infty$. From $z_{k_n} \overset{\star}{\rightharpoonup} z$ and the boundedness of the norms $\|z_{k_n}\|_X$, we now obtain $\hat{z}_n \overset{\star}{\rightharpoonup} z$ with

$$\lim_{k\to\infty} Q_j^{x,g}(z_k) = \lim_{n\to\infty} Q_j^{x,g}(z_{k_n}) = \liminf_{n\to\infty} \frac{j(x + t_n \hat{z}_n) - j(x) - t_n\langle g, \hat{z}_n\rangle}{t_n^2/2} \geq Q_j^{x,g}(z).$$

This establishes (ii) in the case that Y is separable. If Y is not separable but reflexive, we can use standard arguments as employed, e.g., in [Kuttler, 1997, Proof of Theorem 6.24] to resort to the separable case.

It remains to prove (iii). To this end, suppose that a $z \in \mathcal{K}_j^{x,g}$ is given and that z_n is a recovery sequence for some $\{t_n\}$ with $t_n \searrow 0$ as in the definition of the strong second-order epi-differentiability. Then, the Hadamard directional differentiability and the finiteness of $Q_j^{x,g}(z)$ yield

$$0 = \lim_{n\to\infty} \frac{j(x + t_n z_n) - j(x) - t_n\langle g, z_n\rangle}{t_n} = j'(x; z) - \langle g, z\rangle. \qquad \square$$

Remark 2.12. *If j is Hadamard directionally differentiable in $\bar{x}_0$ and $a_0 := -A(0, \bar{x}_0)$, then it is easy to check that $\bar{x}_0 \in S(0)$ implies*

$$j'(\bar{x}_0; z) - \langle a_0, z\rangle \geq 0 \qquad \forall z \in X. \tag{14}$$

In the case that the VI (1) arises from a minimization problem of the form (2), (14) is precisely the necessary optimality condition of first order. Lemma 2.11 (iii) shows that, under the assumptions of Hadamard directional differentiability and strong second-order epi-differentiability, all elements of the set $\mathcal{K}_j^{\bar{x}_0,a_0}$ satisfy the necessary condition (14) with equality. The set $\mathcal{K}_j^{\bar{x}_0,a_0}$ is thus contained in what is typically referred to as the critical cone. We point out that the latter inclusion is in general strict, cf. the examples in Section 5. It therefore makes sense to call the set $\mathcal{K}_j^{\bar{x}_0,a_0}$ the reduced critical cone.

We are now in the position to state the main theorem of this section. It establishes that the second-order epi-differentiability of j and the uniqueness of solutions to (6) are sufficient for the weak-$\star$ convergence of the difference quotients y_t.

Theorem 2.13 (Sufficient Condition for the Convergence of the Difference Quotients)**.** *Suppose that one of the following conditions is satisfied.*

(i) j is weakly-$\star$ twice epi-differentiable in $\bar{x}_0$ for a_0 and A_x is weakly-$\star$ completely continuous in the sense that $y_n \overset{\star}{\rightharpoonup} y$ in X implies $A_x y_n \to A_x y$ in Y.

(ii) j is strongly twice epi-differentiable in $\bar{x}_0$ for a_0 and A_x is such that $y_n \overset{\star}{\rightharpoonup} y$ in X implies $A_x y_n \rightharpoonup A_x y$ in Y and $\liminf_{n\to\infty}\langle A_x y_n, y_n\rangle \geq \langle A_x y, y\rangle$.

Then, the sequence of difference quotients y_t has at least one weak-$\star$ accumulation point for $t \searrow 0$, and if $y \in X$ is such an accumulation point, then it holds

$$\langle A_p q + A_x y, z - y\rangle + \frac{1}{2}Q_j^{\bar{x}_0,a_0}(z) - \frac{1}{2}Q_j^{\bar{x}_0,a_0}(y) \geq 0 \qquad \forall z \in X \tag{15}$$

and $\langle A_x y_{t_n}, y_{t_n}\rangle \to \langle A_x y, y\rangle$ for every sequence $\{t_n\} \subset \mathbb{R}^+$ with $t_n \searrow 0$ and $y_{t_n} \overset{\star}{\rightharpoonup} y$. If, moreover, (15) admits at most one solution, then there exists a unique $y \in X$ with $y_t \overset{\star}{\rightharpoonup} y$ and $\langle A_x y_t, y_t\rangle \to \langle A_x y, y\rangle$ for $t \searrow 0$, and this limit y is a solution to (15).

Proof. Since the family of difference quotients $\{y_t\}$ is bounded by (3), the existence of a weak-$\star$ accumulation point is a direct consequence of the theorem of Banach-Alaoglu. Consider now an arbitrary but fixed $y \in X$ that satisfies $y_n := y_{t_n} \overset{\star}{\rightharpoonup} y$ for some $\{t_n\} \subset \mathbb{R}^+$ with $t_n \searrow 0$ and let $z \in \mathcal{K}_j^{\bar{x}_0,a_0}$ be given. Then, the definitions of weak-$\star$ and strong second-order epi-differentiability imply that in both cases (i) and (ii) we can find a recovery sequence $\{z_n\}$ with

$$z_n \overset{\star}{\rightharpoonup} z, \quad \langle A_x y_n, z_n\rangle \to \langle A_x y, z\rangle, \quad Q_j^{\bar{x}_0,a_0}(z) = \lim_{n\to\infty} \frac{j(\bar{x}_0 + t_n z_n) - j(\bar{x}_0) - t_n\langle a_0, z_n\rangle}{t_n^2/2}.$$

Using the sequence z_n in (5), we find that

$$\begin{aligned}\big\langle A_p q_n + A_x y_n, z_n - y_n\big\rangle + \frac{1}{2}\,\frac{j(\bar{x}_0 + t_n z_n) - j(\bar{x}_0) - t_n\,\langle a_0, z_n\rangle}{t_n^2/2}& \\ - \frac{1}{2}\,\frac{j(\bar{x}_0 + t_n y_n) - j(\bar{x}_0) - t_n\,\langle a_0, y_n\rangle}{t_n^2/2} + \hat{r}_n\,\|z_n - y_n\|_X &\geq 0,\end{aligned}$$

where $q_n := q_{t_n}$ and where $\hat{r}_n$ is a remainder with $\hat{r}_n \searrow 0$ for $n \to \infty$. Letting $n \to \infty$ in the above, it follows

$$\begin{aligned}\big\langle A_p q + A_x y, z\big\rangle - \big\langle A_p q, y\big\rangle + \frac{1}{2}\,Q_j^{\bar{x}_0,a_0}(z)& \\ \geq \limsup_{n\to\infty}\left(\big\langle A_x y_n, y_n\big\rangle + \frac{1}{2}\,\frac{j(\bar{x}_0 + t_n y_n) - j(\bar{x}_0) - t_n\,\langle a_0, y_n\rangle}{t_n^2/2}\right)& \\ \geq \limsup_{n\to\infty}\big\langle A_x y_n, y_n\big\rangle + \frac{1}{2}\,Q_j^{\bar{x}_0,a_0}(y)& \\ \geq \liminf_{n\to\infty}\big\langle A_x y_n, y_n\big\rangle + \frac{1}{2}\,Q_j^{\bar{x}_0,a_0}(y) \geq \big\langle A_x y, y\big\rangle + \frac{1}{2}\,Q_j^{\bar{x}_0,a_0}(y).&\end{aligned}$$

Hence, $y \in \mathcal{K}_j^{\bar{x}_0,a_0}$ and y solves (15). Moreover, by using the test function $z = y$ in the above chain of inequalities, we obtain $\langle A_x y_n, y_n \rangle \to \langle A_x y, y \rangle$. This proves the first claim.

Suppose now that (15) admits at most one solution. Then, the boundedness of the family $\{y_t\}$ implies that for every sequence $\{t_n\} \subset \mathbb{R}^+$ with $t_n \searrow 0$ a subsequence of $\{y_{t_n}\}$ converges weakly-$\star$. From the first part of the theorem and the fact that (15) can have at most one solution, we obtain that the weak-$\star$ limit point is unique. A standard argument now shows that the entire sequence y_t has to be weakly-$\star$ convergent to y with $\langle A_x y_t, y_t \rangle \to \langle A_x y, y \rangle$ for $t \searrow 0$. This completes the proof. □

Note that, as a byproduct of our sensitivity analysis, we obtain that (15) always admits a solution $y \in X$ in the situation of Theorem 2.13.

3 A Tangible Corollary and Some Helpful Results

To make the results of Section 2 more accessible, we state the following self-contained corollary of Theorem 2.13 that covers the case where the solution operator $S : P \rightrightarrows X$ satisfies a generalized local Lipschitz condition.

Corollary 3.1 (Directional Differentiability in the Case of Local Lipschitz Continuity). *Let $S : P \rightrightarrows X$ denote the (potentially set-valued) solution operator of the VI*

$$\bar{x} \in X, \qquad \langle A(p, \bar{x}), x - \bar{x} \rangle + j(x) - j(\bar{x}) \geq 0 \qquad \forall x \in X,$$

where X, A, j are assumed to satisfy the conditions in Assumption 2.1. Denote by $B_r^Z(z)$ the closed ball in a normed space Z with radius $r > 0$ and midpoint $z \in Z$. Suppose that a $p_0 \in P$, an $\bar{x}_0 \in S(p_0)$ and an $R > 0$ are given such that $S(p_0) \cap B_R^X(\bar{x}_0) = \{\bar{x}_0\}$, such that A is Fréchet-differentiable in $(p_0, \bar{x}_0)$ with partial derivatives $A_p \in \mathcal{L}(P, Y)$ and $A_x \in \mathcal{L}(X, Y)$, and such that the solution map S is locally nonempty and upper Lipschitzian at p_0 in the sense that

$$\emptyset \neq S(p) \cap B_R^X(\bar{x}_0) \subset B_{Lt}^X(\bar{x}_0) \quad \forall p \in B_t^P(p_0)$$

for some $L > 0$ and all small enough $t > 0$. Suppose further that the VI

$$y \in X, \qquad \langle A_p q + A_x y, z - y \rangle + \frac{1}{2} Q_j^{\bar{x}_0,a_0}(z) - \frac{1}{2} Q_j^{\bar{x}_0,a_0}(y) \geq 0 \qquad \forall z \in X \tag{16}$$

with $a_0 := -A(p_0, \bar{x}_0)$ admits at most one solution for every $q \in P$ and assume that one of the following conditions is satisfied.

(i) j is weakly-$\star$ twice epi-differentiable in $\bar{x}_0$ for a_0 and A_x is weakly-$\star$ completely continuous in the sense that $y_n \overset{\star}{\rightharpoonup} y$ in X implies $A_x y_n \to A_x y$ in Y.

(ii) j is strongly twice epi-differentiable in $\bar{x}_0$ for a_0 and A_x is such that $y_n \overset{\star}{\rightharpoonup} y$ in X implies $A_x y_n \rightharpoonup A_x y$ in Y and $\liminf_{n\to\infty} \langle A_x y_n, y_n \rangle \geq \langle A_x y, y \rangle$.

Then, (16) is uniquely solvable for all $q \in P$ and S is weakly-$\star$ Hadamard directionally differentiable in the sense that for every family of parameters $\{q_t\}_{0<t<t_0} \subset P$ that satisfies $q_t \to q$ for $t \searrow 0$ with some $q \in P$ and every family of solutions $\{\bar{x}_t\}_{0<t<t_0} \subset X$ that satisfies $\bar{x}_t \in S(p_0 + tq_t) \cap B_R^X(\bar{x}_0)$ for all $0 < t < t_0$, it holds

$$\frac{\bar{x}_t - \bar{x}_0}{t} \stackrel{\star}{\rightharpoonup} y \quad \text{and} \quad \left\langle A_x \left(\frac{\bar{x}_t - \bar{x}_0}{t} \right), \frac{\bar{x}_t - \bar{x}_0}{t} \right\rangle \to \langle A_x y, y \rangle \tag{17}$$

for $t \searrow 0$, where y is the unique solution to (16). If, moreover, $z \mapsto \langle A_x z, z \rangle$ is a Legendre form in the sense of [Christof and G. Wachsmuth, 2017a, Lemma 5.1b)], then the convergence of the difference quotients is even strong.

Proof. If we start with a family of parameters $\{q_t\}_{0<t<t_0} \subset P$ and a family of solutions $\{\bar{x}_t\}_{0<t<t_0} \subset X$ as in the definition of the weak-$\star$ Hadamard directional differentiability, then we are precisely in the situation of Assumption 2.2 (after translation by p_0) and Theorem 2.13 immediately implies (17). To obtain the claim with the strong convergence, we just have to use the definition of the Legendre form. This completes the proof. □

We remark that a special case of the above corollary may be found in [Bonnans and Shapiro, 2000, Theorem 5.5].

In practice, it is typically hard to check whether a given functional j is twice epi-differentiable in a point $x \in \operatorname{dom}(j)$ for some $g \in \partial j(x)$, cf., e.g., the calculations in [Christof and Meyer, 2016, Section 4] and [Christof and G. Wachsmuth, 2017a, Section 6.2]. The following lemma turns out to be helpful in this context not only in practical applications but also for theoretical considerations.

Lemma 3.2 (Criterion for Second-Order Epi-Differentiability). *Let $x \in \operatorname{dom}(j)$ and $g \in \partial j(x)$ be given. Suppose that there exist a set $Z \subset \mathcal{K}_j^{x,g}$ and a functional $Q : \mathcal{K}_j^{x,g} \to [0, \infty)$ such that*

(i) for all $z \in \mathcal{K}_j^{x,g}$ it holds $Q_j^{x,g}(z) \geq Q(z)$,

(ii) for all $z \in Z$ and all $\{t_n\} \subset \mathbb{R}^+$ with $t_n \searrow 0$, there exists a sequence $\{z_n\} \subset X$ satisfying $z_n \stackrel{\star}{\rightharpoonup} z$, $\|z_n\|_X \to \|z\|_X$, and

$$Q(z) = \lim_{n\to\infty} \frac{j(x + t_n z_n) - j(x) - t_n \langle g, z_n \rangle}{t_n^2/2},$$

(iii) for all $z \in \mathcal{K}_j^{x,g}$ there exists a sequence $\{z_k\} \subset Z$ with $z_k \stackrel{\star}{\rightharpoonup} z$, $\|z_k\|_X \to \|z\|_X$ and $Q(z) \geq \liminf_{k\to\infty} Q(z_k)$.

Then, $Q = Q_j^{x,g}$ and j is strictly twice epi-differentiable in x for g. If, moreover, the sequences in (ii) and (iii) can be chosen to be strongly convergent, then j is even strongly twice epi-differentiable in x for g.

Proof. We first prove the strict second-order epi-differentiability. From the properties of Q and the definition of $Q_j^{x,g}$, we immediately obtain $Q = Q_j^{x,g}$ on Z. Assume now that a

$z \in \mathcal{K}_j^{x,g}$ and a sequence $\{t_n\} \subset \mathbb{R}^+$ with $t_n \searrow 0$ are given. Then, (iii) implies that we can find a sequence $\{z_k\} \subset Z$ with $z_k \stackrel{\star}{\rightharpoonup} z$, $\|z_k\|_X \to \|z\|_X$ and $Q(z) \geq \liminf_{k\to\infty} Q(z_k)$. From (ii), we obtain further that for each z_k there exists a sequence $\{z_{k,n}\}$ satisfying

$$z_{k,n} \stackrel{\star}{\rightharpoonup} z_k, \quad \|z_{k,n}\|_X \to \|z_k\|_X, \quad \text{and} \quad \frac{j(x + t_n\, z_{k,n}) - j(x) - t_n\langle g, z_{k,n}\rangle}{t_n^2/2} \to Q(z_k)$$

for $n \to \infty$. Using exactly the same argumentation as in the proof of Lemma 2.11 (ii), we can now construct a sequence $\{\hat{z}_n\}$ with $\hat{z}_n \stackrel{\star}{\rightharpoonup} z$, $\|\hat{z}_n\|_X \to \|z\|_X$, and

$$Q_j^{x,g}(z) \leq \liminf_{n\to\infty} \frac{j(x + t_n\, \hat{z}_n) - j(x) - t_n\langle g, \hat{z}_n\rangle}{t_n^2/2} = \liminf_{k\to\infty} Q(z_k) \leq Q(z) \leq Q_j^{x,g}(z),$$

where the first and the last estimate follow from Definition 2.5 and (i), respectively. The above implies that $\{\hat{z}_n\}$ is a recovery sequence for z as in the definition of the strict second-order epi-differentiability. Since $z \in \mathcal{K}_j^{x,g}$ was arbitrary, the first claim of the lemma now follows immediately, cf. Remark 2.10 (iii).

To obtain the strong second-order epi-differentiability under the assumption of strong convergence in (ii) and (iii), we can proceed along exactly the same lines (just modify the selection argument in the proof of Lemma 2.11 (ii) accordingly). □

Using Lemma 3.2, we obtain, e.g., the following result.

Corollary 3.3 (Indicator Functions of Extended Polyhedric Sets)**.** *Let $K \subset X$ be a closed, convex, nonempty set, and denote by $\delta_K : X \to \{0, \infty\}$ the indicator function of K. Suppose that X is reflexive, and assume that an $x \in K$ and a $g \in \partial\delta_K(x)$ are given such that K is extended polyhedric in x for g in the sense of [Bonnans and Shapiro, 2000, Definition 3.52], i.e., such that*

$$\mathcal{T}_K(x) \cap g^\perp = \operatorname{cl}\big(\{z \in \mathcal{T}_K(x) \mid 0 \in \mathcal{T}_K^2(x,z)\} \cap g^\perp\big)$$

holds, where $\mathcal{T}_K(x) := \operatorname{cl}(\mathbb{R}^+(K - x))$ and

$$\mathcal{T}_K^2(x,z) := \Big\{r \in X : \operatorname{dist}\big(x + t\, z + \tfrac{1}{2}\, t^2\, r, K\big) = o(t^2) \text{ as } t \searrow 0\Big\}$$

denote the tangent cone and the second-order tangent set at x and (x,z), respectively. Then, δ_K is strongly twice epi-differentiable in x for g and it holds

$$\mathcal{K}_{\delta_K}^{x,g} = \mathcal{T}_K(x) \cap g^\perp \quad \text{and} \quad Q_{\delta_K}^{x,g}(z) = 0 \quad \forall z \in \mathcal{K}_{\delta_K}^{x,g}.$$

Proof. We use Lemma 3.2 to prove the claim: Set $Z := \big\{z \in \mathcal{T}_K(x) \mid 0 \in \mathcal{T}_K^2(x,z)\big\} \cap g^\perp$. Then, the definition of $\mathcal{T}_K^2(x,z)$ implies that for every $z \in Z$ and every $\{t_n\} \subset \mathbb{R}^+$ with $t_n \searrow 0$ there exists a sequence $\{r_n\} \subset X$ with $x + t_n z + \frac{1}{2} t_n^2 r_n \in K$ and $r_n \to 0$. The latter yields (cf. Definition 2.5)

$$0 \leq Q_{\delta_K}^{x,g}(z) \leq \liminf_{n\to\infty} \langle -g, r_n\rangle = 0,$$

i.e., $Q^{x,g}_{\delta_K}(z) = 0$ for all $z \in Z$ and $Z \subset \mathcal{K}^{x,g}_{\delta_K}$. From the definition of $Q^{x,g}_{\delta_K}$ and the lemma of Mazur, we obtain further that $\mathcal{K}^{x,g}_{\delta_K} \subset \mathcal{T}_K(x) \cap g^\perp$. This implies in tandem with the extended polyhedricity of K in x for g that the set Z is dense in $\mathcal{K}^{x,g}_{\delta_K}$. Defining $Q : \mathcal{K}^{x,g}_{\delta_K} \to [0, \infty)$, $Q(z) := 0$ for all $z \in \mathcal{K}^{x,g}_{\delta_K}$, we now arrive exactly at the situation of Lemma 3.2 (with strong convergence). This allows us to deduce that $Q = Q^{x,g}_{\delta_K} \equiv 0$ holds on $\mathcal{K}^{x,g}_{\delta_K}$ and that δ_K is strongly twice epi-differentiable in x for g. From Lemma 2.11 (ii), we now obtain that $Q^{x,g}_{\delta_K} : X \to [0, \infty]$ is weakly-$\star$ lower semicontinuous. This yields in combination with the density of Z in $\mathcal{T}_K(x) \cap g^\perp$ and the inclusion $\mathcal{K}^{x,g}_{\delta_K} \subset \mathcal{T}_K(x) \cap g^\perp$ that $\mathcal{K}^{x,g}_{\delta_K} = \mathcal{T}_K(x) \cap g^\perp$ and completes the proof. □

Note that, in the situation of Corollary 3.3, it is always true that

$$\mathbb{R}^+(K - x) \subset \left\{ z \in \mathcal{T}_K(x) \mid 0 \in \mathcal{T}^2_K(x, z) \right\},$$

cf. the definition of the second-order tangent set $\mathcal{T}^2_K(x, z)$. This implies that every set that is polyhedric in the sense of [Haraux, 1977] is also extended polyhedric in the sense of [Bonnans and Shapiro, 2000]. In particular, we may combine Corollary 3.1 with Corollary 3.3 to obtain a generalization of the classical differentiability result of [Mignot, 1976], cf. also the results in [Do, 1992, Example 4.6] in this context.

We point out that there exist closed convex sets that satisfy the condition of extended polyhedricity but violate that of polyhedricity. An easy example is the set

$$\{0\} \cup \operatorname{conv}\left\{(1/n, 1/n^4) \in \mathbb{R}^2 \;\middle|\; n \in \mathbb{Z}\right\} \subset \mathbb{R}^2.$$

4 Elliptic Variational Inequalities in Hilbert Spaces

Having studied the very general setting of Section 2, we now turn our attention to elliptic variational inequalities in Hilbert spaces. What is remarkable about VIs of this type is that the second-order epi-differentiability of the functional j is not only sufficient for the directional differentiability of the solution map S but also necessary. More precisely, we have the following result.

Theorem 4.1 (Directional Differentiability for Elliptic Variational Inequalities). *Let X, A, j be as in Assumption 2.1. Suppose that X is a Hilbert space and assume that a $p_0 \in P$ and an $R > 0$ are given such that the VI*

$$\bar{x} \in X, \qquad \langle A(p, \bar{x}), x - \bar{x} \rangle + j(x) - j(\bar{x}) \geq 0 \qquad \forall x \in X \tag{18}$$

admits a unique solution $S(p) \in X$ for all $p \in B^P_R(p_0)$ and such that there exist constants $c, C > 0$ with

$$c\|x_1 - x_2\|^2_X \leq \langle A(p_0, x_1) - A(p_0, x_2), x_1 - x_2 \rangle \quad \forall x_1, x_2 \in X \tag{19}$$

and

$$\|A(p, x) - A(p_0, x)\|_Y \leq C\|p - p_0\|_P \quad \forall x \in X \quad \forall p \in B^P_R(p_0). \tag{20}$$

Write $\bar{x}_0 := S(p_0)$ and $a_0 := -A(p_0, \bar{x}_0)$, and suppose that A is Fréchet differentiable in $(p_0, \bar{x}_0)$ with partial derivatives $A_p \in \mathcal{L}(P, Y)$ and $A_x \in \mathcal{L}(X, Y)$. Assume that A_p is surjective. Then, the following statements are equivalent:

- *(I) The solution map $S : B_R^P(p_0) \to X$ is strongly Hadamard directionally differentiable in p_0 in all directions $q \in P$.*
- *(II) The functional j is strongly twice epi-differentiable in $\bar{x}_0$ for a_0.*

Moreover, if one of these conditions is satisfied, then the following assertions hold.

- *(i) $Q_j^{\bar{x}_0,a_0} : X \to [0, \infty]$ is proper, weakly lower semicontinuous and positively homogeneous of degree two, and $\mathcal{K}_j^{\bar{x}_0,a_0}$ is a pointed cone.*
- *(ii) The directional derivative $y := S'(p_0; q) \in X$ in p_0 in a direction q is uniquely characterized by the variational inequality*

$$y \in X, \qquad \langle A_p q + A_x y, z - y\rangle + \frac{1}{2}Q_j^{\bar{x}_0,a_0}(z) - \frac{1}{2}Q_j^{\bar{x}_0,a_0}(y) \geq 0 \qquad \forall z \in X. \quad (21)$$

  Moreover,

$$Q_j^{\bar{x}_0,a_0}(y) = \lim_{t \searrow 0} \frac{j(\bar{x}_0 + t\, y_t) - j(\bar{x}_0) - t\, \langle a_0, y_t\rangle}{t^2/2},$$

  where $y_t := (S(p_0 + t\, q) - \bar{x}_0)/t$, i.e., the difference quotients y_t are a recovery sequence for y.
- *(iii) For every $z \in \mathcal{K}_j^{\bar{x}_0,a_0}$ there exists a sequence $\{y_k\} \subset S'(p_0; P)$ with*

$$y_k \to z \quad \text{and} \quad Q_j^{\bar{x}_0,a_0}(y_k) \nearrow Q_j^{\bar{x}_0,a_0}(z)$$

  as $k \to \infty$. In particular, $S'(p_0; P)$ is a dense subset of $\mathcal{K}_j^{\bar{x}_0,a_0}$.

Proof. We first demonstrate that part (ii) of Corollary 3.1 is applicable. Set $\bar{x}_p := S(p)$ for all $p \in B_R^P(p_0)$. Then, (18), (19) and (20) yield

$$\begin{aligned} c\|\bar{x}_p - \bar{x}_0\|_X^2 &\leq \langle A(p_0, \bar{x}_p) - A(p_0, \bar{x}_0), \bar{x}_p - \bar{x}_0\rangle \\ &\leq \langle A(p, \bar{x}_p) - A(p_0, \bar{x}_p), \bar{x}_0 - \bar{x}_p\rangle \leq C\|p - p_0\|_P \|\bar{x}_p - \bar{x}_0\|_X \end{aligned}$$

for all $p \in B_R^P(p_0)$, and we obtain

$$\|\bar{x}_p - \bar{x}_0\|_X \leq \frac{C}{c}\|p - p_0\|_P \quad \forall p \in B_R^P(p_0).$$

This shows that the solution map $S : B_R^P(p_0) \to X$ is Lipschitz at p_0 (in the classical sense). From (19) and the Fréchet differentiability of A in $(p_0, \bar{x}_0)$, it follows further

$$c\|z\|_X^2 \leq \lim_{t \searrow 0} \frac{\langle A(p_0, \bar{x}_0 + tz) - A(p_0, \bar{x}_0), z\rangle}{t} = \langle A_x z, z\rangle \quad (22)$$

for all $z \in X$, i.e., the bilinear form $(z_1, z_2) \mapsto \langle A_x z_1, z_2\rangle$ is elliptic. Note that this ellipticity implies in particular that the map $z \mapsto \langle A_x z, z\rangle$ is a Legendre form. Consider now the VI (21) and assume that there exists a $q \in P$ such that (21) admits two solutions y_1 and y_2. Then, it necessarily holds

$$\langle A_p q + A_x y_1, y_2 - y_1\rangle + \frac{1}{2}Q_j^{\bar{x}_0,a_0}(y_2) - \frac{1}{2}Q_j^{\bar{x}_0,a_0}(y_1) \geq 0 \qquad \text{and}$$
$$\langle A_p q + A_x y_2, y_1 - y_2\rangle + \frac{1}{2}Q_j^{\bar{x}_0,a_0}(y_1) - \frac{1}{2}Q_j^{\bar{x}_0,a_0}(y_2) \geq 0,$$

and we obtain by addition

$$0 \geq \langle A_x y_1 - A_x y_2, y_1 - y_2\rangle \geq c\|y_1 - y_2\|_X^2. \tag{23}$$

This shows that (21) can have at most one solution. If we combine all of the above, we see that the assumptions of part (ii) of Corollary 3.1 are indeed satisfied under condition (II). The implication (II) $\Rightarrow$ (I) now follows immediately.

Next, we check that (i) and (ii) hold under condition (II). First, we note that Lemmas 2.6 and 2.11 yield that (II) implies (i). Further, Corollary 3.1 and Theorem 2.7 show that (II) implies (ii).

It thus only remains to prove that (I) $\Rightarrow$ (II) holds and that one of the conditions (I) and (II) entails (iii).

So let us assume that (I) is satisfied, i.e., suppose that the map $S : B_R^P(p_0) \to X$ is strongly Hadamard directionally differentiable in p_0 in all directions $q \in P$. Then, it follows from Theorem 2.7 that the directional derivative $y := S'(p_0; q)$ of S in p_0 in a direction $q \in P$ solves (21). Since S is directionally differentiable in p_0 in all directions $q \in P$ and since (21) can have at most one solution, the latter implies that (21) possesses a unique solution y for all $q \in P$ and that, if $y \in X$ solves (21) with parameter q, then y is necessarily identical to the directional derivative $S'(p_0; q)$. Consider now for $n \in \mathbb{N}_0$ and

$$\varepsilon := \frac{c}{2\|A_x\|_{\mathcal{L}(X,Y)}} \in (0, \infty)$$

the variational inequality

$$y \in X, \qquad \langle A_p q + (1 + n\varepsilon)A_x y, z - y\rangle + \frac{1}{2}Q_j^{\bar{x}_0,a_0}(z) - \frac{1}{2}Q_j^{\bar{x}_0,a_0}(y) \geq 0 \qquad \forall z \in X. \tag{24}$$

We claim that (24) admits a unique solution $y \in X$ for all $q \in P$ and all $n \in \mathbb{N}_0$. Note that this unique solvability is indeed nontrivial since we do not know anything about the second subderivative $Q_j^{\bar{x}_0,a_0}$ at this moment. To show that (24) has a unique solution for all $q \in P$, we use induction on n. Since (24) with $n = 0$ is precisely (21), the induction basis is trivial, so let us assume that the unique solvability is proved for some $n \in \mathbb{N}_0$. From the surjectivity of A_p, we obtain that for every $u \in X$ and every $q \in P$, there exists a $\tilde{q} \in P$ such that $A_p\tilde{q} = A_p q + \varepsilon A_x u$. The latter implies in combination with the induction hypothesis that the VI

$$y \in X, \qquad \langle A_p q + \varepsilon A_x u + (1 + n\varepsilon)A_x y, z - y\rangle + \frac{1}{2}Q_j^{\bar{x}_0,a_0}(z) - \frac{1}{2}Q_j^{\bar{x}_0,a_0}(y) \geq 0 \qquad \forall z \in X \tag{25}$$

admits a unique solution for all $q \in P$ and all $u \in X$. Fix q and denote the solution operator $X \ni u \mapsto y \in X$ of (25) by T. Then, it follows analogously to the proof of (23) that T is globally Lipschitz with

$$\|T(u_1) - T(u_2)\|_X \leq \frac{\varepsilon}{(1+n\varepsilon)c}\|A_x u_1 - A_x u_2\|_Y \leq \frac{1}{2}\|u_1 - u_2\|_X,$$

where the last estimate follows from the definition of ε. The above shows that T is a contraction and implies, in combination with Banach's fixed-point theorem, that there exists a unique $u \in X$ with $Tu = u$, i.e., with

$$u \in X, \qquad \langle A_p q + (1+(n+1)\varepsilon)A_x u, z - u\rangle + \frac{1}{2}Q_j^{\bar{x}_0,a_0}(z) - \frac{1}{2}Q_j^{\bar{x}_0,a_0}(u) \geq 0 \qquad \forall z \in X.$$

This completes the induction step.

We are now in the position to prove (II). Consider an arbitrary but fixed $\tilde{z} \in \mathcal{K}_j^{\bar{x}_0,a_0}$, and choose a sequence $\{q_n\} \subset P$ with $A_p q_n = -(1+n\varepsilon)A_x\tilde{z}$ for all $n \in \mathbb{N}$ (possible due to surjectivity). Denote by y_n, $n \in \mathbb{N}$, the unique solution to

$$y_n \in X, \qquad \langle A_p q_n + (1+n\varepsilon)A_x y_n, z - y_n\rangle + \frac{1}{2}Q_j^{\bar{x}_0,a_0}(z) - \frac{1}{2}Q_j^{\bar{x}_0,a_0}(y_n) \geq 0 \qquad \forall z \in X. \tag{26}$$

Then, the choice $z = \tilde{z}$, the definition of q_n and (22) yield

$$c(1+n\varepsilon)\|\tilde{z} - y_n\|_X^2 \leq (1+n\varepsilon)\langle A_x\tilde{z} - A_x y_n, \tilde{z} - y_n\rangle \leq \frac{1}{2}Q_j^{\bar{x}_0,a_0}(\tilde{z}) - \frac{1}{2}Q_j^{\bar{x}_0,a_0}(y_n),$$

and we may deduce that

$$Q_j^{\bar{x}_0,a_0}(y_n) \leq Q_j^{\bar{x}_0,a_0}(\tilde{z}) \qquad \text{and} \qquad \|\tilde{z} - y_n\|_X^2 \leq \frac{1}{2c(1+n\varepsilon)}Q_j^{\bar{x}_0,a_0}(\tilde{z})$$

holds for all $n \in \mathbb{N}$. Note that the surjectivity of A_p implies that for every y_n there exists a $\tilde{q}_n \in P$ with $A_p\tilde{q}_n = A_p q_n + n\varepsilon A_x y_n$, and that for each such $\tilde{q}_n$ it necessarily holds $y_n = S'(p_0; \tilde{q}_n)$ by (26) and Theorem 2.7. We may thus conclude that for every $\tilde{z} \in \mathcal{K}_j^{\bar{x}_0,a_0}$ we can find a sequence y_n with

$$y_n \in S'(p_0; P), \qquad y_n \to \tilde{z} \qquad \text{and} \qquad \limsup_{n\to\infty} Q_j^{\bar{x}_0,a_0}(y_n) \leq Q_j^{\bar{x}_0,a_0}(\tilde{z}). \tag{27}$$

Using that for each $y \in S'(p_0; P)$ there exists a recovery sequence as in the definition of the strong second-order epi-differentiability, see (8), and applying Lemma 3.2 with $Q = Q_j^{\bar{x}_0,a_0}$ and $Z = S'(p_0; P)$, it now follows straightforwardly that j is strongly twice epi-differentiable in $\bar{x}_0$ for a_0. This shows that (I) indeed implies (II).

To finally prove (iii), we note that the strong second-order epi-differentiability of j in $\bar{x}_0$ for a_0 yields the weak lower semi-continuity of the functional $Q_j^{\bar{x}_0,a_0}$, see Lemma 2.11(ii). This allows us to continue the last estimate in (27) as follows

$$Q_j^{\bar{x}_0,a_0}(\tilde{z}) \geq \limsup_{n\to\infty} Q_j^{\bar{x}_0,a_0}(y_n) \geq \liminf_{n\to\infty} Q_j^{\bar{x}_0,a_0}(y_n) \geq Q_j^{\bar{x}_0,a_0}(\tilde{z}),$$

i.e., $Q_j^{\bar{x}_0,a_0}(y_n) \to Q_j^{\bar{x}_0,a_0}(\tilde{z})$ as $n \to \infty$. Since $Q_j^{\bar{x}_0,a_0}(y_n) \leq Q_j^{\bar{x}_0,a_0}(\tilde{z})$ holds by the construction of y_n, (iii) follows immediately. This completes the proof. □

Some remarks regarding Theorem 4.1 are in order.

Remark 4.2.

(i) *The VIs in Theorem 4.1 are, in fact, not elliptic variational inequalities in the classical sense since we do not assume, e.g., that j is convex and lower semicontinuous. The classical setting, i.e., the situation where $P = X^\star$ and where (18) takes the form*

$$\bar{x} \in X, \qquad \langle A(\bar{x}), x - \bar{x}\rangle + j(x) - j(\bar{x}) \geq \langle p, x - \bar{x}\rangle \qquad \forall x \in X$$

with a strongly monotone and Fréchet differentiable operator A and a convex, lower semicontinuous and proper functional j (cf. [Adly and Bourdin, 2017, Section 1.1]), is, of course, covered by Theorem 4.1 as one may easily check. Note that, for a classical elliptic variational inequality, the uniqueness and existence of solutions $S(p)$, $p \in X^\star$, follows immediately from the theory of pseudomonotone operators, cf. [Oden and Kikuchi, 1980, Section 1.7].

(ii) *We point out that Theorem 4.1 significantly generalizes [Do, 1992, Theorem 4.3], where the equivalence of (I) and (II) is proved under the assumption that $P = X$, that $\langle A(p, x_1), x_2\rangle = (x_1 - p, x_2)_X$ for all $x_1, x_2, p \in X$ (where $(.,.)_X$ denotes the inner product in X), and that j is convex, proper and lower semicontinuous (see also [Borwein and Noll, 1994, Proposition 6.3] in this context). Note that we have proved Theorem 4.1 without ever using the concept of protodifferentiability, and that Theorem 4.1 also covers those cases where the VI (18) cannot be identified with a minimization problem of the form (2) and where, as a consequence, the method of proof in [Do, 1992] cannot be applied.*

(iii) *Although the proof of Theorem 4.1 does not need the Hilbert space structure of X, it is not useful to assume that X is only a Banach space in the situation of Theorem 4.1 since the existence of a Fréchet differentiable map A with the property (19) already implies that X is Hilbertizable, see (22).*

5 Three Applications

In what follows, we demonstrate by means of three tangible examples that the results in Sections 2 to 4 are not only interesting from a theoretical point of view but also suitable for practical applications which are not covered by the existing literature.

5.1 Static Elastoplasticity in Dual Formulation

In this section, we demonstrate that Corollary 3.1 enables us to differentiate the solution map of static elastoplasticity. In [Herzog et al., 2013], it was shown that this map is weakly directionally differentiable. This result was sharpened in [Betz and Meyer, 2015, Theorem 3.8] to Bouligand differentiability under more restrictive regularity assumptions. Our technique enables us to prove Hadamard directional differentiability under the natural regularity of the problem.

We will work in a slightly abstract setting. However, we roughly keep the notation of [Herzog et al., 2013; Betz and Meyer, 2015] to make it easier to transfer our results to the precise setting of static elastoplasticity.

Assumption 5.1 (Setting of Elastoplasticity)**.** *We assume that V is a Hilbert space, that μ is a σ-finite measure on a set Ω, and that $m, n \in \mathbb{N}$. We further suppose that a bounded, linear, symmetric and coercive map $A : S^2 \to (S^2)^\star$, a bounded linear map $B : S^2 \to V^\star$ and a linear operator $\mathcal{D} : \mathbb{R}^m \to \mathbb{R}^n$ are given, where $S^2 := L^2(\mu)^m$. Via $\mathcal{D}$ we define the set $K := \{\Sigma \in S^2 : |\mathcal{D}\Sigma|_{\mathbb{R}^n} \leq 1 \text{ a.e. in } \Omega\}$. Finally, we assume that the restriction of B to the Hilbert space $H := \{\Sigma \in S^2 : |\mathcal{D}\Sigma|_{\mathbb{R}^n} = 0 \text{ a.e. in } \Omega\}$ is surjective.*

Although we do not need the product-type structure of S^2, we keep this notation for consistency with the above references. For the same reason, we use the symbol A to denote the first operator appearing in Assumption 5.1. (This operator will only be a part of the nonlinearity in (1) and should not be confused with it, cf. (32) below).

By $\mathcal{T}_K(\Sigma), \mathcal{N}_K(\Sigma) \subset S^2$ we denote the tangent cone and the normal cone to K at $\Sigma \in K$ in the sense of convex analysis, respectively.

Within the above framework, we consider for a given datum $\ell \in V^\star$ the minimization problem

$$\text{Minimize} \quad \frac{1}{2}\langle A\,\Sigma, \Sigma\rangle \quad \text{such that} \quad B\Sigma = \ell \quad \text{and} \quad \Sigma \in K. \tag{28}$$

First, we provide a result concerning the solvability of (28).

Lemma 5.2. *Problem (28) admits a unique solution Σ_ℓ for every $\ell \in V^\star$. Moreover, for every $\ell \in V^\star$ with associated solution Σ_ℓ, there exists a unique multiplier $u_\ell \in V$ with*

$$\langle A\Sigma_\ell + B^\star u_\ell, T - \Sigma_\ell\rangle - \langle B\Sigma_\ell - \ell, v - u_\ell\rangle \geq 0 \qquad \forall (T, v) \in K \times V, \tag{29}$$

and the solution map $V^\star \ni \ell \mapsto (\Sigma_\ell, u_\ell) \in S^2 \times V$ is globally Lipschitz continuous.

Proof. By our assumptions, the feasible set of (28) is closed, convex and nonempty. Together with the continuity, radial unboundedness and strict convexity of the objective, this yields the existence of a unique solution Σ_ℓ for all $\ell \in V^\star$. Further, the CQ of Zowe and Kurcyusz is satisfied by (28). Thus, there exist $u_\ell \in V$ and $\Xi_\ell \in \mathcal{N}_K(\Sigma_\ell)$ with

$$A\Sigma_\ell + B^\star u_\ell + \Xi_\ell = 0.$$

The above and the equality $B\Sigma_\ell = \ell$ immediately give (29). Next, we prove the Lipschitz continuity of the solution map. By using that $B : H \to V^\star$ is surjective and that H is a Hilbert space, we find that B admits a bounded linear right inverse $\tilde{B} : V^\star \to H$, i.e., $B\tilde{B} = \mathrm{Id}_{V^\star}$. Consider now two right-hand sides $\ell_1, \ell_2 \in V^\star$ with associated solutions $\Sigma_{\ell_1}, \Sigma_{\ell_2}$ and multipliers u_{ℓ_1}, u_{ℓ_2}. Then, we may choose the tuple $(T, v) := (\Sigma_{\ell_2} + \tilde{B}(\ell_1 - \ell_2), u_{\ell_1})$ in the VI for ℓ_1 and the tuple $(T, v) := (\Sigma_{\ell_1} + \tilde{B}(\ell_2 - \ell_1), u_{\ell_2})$ in the VI for ℓ_2 to obtain

$$\langle A\Sigma_{\ell_1}, \Sigma_{\ell_2} - \Sigma_{\ell_1} + \tilde{B}(\ell_1 - \ell_2)\rangle \geq 0 \quad \text{and} \quad \langle A\Sigma_{\ell_2}, \Sigma_{\ell_1} - \Sigma_{\ell_2} + \tilde{B}(\ell_2 - \ell_1)\rangle \geq 0.$$

If we add the above, it follows immediately that the solution map to (28) is Lipschitz in the Σ-component. If, on the other hand, we choose tuples of the form $(\Sigma_{\ell_1} + \tilde{B}\hat{\ell}, u_{\ell_1})$ and $(\Sigma_{\ell_2} + \tilde{B}\hat{\ell}, u_{\ell_2})$, $\hat{\ell} \in V^\star$, in the VIs for ℓ_1 and ℓ_2, respectively, then we obtain

$$\langle A\Sigma_{\ell_1} + B^\star u_{\ell_1}, \tilde{B}\hat{\ell}\rangle \geq 0 \quad \text{and} \quad \langle A\Sigma_{\ell_2} + B^\star u_{\ell_2}, \tilde{B}\hat{\ell}\rangle \geq 0 \qquad \forall \hat{\ell} \in V^\star$$

and, consequently,

$$\langle u_{\ell_1} - u_{\ell_2}, \hat{\ell}\rangle = \langle A\Sigma_{\ell_2} - A\Sigma_{\ell_1}, \tilde{B}\hat{\ell}\rangle \qquad \forall \hat{\ell} \in V^\star. \tag{30}$$

The above yields that the multiplier u_ℓ is unique for each ℓ and that the map $\ell \mapsto u_\ell$ is globally Lipschitz continuous, too. □

We emphasize that the linear operator which defines the VI (29) has saddle-point structure. Thus, it is not coercive and the VI cannot be identified with a projection problem. In particular, we cannot apply the classical results of, e.g., [Do, 1992] to obtain the directional differentiability of the solution operator.

As a preparation for our differentiability result, we give an expression for the second subderivative of the indicator function of K.

Lemma 5.3. *Let $\Sigma \in K$ and $\Xi \in \mathcal{N}_K(\Sigma)$ be given. Then, there exists a $\lambda \in L^2(\mu)$ with $\lambda \geq 0$ such that $\Xi = \lambda\,\mathcal{D}^\star\mathcal{D}\Sigma$. Moreover, the indicator function δ_K is strongly twice epi-differentiable in Σ for Ξ with*

$$Q_{\delta_K}^{\Sigma,\Xi}(T) = \int_\Omega \lambda\,|\mathcal{D}T|_{\mathbb{R}^n}^2 \,\mathrm{d}\mu \in [0,\infty]$$

for all $T \in \mathcal{T}_K(\Sigma) \cap \Xi^\perp$ and $Q_{\delta_K}^{\Sigma,\Xi}(T) = +\infty$ otherwise. In particular,

$$\mathcal{K}_{\delta_K}^{\Sigma,\Xi} = \Big\{ T \in \mathcal{T}_K(\Sigma) \cap \Xi^\perp \;\Big|\; \int_\Omega \lambda\,|\mathcal{D}T|_{\mathbb{R}^n}^2 \,\mathrm{d}\mu < \infty \Big\}.$$

Proof. It is easy to check that $\Xi(x)$ is a nonnegative multiple of $\mathcal{D}^\star\mathcal{D}\Sigma(x)$ for a.e. $x \in \Omega$ with $|\mathcal{D}\Sigma(x)|_{\mathbb{R}^n} = 1$, and zero otherwise. Moreover, it is easy to see that there exists a constant $c > 0$ with $|\mathcal{D}^\star\mathcal{D}z|_{\mathbb{R}^m} \geq c|\mathcal{D}z|_{\mathbb{R}^n}$ for all $z \in \mathbb{R}^m$. Combining these facts, we obtain that the function

$$\lambda(x) := \begin{cases} (\Xi(x), \mathcal{D}^\star\mathcal{D}\Sigma(x))_{\mathbb{R}^m} / |\mathcal{D}^\star\mathcal{D}\Sigma(x)|_{\mathbb{R}^m}^2 & \text{if } |\mathcal{D}\Sigma(x)|_{\mathbb{R}^m} = 1 \\ 0 & \text{else} \end{cases}$$

satisfies $0 \leq \lambda \in L^2(\mu)$ and $\Xi = \lambda\,\mathcal{D}^\star\mathcal{D}\Sigma$ as claimed. The formula for the second subderivative follows from [Do, 1992, Section 5] and [Rockafellar and Wets, 1998, Exercise 13.17]. In particular, [Do, 1992, Theorem 5.5] implies that δ_K is strongly twice epi-differentiable. □

We are now in the position to prove the main result of this section.

Theorem 5.4. *Let $\ell_0 \in V^\star$ be given. Denote by $\lambda_0 \in L^2(\mu)$ the function from Lemma 5.3 such that*

$$A\Sigma_{\ell_0} + B^\star u_{\ell_0} + \lambda_0\, \mathcal{D}^\star \mathcal{D}\Sigma_{\ell_0} = 0.$$

Then, the mapping $V^\star \ni \ell \mapsto (\Sigma_\ell, u_\ell) \in S^2 \times V$ is strongly Hadamard directionally differentiable in ℓ_0. Moreover, the directional derivative $(\Sigma', u') \in \mathcal{K}_{\delta_K}^{\Sigma_{\ell_0},\Xi_{\ell_0}} \times V$ in direction $\delta\ell \in V^\star$ is given by the unique solution of

$$\begin{aligned} \langle A\Sigma' + B^\star u', T - \Sigma'\rangle + \int_\Omega \lambda_0 \left(\mathcal{D}\Sigma', \mathcal{D}(T - \Sigma')\right)_{\mathbb{R}^n} \mathrm{d}\mu - \langle B\Sigma' - \delta\ell, v - u'\rangle \geq 0 \\ \forall (T, v) \in \mathcal{K}_{\delta_K}^{\Sigma_{\ell_0},\Xi_{\ell_0}} \times V. \end{aligned} \tag{31}$$

Proof. In what follows, our aim is to apply Corollary 3.1 under its assumption (ii). To this end, we note that, if we set $X := S^2 \times V$, $Y := S^2 \times V^\star$, $P := V^\star$ and

$$\mathcal{A} : P \times X \to Y, \qquad \mathcal{A}(\ell, (\Sigma, u)) := (A\Sigma + B^\star u, -B\Sigma + \ell), \tag{32}$$

and if we define j to be the indicator function of $K \times V$, then (29) takes exactly the form (1). In this setting, the linearized VI (16) becomes

$$\begin{aligned} \langle A\Sigma' + B^\star u', T - \Sigma'\rangle + \int_\Omega \frac{\lambda_0}{2} \left[|\mathcal{D}T|^2_{\mathbb{R}^n} - |\mathcal{D}\Sigma'|^2_{\mathbb{R}^n}\right] \mathrm{d}\mu - \langle B\Sigma' - \delta\ell, v - u'\rangle \geq 0 \\ \forall (T, v) \in \mathcal{K}_{\delta_K}^{\Sigma_{\ell_0},\Xi_{\ell_0}} \times V. \end{aligned} \tag{33}$$

Using exactly the same arguments as in the proof of Lemma 5.2, we obtain that (33) can have at most one solution. Finally, (ii) in Corollary 3.1 follows from Lemma 5.3 and the fact that $(T, v) \mapsto \langle \mathcal{A}_x(T, v), (T, v)\rangle = \langle AT, T\rangle$ is convex and continuous, thus weakly lower semicontinuous. Corollary 3.1 now yields that the solution map is weakly Hadamard directionally differentiable and that the directional derivatives are uniquely characterized by (33). Moreover, (17) together with $\langle \mathcal{A}_x(T, v), (T, v)\rangle = \langle AT, T\rangle$ implies that the difference quotients associated with the S^2-component converge strongly. The strong convergence of the difference quotients associated with the V-component now follows from (30). Finally, the equivalence of (33) and (31) is easy to check. □

5.2 Projection onto a Prox-Regular Set

Next, we show that Theorem 4.1 can also be used to study the differentiability properties of the projection onto a prox-regular set. The notion of prox-regular sets generalizes the concept of convexity and has been introduced many times with different names. Some unification was performed in [Poliquin et al., 2000; Colombo and Thibault, 2010]. Throughout this section, K denotes a closed nonempty subset of a real Hilbert space H.

Definition 5.5 (Prox-Regularity).

(i) The set-valued projection of $y \in H$ onto K is given by

$$\pi_K(y) := \left\{x \in K \;\middle|\; \|x - y\|_H = \inf_{x' \in K} \|x' - y\|_H\right\}.$$

(ii) For $x \in K$ the proximal normal cone is given by

$$\mathcal{N}_K^P(x) := \big\{ v \in H \;\big|\; \exists \lambda \geq 0, y \in H : x \in \pi_K(y) \text{ and } v = \lambda\,(y - x) \big\}.$$

(iii) For $r > 0$, the set K is called r-prox-regular, if

$$\frac{1}{2}\,\|v\|_H\,\|x - \bar{x}\|_H^2 \geq r\,(v, x - \bar{x})_H \qquad \forall \bar{x}, x \in K, v \in \mathcal{N}_K^P(\bar{x}). \tag{34}$$

Note that the set K is convex if and only if K is r-prox-regular for all $r > 0$. We give a geometric interpretation of (34). For a given $\bar{x} \in K$ and a $v \in \mathcal{N}_K^P(\bar{x})$ with $\|v\|_H < r$, this condition implies

$$\begin{aligned} \big\|x - (\bar{x} + v)\big\|_H^2 &= \Big(\frac{\|v\|_H}{r} + 1 - \frac{\|v\|_H}{r}\Big)\,\|x - \bar{x}\|_H^2 + \|v\|_H^2 - 2\,(v, x - \bar{x})_H \\ &\geq \Big(1 - \frac{\|v\|_H}{r}\Big)\,\|x - \bar{x}\|_H^2 + \|v\|_H^2 \geq \|v\|_H^2 \qquad \forall x \in K, \end{aligned} \tag{35}$$

where the last inequality is strict for $x \neq \bar{x}$. Hence, the intersection of the closed ball $B^H_{\|v\|_H}(\bar{x} + v)$ with K is precisely the point $\bar{x}$. In particular, $\pi_K(\bar{x} + v) = \{\bar{x}\}$.

The above statement can be strengthened as follows, see [Poliquin et al., 2000, Theorem 4.1, Lemma 4.2], [Colombo and Thibault, 2010, Theorem 0.16].

Theorem 5.6. *Assume that the closed set $K \subset H$ is r-prox-regular for some $r > 0$. We fix $\rho \in (0, r)$ and consider the ρ-enlargement*

$$K_\rho := \{ y \in H \mid \exists x \in K : \|x - y\|_H < \rho \}.$$

Then, π_K is single-valued on K_ρ and we define $\operatorname{proj}_K(y)$ to be the single element in $\pi_K(y)$ for all $y \in K_\rho$. Moreover, proj_K is Lipschitz continuous on K_ρ with rank $r/(r - \rho)$.

For later use, we recast (35) in the setting of Theorem 5.6. Let us fix $p \in K_\rho$, $\rho \in (0, r)$. We set $\bar{x} = \operatorname{proj}_K(p)$ and $v = p - \bar{x} \in \mathcal{N}_K^P(\bar{x})$. Applying (35) yields

$$\begin{aligned} \|x - p\|_H^2 &\geq \Big(1 - \frac{\|p - \bar{x}\|_H}{r}\Big)\,\|x - \bar{x}\|_H^2 + \|p - \bar{x}\|_H^2 \\ &\geq \Big(1 - \frac{\rho}{r}\Big)\,\|x - \bar{x}\|_H^2 + \|p - \bar{x}\|_H^2 \qquad \forall x \in K. \end{aligned} \tag{36}$$

In what follows, we first link the second-order epi-differentiability of δ_K to the differentiability of an auxiliary function.

Lemma 5.7. *Assume that the closed set $K \subset H$ is r-prox-regular for some $r > 0$. We fix $p_0 \in K_r$ and set $\bar{x} = \operatorname{proj}_K(p_0)$. If δ_K is strongly twice epi-differentiable in $\bar{x}$ for $p_0 - \bar{x}$, then the function $j := \frac{1}{2}\,\|\cdot\|_H^2 + \delta_K$ is strongly twice epi-differentiable in $\bar{x}$ for p_0.*

Proof. By invoking the definition of $Q_j^{\bar{x},p_0}$, for every $z \in H$, we find

$$
\begin{aligned}
&Q_j^{\bar{x},p_0}(z) \\
&= \inf\Big\{ \liminf_{n\to\infty} \frac{\delta_K(\bar{x}+t_n\, z_n) - \delta_K(\bar{x}) - t_n\,(p_0 - \bar{x}, z_n)_H}{t_n^2/2} + \|z_n\|_H^2 \;\Big|\; t_n \searrow 0, z_n \rightharpoonup z \Big\} \\
&\geq \|z\|_H^2 + \inf\Big\{ \liminf_{n\to\infty} \frac{\delta_K(\bar{x}+t_n\, z_n) - \delta_K(\bar{x}) - t_n\,(p_0 - \bar{x}, z_n)_H}{t_n^2/2} \;\Big|\; t_n \searrow 0, z_n \rightharpoonup z \Big\} \\
&= \|z\|_H^2 + Q_{\delta_K}^{\bar{x},p_0-\bar{x}}(z).
\end{aligned}
$$

On the other hand, by the strong second-order epi-differentiability of δ_K in $\bar{x}$ for $p_0 - \bar{x}$, given $t_n \searrow 0$, we find for every $z \in H$ a sequence z_n with $z_n \to z$ and

$$
\begin{aligned}
\|z\|_H^2 + Q_{\delta_K}^{\bar{x},p_0-\bar{x}}(z) &= \|z\|_H^2 + \lim_{n\to\infty} \frac{\delta_K(\bar{x}+t_n\, z_n) - \delta_K(\bar{x}) - t_n\,(p_0 - \bar{x}, z_n)_H}{t_n^2/2} \\
&= \lim_{n\to\infty} \frac{j(\bar{x}+t_n\, z_n) - j(\bar{x}) - t_n\,(p_0, z_n)_H}{t_n^2/2} \geq Q_j^{\bar{x},p_0}(z).
\end{aligned}
$$

These two inequalities show that j is strongly twice epi-differentiable at $\bar{x}$ for p_0 with

$$
Q_j^{\bar{x},p_0}(z) = \|z\|_H^2 + Q_{\delta_K}^{\bar{x},p_0-\bar{x}}(z) \qquad \forall z \in H. \tag{37}
$$

□

Now, we present the main theorem of this section.

Theorem 5.8. *Assume that the closed set $K \subset H$ is r-prox-regular for some $r > 0$. We fix $p_0 \in K_r$ and set $\bar{x} := \operatorname{proj}_K(p_0)$. Then the following assertions are equivalent.*

(i) For all $\rho \in (\|p_0 - \bar{x}\|_H, r)$, the function $j := \frac{1}{2}\|\cdot\|_H^2 + \delta_K$ is strongly twice epi-differentiable in $\bar{x}$ for $p_0 + (1 - \frac{\rho}{r})(p_0 - \bar{x})$.

(ii) For some $\rho \in (\|p_0 - \bar{x}\|_H, r)$, the function j is strongly twice epi-differentiable in $\bar{x}$ for $p_0 + (1 - \frac{\rho}{r})(p_0 - \bar{x})$.

(iii) The projection proj_K is Hadamard directionally differentiable at p_0.

Proof. We show that for each fixed $\rho \in (\|p_0 - \bar{x}\|_H, r)$, assertion (iii) is equivalent to the strong second-order epi-differentiability of j in $\bar{x}$ for $p_0 + (1 - \frac{\rho}{r})(p_0 - \bar{x})$.

We are going to apply Theorem 4.1 with the setting $X = Y = P = H$,

$$
A(p, x) = \big(1 - \frac{\rho}{r}\big)(x - p) - p, \quad j(x) = \frac{1}{2}\|x\|_H^2 + \delta_K(x).
$$

We first check that the VI (18) has a unique solution for all $p \in K_\rho$ and that this solution is exactly the projection of p onto K. To this end, we set $\bar{x}_p := \operatorname{proj}_K(p)$, and note that (34) and (36) yield

$$
\begin{aligned}
\langle A(p, \bar{x}_p), x - \bar{x}_p \rangle + j(x) - j(\bar{x}_p) &= \big(1 - \frac{\rho}{r}\big)\,(\bar{x}_p - p, x - \bar{x}_p)_H + \frac{\|x - p\|_H^2}{2} - \frac{\|\bar{x}_p - p\|_H^2}{2} \\
&\geq -\big(1 - \frac{\rho}{r}\big)\,\frac{\|\bar{x}_p - p\|_H}{2\,r}\|x - \bar{x}_p\|_H^2 + \big(1 - \frac{\rho}{r}\big)\,\frac{\|x - \bar{x}_p\|_H^2}{2} \\
&\geq 0 \qquad \forall x \in K.
\end{aligned}
$$

By the ellipticity of A, the solution of (18) is also unique. Hence, $\bar{x}_p = \operatorname{proj}_K(p)$ is the unique solution of (18) with parameter p for all $p \in K_\rho$. The equivalence of (ii) and (iii) now follows immediately from Theorem 4.1. $\square$

From the last theorem, we easily get the following corollary, which is proven in [Noll, 1995, Proposition 2.2] in the case that K is convex.

Corollary 5.9. *Assume that the closed set $K \subset H$ is r-prox-regular for some $r > 0$. We fix $p_0 \in K_r$ and set $\bar{x} := \operatorname{proj}_K(p_0)$, $v = (p_0 - \bar{x})/\|p_0 - \bar{x}\|_H$. Then the following assertions are equivalent.*

(i) The projection proj_K is Hadamard differentiable in $\bar{x} + \rho\, v$ for one $\rho \in (0, r)$.

(ii) The projection proj_K is Hadamard differentiable in $\bar{x} + \rho\, v$ for all $\rho \in (0, r)$.

(iii) For one $\tilde{\rho} \in (0, r)$, the function j is strongly twice epi-differentiable in $\bar{x}$ for $\bar{x} + \tilde{\rho}\, v$.

(iv) For all $\tilde{\rho} \in (0, r)$, the function j is strongly twice epi-differentiable in $\bar{x}$ for $\bar{x} + \tilde{\rho}\, v$.

Proof. Let us fix $\rho \in (0, r)$ and $\tilde{\rho} \in \big(\rho, (2 - \frac{\rho}{r})\, \rho\big)$. Then, Theorem 5.8 shows that (i) holds for ρ if and only if (iii) holds for $\tilde{\rho}$.

Now, any two points in the set $\big\{(\rho, \tilde{\rho}) \in (0, r)^2 \;\big|\; \rho < \tilde{\rho} < (2 - \frac{\rho}{r})\, \rho\big\}$ can be connected by a finite polygonal path whose edges are parallel to the coordinate axes. This yields the claim. $\square$

Before we conclude this section, we remark that the converse of Lemma 5.7 can be easily established if the space H is finite dimensional. In infinite dimensions, however, this reverse implication does not hold anymore in general. Consider, for example, the set $K = \{x \in H \mid \|x\|_H \geq 1\}$ in an infinite-dimensional Hilbert space H with orthonormal system $\{e_i\}_{i \in \mathbb{N}}$. It is easy to check that K is 1-prox-regular and that the projection onto K is directionally differentiable on $H \setminus \{0\}$. Define $p_0 := \frac{1}{2} e_1$ and $\bar{x} := e_1 = \operatorname{proj}_K(p_0)$. Then Corollary 5.9 and standard arguments yield that $j := \frac{1}{2}\|\cdot\|_H^2 + \delta_K$ is strongly twice epi-differentiable in $\bar{x}$ for p_0 and that $p_0 \in \partial j(\bar{x})$. In particular, it holds $Q_j^{\bar{x}, p_0}(z) \geq 0$ for all $z \in H$ by Lemma 2.6. For a fixed $L > 2$, on the other hand, we have $-L\, t_n\, e_1 + e_2 + L\, e_n \rightharpoonup e_2$, $\bar{x} + t_n\, (-L\, t_n\, e_1 + e_2 + L\, e_n) \in K$ for $n > 2$ and

$$-\frac{2}{t_n}(p_0 - \bar{x}, -L\, t_n\, e_1 + e_2 + L\, e_n)_H = -L.$$

The above implies $Q_{\delta_K}^{\bar{x}, p_0 - \bar{x}}(e_2) = -\infty$. This shows that (37) indeed cannot be satisfied.

5.3 Bang-Bang Optimal Control Problems

In what follows, we consider optimal control problems of the form

$$\begin{aligned} \text{Minimize} \quad & F(u) := \int_\Omega L(\cdot, G(u))\, \mathrm{d}x \\ \text{such that} \quad & -1 \leq u \leq 1 \text{ a.e. in } \Omega. \end{aligned} \tag{38}$$

Here, $\Omega \subset \mathbb{R}^d$ is a bounded domain and $G : L^2(\Omega) \to L^2(\Omega)$ is the control-to-state map. The function $F : L^2(\Omega) \to \mathbb{R}$ is called the reduced objective functional. We are interested in stability properties of a bang-bang solution $\bar{u}$ of (38), i.e., a local solution of (38) which satisfies $\bar{u} \in \{-1, 1\}$ a.e. on Ω.

In order to give a more tangible example, we will mainly focus on the optimal control of a semilinear partial differential equation (PDE) in the case $d \in \{1, 2, 3\}$. We emphasize that our arguments also apply to a broader class of problems, see Remark 5.12 below.

So let us consider a bounded domain $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$, with a Lipschitz boundary. We assume that the state $G(u)$ associated with the control $u \in U_{\mathrm{ad}} := \{v \in L^\infty(\Omega) \mid -1 \le v \le 1 \text{ a.e. in } \Omega\}$ is defined to be the weak solution $y \in H_0^1(\Omega)$ of the PDE

$$-\Delta y + f(\cdot, y) = u \text{ in } \Omega, \qquad y = 0 \text{ on } \partial\Omega. \tag{39}$$

We further suppose that the functions f and L appearing in (38) and (39) satisfy the following common assumptions, cf. [Casas, 2012; Casas et al., 2017; Nguyen and D. Wachsmuth, 2017].

Assumption 5.10. *The functions $L, f : \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory mappings that are C^2 w.r.t. their second argument. Moreover, the following conditions hold true.*

(i) We have $L(\cdot, 0) \in L^1(\Omega)$. For each $M > 0$, there is a constant $C_{L,M}$ and a function $\psi_M \in L^2(\Omega)$ such that

$$\Big|\frac{\partial L}{\partial y}(x, y)\Big| \le \psi_M(x), \qquad \Big|\frac{\partial^2 L}{\partial y^2}(x, y)\Big| \le C_{L,M},$$
$$\Big|\frac{\partial^2 L}{\partial y^2}(x, y_1) - \frac{\partial^2 L}{\partial y^2}(x, y_2)\Big| \le C_{L,M}\,|y_1 - y_2|$$

hold for a.a. $x \in \Omega$ and all $|y|, |y_2|, |y_2| \le M$.

(ii) We have $f(\cdot, 0) \in L^2(\Omega)$ and $\frac{\partial f}{\partial y}(x, y) \ge 0$ for a.a. $x \in \Omega$ and all $y \in \mathbb{R}$. For each $M > 0$, there is a constant $C_{f,M}$ such that

$$\Big|\frac{\partial f}{\partial y}(x, y)\Big| + \Big|\frac{\partial^2 f}{\partial y^2}(x, y)\Big| \le C_{f,M}, \quad \Big|\frac{\partial^2 f}{\partial y^2}(x, y_1) - \frac{\partial^2 f}{\partial y^2}(x, y_2)\Big| \le C_{f,M}\,|y_1 - y_2|$$

hold for a.a. $x \in \Omega$ and all $|y|, |y_2|, |y_2| \le M$.

We remark that it is possible to weaken the Lipschitz assumption on the second derivatives of f and L in the above to a kind of uniform continuity, see, e.g., [Nguyen and D. Wachsmuth, 2017, Assumptions (A1), (A2)]. We will not pursue this approach here.

Note that the conditions in Assumption 5.10 imply the following differentiability results for the control-to-state map G and the reduced objective F.

Lemma 5.11. *The control-to-state map $G : L^2(\Omega) \to L^2(\Omega)$ and the reduced objective $F : L^2(\Omega) \to \mathbb{R}$ are well-defined and twice continuously Fréchet differentiable. For all $u \in L^2(\Omega)$, the first derivative of F satisfies $F'(u) \in C_0(\Omega)$ and the second derivative*

$F''(u)$ can be extended from $L^2(\Omega)$ to a continuous bilinear form on $\mathcal{M}(\Omega)$. Moreover, it holds $F''(u)\mu \in C_0(\Omega)$ for all $\mu \in \mathcal{M}(\Omega)$ and $\mu_k \stackrel{\star}{\rightharpoonup} \mu$ in $\mathcal{M}(\Omega)$ implies $F''(u)\,\mu_k \to F''(u)\,\mu$ in $C_0(\Omega)$. For all $u \in U_{\mathrm{ad}}$, the derivative $G'(u) : L^2(\Omega) \to L^2(\Omega)$ can be extended to a mapping $G'(u) : \mathcal{M}(\Omega) \to L^2(\Omega)$, which maps weakly-$\star$ convergent sequences to strongly convergent sequences, and there exists an adjoint $G'(u)^\star : L^2(\Omega) \to C_0(\Omega)$ satisfying

$$(p, G'(u)\,\mu)_{L^2(\Omega)} = \langle G'(u)^\star p, \mu\rangle \qquad \forall p \in L^2(\Omega), \mu \in \mathcal{M}(\Omega). \tag{40}$$

The mappings F and G are compact in the sense that for all small $p_1 \in L^2(\Omega)$ and all sequences $\{u_k\} \subset U_{\mathrm{ad}}$ with $u_k \stackrel{\star}{\rightharpoonup} u$ in $L^\infty(\Omega)$, we have $F(u_k) \to F(u)$ in $\mathbb{R}$ and $G(u_k + p_1) \rightharpoonup G(u + p_1)$ in $L^2(\Omega)$.

There is a $\delta > 0$ such that for some $C > 0$, the estimates

$$\|G'(u_1 + p_1)v - G'(u_2)v\|_{L^2(\Omega)} \le C\,\|u_1 + p_1 - u_2\|_{L^2(\Omega)}\,\|v\|_{L^1(\Omega)} \tag{41}$$

$$\|G'(u_1)v\|_{L^2(\Omega)} \le C\,\|v\|_{L^1(\Omega)} \tag{42}$$

$$\left|F''(u_1 + p_1)[v_1, v_2]\right| \le C\,\|v_1\|_{L^1(\Omega)}\,\|v_2\|_{L^1(\Omega)} \tag{43}$$

hold for all $u_1, u_2 \in U_{\mathrm{ad}}$, $v, v_1, v_2 \in L^2(\Omega)$ and $p \in L^2(\Omega)$ with $\|p\|_{L^2(\Omega)} \le \delta$.

For every $\varepsilon > 0$, there is $\delta > 0$ such that

$$\left|F''(u_1 + p_1)[v_1, v_2] - F''(u_2)[v_1, v_2]\right| \le \varepsilon\,\|v_1\|_{L^1(\Omega)}\,\|v_2\|_{L^1(\Omega)} \tag{44}$$

for all $u_1, u_2 \in U_{\mathrm{ad}}$, $v_1, v_2 \in L^2(\Omega)$, $p \in L^2(\Omega)$ with $\|u_1 - u_2\|_{L^2(\Omega)} \le \delta$ and $\|p\|_{L^2(\Omega)} \le \delta$.

Proof. The well-definedness and the differentiability of G and F follow from standard arguments, see, e.g., [Casas, 2012, p. 2357]. In particular, the derivatives of the reduced objective are given by

$$F'(u)v = \int_\Omega \varphi_u v\,\mathrm{d}x, \tag{45}$$

$$F''(u)(v_1, v_2) = \int_\Omega \left[\frac{\partial^2 L}{\partial y^2}(\cdot, G(u)) - \frac{\partial^2 f}{\partial y^2}(\cdot, G(u))\,\varphi_u\right] G'(u)v_1\,G'(u)v_2\,\mathrm{d}x, \tag{46}$$

where $\varphi_u \in H_0^1(\Omega) \cap C_0(\Omega)$ is the adjoint state associated with u, i.e., the weak solution of

$$-\Delta\varphi_u + \frac{\partial f}{\partial y}(\cdot, y_u)\,\varphi_u = \frac{\partial L}{\partial y}(\cdot, y_u) \text{ in } \Omega, \qquad \varphi_u = 0 \text{ on } \partial\Omega. \tag{47}$$

This implies $F'(u) = \varphi_u \in C_0(\Omega)$.

Next, we discuss the properties of $G'(u)$. For $v \in L^2(\Omega)$, the derivative $z_{u,v} := G'(u)\,v$ is the unique weak solution of the linearized equation

$$-\Delta z_{u,v} + \frac{\partial f}{\partial y}(\cdot, y_u)\,z_{u,v} = v \text{ in } \Omega, \qquad z_{u,v} = 0 \text{ on } \partial\Omega. \tag{48}$$

It follows from classical arguments that this PDE with right-hand side $\mu \in \mathcal{M}(\Omega)$ has a unique solution $z_{u,\mu} \in L^2(\Omega)$ for every $\mu \in \mathcal{M}(\Omega)$ and that $\mu_n \stackrel{\star}{\rightharpoonup} \mu$ in $\mathcal{M}(\Omega)$ implies

$z_{u,\mu_n} \to z_{u,\mu}$ in $L^2(\Omega)$, see [Casas et al., 2017, Section 2.5]. Finally, $G'(u)^\star p \in C_0(\Omega)$ can be defined as the weak solution of (48) with right-hand side $p \in L^2(\Omega)$ and the desired formula (40) follows easily.

Now, from (40) and (46) it can be seen that $F''(u)$ satisfies

$$F''(u)(v_1, v_2) = \int_\Omega \zeta_{u,v_1} \, v_2 \, \mathrm{d}x,$$

where

$$\zeta_{u,v_1} = G'(u)^\star \Big(\Big[\frac{\partial^2 L}{\partial y^2}(\cdot, G(u)) - \frac{\partial^2 f}{\partial y^2}(\cdot, G(u)) \, \varphi_u \Big] \, G'(u) v_1 \Big).$$

Hence, $F''(u)\,\mu = \zeta_{u,\mu} \in C_0(\Omega)$ holds for all $\mu \in \mathcal{M}(\Omega)$ and $\mu_k \overset{\star}{\rightharpoonup} \mu$ implies $F''(u)\,\mu_k = \zeta_{u,\mu_k} \to \zeta_{u,\mu} = F''(u)\,\mu$ in $C_0(\Omega)$.

The asserted compactness of F and G can be shown as in [Nguyen and D. Wachsmuth, 2017, Theorem 4.1].

The estimates (41)–(42) are proven in [Nguyen and D. Wachsmuth, 2017, Lemma 4.2]. Estimate (43) follows from (46), (42) and the fact that $G(u_1 + p_1)$ and $\varphi_{u_1+p_1}$ can be uniformly bounded in $L^\infty(\Omega)$, see [Nguyen and D. Wachsmuth, 2017, Lemma 4.1]. In [Nguyen and D. Wachsmuth, 2017, Lemma 4.3], (44) is shown for $v_1 = v_2$. The general case follows by polarization. Indeed, with $B := F''(u_1 + p_1) - F''(u_2)$, we obtain

$$\begin{aligned} |B(v_1, v_2)| &= \frac{1}{4} \big| B(v_1 + v_2, v_1 + v_2) - B(v_1 - v_2, v_1 - v_2) \big| \\ &\le \frac{\varepsilon}{4} \big(\|v_1 + v_2\|^2_{L^1(\Omega)} + \|v_1 - v_2\|^2_{L^1(\Omega)} \big) = \frac{\varepsilon}{2} \big(\|v_1\|^2_{L^1(\Omega)} + \|v_2\|^2_{L^1(\Omega)} \big). \end{aligned}$$

With scaling, we can resort to the case $\|v_1\|_{L^1(\Omega)} = \|v_2\|_{L^1(\Omega)} = 1$ and this yields (44). □

Remark 5.12. *In the sequel, we will only work with the results of Lemma 5.11. This means that the following theory is applicable to all problems for which the same estimates are available. In particular, in all of the following results, Assumption 5.10 can be substituted by the assertions of Lemma 5.11.*

Next, we are going to apply the second-order theory of [Christof and G. Wachsmuth, 2017a] to the optimal control problem (38). Therefore, let $\bar u \in U_{\mathrm{ad}}$ be a stationary point of (38), i.e., $\bar\varphi := F'(\bar u) \in C_0(\Omega)$ satisfies

$$(\bar\varphi, u - \bar u)_{L^2(\Omega)} \ge 0 \qquad \forall u \in U_{\mathrm{ad}}.$$

Note that, in the setting of the semilinear PDE (39), $\bar\varphi$ is the solution of the adjoint equation (47) with y_u replaced by $\bar y = G(\bar u)$, see (45).

Let us introduce some notation in order to comply with the setting of [Christof and G. Wachsmuth, 2017a] and of Section 2. We define the separable space

$$Y := C_0(\Omega) = \mathrm{cl}_{\|\cdot\|_\infty} (C_c(\Omega))$$

endowed with the usual supremum norm. Its dual space can be identified with

$$X := \mathcal{M}(\Omega)$$

which is the space of signed finite Radon measures on the domain Ω endowed with the norm $\|\mu\|_{\mathcal{M}(\Omega)} := |\mu|(\Omega)$, cf. [Ambrosio, 2000, Theorem 1.54]. The space $L^1(\Omega)$ is identified with a closed subspace of $\mathcal{M}(\Omega)$ via the isometric embedding $h \mapsto h\mathcal{L}^d$, where $\mathcal{L}^d$ is Lebesgue's measure. In the same way, U_{ad} is considered as a subset of $\mathcal{M}(\Omega)$.

Assumption 5.13 (Assumptions for the Calculation of the Second Subderivative)**.** *We require $\bar{\varphi} \in C_0(\Omega) \cap C^1(\Omega)$ and define $\mathcal{Z} := \{z \in \Omega : \bar{\varphi}(z) = 0\}$. We assume $\mathcal{Z} \subset \{z \in \Omega : |\nabla\bar{\varphi}(z)| \neq 0\}$. Here and in the sequel, $|\nabla\bar{\varphi}(z)|$ denotes the Euclidean norm of $\nabla\bar{\varphi}(z) \in \mathbb{R}^d$. We further require*

$$c := \liminf_{s \searrow 0} \left(\frac{s}{\mathcal{L}^d(\{|\bar{\varphi}| \leq s\})} \right) > 0. \tag{49}$$

Note that (49) is in particular satisfied (with $c \geq C^{-1}$) if

$$\mathcal{L}^d(\{|\bar{\varphi}| \leq s\}) \leq C\, s \qquad \forall s > 0 \tag{50}$$

holds for some $C > 0$. Such an assumption was previously used in, e.g., [G. Wachsmuth and D. Wachsmuth, 2011; Deckelnick and Hinze, 2012; Casas et al., 2017].

We are now in the position to study the second-order epi-differentiability of the indicator function of U_{ad}. In what follows, we denote by $\mathcal{H}^{d-1}$ the $(d-1)$-dimensional Hausdorff measure, which is scaled as in [Evans and Gariepy, 2015, Definition 2.1].

Theorem 5.14. *Under Assumption 5.13, the indicator function $j := \delta_{U_{\mathrm{ad}}}$ of U_{ad} is strictly twice epi-differentiable in $\bar{u}$ for $-\bar{\varphi}$ with*

$$\mathcal{K}_j^{\bar{u},-\bar{\varphi}} = \left\{ g\mathcal{H}^{d-1}|_{\mathcal{Z}} \;\middle|\; g \in L^1\left(\mathcal{Z}, \mathcal{H}^{d-1}\right) \cap L^2\left(\mathcal{Z}, |\nabla\bar{\varphi}|\mathcal{H}^{d-1}\right) \right\}$$

and for every element $h = g\mathcal{H}^{d-1}|_{\mathcal{Z}}$ of the above set we have

$$Q_j^{\bar{u},-\bar{\varphi}}(h) = \frac{1}{2} \int_{\mathcal{Z}} g^2 |\nabla\bar{\varphi}| \mathrm{d}\mathcal{H}^{d-1}.$$

Proof. In view of

$$\begin{aligned} Q_j^{\bar{u},-\bar{\varphi}}(z) &= \inf\left\{ \liminf_{n\to\infty} \frac{j(\bar{u} + t_n z_n) - j(\bar{u}) + t_n\langle\bar{\varphi}, z_n\rangle}{t_n^2/2} \;\middle|\; t_n \searrow 0, z_n \xrightharpoonup{\star} z \right\} \\ &= \inf\left\{ \liminf_{n\to\infty} \frac{\langle\bar{\varphi}, z_n\rangle}{t_n/2} \;\middle|\; t_n \searrow 0, z_n \xrightharpoonup{\star} z, \bar{u} + t_n z_n \in U_{\mathrm{ad}} \right\} \\ &= \inf\left\{ \liminf_{n\to\infty} \langle\bar{\varphi}, 2z/t_n + r_n\rangle \;\middle|\; t_n \searrow 0, r_n \xrightharpoonup{\star} 0, \bar{u} + t_n z + \frac{1}{2} t_n^2 r_n \in U_{\mathrm{ad}} \right\}, \end{aligned}$$

we find that $Q_j^{\bar{u},-\bar{\varphi}}(z) = +\infty$ for $z \notin \mathcal{T}_{U_{\mathrm{ad}}}^{\star}(\bar{u}) \cap \bar{\varphi}^{\perp}$, where

$$\mathcal{T}_{U_{\mathrm{ad}}}^{\star}(\bar{u}) := \left\{ h \in \mathcal{M}(\Omega) \;\middle|\; \exists t_k \searrow 0, \exists u_k \in U_{\mathrm{ad}} \text{ such that } \frac{u_k - \bar{u}}{t_k} \xrightharpoonup{\star} h \right\}.$$

Moreover, for all $z \in \mathcal{T}^\star_{U_{\text{ad}}}(\bar u) \cap \bar\varphi^\perp$ the value $Q_j^{\bar u,-\bar\varphi}(z)$ coincides with the value of the curvature functional introduced in [Christof and G. Wachsmuth, 2017a], see in particular the comment after [Christof and G. Wachsmuth, 2017a, Definition 3.1]. Now, [Christof and G. Wachsmuth, 2017a, Theorem 6.11] shows that the weak-$\star$ second subderivative of j together with its domain $\mathcal{K}_j^{\bar u,-\bar\varphi}$ is given as in the assertion of the theorem.

The strict second-order epi-differentiability of j in $(\bar u, -\bar\varphi)$ follows from [Christof and G. Wachsmuth, 2017a, Lemma 6.10] combined with Lemma 3.2, cf. the proof of [Christof and G. Wachsmuth, 2017a, Theorem 6.11]. □

For later use, we remark that the reduced critical cone $\mathcal{K}_j^{\bar u,-\bar\varphi}$ is a linear subspace of $\mathcal{M}(\Omega)$. Moreover, $Q_j^{\bar u,-\bar\varphi}$ is a quadratic functional on this subspace. In particular, we can define the bilinear form associated with $Q_j^{\bar u,-\bar\varphi}$ via

$$\hat Q_j^{\bar u,-\bar\varphi}[h_1,h_2] = \frac{1}{2}\int_{\mathcal{Z}} g_1\, g_2 |\nabla\bar\varphi| \mathrm{d}\mathcal{H}^{d-1} \in \mathbb{R}$$

for all $h_1 = g_1\mathcal{H}^{d-1}|_{\mathcal{Z}} \in \mathcal{K}_j^{\bar u,-\bar\varphi}$ and $h_2 = g_2\mathcal{H}^{d-1}|_{\mathcal{Z}} \in \mathcal{K}_j^{\bar u,-\bar\varphi}$.

Finally, in order to apply the second-order theory, we have to verify [Christof and G. Wachsmuth, 2017a, Assumption 4.1]. This amounts to the verification of

$$\lim_{k\to\infty} \frac{F(\bar u + t_k\, h_k) - F(\bar u) - t_k\, F'(\bar u)\, h_k - \frac{1}{2}\, t_k^2\, F''(\bar u)\, h_k^2}{t_k^2} = 0 \tag{51}$$

for all $\{h_k\} \subset L^\infty(\Omega)$, $\{t_k\} \subset \mathbb{R}^+$ satisfying $t_k \searrow 0$, $h_k \overset{\star}{\rightharpoonup} h \in \mathcal{M}(\Omega)$ and $\bar u + t_k h_k \in U_{\text{ad}}$. In order to verify (51), we use the Taylor expansion

$$\begin{aligned} 0 &= F(\bar u + t_k\, h_k) - F(\bar u) - t_k\, F'(\bar u)\, h_k - \frac{1}{2}\, t_k^2\, F''(u_k)\, h_k^2 \\ &= F(\bar u + t_k\, h_k) - F(\bar u) - t_k\, F'(\bar u)\, h_k - \frac{1}{2}\, t_k^2\, F''(\bar u)\, h_k^2 + \frac{1}{2}\, t_k^2 \left[F''(\bar u)\, h_k^2 - F''(u_k)\, h_k^2\right] \end{aligned}$$

with $u_k = \bar u + \tau_k\, t_k\, h_k \in U_{\text{ad}}$ for some $\tau_k \in [0,1]$. We recall that $L^\infty(\Omega) \ni h_k \overset{\star}{\rightharpoonup} h$ in $\mathcal{M}(\Omega)$ implies that h_k is bounded in $L^1(\Omega)$. Using that $u_k - \bar u = \tau_k\, t_k\, h_k$ is a null sequence in $L^1(\Omega)$ and bounded in $L^\infty(\Omega)$, we have $\|u_k - \bar u\|_{L^2(\Omega)} \to 0$. For $\varepsilon > 0$ by using (44) we obtain

$$\frac{\left|F(\bar u + t_k\, h_k) - F(\bar u) - t_k\, F'(\bar u)\, h_k - \frac{1}{2}\, t_k^2\, F''(\bar u)\, h_k^2\right|}{t_k^2} \le \varepsilon\, \|h_k\|_{L^1(\Omega)}^2$$

for k large enough. Due to the boundedness of h_k in $L^1(\Omega)$, this implies (51).

Now we can provide a second-order condition.

Theorem 5.15. *Under Assumptions 5.10 and 5.13, the condition*

$$F''(\bar u)h^2 + Q_j^{\bar u,-\bar\varphi}(h) > 0 \quad \forall h \in \mathcal{K}_j^{\bar u,-\bar\varphi} \setminus \{0\} \tag{52}$$

is equivalent to the quadratic growth condition

$$F(u) \geq F(\bar{u}) + \frac{c}{2} \|u - \bar{u}\|^2_{L^1(\Omega)} \quad \forall u \in U_{\text{ad}}, \|u - \bar{u}\|_{L^1(\Omega)} \leq \varepsilon \tag{53}$$

with constants $c > 0$ and $\varepsilon > 0$. Further, (53) implies

$$F''(\bar{u})h^2 + Q_j^{\bar{u},-\bar{\varphi}}(h) \geq c\,\|h\|^2_{\mathcal{M}(\Omega)} \quad \forall h \in \mathcal{K}_j^{\bar{u},-\bar{\varphi}}. \tag{54}$$

Proof. The result follows from [Christof and G. Wachsmuth, 2017a, Theorem 6.12] since all assumptions are verified in our situation, see also [Christof and G. Wachsmuth, 2017a, Theorem 4.3]. □

We mention that the appearance of $Q_j^{\bar{u},-\bar{\varphi}}(h)$ in (52) and (54) accounts for the curvature of the set U_{ad} w.r.t. the weak-$\star$ topology of $\mathcal{M}(\Omega)$. It is surprising that U_{ad} possesses curvature in this situation, since it is well known that U_{ad} is a polyhedric set in the function spaces $L^p(\Omega)$, i.e., it does not possess any curvature in these stronger topologies.

In case that the growth condition (53) is satisfied, we expect that the solution $\bar{u}$ is stable w.r.t. small perturbations of the objective F and we are interested in its sensitivity properties. To this end, we consider a class of perturbations as in [Nguyen and D. Wachsmuth, 2017]. We fix the perturbation space

$$P = L^2(\Omega) \times L^2(\Omega)$$

and for a perturbation $p = (p_1, p_2) \in P$, we define the perturbed objective

$$J(p, u) = F(u + p_1) + (p_2, G(u + p_1))_{L^2(\Omega)}.$$

In what follows, we first address the solvability of the perturbed problem and the stability of $\bar{u}$ w.r.t. perturbations.

Theorem 5.16. *Suppose that Assumption 5.10 and (53) hold with some $c, \varepsilon > 0$. Then, there is $\delta > 0$ such that for all $\|p\|_P \leq \delta$ the perturbed problem*

$$\text{Minimize } J(p, u) \text{ w.r.t. } u \in U_{\text{ad}} \tag{55}$$

has a local solution $\bar{u}_p$ satisfying $\|\bar{u}_p - \bar{u}\|_{L^1(\Omega)} \leq \varepsilon$ and $J(p, \bar{u}_p) \leq J(p, \bar{u})$. Moreover, there exists a constant $L > 0$ such that for all $\bar{u}_p$ with the latter two properties, it holds

$$\|\bar{u}_p - \bar{u}\|_{L^1(\Omega)} \leq L\,\|p\|_P.$$

Proof. Let us take some $\delta \leq 1$ and $p \in P$ with $\|p\|_P \leq \delta$. Using a Taylor expansion, we find

$$J(p, u) = F(u + p_1) + (p_2, G(u + p_1)) = F(u) + F'(u + \tau\, p_1)\, p_1 + (p_2, G(u + p_1))$$

for all $u \in U_{\mathrm{ad}}$ with a $\tau \in [0,1]$. We use an analogous expansion for $J(p,\bar{u})$. Together with (53), this yields

$$\begin{aligned} J(p,u) - J(p,\bar{u}) &= F(u) + F'(u + \tau\, p_1)\, p_1 - F(\bar{u}) - F'(\bar{u} + \bar{\tau}\, p_1)\, p_1 \\ &\qquad + (p_2, G(u + p_1) - G(\bar{u} + p_1)) \\ &\geq \frac{c}{2}\, \|u - \bar{u}\|^2_{L^1(\Omega)} - \big|F'(u + \tau\, p_1)\, p_1 - F'(\bar{u} + \bar{\tau}\, p_1)\, p_1\big| \\ &\qquad - \|p_2\|_{L^2(\Omega)}\, \|G(u + p_1) - G(\bar{u} + p_1)\|_{L^2(\Omega)} \end{aligned}$$

for all $u \in U_{\mathrm{ad}}$ with $\|u - \bar{u}\|_{L^1(\Omega)} \leq \varepsilon$. Now, we further suppose that δ is small enough such that (42) applies. This inequality yields

$$\|G(u + p_1) - G(\bar{u} + p_1)\|_{L^2(\Omega)} \leq \|G'(\lambda\, u + (1 - \lambda)\, \bar{u} + p_1)(u - \bar{u})\|_{L^2(\Omega)} \leq C\, \|u - \bar{u}\|_{L^1(\Omega)}$$

with some $\lambda \in [0,1]$. Using (43), the term involving F' gives

$$\begin{aligned} \big|F'(u + \tau\, p_1)\, p_1 - F'(\bar{u} + \bar{\tau}\, p_1)\, p_1\big| &= \big|F''(\hat{u})[u - \bar{u} + (\tau - \bar{\tau})\, p_1, p_1]\big| \\ &\leq C\, \|u - \bar{u} + (\tau - \bar{\tau})\, p_1\|_{L^1(\Omega)}\, \|p_1\|_{L^2(\Omega)}. \end{aligned}$$

Together with the above estimates and Young's inequality we arrive at

$$J(p,u) - J(p,\bar{u}) \geq \frac{c}{4}\, \|u - \bar{u}\|^2_{L^1(\Omega)} - C\, \|p\|_P^2.$$

Now, we further suppose that $\delta \leq \varepsilon\sqrt{c/(4C)}$. For all $u \in U_{\mathrm{ad}}$ with $\|u - \bar{u}\|_{L^1(\Omega)} \in (\sqrt{(4C)/c}\, \|p\|, \varepsilon]$ the above calculation shows $J(p,u) > J(p,\bar{u})$. Hence, we can utilize the compactness of F and G, see Lemma 5.11, and a standard argument shows that the perturbed problem has at least one solution $\bar{u}_p$ in the ε-ball centered at $\bar{u}$. Moreover, all these solutions satisfy the desired inequalities with $L = \sqrt{(4C)/c}$. □

Similarly, we can provide a stability result for stationary points.

Theorem 5.17. *Suppose that Assumptions 5.10, 5.13 and (53) hold with some $c, \varepsilon > 0$. Then, there exist $\delta, \eta, L > 0$ such that for all $\|p\|_P \leq \delta$ and all stationary points $\bar{u}_p$ for the perturbed problem*

$$\text{Minimize } J(p,u) \text{ w.r.t. } u \in U_{\mathrm{ad}}$$

with $\|\bar{u}_p - \bar{u}\|_{L^1(\Omega)} \leq \eta$ we have

$$\|\bar{u}_p - \bar{u}\|_{L^1(\Omega)} \leq L\, \|p\|_P.$$

Proof. We proceed by contradiction. This yields sequences $p_k \to 0$ in P, $\bar{u}_k \to \bar{u}$ in $L^1(\Omega)$ such that $\bar{u}_k$ is a stationary point for the perturbed problem with $p = p_k$ and $\|\bar{u}_k - \bar{u}\|_{L^1(\Omega)} \geq k\, \|p_k\|_P$. We set $t_k := \|\bar{u}_k - \bar{u}\|_{L^1(\Omega)}$ and w.l.o.g. $v_k := (\bar{u}_k - \bar{u})/t_k \stackrel{\star}{\rightharpoonup} \mu$ in $\mathcal{M}(\Omega)$. For arbitrary $\hat{\mu} \in \mathcal{K}_j^{\bar{u},-\bar{\varphi}}$, let $\hat{\mu}_k$ be a recovery sequence, i.e., $\bar{u} + t_k\, \hat{\mu}_k \in U_{\mathrm{ad}}$, $\hat{\mu}_k \stackrel{\star}{\rightharpoonup} \hat{\mu}$ in $\mathcal{M}(\Omega)$ and $Q_j^{\bar{u},-\bar{\varphi}}(\hat{\mu}) = \lim_{k\to\infty} \frac{\langle \bar{\varphi}, \hat{\mu}_k \rangle}{t_k/2}$. Since $\bar{u}_k$ is stationary for p_k, we have

$$\langle J_u(p_k, \bar{u}_k), \bar{u} + t_k\, \hat{\mu}_k - \bar{u}_k \rangle \geq 0.$$

Using (41) and (44), a tedious computation shows $\|J_u(p_k, \bar{u}_k) - J_u(0, \bar{u}_k)\|_{C_0(\Omega)} \le C \, \|p_k\|_P$ for some $C > 0$ and all k large enough. Note that the same estimate was derived in [Nguyen and D. Wachsmuth, 2017, proof of Theorem 4.5] in the setting of the semilinear PDE. Using $J_u(0, \bar{u}_k) = F'(\bar{u}_k)$ together with a Taylor expansion of F' and (44), this leads to

$$0 \le \langle F'(\bar{u}), \bar{u} + t_k \, \hat{\mu}_k - \bar{u}_k \rangle + F''(\bar{u})[\bar{u} + t_k \, \hat{\mu}_k - \bar{u}_k, \bar{u}_k - \bar{u}] + \varepsilon_k \, t_k^2 + \frac{C}{k} \, t_k^2,$$

where $\varepsilon_k \to 0$. Dividing by t_k^2 yields

$$\frac{1}{2} \frac{\langle F'(\bar{u}), \frac{\bar{u}_k - \bar{u}}{t_k} \rangle}{t_k/2} - \frac{1}{2} \frac{\langle F'(\bar{u}), \hat{\mu}_k \rangle}{t_k/2} + F''(\bar{u}) \left[\frac{\bar{u}_k - \bar{u}}{t_k} - \hat{\mu}_k, \frac{\bar{u}_k - \bar{u}}{t_k} \right] \le \varepsilon_k + \frac{C}{k}. \tag{56}$$

By passing to the limit $k \to \infty$, we obtain

$$\frac{1}{2} \, Q_j^{\bar{u}, -\bar{\varphi}}(\mu) - \frac{1}{2} \, Q_j^{\bar{u}, -\bar{\varphi}}(\hat{\mu}) + F''(\bar{u}) \, [\mu - \hat{\mu}, \mu] \le 0 \qquad \forall \hat{\mu} \in \mathcal{K}_j^{\bar{u}, -\bar{\varphi}}.$$

Since $\mathcal{K}_j^{\bar{u}, -\bar{\varphi}}$ is a subspace, we can choose $\hat{\mu} = s \, \mu$ for $s \in (0, 1) \cup (1, 2)$. Dividing the above inequality by $s - 1$ and passing to the limits $s \nearrow 1$ and $s \searrow 1$, this shows

$$Q_j^{\bar{u}, -\bar{\varphi}}(\mu) + F''(\bar{u}) \, \mu^2 \le 0.$$

Now, (52) implies $\mu = 0$. From [Christof and G. Wachsmuth, 2017a, Lemma 6.3], we know that (49) implies

$$\left\langle F'(\bar{u}), \frac{\bar{u}_k - \bar{u}}{t_k} \right\rangle = \frac{\langle F'(\bar{u}), \bar{u}_k - \bar{u} \rangle}{t_k} \ge c \, \frac{\|\bar{u}_k - \bar{u}\|^2}{t_k} = c \, t_k$$

for some $c > 0$. Using this information, we reconsider (56) with $\hat{\mu}_k \equiv 0$ and together with $F''(\bar{u})[(\bar{u}_k - \bar{u})/t_k]^2 \to F''(\bar{u}) 0^2 = 0$ this yields a contradiction. ☐

Note that, if we set $p = 0$, then the above theorem yields that $\bar{u}$ is the unique stationary point for the unperturbed problem in $U_{\mathrm{ad}} \cap B_\eta^{L^1(\Omega)}(\bar{u})$.

We compare Theorem 5.16 with [Nguyen and D. Wachsmuth, 2017, Theorem 4.5]. Therein, the authors used a relaxed version of the measure assumption (50), in which the right-hand side $C \, s$ is replaced by $C \, s^{æ}$ for some $æ > 0$. However, their CQ [Nguyen and D. Wachsmuth, 2017, Eq. (3.22)] (in conjunction with $æ = 1$) implies our growth condition, see [Nguyen and D. Wachsmuth, 2017, Theorem 3.1].

The next theorem is the main theorem of this section.

Theorem 5.18. *Let Assumptions 5.10 and 5.13 be satisfied. Suppose that (53) holds for some $c, \varepsilon > 0$. Let $q_t \to q$ in P be given. For $t > 0$ small enough, denote by $\bar{u}_t$ a stationary point of (55) with perturbation $p = t \, q_t$ satisfying $\bar{u}_t \to \bar{u}$ in $L^1(\Omega)$. Then, there is a measure $\mu \in \mathcal{K}_j^{\bar{u}, -\bar{\varphi}}$ such that $(\bar{u}_t - \bar{u})/t \overset{\star}{\rightharpoonup} \mu$ in $\mathcal{M}(\Omega)$ as $t \searrow 0$. Moreover, μ is given by the unique solution in $\mathcal{K}_j^{\bar{u}, -\bar{\varphi}}$ of*

$$\langle J_{up}(0, \bar{u}) p + J_{uu}(0, \bar{u}) \, \mu, \zeta \rangle + \hat{Q}_j^{\bar{u}, -\bar{\varphi}}[\mu, \zeta] = 0 \qquad \forall \zeta \in \mathcal{K}_j^{\bar{u}, -\bar{\varphi}}, \tag{57}$$

and the mapping $p \mapsto \mu$ is linear.

Proof. In a first step, we use the first-order necessary conditions to identify the optimizers $\bar{u}$ and $\bar{u}_t$ with a solution of a variational inequality. To this end, we use the definitions

$$A(p,u) := J_u(p,u), \qquad j := \delta_{U_{\mathrm{ad}}}.$$

In particular, for $h \in \mathcal{M}(\Omega)$ we have

$$A(p,u)\,h = J_u(p,u)\,h = F'(u+p_1)\,h + (p_2, G'(u+p_1)\,h)_{L^2(\Omega)}.$$

Now, the VI (1) is equivalent to

$$\bar{u}_p \in U_{\mathrm{ad}}, \qquad \langle J_u(p,\bar{u}_p), v - \bar{u}_p\rangle \ge 0 \quad \forall v \in U_{\mathrm{ad}}.$$

That is, every stationary point of the perturbed problem (55) is a solution of this VI.

Let us check that the assumptions of Theorem 2.13 are satisfied. The standing Assumption 2.1 is fulfilled by the above choices of $X = \mathcal{M}(\Omega)$, $Y = C_0(\Omega)$, $P = L^2(\Omega)^2$, and $A : P \times X \to Y$.

Part (i) of Assumption 2.2 is exactly our requirement $q_t \to q$ in P. Since $\bar{u}_t$ is a local solution of (55), it is a solution of the VI (1) with parameter $p = t\,q_t$ and the Lipschitz estimate (3) in Assumption 2.2 (ii) follows from Theorem 5.17.

Finally, it remains to check the differentiability assumption Assumption 2.2 (iii). To this end, we define

$$a_0 := -J_u(0,\bar{u}), \qquad \langle A_p p, h\rangle := F''(\bar{u})[h,p_1] + (p_2, G'(\bar{u})\,h)_{L^2(\Omega)},$$
$$\langle A_x v, h\rangle := F''(\bar{u})[h,v]$$

for all $v, h \in \mathcal{M}(\Omega)$ and $p \in P$. Note that these mappings satisfy the mapping properties of Assumption 2.2 (iii), see Lemma 5.11. Moreover, we define the difference quotient

$$v_t := \frac{\bar{u} - \bar{u}_t}{t} \in L^\infty(\Omega)$$

which is bounded in $L^1(\Omega)$. Now, the residual $r(t) \in Y$ from (4) is given by

$$\begin{aligned}
\langle r(t), h\rangle &:= A(t\,q_t, \bar{u} + t\,v_t)\,h - A(0,\bar{u})\,h - t\,\langle A_p q_t, h\rangle - t\,\langle A_x v_t, h\rangle \\
&= F'(\bar{u} + t\,(q_{t,1} + v_t))\,h - F'(\bar{u})\,h - t\,F''(\bar{u})[h, q_{t,1} + v_t] \\
&\quad + \big(t\,q_{t,2}, G'(\bar{u} + t\,(q_{t,1} + v_t))\,h\big)_{L^2(\Omega)} - t\,\big(q_{t,2}, G'(\bar{u})\,h\big)_{L^2(\Omega)},
\end{aligned}$$

where $h \in X$. In order to obtain an estimate of $\|r(t)\|_Y$, it is sufficient to test $r(t) \in Y = C_0(\Omega)$ with functions $h \in L^1(\Omega)$. Using a Taylor expansion of F, we find

$$\begin{aligned}
t^{-1}\,\langle r(t), h\rangle &= F''\big(\bar{u} + \tau_t\,t\,(q_{t,1} + v_t)\big)[h, q_{t,1} + v_t] - F''(\bar{u})[h, q_{t,1} + v_t] \\
&\quad + \big(q_{t,2}, G'(\bar{u} + t\,(q_{t,1} + v_t))\,h - G'(\bar{u})\,h\big)_{L^2(\Omega)} \\
&= \Big[F''\big(\bar{u} + \tau_t\,t\,(q_{t,1} + v_t)\big) - F''(\bar{u})\Big][h, q_{t,1} + v_t] \\
&\quad + \big(q_{t,2}, G'(\bar{u} + t\,(q_{t,1} + v_t))\,h - G'(\bar{u})\,h\big)_{L^2(\Omega)},
\end{aligned}$$

where $\tau_t \in [0,1]$. Now, we are going to use the estimates (41) and (44). Therefore, we utilize $\bar{u} + \tau_t\, t\,(q_{t,1} + v_t) \to \bar{u}$ in $L^2(\Omega)$. In particular, for every $\varepsilon > 0$, there is $\hat{t} > 0$, such that we can apply (44) with $u_1 = \bar{u} + \tau_t\, t\, v_t$, $u_2 = \bar{u}$, and $p_1 = \tau_t\, t\, q_{t,1}$. This leads to the estimate

$$t^{-1}\left|\langle r(t), h\rangle\right| \le \varepsilon\, \|h\|_{L^1(\Omega)}\, \|q_{t,1} + v_t\|_{L^1(\Omega)} + C\, \|q_{t,2}\|_{L^2(\Omega)}\, \|\tau_t\, t\,(q_{t,1} + v_t)\|_{L^2(\Omega)}\, \|h\|_{L^1(\Omega)}.$$

Taking the supremum w.r.t. all $h \in L^1(\Omega)$ with $\|h\|_{L^1(\Omega)} \le 1$ leads to the desired estimate $\|r(t)\|_Y = o(t)$. Hence, Assumption 2.2 holds.

Next, we check that the requirement (i) of Theorem 2.13 is satisfied. The weak-$\star$ second-order epi-differentiability of $j = \delta_{U_{\mathrm{ad}}}$ was proved in Theorem 5.14. The compactness assumption on $A_x = F''(\bar{u})$ was shown in Lemma 5.11.

Finally, we study the linearized VI (15). In our setting, it reads

$$\langle J_{up}(0,\bar{u})p + J_{uu}(0,\bar{u})\,\mu, \zeta - \mu\rangle + \frac{1}{2}Q_j^{\bar{u},-\bar{\varphi}}(\zeta) - \frac{1}{2}Q_j^{\bar{u},-\bar{\varphi}}(\mu) \ge 0 \qquad \forall \zeta \in \mathcal{K}_j^{\bar{u},-\bar{\varphi}}, \quad (58)$$

where J_{up}, J_{uu} denote the second partial derivatives of J. Thus, it remains to show that this linearized VI has at most one solution and that this solution is given by the solution of (57). The condition (52) implies that (57) possesses at most one solution. Hence, it is sufficient to prove that every solution $\mu \in \mathcal{K}_j^{\bar{u},-\bar{\varphi}}$ of (58) also solves (57). By using $\hat{\zeta} = \mu \pm t\,\zeta$ and

$$\frac{1}{2}\, Q_j^{\bar{u},-\bar{\varphi}}(\mu \pm t\,\zeta) - \frac{1}{2}\, Q_j^{\bar{u},-\bar{\varphi}}(\mu) = \pm t\, \hat{Q}_j^{\bar{u},-\bar{\varphi}}[\mu,\zeta] + \frac{t^2}{2}\, Q_j^{\bar{u},-\bar{\varphi}}(\zeta),$$

this follows immediately. Applying Theorem 2.13 finishes the proof. □

Note that a sequence $\bar{u}_t$ of stationary points with the properties in Theorem 5.18 can always be found by Theorem 5.16.

We remark that differentiability results for bang-bang optimal control problems governed by ordinary differential equations can be found frequently in the literature. See, e.g., [Felgenhauer, 2010] and [Jang-Ho and Maurer, 2004] for some examples. The result in Theorem 5.18, however, seems to be new and is, at least in the authors' opinion, quite remarkable as it allows to precisely track how the sensitivity of a bang-bang solution $\bar{u}$ is related to the curvature properties of the set $U_{\mathrm{ad}} \subset \mathcal{M}(\Omega)$ and the no-gap optimality condition in Theorem 5.15.

6 Concluding Remarks

As we have demonstrated in Sections 3 to 5, our approach to the sensitivity analysis of variational inequalities allows not only to recover and extend known results as those of [Mignot, 1976; Haraux, 1977] and [Do, 1992], cf. Corollary 3.3 and Theorem 4.1, but also to tackle problems that are beyond the scope of the classical theory, cf. the examples in Sections 5.1 to 5.3. We hope that, because of the generality of our theorems and the self-containedness and elementary nature of our proofs, our results will prove helpful to all those who are interested in the differentiability properties of solution operators to VIs of the first and the second kind and the optimal control of variational inequalities.

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