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Comparison of optimality systems for the optimal control of the obstacle problem

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We consider stationarity systems for the optimal control of the obstacle problem. The focus is on the comparison of several different systems which are provided in the literature. We obtain some novel results concerning the relations between these stationarity concepts.

Keywords: obstacle problem, complementarity constraints, weak stationarity, M-stationarity, strong stationarity

MSC: 49K20, 90C33

1 Introduction

We consider the optimal control problem

$$\begin{aligned} \text{minimize} \quad & J(y,u) \\ \text{w.r.t.} \quad & (y,u,\lambda) \in H_0^1(\Omega) \times L^2(\Omega) \times H^{-1}(\Omega) \\ \text{such that} \quad & (y,\lambda) \in \mathbb{K}, \\ & -\Delta y + \lambda = u + f, \\ & u \in U_{\text{ad}}. \end{aligned} \tag{1.1}$$

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Here, $\Omega \subset \mathbb{R}^d$ is open and bounded, $J : H_0^1(\Omega) \times L^2(\Omega) \to \mathbb{R}$ is assumed to be continuously Fréchet differentiable, $f \in H^{-1}(\Omega) := H_0^1(\Omega)^\star$, and $U_{\text{ad}} \subset L^2(\Omega)$ is assumed to be closed and convex. Moreover, the *non-convex* set $\mathbb{K}$ is given by

$$\mathbb{K} := \{(v, \mu) \in H_0^1(\Omega) \times H^{-1}(\Omega) \mid v \geq 0, \mu \leq 0, \langle \mu, v \rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} = 0\}. \tag{1.2}$$

Here, $v \geq 0$ is to be understood in a pointwise a.e. sense and $\mu \leq 0$ for $\mu \in H^{-1}(\Omega)$ is defined via duality, i.e., $\langle \mu, w \rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} \leq 0$ for all $w \in H_0^1(\Omega)$ with $w \geq 0$. Note that the state equation

$$\text{find } (y, \lambda) \in \mathbb{K} \text{ with } -\Delta y + \lambda = u + f$$

in (1.1) is the obstacle problem.

The task of providing *necessary* optimality conditions, i.e., conditions which are satisfied for all local minimizers of (1.1), received great interest in the last forty years, we refer exemplarily to [Mignot, 1976; Barbu, 1984; Jarušek, Outrata, 2007; Hintermüller, Kopacka, 2009; Hintermüller, Surowiec, 2011; Outrata, Jarušek, Stará, 2011; Schiela, D. Wachsmuth, 2013; Hintermüller, Mordukhovich, Surowiec, 2014; G. Wachsmuth, 2014; 2016a; b].

In these references, many optimality systems were introduced and all these systems can be written in the form

$$J_y(\bar{y}, \bar{u}) + \nu - \Delta p = 0, \qquad \mu \in \mathcal{N}_{U_{\text{ad}}}(\bar{u}), \tag{1.3a}$$

$$J_u(\bar{y}, \bar{u}) + \mu - p = 0, \qquad (\nu, -p) \in \mathcal{N}_{\mathbb{K}}^\sharp(\bar{y}, \bar{\lambda}). \tag{1.3b}$$

Here, J_y and J_u denote the partial derivatives of J, and $\mathcal{N}_{U_{\text{ad}}}(\bar{u})$ is the usual normal cone of the convex set U_{ad}. Moreover, $\mathcal{N}_{\mathbb{K}}^\sharp(\bar{y}, \bar{\mu}) \subset H^{-1}(\Omega) \times H_0^1(\Omega)$ is a certain admissible set of multipliers and this set can be understood as a generalized normal cone to the non-convex set $\mathbb{K}$. In the current contribution, we will provide a systematic comparison of many common optimality systems. Since all these systems only differ in the replacement for $\mathcal{N}_{\mathbb{K}}^\sharp(\bar{y}, \bar{\lambda})$ in (1.3), it is enough to compare these replacements and this is the focus of our paper. We mention that many of the provided comparisons are novel results, in particular, we refer to Theorems 5.1 to 5.7.

Let us give a brief outline of this paper. In Section 2 we introduce some notation from convex analysis and variational calculus. Next, we give an introduction to capacity theory in Section 3, where we collect useful results of capacity theory, introduce the fine support of a functional $\xi \in H^{-1}(\Omega)$ and give a first application of these results in Section 3.3. In order to provide more context for stationarity systems of the obstacle problem, we briefly discuss stationarity systems in the finite dimensional setting and in general reflexive Banach spaces in Section 4. In Section 5 we address the different optimality systems of the obstacle problem. We discuss weak stationarity, C- and M-stationarity, and strong stationarity in Sections 5.1 to 5.3 and 5.6. In Sections 5.4 and 5.5 we compare the limiting normal cone with weak and M-stationarity. Next, we give a counterexample to stationarity systems that use pointwise almost-everywhere conditions instead of pointwise quasi-everywhere conditions. Finally, in Section 5.8 we summarize the results by comparing the stationarity systems and their corresponding cones.

2 Notation and preliminaries

We fix some notation. Throughout the paper, $\Omega \subset \mathbb{R}^d$ is assumed to be open and bounded.

We define K to be the set of non-negative functions in $H_0^1(\Omega)$, i.e.,

$$K := H_0^1(\Omega)_+ := \{v \in H_0^1(\Omega) \mid v \geq 0 \text{ a.e. in } \Omega\}.$$

The non-positive and non-negative functionals in $H^{-1}(\Omega)$ are defined via duality, i.e.,

$$\begin{aligned} H^{-1}(\Omega)_- &:= K^\circ := \{\mu \in H^{-1}(\Omega) \mid \langle \mu, v\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} \leq 0 \; \forall v \in K\}, \\ H^{-1}(\Omega)_+ &:= -H^{-1}(\Omega)_-. \end{aligned}$$

We mention that we use

$$\|y\|^2_{H_0^1(\Omega)} := \int_\Omega |\nabla y|^2 \,\mathrm{d}x$$

as a norm in $H_0^1(\Omega)$ and the norm in $H^{-1}(\Omega)$ is defined via duality. For a function $v \in L^1(\Omega)$, we use $v^+ := \max(v, 0)$ and $v^- := \max(-v, 0)$, i.e., $v = v^+ - v^-$. Recall that $v^+, v^- \in H_0^1(\Omega)$ for all $v \in H_0^1(\Omega)$, see, e.g., [Kinderlehrer, Stampacchia, 1980, Theorem II.A.1].

Concepts of convex analysis The radial cone and the tangent cone (in the sense of convex analysis) to K at $v \in K$ are defined via

$$\mathcal{R}_K(v) := \bigcup_{\lambda>0} \lambda(K - v) \qquad \text{and} \qquad \mathcal{T}_K(v) := \overline{\mathcal{R}_K(v)},$$

respectively. Recall that the set K is polyhedric, i.e.,

$$\mathcal{T}_K(v) \cap \mu^\perp = \overline{\mathcal{R}_K(v) \cap \mu^\perp}$$

holds for all $v \in K$ and $\mu \in \mathcal{T}_K(v)^\circ$, see [Mignot, 1976, Théorème 3.2]. Note that $v \in K$, $\mu \in \mathcal{T}_K(v)^\circ$ is equivalent to $(v, \mu) \in \mathbb{K}$, i.e., $\mathbb{K}$ is the graph of the normal cone mapping of K. Associated to $(v, \mu) \in \mathbb{K}$, we define the critical cone

$$\mathcal{K}_K(v, \mu) := \mathcal{T}_K(v) \cap \mu^\perp = \{w \in \mathcal{T}_K(v) \mid \langle \mu, w\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} = 0\}.$$

Concepts of variational calculus We mention two basic concepts of variational calculus. First, we recall that the Fréchet normal cone $\widehat{\mathcal{N}}_C(\bar{x})$ of a subset $C \subset X$ of a Banach space X is defined via

$$\widehat{\mathcal{N}}_C(\bar{x}) := \left\{\eta \in X^\star \;\middle|\; \limsup_{x\to\bar{x}, x\in C} \frac{\langle \eta, x - \bar{x}\rangle}{\|x - \bar{x}\|_X} \leq 0\right\}.$$

If the Banach space X is reflexive, the limiting normal cone (or Mordukhovich normal cone) to a closed set $C \subset X$ at a point $\bar{x} \in C$ can be defined via

$$\mathcal{N}_C^{\lim}(\bar{x}) := \{\eta \in X^\star \mid \exists \{x_n\}_{n\in\mathbb{N}} \subset C, \{\eta_n\}_{n\in\mathbb{N}} \subset X^\star : \eta_n \in \widehat{\mathcal{N}}_C(x_n), x_n \to \bar{x}, \eta_n \rightharpoonup \eta\},$$

see [Mordukhovich, 2006, Definition 1.1, Theorem 2.35]. In the same setting, the Clarke normal cone can be defined as

$$\mathcal{N}_C^{\mathrm{Clarke}}(\bar{x}) := \overline{\operatorname{conv}}\, \mathcal{N}_C^{\lim}(\bar{x}),$$

see [Mordukhovich, 2006, p. 17, Theorem 3.57], where $\overline{\operatorname{conv}}(\cdot)$ denotes the closed convex hull. We note that in the case that C is a closed convex subset of X we have the equality of the aforementioned normal cones, i.e. $\widehat{\mathcal{N}}_C(\bar{x}) = \mathcal{N}_C^{\lim}(\bar{x}) = \mathcal{N}_C^{\mathrm{Clarke}}(\bar{x}) = \mathcal{T}_C(\bar{x})^\circ$ and this cone will be denoted by $\mathcal{N}_C(\bar{x})$.

Now, we are going to apply these definitions to the non-convex set $\mathbb{K}$. Due to the polyhedricity of $K \subset H_0^1(\Omega)$, we have

$$\widehat{\mathcal{N}}_{\mathbb{K}}(y, \lambda) = \mathcal{K}_K(y, \lambda)^\circ \times \mathcal{K}_K(y, \lambda), \tag{2.1}$$

cf. [Franke, Mehlitz, Pilecka, 2016, Lemma 4.1] and [G. Wachsmuth, 2015, Lemma 5.2]. Hence, the limiting normal cone of $\mathbb{K}$ can be written as

$$\mathcal{N}_{\mathbb{K}}^{\lim}(y, \lambda) = \left\{ (\nu, w) \in H^{-1}(\Omega) \times H_0^1(\Omega) \;\middle|\; \begin{array}{l} \exists \{y_n\}_{n\in\mathbb{N}}, \{\lambda_n\}_{n\in\mathbb{N}}, \{\nu_n\}_{n\in\mathbb{N}}, \{w_n\}_{n\in\mathbb{N}} : \\ \quad (y_n, \lambda_n) \in \mathbb{K}, y_n \to y, \lambda_n \to \lambda, \\ \quad \nu_n \in \mathcal{K}_K(y_n, \lambda_n)^\circ, w_n \in \mathcal{K}_K(y_n, \lambda_n), \\ \quad \nu_n \rightharpoonup \nu, w_n \rightharpoonup w \end{array} \right\}.$$

3 Introduction to capacity theory

In this section, we recall some facts about capacity theory, which will be needed in the sequel. We do not claim that any of these results is original, and for the convenience of the reader we give some simple and enlightening proofs.

3.1 Definition and basic properties

We start with the definition of the capacity of an arbitrary subset of Ω, see, e.g., [Attouch, Buttazzo, Michaille, 2006, Section 5.8.2], [Bonnans, Shapiro, 2000, Definition 6.47], [Delfour, Zolésio, 2001, Section 8.6.1], and [Fukushima, Oshima, Takeda, 2011, Section 2].

Definition 3.1. The *capacity* (w.r.t. $H_0^1(\Omega)$) of a set $O \subset \Omega$ is defined as

$$\operatorname{cap}(O) := \inf\{\|v\|_{H_0^1(\Omega)}^2 \mid v \in H_0^1(\Omega) \text{ and } v \geq 1 \text{ a.e. in a neighborhood of } O\}.$$

We remark that the infimum over an empty set is ∞, e.g., $\operatorname{cap}(\Omega) = \infty$.

Similar to the expression that a property holds almost everywhere, we say that a property P (depending on $x \in \Omega$) holds quasi-everywhere (q.e.) on a subset $S \subset \Omega$, if and only if $\operatorname{cap}(\{x \in S \mid P(x) \text{ does not hold}\}) = 0$.

The notion of "quasi-everywhere" is more restrictive than "almost everywhere". In particular, for $d \geq 2$ it can be shown that sets of Hausdorff dimension smaller than $d-2$ have capacity zero and, conversely, sets with capacity zero have a Hausdorff dimension of at most $d-2$, cf., e.g., [Heinonen, Kilpeläinen, Martio, 1993, Theorems 2.26, 2.27], [Ziemer, 1989, Theorem 2.6.16], [Adams, Hedberg, 1996, Theorem 5.1.9]. In dimension $d = 1$, $\operatorname{cap}(A) = 0$ if and only if $A = \emptyset$.

We collect some basic properties of capacities in the following lemma.

Lemma 3.2. (a) $\operatorname{cap}(\emptyset) = 0$ and $\operatorname{cap}(A) \leq \operatorname{cap}(B)$ if $A \subset B$.

(b) $\operatorname{meas}(A) \leq C \operatorname{cap}(A)$ for every measurable set $A \subset \Omega$ and a constant $C > 0$, where $\operatorname{meas}(\cdot)$ denotes the Lebesgue measure. Moreover, sets of zero capacity are Lebesgue measurable and have Lebesgue measure zero.

(c) A compact subset $K \subset \Omega$ has finite capacity.

(d) $\operatorname{cap}(\bigcup_{k\in\mathbb{N}} A_k) \leq \sum_{k\in\mathbb{N}} \operatorname{cap}(A_k)$ for all families $(A_k)_{k\in\mathbb{N}}$ with $A_k \subset \Omega$.

Proof. Part (a) is obviously true. For part (b), consider functions $v \in H_0^1(\Omega)$ with $v \geq 1$ a.e. on an open neighborhood of A. Using Poincaré's inequality, we have

$$\operatorname{meas}(A) \leq \|v\|_{L^2(\Omega)}^2 \leq C\, \|v\|_{H_0^1(\Omega)}^2$$

and taking the infimum over all these functions v yields the result. If the set A has zero capacity, it is contained in a sequence of open sets with arbitrarily small capacity. Since their measure also has to be arbitrarily small, A is contained in a set of measure zero. Thus, it is measurable and $\operatorname{meas}(A) = 0$.

For (c) it suffices to find a function in $v \in H_0^1(\Omega)$ such that $v \geq 1$ a.e. on an open neighborhood of K. However, it is well known that such a function exists even in $C_c^\infty(\Omega)$.

A proof of part (d) can be found in [Heinonen, Kilpeläinen, Martio, 1993, Theorem 2.2].

We continue with the definition of quasi-open sets and quasi-continuous functions, which are important concepts of capacity theory.

Definition 3.3. A set $O \subset \Omega$ is called *quasi-open* if for all $\varepsilon > 0$ there exists an open set $G_\varepsilon \subset \Omega$, such that $\operatorname{cap}(G_\varepsilon) < \varepsilon$ and $O \cup G_\varepsilon$ is open. A set $F \subset \Omega$ is called *quasi-closed* if $\Omega \setminus F$ is quasi-open.

A function $v : \Omega \to \mathbb{R}$ is called *quasi-continuous* if for all $\varepsilon > 0$, there exists an open set $G_\varepsilon \subset \Omega$, such that $\operatorname{cap}(G_\varepsilon) < \varepsilon$ and v is continuous on $\Omega \setminus G_\varepsilon$.

An important application of the concept of a quasi-continuous function is that for every function $v \in H_0^1(\Omega)$ there exists a quasi-continuous representative $\tilde{v}$ such that $v = \tilde{v}$ a.e., see, e.g. [Bonnans, Shapiro, 2000, Lemma 6.50]. The quasi-continuous representative is uniquely determined up to sets of zero capacity. Due to this result we will always refer to the quasi-continuous representative when we speak about a function $v \in H_0^1(\Omega)$. This allows us to state more properties that relate to capacity theory and functions in $H_0^1(\Omega)$.

Lemma 3.4. (a) A function $v : \Omega \to \mathbb{R}$ is quasi-continuous if and only if the preimages of open subsets of $\mathbb{R}$ are quasi-open.

(b) For a quasi-continuous function v and a quasi-open set $O \subset \Omega$, we have

$$v \geq 0 \text{ q.e. on } O \quad \Leftrightarrow \quad v \geq 0 \text{ a.e. on } O$$

(c) Every sequence which converges in $H_0^1(\Omega)$ possesses a pointwise quasi-everywhere convergent subsequence.

(d) For every subset $A \subset \Omega$ we have the identity

$$\operatorname{cap}(A) = \inf\{\|v\|^2_{H_0^1(\Omega)} \mid v \in H_0^1(\Omega) \text{ and } v \geq 1 \text{ q.e. on } A\}$$

and in case $\operatorname{cap}(A) < \infty$, this infimum is attained by a non-negative function v with $v = 1$ q.e. on A.

(e) For all open subsets $\Omega_1 \subset \Omega$, we have the characterization

$$u \in H_0^1(\Omega_1) \quad \Leftrightarrow \quad u \in H_0^1(\Omega) \text{ and } u = 0 \text{ q.e. on } \Omega \setminus \Omega_1.$$

Proof. For part (a) we refer to [Kilpeläinen, Malý, 1992, Theorem 1.4].

The implication "⇒" in (b) follows directly from Lemma 3.2 (b). The other direction can be found in [Fukushima, Oshima, Takeda, 2011, Lemma 2.1.5], see also [G. Wachsmuth, 2014, Lemma 2.3].

For a proof of part (c) we refer to [Bonnans, Shapiro, 2000, Lemma 6.52].

For the proof of part (d) we will write $\operatorname{cap}_1(A)$ as an abbreviation of the right-hand side. First, let $v \in H_0^1(\Omega)$ be given such that $v \geq 1$ a.e. on an open neighborhood of A. By part (b) we have $v \geq 1$ q.e. on an open neighborhood of A. Taking the infimum yields $\operatorname{cap}_1(A) \leq \operatorname{cap}(A)$.

For the other inequality, let $\varepsilon > 0$ and $v_\varepsilon \in H_0^1(\Omega)_+$ be given such that $v_\varepsilon \geq 1$ q.e. on A and $\|v_\varepsilon\|^2_{H_0^1(\Omega)} \leq \operatorname{cap}_1(A) + \varepsilon$. Then $\{x \in \Omega \mid v_\varepsilon(x) > 1 - \varepsilon\}$ is quasi-open by part (a). Let $A_\varepsilon := \{x \in \Omega \mid v_\varepsilon(x) > 1 - \varepsilon\} \cup G_\varepsilon$ be an open set where G_ε is open with $\operatorname{cap}(G_\varepsilon) < \varepsilon$. Hence, there exists $w_\varepsilon \in H_0^1(\Omega)_+$ with $\|w_\varepsilon\|^2_{H_0^1(\Omega)} \leq 2\varepsilon$ and $w_\varepsilon \geq 1$ a.e. on G_ε. Then it follows that $\frac{1}{(1-\varepsilon)} v_\varepsilon + w_\varepsilon \geq 1$ a.e. on A_ε which is an open neighborhood of

A. Then, taking the limit $\varepsilon \to 0$, the result follows from

$$\operatorname{cap}(A) \le \left\| \frac{v_\varepsilon}{1-\varepsilon} + w_\varepsilon \right\|^2_{H_0^1(\Omega)} \le \left[\frac{(\operatorname{cap}_1(A) + \varepsilon)^{1/2}}{1-\varepsilon} + \sqrt{2\,\varepsilon} \right]^2 \to \operatorname{cap}_1(A).$$

Since the set $V := \{v \in H_0^1(\Omega) \mid v \ge 1 \text{ q.e. on } A\}$ is closed by part (c), it is clear that the infimum is attained by the norm-minimal element v in V and it is non-negative by [Kinderlehrer, Stampacchia, 1980, Theorem II.A.1]. Similarly, $\min(v, 1) \in V$ implies $v = 1$ q.e. on A.

Finally, (e) follows from [Heinonen, Kilpeläinen, Martio, 1993, Theorem 4.5].

An important application of capacity theory is that we are able to describe functionals $\xi \in H^{-1}(\Omega)_\pm = \mp K^\circ$ as measures.

Lemma 3.5. Let $\xi \in H^{-1}(\Omega)_+$ be given. We can identify ξ with a regular Borel measure on Ω with the following properties:

(a) The measure ξ does not charge sets of capacity zero, i.e., if $A \subset \Omega$ is a Borel set with $\operatorname{cap}(A) = 0$, then $\xi(A) = 0$.

(b) If $v \in H_0^1(\Omega)$, then v is ξ-integrable and we have

$$\langle \xi, v \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} = \int_\Omega v \,\mathrm{d}\xi.$$

(c) For a Borel subset $A \subset \Omega$ we have $\xi(A) \le \operatorname{cap}(A)^{\frac{1}{2}} \, \|\xi\|_{H^{-1}(\Omega)}$.

(d) The measure ξ is locally finite, i.e., $\xi(K) < \infty$ for all compact sets $K \subset \Omega$.

Proof. For representation of ξ as a regular Borel measure see [Bonnans, Shapiro, 2000, p. 564]. Parts (a) and (b) can be found in [Bonnans, Shapiro, 2000, Lemmas 6.55, 6.56]. For (b), it is crucial that v is interpreted as a quasi-continuous function.

For part (c), assume w.l.o.g. $\operatorname{cap}(A) < \infty$. According to Lemma 3.4 (d), there exists $v \in H_0^1(\Omega)$ with $v \ge 1$ q.e. on A, $v \ge 0$ q.e. and $\operatorname{cap}(A) = \|v\|^2_{H_0^1(\Omega)}$. Together with part (a) this implies $v \ge \chi_A$ ξ-a.e. Then by part (b) have

$$\xi(A) \le \int_\Omega v \,\mathrm{d}\xi = \langle \xi, v \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} \le \|v\|_{H_0^1(\Omega)} \|\xi\|_{H^{-1}(\Omega)} = \operatorname{cap}(A)^{\frac{1}{2}} \, \|\xi\|_{H^{-1}(\Omega)}.$$

Part (d) follows from the combination of part (c) and Lemma 3.2 (c).

An important consequence of Lemma 3.5 (a) is that properties holding q.e. are also valid ξ-a.e., and this was already used in the proof of Lemma 3.5 (c).

Of course the results of the previous lemma can also be applied to functionals $\xi \in H^{-1}(\Omega)_-$ with the obvious changes.

3.2 The fine support

Consider a functional (which is a Borel measure by Lemma 3.5) $\xi \in H^{-1}(\Omega)_\pm$. In this section we want to generalize the notion of the support of the measure ξ in a way which is more useful for our setting. The resulting set will be called the fine support, denoted by $\text{f-supp}(\xi)$. In the context of Dirichlet spaces, this is known as quasi-support, see [Fukushima, Oshima, Takeda, 2011, Section 4.6].

The fine support is usually defined as the complement of the largest finely-open set O with $\xi(O) = 0$, see [G. Wachsmuth, 2014, Appendix A] for details. This approach uses, among other things, the concept of the fine topology on Ω. In this section however we will present an alternative approach for deriving the fine support of ξ, which does not use the fine topology but only the concepts of capacity theory that we discussed so far.

Lemma 3.6. Let $\xi \in H^{-1}(\Omega)_+$ be a measure. Then there exists a largest (up to a set of capacity zero) quasi-closed set $A \subset \Omega$ such that

$$\xi(\{v \neq 0\}) = 0 \quad \Leftrightarrow \quad v = 0 \text{ q.e. on } A \tag{3.1}$$

holds for all $v \in H_0^1(\Omega)$.

Proof. It can be shown that the set of all v that satisfy the left-hand side of (3.1) is a closed lattice ideal in $H_0^1(\Omega)$. By [Stollmann, 1993, Theorem 1] there exists a set A, such that (3.1) is satisfied for all $v \in H_0^1(\Omega)$. By the proof of [Stollmann, 1993, Theorem 1] it can be seen that A can be chosen such that $A = \{f = 0\}$ for a function $f \in H_0^1(\Omega)$.

It remains to show that $\{f = 0\}$ is the largest quasi-closed set A with that property. Let $\tilde{A}$ be another quasi-closed set that satisfies (3.1). Since $\xi(\{f \neq 0\}) = 0$ it should hold that $f = 0$ q.e. on $\tilde{A}$, i.e. $\tilde{A} \subset \{f = 0\}$ up to a set of capacity zero.

We note that the proof of [Stollmann, 1993, Theorem 1] does not use the fine topology.

Clearly, the set A from Lemma 3.6 is unique up to a set of capacity zero. We define the *fine support* of a functional $\xi \in H^{-1}(\Omega)_+$ as $\text{f-supp}(\xi) := A$ where A is the largest quasi-closed set $A \subset \Omega$ that satisfies (3.1). For a functional $\xi \in H^{-1}(\Omega)_-$ we define the fine support as $\text{f-supp}(\xi) := \text{f-supp}(-\xi)$.

We will argue that our definition of the fine support coincides with the one given in [G. Wachsmuth, 2014, Appendix A]. This will be done with the help of the following lemma, which follows from [Kilpeläinen, Malý, 1992, Theorem 1.5].

Lemma 3.7. For every quasi-open set $O \subset \Omega$ there exists a function $v \in H_0^1(\Omega)_+$ such that $O = \{f > 0\}$ up to zero capacity.

Note that the proof of [Kilpeläinen, Malý, 1992, Theorem 1.5] uses the fine topology on $\mathbb{R}^d$ and its quasi-Lindelöf property. A different proof of Lemma 3.7 based on probabilistic arguments can be found in [Fukushima, Oshima, Takeda, 2011, Lemma 4.6.1].

With the help of Lemma 3.7 we can provide further properties of of the fine support.

Lemma 3.8. Let $\xi \in H^{-1}(\Omega)_+$ be given.

(a) For every quasi-open set $O \subset \Omega$ the equivalence

$$\xi(O) = 0 \quad \Leftrightarrow \quad \operatorname{cap}(O \cap \text{f-supp}(\xi)) = 0 \tag{3.2}$$

holds, i.e., ξ charges O if and only if O intersects the fine support of ξ.

(b) Every quasi-closed set A that satisfies (3.1) is equal to f-supp(ξ) up to a set of capacity zero.

Proof. For part (a) let $O \subset \Omega$ be a quasi-open set. By Lemma 3.7, there is $v \in H_0^1(\Omega)_+$ with $O = \{v > 0\}$. In particular, $v > 0$ ξ-a.e. on O. Using (3.1), this yields the chain of equivalencies

$$\begin{aligned} \xi(O) = 0 \quad &\Leftrightarrow \quad \xi(\{v \neq 0\}) = 0 \quad \Leftrightarrow \quad v = 0 \text{ q.e. on f-supp}(\xi) \\ &\Leftrightarrow \quad \operatorname{cap}(\{v \neq 0\} \cap \text{f-supp}(\xi)) = 0 \quad \Leftrightarrow \quad \operatorname{cap}(O \cap \text{f-supp}(\xi)) = 0. \end{aligned}$$

For part (b) let $f, g \in H_0^1(\Omega)_+$ be such that $A = \{f = 0\}$ and f-supp$(\xi) = \{g = 0\}$. Then it follows from (3.1) that $f = 0$ q.e. on $\{g = 0\}$ and vice versa, thus $\{f = 0\} = \{g = 0\}$ up to a set of capacity zero.

Note that the proof of Lemma 3.8 (a) even reveals that the satisfaction of (3.1) for all $v \in H_0^1(\Omega)$ is equivalent to the satisfaction of (3.2) (with f-supp(ξ) replaced by A) for all quasi-open $O \subset \Omega$. Hence, (3.2) is also a characterization of the fine support.

It can be shown that the fine support as defined in [G. Wachsmuth, 2014, Appendix A] is quasi-closed (this follows from the fact that finely open sets are quasi-open). Therefore, by using Lemma 3.8 (b) this definition of the fine support coincides with our definition given by Lemma 3.6.

The following lemma will prove to be useful later.

Lemma 3.9. (a) Let $O \subset F \subset \Omega$ be given such that O is quasi-open and F is quasi-closed. Then, $O \subset \text{f-supp}(\chi_O) \subset F$.

(b) Let $v \in H_0^1(\Omega)_+$ be given. Then, $\bigcap_{n=1}^{\infty} \text{f-supp}(\chi_{\{v<1/n\}}) = \{v = 0\}$.

Proof. (a): The set $U := \Omega \setminus \text{f-supp}(\chi_O)$ is quasi-open and does not intersect f-supp(χ_O). Hence, Lemma 3.8 (a) implies $0 = \chi_O(U) = \operatorname{meas}(U \cap O)$. Since $U \cap O$ is quasi-open, this implies that $\operatorname{cap}(U \cap O) = 0$, see [Fukushima, Ōshima, Takeda, 1994, Lemma 2.1.7] and [G. Wachsmuth, 2016a, Lemma 2.1]. Hence, $O \subset \text{f-supp}(\chi_O)$ up to capacity zero.

Similarly, $\Omega \setminus F$ is quasi-open and $O \cap (\Omega \setminus F) = 0$. Hence, $\chi_O(\Omega \setminus F) = 0$ and Lemma 3.8 (a) implies $\operatorname{cap}(\text{f-supp}(\chi_O) \setminus F) = 0$. Hence, f-supp$(\chi_O) \subset F$ up to capacity zero.

(b): From (a) we infer $\{v < 1/n\} \subset \operatorname{f-supp}(\chi_{\{v<1/n\}}) \subset \{v \le 1/n\}$. This implies the claim.

Finally, we provide a further auxiliary lemma which shows that each Borel set of non-zero capacity contains the fine support of a non-zero functional.

Lemma 3.10. Let $A \subset \Omega$ be a Borel set with $0 < \operatorname{cap}(A) < \infty$. Then, for all $c \in (0,1)$, there is a $\nu \in H^{-1}(\Omega)_+$ with

$$\nu(A) \ge c \operatorname{cap}(A)^{1/2} \|\nu\|_{H^{-1}(\Omega)}$$

and $\operatorname{f-supp}(\nu) \subset A$ up to a set of capacity zero. If A is compact, we obtain additionally

$$\nu(A) = \operatorname{cap}(A)^{1/2} \|\nu\|_{H^{-1}(\Omega)}.$$

In the compact case, one can choose the so-called capacitary measure and the Borel case follows with an exhaustion of compact sets from Choquet's theorem. For convenience, we provide the proof, see also [Fukushima, Ōshima, Takeda, 1994, Theorem 2.2.3].

Proof. Let $M \subset \Omega$ be compact. We define the closed and convex set $\mathcal{M} := \{v \in H_0^1(\Omega) \mid v \ge 1 \text{ q.e. on } M\}$ and consider the minimization problem

$$\text{minimize } \frac{1}{2} \|v\|_{H_0^1(\Omega)}^2 \text{ w.r.t. } v \in \mathcal{M}.$$

From Lemma 3.4 (d) it follows that this problem has a non-negative solution $v_M \in \mathcal{M}$ with $v_M = 1$ q.e. on M and the solution is characterized by

$$\nu_M := -\Delta v_M \in -\mathcal{N}_{\mathcal{M}}(v_M).$$

Since $v_M + v \in \mathcal{M}$ for all $v \in H_0^1(\Omega)_+$, we find

$$\langle \nu_M, v \rangle = \langle \nu_M, (v_M + v) - v_M \rangle \ge 0,$$

i.e., $\nu_M \in H^{-1}(\Omega)_+$. Next, we check that the fine support of ν_M is contained in M. The set $\Omega \setminus M$ is open, hence, there is a non-negative $\hat{v} \in H_0^1(\Omega)$ with $\hat{v} > 0$ q.e. in $\Omega \setminus M$, see Lemma 3.7. From $v_M \pm \hat{v} \in \mathcal{M}$, we infer $\int_\Omega \hat{v} \, \mathrm{d}\nu_M = 0$, hence, $\nu_M(\{\hat{v} \ne 0\}) = 0$. By the definition (3.1) of the fine support, this implies $\hat{v} = 0$ q.e. on $\operatorname{f-supp}(\nu_M)$, hence, $\operatorname{f-supp}(\nu_M) \subset M$ up to a set of capacity zero.

Now, we use Riesz's representation theorem and Lemma 3.4 (d) to obtain

$$\begin{aligned} \operatorname{cap}(M) &= \|v_M\|_{H_0^1(\Omega)}^2 = \|\nu_M\|_{H^{-1}(\Omega)}^2 = \langle \nu_M, v_M \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} \\ &= \int_\Omega v_M \, \mathrm{d}\nu_M = \int_M v_M \, \mathrm{d}\nu_M = \int_M \mathrm{d}\nu_M = \nu_M(M). \end{aligned}$$

Now, let a Borel set $A \subset \Omega$ with $0 < \operatorname{cap}(A) < \infty$ be given. From the famous capacity theorem of Choquet, see, e.g., [Heinonen, Kilpeläinen, Martio, 1993, Theorem 2.5] or [Fukushima, Oshima, Takeda, 2011, Theorem A.1.1], we obtain that A is capacitable. Hence, there exists a compact set $M \subset A$ with $\operatorname{cap}(M) \geq c^2 \operatorname{cap}(A)$. By applying the first part of the proof to M, we obtain $\nu_M \in H^{-1}(\Omega)_+$, $\operatorname{f-supp}(\nu_M) \subset M \subset A$ (up to a polar set) and

$$\nu_M(A) \geq \nu_M(M) = \operatorname{cap}(M)^{1/2} \, \|\nu_M\|_{H^{-1}(\Omega)} \geq c \operatorname{cap}(A)^{1/2} \, \|\nu_M\|_{H^{-1}(\Omega)}.$$

3.3 Tangent and normal cones in $H_0^1(\Omega)$

With the help of capacity and the fine support, it is possible to give descriptions for the normal, tangent, and critical cone of K. Those descriptions give us a pointwise q.e. conditions for functions that are in these cones. With this formulation they will be useful later.

Lemma 3.11. Let $v \in K = H_0^1(\Omega)_+$ be given.

(a) Then the normal cone at v is given by

$$\begin{aligned} \mathcal{N}_K(v) &:= \mathcal{T}_K(v)^\circ = K^\circ \cap \xi^\perp \\ &= \{\xi \in H^{-1}(\Omega)_- \mid \xi(\{v > 0\}) = 0\} \\ &= \{\xi \in H^{-1}(\Omega)_- \mid v = 0 \text{ q.e. on } \operatorname{f-supp}(\xi)\}. \end{aligned}$$

(b) The tangent cone at v can be written as

$$\mathcal{T}_K(v) = \{w \in H_0^1(\Omega) \mid w \geq 0 \text{ q.e. on } \{v = 0\}\}. \tag{3.3}$$

(c) For a functional $\mu \in \mathcal{N}_K(v)$ the critical cone at (v, μ) can be expressed as

$$\mathcal{K}_K(v, \mu) = \{w \in H_0^1(\Omega) \mid w \geq 0 \text{ q.e. on } \{v = 0\}, w = 0 \text{ q.e. on } \operatorname{f-supp}(\mu)\}. \tag{3.4}$$

Proof. For $\xi \in K^\circ$ and $v \in H_0^1(\Omega)$ with $v \geq 0$ q.e. on $\operatorname{f-supp}(\xi)$ the equivalences

$$\langle \xi, v \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} = 0 \quad \Leftrightarrow \quad \xi(\{v \neq 0\}) = 0 \quad \Leftrightarrow \quad v = 0 \text{ q.e. on } \operatorname{f-supp}(\xi) \tag{3.5}$$

can be shown with the help of Lemma 3.5 (b) and Lemma 3.6. With this information, the result for the normal cone can be seen directly.

For proving the expression of the tangent cone, we denote by $\widehat{\mathcal{T}}$ the right-hand side of (3.3). Using Lemma 3.4 (b) it is easy to see that $\mathcal{R}_K(v) \subset \widehat{\mathcal{T}}$. Now let $w \in \widehat{\mathcal{T}}$ and $\xi \in \mathcal{N}_K(v)$ be given. From the respective definitions it follows that $w \geq 0$ q.e. on f-supp(ξ). Therefore, $\langle \xi, w\rangle = \int_\Omega v \,\mathrm{d}\xi \le 0$. Since $w \in \widehat{\mathcal{T}}$ and $\xi \in \mathcal{N}_K(v)$ are arbitrary we have $\mathcal{R}_K(v) \subset \widehat{\mathcal{T}} \subset \mathcal{N}_K(v)^\circ$. Taking the closure of all sets and using the bipolar theorem yields the result.

The expression for the critical cone follows from part (b) and (3.5).

4 Stationarity systems for general MPCCs

In this section, we give a brief overview on optimality systems for general classes of MPCCs. First, we review the classical finite-dimensional case in Section 4.1 and address the case of abstract MPCCs in reflexive Banach spaces afterwards in Section 4.2.

4.1 Stationarity systems for finite-dimensional MPCCs

We consider the problem

$$\begin{aligned} \text{minimize} \quad & f(x), \\ \text{w.r.t.} \quad & x \in \mathbb{R}^n, \\ \text{s.t.} \quad & h(x) = 0, \\ & G(x) \ge 0,\ H(x) \le 0,\ G(x)^\top H(x) = 0. \end{aligned} \tag{4.1}$$

Here, $f : \mathbb{R}^n \to \mathbb{R}$, $h : \mathbb{R}^n \to \mathbb{R}^k$, $G, H : \mathbb{R}^n \to \mathbb{R}^m$ are assumed to be continuously differentiable and $n, m \ge 1$, $k \ge 0$. For simplicity, we did not include any additional inequality constraints. They can, however, be added in a straightforward way. Typically, one formulates an MPCC with $H(x) \ge 0$, but in order to increase the similarity with (1.1), we used $H(x) \le 0$. This will have an impact on the sign conditions of the multipliers associated with this constraint in the stationarity systems below.

It can be easily checked that the constraint qualification of Mangasarian-Fromovitz is violated in all feasible points of (4.1). However, under some conditions, the Guignard constraint qualification is satisfied, see [Flegel, Kanzow, 2005b]. Therefore, several constraint qualifications tailored to MPCCs have been introduced, we refer exemplarily to [Luo, Pang, Ralph, 1996; Pang, Fukushima, 1999; Scheel, Scholtes, 2000; Flegel, Kanzow, 2005a; b; Hoheisel, Kanzow, Schwartz, 2013].

Due to this violation of standard constraint qualifications, the classical Karush-Kuhn-Tucker conditions fail to be satisfied for some problems of type (4.1). Moreover, since the Mangasarian-Fromovitz condition is inevitably violated, the set of standard Lagrange multipliers for (4.1) is always unbounded (or empty). This is caused by some redundancy in the Karush-Kuhn-Tucker system.

In order to formulate stationarity systems for (4.1), we introduce the so-called MPCC-Lagrangian $\mathcal{L} : \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$,

$$\mathcal{L}(x, \mu, \nu, \rho) = f(x) + (h(x),\, \mu)_{\mathbb{R}^k} + (G(x),\, \nu)_{\mathbb{R}^m} + (H(x),\, \rho)_{\mathbb{R}^m}.$$

Note that this MPCC-Lagrangian does not include a multiplier for the complementarity constraint $G(x)^\top H(x) = 0$. This may help to get rid of the redundancy in the stationarity system. Given a feasible point $\bar{x} \in \mathbb{R}^n$ of (4.1), we define the index sets (suppressing the dependence on $\bar{x}$)

$$\begin{aligned} I^{+0} &:= \{i \in \{1, \dots, m\} \mid G_i(\bar{x}) > 0\}, \\ I^{0-} &:= \{i \in \{1, \dots, m\} \mid H_i(\bar{x}) < 0\}, \\ I^{00} &:= \{i \in \{1, \dots, m\} \mid G_i(\bar{x}) = H_i(\bar{x}) = 0\}. \end{aligned}$$

Note that these index sets form a partition of $\{1, \dots, m\}$.

A feasible point $\bar{x}$ of (4.1) is said to be *weakly stationary*, if and only if there exist multipliers $\mu \in \mathbb{R}^k$, $\nu, \rho \in \mathbb{R}^m$ such that

$$0 = \nabla_x \mathcal{L}(\bar{x}, \mu, \nu, \rho) = f'(\bar{x}) + h'(\bar{x})^\top \mu + G'(\bar{x})^\top \nu + H'(\bar{x})^\top \rho, \tag{4.2a}$$

$$\nu_i = 0 \quad \forall i \in I^{+0}, \qquad \rho_i = 0 \quad \forall i \in I^{0-} \tag{4.2b}$$

is satisfied. This is the weakest stationarity system associated with (4.1) and it is satisfied under rather weak conditions on the regularity of (4.1), see [Scheel, Scholtes, 2000, Section 2.2].

Stronger optimality conditions contain additional conditions for ν, ρ on the bi-active set I^{00}. The system of *C(larke)-stationarity* requires that, in addition to (4.2), one has

$$\nu_i \, \rho_i \le 0 \qquad \forall i \in I^{00}. \tag{4.3}$$

We emphasize that, since we do not have any sign conditions on the multipliers ν, ρ on I^{00}, that (4.3) is significantly stronger than

$$\sum_{i \in I^{00}} \nu_i \, \rho_i = \sum_{i=1}^{m} \nu_i \, \rho_i \le 0. \tag{4.4}$$

Typically, the system of C-stationarity (4.2)–(4.3) can be obtained via a regularization of the complementarity constraint in (4.1), see [Scholtes, 2001; Hoheisel, Kanzow, Schwartz, 2013].

A stronger optimality system is the so-called *M(ordukhovich)-stationarity*, in which

$$\nu_i \, \rho_i = 0 \text{ or } (\nu_i \le 0 \text{ and } \rho_i \ge 0) \qquad \forall i \in I^{00}. \tag{4.5}$$

is required in addition to (4.2). In comparison to (4.3), the case $\nu_i > 0$, $\rho_i < 0$ is ruled out. These conditions (4.2), (4.5) of M-stationarity can be obtained via techniques of variational analysis, in particular by using the limiting normal cone, see [Outrata, 1999], or by a direct proof, see [Flegel, Kanzow, 2006].

This formulation for M-stationarity can, however, not be transferred to our problem (1.1). Therefore, we give an alternative description, see [G. Wachsmuth, 2016b]: there is a disjoint decomposition of the biactive set $I^{00} = \hat{I}^{+0} \cup \hat{I}^{00} \cup \hat{I}^{0-}$, such that

$$\nu_i = 0 \text{ for } i \in I^{+0} \cup \hat{I}^{+0}, \tag{4.6a}$$

$$\rho_i = 0 \text{ for } i \in I^{0-} \cup \hat{I}^{0-}, \tag{4.6b}$$

$$\nu_i \le 0, \rho_i \ge 0 \text{ for } i \in \hat{I}^{00} \tag{4.6c}$$

are satisfied. It is easy to see that the existence of a disjoint decomposition $I^{00} = \hat{I}^{+0} \cup \hat{I}^{00} \cup \hat{I}^{0-}$ satisfying (4.6) is equivalent to the conditions (4.2b), (4.5) from the system of M-stationarity. Moreover, it is possible to transfer this definition to our problem (1.1), see Section 5.3 below.

Finally, the strongest optimality system is called *strong stationarity* and this system additionally contains

$$\nu_i \le 0 \text{ and } \rho_i \ge 0 \qquad \forall i \in I^{00}. \tag{4.7}$$

It is easy to verify that the conditions (4.2), (4.7) are equivalent to $\bar{x}$ being a (classical) Karush-Kuhn-Tucker point of (4.1) itself. Such a system is satisfied only under quite restrictive assumptions, see [Scheel, Scholtes, 2000; Flegel, Kanzow, 2005b].

For a more detailed account of this finite-dimensional situation, we refer to [Luo, Pang, Ralph, 1996; Scheel, Scholtes, 2000; Hoheisel, Kanzow, Schwartz, 2013] and the references therein.

4.2 Stationarity systems for abstract MPCCs in reflexive Banach spaces

Next, we discuss the problem

$$\begin{aligned} \text{minimize} \quad & f(x), \\ \text{s.t.} \quad & g(x) \in C,\ G(x) \in K,\ H(x) \in K^\circ,\ \langle G(x),\, H(x) \rangle = 0, \end{aligned} \tag{4.8}$$

which is posed in Banach spaces. More precisely, $f : X \to \mathbb{R}$ is Fréchet differentiable, $g : X \to Y$, $G : X \to Z$ and $H : X \to Z^\star$ are continuously Fréchet differentiable, X, Y, Z are (real) Banach spaces and Z is assumed to be reflexive. Moreover, $C \subset Y$ is a closed, convex set and $K \subset Z$ is a closed, convex cone.

Due to the reflexivity of Z, the problem (4.8) is symmetric w.r.t. G and H.

A straightforward computation shows that the Robinson-Zowe-Kurcyusz constraint qualification cannot be satisfied at any feasible point, see [Mehlitz, G. Wachsmuth, 2016b, Lemma 3.1]. This is similar to the violation of Mangasarian-Fromovitz constraint qualification for (4.1). Hence, the Karush-Kuhn-Tucker conditions may fail to be necessary for optimality. Therefore, the aim of this section is to provide alternative stationarity concepts for (4.8).

Stationarity systems for (4.8) have been proposed in [G. Wachsmuth, 2015; Mehlitz, G. Wachsmuth, 2016b; Mehlitz, 2017; G. Wachsmuth, 2017]. In [G. Wachsmuth, 2015] it was suggested to use the so-called local decomposition approach, which is well known for the finite-dimensional problem (4.1), see [Luo, Pang, Ralph, 1996; Pang, Fukushima, 1999; Scheel, Scholtes, 2000; Flegel, Kanzow, 2005a; b]. This leads to the following stationarity concepts. A feasible point $\bar{x}$ of (4.8) is called weakly stationary if there exist Lagrange multipliers $\mu \in Y^\star$, $\nu \in Z^\star$ and $\rho \in Z$, such that

$$0 = f'(\bar{x}) + g'(\bar{x})^\star \mu + G'(\bar{x})^\star \nu + H'(\bar{x})^\star \rho, \tag{4.9a}$$

$$\mu \in \mathcal{T}_C(g(\bar{x}))^\circ, \tag{4.9b}$$

$$\nu \in \operatorname{cl}\big(K^\circ - K^\circ \cap G(\bar{x})^\perp\big) \cap G(\bar{x})^\perp, \tag{4.9c}$$

$$\rho \in \operatorname{cl}\big(K - K \cap H(\bar{x})^\perp\big) \cap H(\bar{x})^\perp. \tag{4.9d}$$

The point $\bar{x}$ is called strongly stationary, if the above multipliers satisfy additionally

$$\nu \in \mathcal{T}_{K^\circ}(H(\bar{x})) \cap G(\bar{x})^\perp = \mathcal{K}_{K^\circ}(H(\bar{x}), G(\bar{x})), \tag{4.10a}$$

$$\rho \in \mathcal{T}_K(G(\bar{x})) \cap H(\bar{x})^\perp \ = \mathcal{K}_K(G(\bar{x}), H(\bar{x})). \tag{4.10b}$$

We refer to [Mehlitz, G. Wachsmuth, 2016b, Definition 3.3] and [G. Wachsmuth, 2015, Definition 5.1]. It can be easily checked that these conditions (4.9) and (4.10) applied to (4.1) are precisely equivalent to (4.2) and (4.7), respectively. However, the classical Karush-Kuhn-Tucker conditions for (4.8) are, in general, slightly stronger than the above definition of strong stationarity for (4.8), see [G. Wachsmuth, 2015, Lemma 5.1]. Finally, we mention that (4.10) implies (4.9c), (4.9d), i.e., strong stationarity implies weak stationarity, see [Mehlitz, G. Wachsmuth, 2016b, Lemma 3.4].

Constraint qualifications ensuring that local minimizers of (4.8) are strongly or weakly stationary can be found in [G. Wachsmuth, 2015, Section 5.3], [Mehlitz, G. Wachsmuth, 2016b, Theorem 3.6], and [Mehlitz, 2017, Proposition 3.4].

In the case that the cone K is polyhedric, it has been shown in [G. Wachsmuth, 2017, Section 5.2] that the above system of strong stationarity is of reasonable strength. In particular, if K is polyhedric and $\bar{x}$ is a strongly stationary point, then it is first-order stationary w.r.t. a linearized feasible set, i.e.,

$$f'(\bar{x})\, h \ge 0 \qquad \forall h \in X : g'(\bar{x})\, h \in \mathcal{T}_C(g(\bar{x})),\, G'(\bar{x})\, h \in \mathcal{K}_K(G(\bar{x}), H(\bar{x})),\\ H'(\bar{x})\, h \in \mathcal{K}_{K^\circ}(H(\bar{x}), G(\bar{x})),$$

see [G. Wachsmuth, 2017, Theorem 5.1].

Finally, we briefly comment on the case that the cone K is not polyhedric. In this case, the above stationarity systems are too weak since they do not take into account the curvature of the boundary of K. Stronger optimality conditions can be obtained by an additional linearization approach, see [G. Wachsmuth, 2015, Section 6.2] and [G. Wachsmuth, 2017]. This has been successfully applied to the case in which K is the second-order cone or the cone of positive semi-definite matrices.

5 Stationarity systems for (1.1)

In this section, we are going to review the different optimality systems by using the notations of capacity theory and variational calculus. To this end, let $(\bar{y}, \bar{u}, \bar{\lambda})$ be a locally optimal solution of (1.1). We state certain replacements for $\mathcal{N}_{\mathbb{K}}^{\sharp}(\bar{y}, \bar{\lambda})$, such that the system (1.3) becomes a necessary optimality system, under certain assumptions on the data of the problem (1.1). Further, we fix the sets

$$\begin{aligned} \mathcal{A} &:= \{x \in \Omega \mid \bar{y}(x) = 0\}, & \mathcal{A}_s &:= \text{f-supp}(\bar{\lambda}), \\ \mathcal{I} &:= \{x \in \Omega \mid \bar{y}(x) > 0\}, & \mathcal{B} &:= \mathcal{A} \setminus \mathcal{A}_s, \end{aligned}$$

which are called active set, strictly active set, inactive set, and biactive set, respectively. Note that $\mathcal{I}$, $\mathcal{B}$ and $\mathcal{A}_s$ correspond to the index sets I^{+0}, I^{00} and I^{0+} in the finite-dimensional setting in Section 4.1. We emphasize that these sets are defined up to sets of zero capacity.

To see the relation between (1.3) and (4.9), we associate with (1.1) the Lagrangian

$$\mathcal{L}(y, u, \lambda, p, \mu, \nu, \rho) := J(y, u) + \langle -\Delta y + \lambda - u - f,\, p\rangle + (\mu,\, u) + \langle \nu,\, y\rangle + \langle \lambda,\, \rho\rangle$$

for $p, \rho \in H_0^1(\Omega)$, $\nu \in H^{-1}(\Omega)$ and $\mu \in L^2(\Omega)$. Taking derivatives w.r.t. (y, u, λ), we arrive at the optimality system

$$\begin{aligned} J_y(\bar{y}, \bar{u}) - \Delta p + \nu &= 0, & \nu &\in \operatorname{cl}(K^\circ - K^\circ \cap \bar{y}^\perp) \cap \bar{y}^\perp, \\ J_u(\bar{y}, \bar{u}) - p + \mu &= 0, & \mu &\in \mathcal{N}_{U_{\text{ad}}}(\bar{u}), \\ p + \rho &= 0, & \rho &\in \operatorname{cl}(K - K \cap \bar{\lambda}^\perp) \cap \bar{\lambda}^\perp, \end{aligned}$$

cf. (4.9). By substituting ρ with $-p$ we arrive at (1.3) with a special replacement for $\mathcal{N}_{\mathbb{K}}^{\sharp}(y, \lambda)$.

5.1 Weak stationarity

The system of weak stationarity is obtained by using

$$\mathcal{N}_{\mathbb{K}}^{\text{weak}}(\bar{y}, \bar{\lambda}) := \{z \in H_0^1(\Omega) \mid z = 0 \text{ q.e. on } \mathcal{A}\}^\circ \times \{w \in H_0^1(\Omega) \mid w = 0 \text{ q.e. on } \mathcal{A}_s\} \quad (5.1)$$

instead of $\mathcal{N}_{\mathbb{K}}^{\sharp}(\bar{y}, \bar{\lambda})$ in (1.3). This system is satisfied for all local minimizers under very weak assumptions on the data, cf. [G. Wachsmuth, 2016b, Lemma 4.4]. Moreover, we mention that the above cone $\mathcal{N}_{\mathbb{K}}^{\text{weak}}(\bar{y}, \bar{\lambda})$ provides an appropriate generalization of (4.2b). In particular, the condition on the multiplier $\nu \in H^{-1}(\Omega)$ is formulated by duality, since ν is, in general, not a proper function.

The next lemma demonstrates that this system coincides with the weak stationarity system (4.9) of Section 4.2.

Theorem 5.1. We have

$$\operatorname{cl}\big(K - K \cap \bar\lambda^\perp\big) \cap \bar\lambda^\perp = \{v \in H_0^1(\Omega) \mid v = 0 \text{ q.e. on } \mathcal{A}_s\},$$
$$\operatorname{cl}\big(K^\circ - K^\circ \cap \bar y^\perp\big) \cap \bar y^\perp = \{v \in H_0^1(\Omega) \mid v = 0 \text{ q.e. on } \mathcal{A}\}^\circ.$$

Proof. To show the first equality, we first observe that

$$K - K \cap \bar\lambda^\perp = \{v \in H_0^1(\Omega) \mid v \ge 0 \text{ q.e. on } \operatorname{f-supp}(\bar\lambda)\}$$

and this set is closed. Hence,

$$\begin{aligned}\operatorname{cl}\big(K - K \cap \bar\lambda^\perp\big) \cap \bar\lambda^\perp &= \big(K - K \cap \bar\lambda^\perp\big) \cap \bar\lambda^\perp = K \cap \bar\lambda^\perp - K \cap \bar\lambda^\perp \\ &= \{v \in H_0^1(\Omega) \mid v = 0 \text{ q.e. on } \mathcal{A}_s\},\end{aligned}$$

which has been claimed.

In order to prove the second equality, we proceed in two steps. First, we show "$\supset$". Using the characterization of the tangent cone, we have

$$\mathcal{T}_K(\bar y) \cap -\mathcal{T}_K(\bar y) = \{v \in H_0^1(\Omega) \mid v = 0 \text{ q.e. on } \mathcal{A}\}.$$

Taking the polar and using [Bonnans, Shapiro, 2000, (2.32)], we find

$$\begin{aligned}\operatorname{cl}\big(K^\circ - K^\circ \cap \bar y^\perp\big) \cap \bar y^\perp &\supset \operatorname{cl}\big(K^\circ \cap \bar y^\perp - K^\circ \cap \bar y^\perp\big) = \operatorname{cl}\big(\mathcal{T}_K(\bar y)^\circ - \mathcal{T}_K(\bar y)^\circ\big) \\ &= \big(\mathcal{T}_K(\bar y) \cap -\mathcal{T}_K(\bar y)\big)^\circ = \{v \in H_0^1(\Omega) \mid v = 0 \text{ q.e. on } \mathcal{A}\}^\circ.\end{aligned}$$

In order to prove "$\subset$", we choose an arbitrary $\mu \in \operatorname{cl}\big(K^\circ - K^\circ \cap \bar y^\perp\big) \cap \bar y^\perp$. By definition, there are sequences $\{\mu_n\}_{n\in\mathbb{N}} \subset K^\circ$ and $\{\nu_n\}_{n\in\mathbb{N}} \subset K^\circ \cap \bar y^\perp$ with $\mu_n - \nu_n \to \mu$. Next, let $v \in H_0^1(\Omega)$ with $-\bar y \le v \le \bar y$ be given. Then,

$$\begin{aligned}\langle \mu, \pm v\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} &= \langle \mu, \bar y \pm v\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} = \lim_{n\to\infty} \langle \mu_n - \nu_n, \bar y \pm v\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} \\ &= \lim_{n\to\infty} \big(\langle \mu_n, \bar y \pm v\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} - \langle \nu_n, \bar y \pm v\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}\big).\end{aligned}$$

Since $\operatorname{f-supp}(\nu_n) \subset \{\bar y = 0\} \subset \{\bar y \pm v = 0\}$ and $\bar y \pm v \ge 0$ q.e., we have $\langle \nu_n, \bar y \pm v\rangle = 0$. Moreover, $\mu_n \in K^\circ$ and $\bar y \pm v \in K$ implies $\langle \mu_n, \bar y \pm v\rangle \le 0$. This shows $\langle \mu, \pm v\rangle \le 0$. Hence,

$$\mu \in \{v \in H_0^1(\Omega) \mid -\bar y \le v \le \bar y\}^\perp.$$

Thus, it remains to show

$$\operatorname{cl}\operatorname{cone}\{v \in H_0^1(\Omega) \mid -\bar y \le v \le \bar y\} = \{v \in H_0^1(\Omega) \mid v = 0 \text{ q.e. on } \mathcal{A}\}, \tag{5.2}$$

where cl cone denotes the closed conic hull. The inclusion "$\subset$" is clear. To prove "$\supset$", let $v \in H_0^1(\Omega)$ with $v = 0$ q.e. on $\mathcal{A}$ be given. Then, $v, -v \in \mathcal{T}_K(\bar y)$ and, thus, there exist sequences $\{w_n\}_{n\in\mathbb{N}} \subset \mathcal{R}_K(\bar y)$ and $\{z_n\}_{n\in\mathbb{N}} \subset -\mathcal{R}_K(\bar y)$ with $w_n \to v$ and $z_n \to v$. This implies $z_n^+ - w_n^- \to v$ and follows from the definition that $z_n^+ - w_n^-$ belongs to the conic hull of $\{v \in H_0^1(\Omega) \mid -\bar y \le v \le \bar y\}$. This finishes the proof of (5.2). Thus,

$$\mu \in \{v \in H_0^1(\Omega) \mid -\bar y \le v \le \bar y\}^\perp = \{v \in H_0^1(\Omega) \mid v = 0 \text{ q.e. on } \mathcal{A}\}^\circ$$

and this proves "$\subset$" in the second equality.

For standard finite-dimensional problems with complementarity constraints, the system of weak stationarity can be written by using the Clarke normal cone of the complementarity set. In the current situation, this relation is shown in the next lemma.

Theorem 5.2. Let $(\bar y, \bar\lambda) \in \mathbb{K}$ be arbitrary. Then,

$$\mathcal{N}_{\mathbb{K}}^{\text{weak}}(\bar y, \bar\lambda) = \mathcal{N}_{\mathbb{K}}^{\text{Clarke}}(\bar y, \bar\lambda).$$

Proof. "$\supset$": From (5.5) below, we have

$$\mathcal{N}_{\mathbb{K}}^{\lim}(\bar y, \bar\lambda) \subset \mathcal{N}_{\mathbb{K}}^{\text{weak}}(\bar y, \bar\lambda).$$

Since $\mathcal{N}_{\mathbb{K}}^{\text{weak}}(\bar y, \bar\lambda)$ is closed and convex, this implies

$$\mathcal{N}_{\mathbb{K}}^{\text{Clarke}}(\bar y, \bar\lambda) = \overline{\operatorname{conv}} \, \mathcal{N}_{\mathbb{K}}^{\lim}(\bar y, \bar\lambda) \subset \mathcal{N}_{\mathbb{K}}^{\text{weak}}(\bar y, \bar\lambda).$$

"$\subset$": Now, let $(\nu, w) \in \mathcal{N}_{\mathbb{K}}^{\text{weak}}(\bar y, \bar\lambda)$ be given. Since $\operatorname{conv} \mathcal{N}_{\mathbb{K}}^{\lim}(\bar y, \bar\lambda) \subset \mathcal{N}^{\text{Clarke}}(\bar y, \bar\lambda)$, and since $\mathcal{N}_{\mathbb{K}}^{\lim}(\bar y, \bar\lambda)$ is a cone, it is enough to show $(\nu, 0), (0, w) \in \mathcal{N}_{\mathbb{K}}^{\lim}(\bar y, \bar\lambda)$.

In order to verify $(0, w) \in \mathcal{N}_{\mathbb{K}}^{\lim}(\bar y, \bar\lambda)$, we set $y_n = \bar y + \frac{1}{n} w^- \in K$ and $\lambda_n = \bar\lambda \in K^\circ$. The orthogonality $\langle y_n, \lambda_n \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} = 0$ follows immediately from $(\bar y, \bar\lambda) \in \mathbb{K}$ and $w = 0$ q.e. on $\mathcal{A}_s = \text{f-supp}(\bar\lambda)$, cf. (5.1). Thus, $(y_n, \lambda_n) \in \mathbb{K}$ and the convergence $(y_n, \lambda_n) \to (\bar y, \bar\lambda)$ is clear. Next, we check that $w \in \mathcal{K}_K(y_n, \lambda_n)$. The condition $w = 0$ q.e. on $\text{f-supp}(\lambda_n) = \text{f-supp}(\bar\lambda)$ is clear, and $w \ge 0$ q.e. on $\{y_n = 0\}$ follows from

$$y_n(x) = 0 \quad \Rightarrow \quad \bar y(x) + \frac{1}{n} w^-(x) = 0 \quad \Rightarrow \quad w^-(x) = 0 \quad \Rightarrow \quad w(x) \ge 0$$

for q.a. $x \in \Omega$. Thus, we have $(0, w) \in \widehat{\mathcal{N}}_{\mathbb{K}}(y_n, \lambda_n)$, cf. (2.1), and $(0, w) \in \mathcal{N}_{\mathbb{K}}^{\lim}(\bar y, \bar\lambda)$ follows.

It remains to verify $(\nu, 0) \in \mathcal{N}_{\mathbb{K}}^{\lim}(\bar y, \bar\lambda)$. To this end, we set $y_n = (\bar y - \frac{1}{n})^+ \in K$ and $\lambda_n = \bar\lambda - \frac{1}{n} \chi_{\{\bar y < 1/n\}} \in K^\circ$. By continuity of $(\cdot)^+$ in $H^1(\Omega)$, it is clear that $(y_n, \lambda_n) \to (\bar y, \bar\lambda)$. From Lemma 3.9 (a) we have $\text{f-supp}(\lambda_n) \subset \{\bar y \le 1/n\} = \{y_n = 0\}$, thus $\langle \lambda_n, y_n \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} = 0$, i.e., $(y_n, \lambda_n) \in \mathbb{K}$. From Lemma 3.9 (b), we find $\{\bar y = 0\} \subset \text{f-supp}(\chi_{\{y < 1/n\}}) \subset \text{f-supp}(\lambda_n)$ (up to a set of zero capacity). Hence,

$$\begin{aligned} \nu &\in \{z \in H_0^1(\Omega) \mid z = 0 \text{ q.e. on } \{\bar y = 0\}\}^\circ \\ &\subset \{z \in H_0^1(\Omega) \mid z = 0 \text{ q.e. on } \text{f-supp}(\lambda_n)\}^\circ \subset \mathcal{K}_K(y_n, \lambda_n)^\circ. \end{aligned}$$

Thus, $(\nu, 0) \in \widehat{\mathcal{N}}_{\mathbb{K}}(y_n, \lambda_n)$ and this yields $(\nu, 0) \in \mathcal{N}_{\mathbb{K}}^{\lim}(\bar y, \bar\lambda)$.

Thus, the previous two lemmas show that using the definition (5.1) of $\mathcal{N}_{\mathbb{K}}^{\text{weak}}$ in (1.3) seems to be the correct generalization of weak stationarity for (1.1). In particular,

- it is a pointwise version of the coefficientwise conditions (4.2);
- it coincides with the system (4.9) arising from the abstract theory for (4.8); and
- we have $\mathcal{N}^{\text{weak}}(\bar{y}, \bar{\lambda}) = \mathcal{N}^{\text{Clarke}}(\bar{y}, \bar{\lambda})$.

5.2 C-Stationarity

The next stronger system in the hierarchy of stationarity systems for (4.1) was the system of C-stationarity (4.2), (4.3). In [Schiela, D. Wachsmuth, 2013, Definition 1.1], it was proposed to check the product of $\nu \in H^{-1}(\Omega)$, $p \in H_0^1(\Omega)$ and a suitable, non-negative test function for non-negativity. This definition is motivated by the observation that, under the condition (4.2b), (4.3) is equivalent to

$$(\mu,\, \nu \circ \varphi)_{\mathbb{R}^m} \geq 0 \qquad \forall \varphi \in \mathbb{R}^m, \varphi \geq 0. \tag{5.3}$$

Here, $\nu \circ \varphi \in \mathbb{R}^m$ denotes the Hadamard product of the vectors $\nu, \varphi \in \mathbb{R}^m$.

We define the cone

$$\mathcal{N}_{\mathbb{K}}^{\mathrm{C}}(\bar{y}, \bar{\lambda}) := \{(\nu, w) \in \mathcal{N}_{\mathbb{K}}^{\text{weak}}(\bar{y}, \bar{\lambda}) \mid \langle \nu, w\, \varphi \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} \leq 0 \; \forall \varphi \in W^{1,\infty}(\Omega)_+\},$$

and the system of C-stationarity can be written by using $\mathcal{N}_{\mathbb{K}}^{\mathrm{C}}(\bar{y}, \bar{\lambda})$ instead of $\mathcal{N}_{\mathbb{K}}^{\sharp}(\bar{y}, \bar{\lambda})$ in (1.3). Note that the above system slightly differs from the system of C-stationarity in [Schiela, D. Wachsmuth, 2013], since therein, higher regularity of the data of (1.1) has been utilized. However, it has been shown in [G. Wachsmuth, 2016b, Lemma 4.6], that the system (1.3) with $\mathcal{N}_{\mathbb{K}}^{\mathrm{C}}$ coincides with the system of C-stationarity in [Schiela, D. Wachsmuth, 2013].

Finally, we mention that often a weaker variant similarly to (4.4) is used, e.g., one only requires

$$\langle \nu,\, p \rangle \leq 0.$$

5.3 M-Stationarity

Next, we will state the definition of M-stationarity from [G. Wachsmuth, 2016b] which parallels (4.6) in the finite-dimensional case. Let $\mathcal{B} = \hat{\mathcal{I}} \cup \hat{\mathcal{B}} \cup \hat{\mathcal{A}}_s$ be a disjoint decomposition of the biactive set and we define

$$\hat{\mathcal{K}}(\hat{\mathcal{B}}, \hat{\mathcal{A}}_s) := \{v \in H_0^1(\Omega) \mid v \geq 0 \text{ q.e. on } \hat{\mathcal{B}} \text{ and } v = 0 \text{ q.e. on } \mathcal{A}_s \cup \hat{\mathcal{A}}_s\}. \tag{5.4}$$

Note that the critical cone satisfies $\mathcal{K}_K(\bar{y}, \bar{\lambda}) = \hat{\mathcal{K}}(\mathcal{B}, \emptyset)$. Then, the M-stationarity conditions of [G. Wachsmuth, 2016b] are obtained by replacing $\mathcal{N}_{\mathbb{K}}^{\sharp}(\bar{y}, \bar{\lambda})$ in (1.3) with

$$\mathcal{N}_{\mathbb{K}}^{\mathrm{M}}(\bar{y}, \bar{\lambda}) = \left\{ (\nu, w) \in H^{-1}(\Omega) \times H_0^1(\Omega) \;\middle|\; \begin{array}{l} \text{there is a decomposition } \mathcal{B} = \hat{\mathcal{I}} \cup \hat{\mathcal{B}} \cup \hat{\mathcal{A}}_s \\ \text{with } \nu \in \hat{\mathcal{K}}(\hat{\mathcal{B}}, \hat{\mathcal{A}}_s)^\circ, w \in \hat{\mathcal{K}}(\hat{\mathcal{B}}, \hat{\mathcal{A}}_s) \end{array} \right\}.$$

In finite dimensions, the system (4.2), (4.6) of M-stationarity can be shown by using the limiting normal cone associated with the complementarity set

$$\{(x, y) \in (\mathbb{R}^m)^2 \mid x \geq 0, y \leq 0, x^\top y = 0\}.$$

However, this is not known for the problem (1.1) and we will distinguish between the above notion of M-stationarity and a system obtained by the limiting normal cone to $\mathbb{K}$. In Section 5.5 it will be shown that both systems coincide in the one-dimensional case $d = 1$.

The case $d \geq 2$ is different. In particular, it is currently not known, whether the system of M-stationarity (1.3) with $\mathcal{N}_{\mathbb{K}}^{\mathrm{M}}$ is a necessary optimality condition. The results in [G. Wachsmuth, 2016b, Section 5] show that such a system can be obtained by a penalization of the control constraints in (1.1) under a very mild condition on the sequence of regularized multipliers. However, it is neither clear whether this condition is always fulfilled nor whether there exists an instance of (1.1), in which a local minimizer is not M-stationary. This is subject to further research.

5.4 The limiting normal cone in dimension $d \geq 2$

One can show that the system (1.3) with $\mathcal{N}_{\mathbb{K}}^{\sharp}$ replaced by the limiting normal cone $\mathcal{N}_{\mathbb{K}}^{\mathrm{lim}}$ is a necessary optimality condition under quite general assumptions on the data of (1.1), see [Hintermüller, Mordukhovich, Surowiec, 2014, Section 3] and [G. Wachsmuth, 2016b, Proof of Lemma 4.4], see also [Outrata, Jarušek, Stará, 2011, Proof of Theorem 16] in case of controls from $H^{-1}(\Omega)$. Note that in the last two references, the optimality system was not stated explicitly by means of the limiting normal cone, but it can be easily extracted from the referenced proofs.

However, there is no precise characterization of this limiting normal cone available. As in the proof of [G. Wachsmuth, 2016b, Lemma 4.4] one can show

$$\mathcal{N}_{\mathbb{K}}^{\mathrm{lim}}(\bar{y}, \bar{\lambda}) \subset \mathcal{N}_{\mathbb{K}}^{\mathrm{weak}}(\bar{y}, \bar{\lambda}). \tag{5.5}$$

We mention that this statement has been generalized to other cones K inducing a lattice with further continuity properties in [Mehlitz, 2017, Lemma 3.10]. Unfortunately, this upper estimate is really large. Even more disappointing, in dimensions $d \geq 2$, the lower estimate

$$\mathcal{N}_{\mathbb{K}}^{\mathrm{weak}}(\bar{y}, \bar{\lambda}) \cap (L^p(\Omega) \times H_0^1(\Omega)) \subset \mathcal{N}_{\mathbb{K}}^{\mathrm{lim}}(\bar{y}, \bar{\lambda})$$

has been shown recently in [Harder, G. Wachsmuth, 2017], where $p \in (1, \infty)$ is chosen such that $L^p(\Omega) \hookrightarrow H^{-1}(\Omega)$. Together with (5.5), this shows that we have the equivalence

$$\forall \nu \in L^p(\Omega), w \in H_0^1(\Omega): \qquad (\nu, w) \in \mathcal{N}_{\mathbb{K}}^{\mathrm{lim}}(\bar{y}, \bar{\lambda}) \quad \Longleftrightarrow \quad (\nu, w) \in \mathcal{N}_{\mathbb{K}}^{\mathrm{weak}}(\bar{y}, \bar{\lambda}) \tag{5.6}$$

in dimension $d \geq 2$ with $L^p(\Omega) \hookrightarrow H^{-1}(\Omega)$.

The following lemma provides another lower bound which is valid for $d \geq 1$.

Theorem 5.3. Let $d \geq 1$. We have

$$\mathcal{N}^{\mathrm{M}}_{\mathbb{K}}(\bar{y}, \bar{\lambda}) \subset \mathcal{N}^{\lim}_{\mathbb{K}}(\bar{y}, \bar{\lambda}).$$

Proof. Let $(\nu, w) \in \mathcal{N}^{\mathrm{M}}_{\mathbb{K}}(\bar{y}, \bar{\lambda})$ be given. We define

$$\hat{\mathcal{A}}_s := \mathcal{B} \cap \{w = 0\}, \quad \hat{\mathcal{I}} := \mathcal{B} \cap \{w < 0\}, \quad \hat{\mathcal{B}} := \mathcal{B} \cap \{w > 0\}.$$

Obviously, this is a partition of the set $\mathcal{B}$ and we can check $\nu \in \hat{\mathcal{K}}(\hat{\mathcal{B}}, \hat{\mathcal{A}}_s)^\circ$, $w \in \hat{\mathcal{K}}(\hat{\mathcal{B}}, \hat{\mathcal{A}}_s)$. Further,

$$\mathcal{A}_s \cup \hat{\mathcal{A}}_s = \text{f-supp}(\bar{\lambda}) \cup (\{\bar{y} = 0\} \cap \{w = 0\} \setminus \text{f-supp}(\bar{\lambda})) = \text{f-supp}(\bar{\lambda}) \cup (\{\bar{y} = 0\} \cap \{w = 0\})$$

is quasi-closed. Hence, there is a $v \in H^1_0(\Omega)_+$ with $\mathcal{A}_s \cup \hat{\mathcal{A}}_s = \{v = 0\}$. Because $w = 0, \bar{y} = 0$ q.e. on $\mathcal{A}_s \cup \hat{\mathcal{A}}_s$, we can assume that $\bar{y} + w^+ + w^- \leq v$. We set $w_n := \min((v - \frac{1}{n})^+, w^+) - \min((v - \frac{1}{n})^-, w^-)$. Then, it is easy to check that $w_n \to w$ in $H^1_0(\Omega)$, $w_n \geq 0$ q.e. on $\hat{\mathcal{B}}$ and $w_n = 0$ q.e. on $\{v \leq 1/n\}$.

Now, we note that

$$\mathcal{I} \cup \hat{\mathcal{I}} = \{\bar{y} > 0\} \cup (\{\bar{y} = 0\} \cap \{w < 0\} \setminus \text{f-supp}(\bar{\lambda})) = \{\bar{y} > 0\} \cup (\{w < 0\} \setminus \text{f-supp}(\bar{\lambda}))$$

is quasi-open. Hence, there is $\hat{y}_n \in H^1_0(\Omega)_+$ with $\{\hat{y}_n > 0\} = \mathcal{I} \cup \hat{\mathcal{I}} \setminus \{v \leq 1/n\}$. Now, we define

$$y_n := \min(\bar{y}, (v - \tfrac{1}{n})^+) + \frac{\hat{y}_n}{n + n\|\hat{y}_n\|_{H^1_0(\Omega)}} \geq 0, \quad \lambda_n := \bar{\lambda} - \frac{1}{n} \chi_{\{v < 1/n\}} \leq 0.$$

This yields

$$\{y_n > 0\} = \mathcal{I} \cup \hat{\mathcal{I}} \setminus \{v \leq 1/n\}, \qquad \{v < 1/n\} \subset \text{f-supp}(\lambda_n) \subset \{v \leq 1/n\},$$

see Lemma 3.9. Hence, $(y_n, \lambda_n) \in \mathbb{K}$ and we also have $y_n \to \bar{y}$ and $\lambda_n \to \bar{\lambda}$.

Due to $w_n = 0$ q.e. on $\{v \leq 1/n\}$ and $w_n \geq 0$ q.e. on $\hat{\mathcal{B}}$, we find $w_n \in \mathcal{K}_K(y_n, \lambda_n)$.

For $z \in \mathcal{K}_K(y_n, \lambda_n)$, we have $z = 0$ q.e. on f-supp(λ_n), hence, $z = 0$ q.e. on $\mathcal{A}_s \cup \hat{\mathcal{A}}_s$. Further, $z \geq 0$ q.e. on $\{y_n = 0\} = \{v \leq 1/n\} \cup \hat{\mathcal{B}} \cup \hat{\mathcal{A}}_s \cup \mathcal{A}$. Hence, $z \in \hat{\mathcal{K}}(\hat{\mathcal{B}}, \hat{\mathcal{A}}_s)$. This implies $\nu \in \hat{\mathcal{K}}(\hat{\mathcal{B}}, \hat{\mathcal{A}}_s)^\circ \subset \mathcal{K}_K(y_n, \lambda_n)^\circ$.

Thus, $(\nu, w_n) \in \widehat{\mathcal{N}}_{\mathbb{K}}(y_n, \lambda_n)$ and the convergences $w_n \to w$, $y_n \to \bar{y}$ and $\lambda_n \to \bar{\lambda}$ yield $(\nu, w) \in \mathcal{N}^{\lim}_{\mathbb{K}}(\bar{y}, \bar{\lambda})$.

In fact, the proof even shows a stronger statement, since the approximating sequence (ν, w_n) for the multipliers converges strongly. Together with the results of [G. Wachsmuth, 2016b, Sections 2, 5], this implies

$$\mathcal{N}_{\mathbb{K}}^{\mathrm{M}}(\bar y, \bar\lambda) = \left\{ (\nu, w) \in H^{-1}(\Omega) \times H_0^1(\Omega) \;\middle|\; \begin{array}{l} \exists \{y_n\}_{n\in\mathbb{N}}, \{\lambda_n\}_{n\in\mathbb{N}}, \{\nu_n\}_{n\in\mathbb{N}}, \{w_n\}_{n\in\mathbb{N}} : \\ \quad (y_n, \lambda_n) \in \mathbb{K}, y_n \to y, \lambda_n \to \lambda, \\ \quad \nu_n \in \mathcal{K}_K(y_n, \lambda_n)^\circ, w_n \in \mathcal{K}_K(y_n, \lambda_n), \\ \quad \nu_n \to \nu, w_n \to w \end{array} \right\}.$$

Note that in difference to the definition of the limiting normal cone $\mathcal{N}_{\mathbb{K}}^{\lim}(\bar y, \bar\lambda)$, we have used strong convergence of (ν_n, w_n). A similar phenomenon was observed for pointwise defined sets in Lebesgue spaces in [Mehlitz, G. Wachsmuth, 2016a; 2017].

We also mention that the proof of Theorem 5.3 could be drastically simplified, if we would know that for every quasi-closed set $A \subset \Omega$, there is a $\xi \in H^{-1}(\Omega)_+$ with f-supp$(\xi) = A$. Indeed, in this case we could choose $\hat\lambda \in H^{-1}(\Omega)_+$ with f-supp$(\hat\lambda) = \mathcal{A}_s \cup \hat{\mathcal{A}}_s$ and use the sequences

$$y_n := \bar y + \frac{1}{n}\,\hat y, \quad \lambda_n := \bar\lambda - \frac{1}{n}\,\hat\lambda, \quad w_n := w, \quad \nu_n := \nu,$$

where $\hat y \in H_0^1(\Omega)_+$ is chosen such that $\{\hat y > 0\} = \mathcal{I} \cup \hat{\mathcal{I}}$.

5.5 The limiting normal cone in dimension $d = 1$

In this section we will have a look at the case $\Omega \subset \mathbb{R}$, i.e., $d = 1$. This case is fundamentally different from $d \ge 2$ and this is manly due to the (compact) embedding $H_0^1(\Omega) \hookrightarrow C_0(\Omega)$. This embedding has been exploited in [Outrata, Jarušek, Stará, 2011; G. Wachsmuth, 2016b] to prove that the system of M-stationarity is a necessary optimality condition in $d = 1$.

In this section, we provide the new result that the above system of M-stationarity is indeed equivalent to the formulation using the limiting normal cone in dimension $d = 1$.

Theorem 5.4. Assume that $\Omega \subset \mathbb{R}^1$. Then,

$$\mathcal{N}_{\mathbb{K}}^{\lim}(\bar y, \bar\lambda) = \mathcal{N}_{\mathbb{K}}^{\mathrm{M}}(\bar y, \bar\lambda).$$

Proof. As already said, the inclusion "$\subset$" follows from [G. Wachsmuth, 2016b, Lemma 2.3 and Section 5] and "$\supset$" was shown in Theorem 5.3.

Finally, we prove that the M-stationarity system from Section 5.3 is equivalent to the optimality conditions obtained in [Jarušek, Outrata, 2007, Theorem 11].

Theorem 5.5. Let $\Omega \subset \mathbb{R}^1$ and $(\nu, w) \in H^{-1}(\Omega) \times H_0^1(\Omega)$ be given. Then, $(\nu, w) \in \mathcal{N}_{\mathbb{K}}^{\mathrm{M}}(\bar{y}, \bar{\lambda})$ is equivalent to the satisfaction of the system

$$\langle \nu,\, z \rangle = 0 \text{ for all } z \in H_0^1(\Omega), z = 0 \text{ on } \mathcal{A}, \tag{5.7a}$$
$$w(s) = 0 \text{ for all } s \in \operatorname{supp}(\bar{\lambda}), \tag{5.7b}$$
$$\langle \nu,\, z \rangle = 0 \text{ for all open intervals } U \subset \{w < 0\} \text{ and } z \in H_0^1(\Omega), z = 0 \text{ on } \Omega \setminus U, \tag{5.7c}$$
$$\langle \nu,\, z \rangle \le 0 \text{ for all open intervals } U \subset \{w > 0\} \text{ and } z \in H_0^1(\Omega)_+, z = 0 \text{ on } \Omega \setminus U. \tag{5.7d}$$

Here, $\operatorname{supp}(\bar{\lambda})$ is the support of the measure $\bar{\lambda}$.

Note that in case $d = 1$, we have $\operatorname{supp}(\bar{\lambda}) = \text{f-supp}(\bar{\lambda})$.

Proof. "$\Rightarrow$": Let $(\nu, w) \in \mathcal{N}_{\mathbb{K}}^{\mathrm{M}}(\bar{y}, \bar{\lambda})$ be given. The first two conditions (5.7a) and (5.7b) follow from $(\nu, w) \in \mathcal{N}_{\mathbb{K}}^{\mathrm{M}}(\bar{y}, \bar{\lambda}) \subset \mathcal{N}_{\mathbb{K}}^{\mathrm{weak}}(\bar{y}, \bar{\lambda})$ and the embedding $H_0^1(\Omega) \hookrightarrow C_0(\Omega)$. Now, let $\mathcal{B} = \hat{\mathcal{I}} \cup \hat{\mathcal{B}} \cup \hat{\mathcal{A}}_s$ be the decomposition of the biactive set associated with $(\nu, w) \in \mathcal{N}_{\mathbb{K}}^{\mathrm{M}}(\bar{y}, \bar{\lambda})$.

Let an open intervals $U \subset \{w < 0\}$ and $z \in H_0^1(\Omega)$ with $z = 0$ on $\Omega \setminus U$ be given. Since $\{w < 0\} \subset \mathcal{I} \cup \hat{\mathcal{I}}$, we have $\pm z \in \hat{\mathcal{K}}(\hat{\mathcal{B}}, \hat{\mathcal{A}}_s)$, hence $\langle \nu, z\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} = 0$. Thus, (5.7c) holds.

Finally, if $U \subset \{w > 0\}$ is an open interval and $z \in H_0^1(\Omega)_+$ with $z = 0$ on $\Omega \setminus U$, we find $z \in \hat{\mathcal{K}}(\hat{\mathcal{B}}, \hat{\mathcal{A}}_s)$ due to $\{w > 0\} \subset \mathcal{I} \cup \hat{\mathcal{I}} \cup \hat{\mathcal{B}}$. Hence, $\langle \nu, z\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} \le 0$. This shows (5.7d).

"$\Leftarrow$": Let $(\nu, w) \in H^{-1}(\Omega) \times H_0^1(\Omega)$ be given, such that the system (5.7) is satisfied. The first two conditions imply $(\nu, w) \in \mathcal{N}_{\mathbb{K}}^{\mathrm{weak}}(\bar{y}, \bar{\lambda})$. Now, we define the sets

$$\hat{\mathcal{I}} := \{w < 0\} \cap \{\bar{y} = 0\}, \quad \hat{\mathcal{B}} := \{w > 0\} \cap \{\bar{y} = 0\}, \quad \hat{\mathcal{A}}_s := \{w = 0\} \cap \{\bar{y} = 0\} \cap \mathcal{B}.$$

As in [G. Wachsmuth, 2016b, Section 5], we can check that this is a decomposition of $\mathcal{B}$, and $w \in \hat{\mathcal{K}}(\hat{\mathcal{B}}, \hat{\mathcal{A}}_s)$ follows directly from this definition.

It remains to verify $\nu \in \hat{\mathcal{K}}(\hat{\mathcal{B}}, \hat{\mathcal{A}}_s)^\circ$. To this end, let $z \in \hat{\mathcal{K}}(\hat{\mathcal{B}}, \hat{\mathcal{A}}_s)$ be given. Then, $\{z^- > 0\} \subset \mathcal{I} \cup \hat{\mathcal{I}}$. Note that $\mathcal{U} := \mathcal{I} \cup \hat{\mathcal{I}} = \{\bar{y} > 0\} \cup \{w < 0\}$ is open. We denote by $\{V_i\}_{i\in\mathbb{N}}$ the (at most countably many) connected components of $\{w < 0\}$. Since $z^- \in H_0^1(\mathcal{U})$, we find a sequence $\{z_n\}_{n\in\mathbb{N}} \subset C_c^\infty(\mathcal{U})_+$ such that $z_n \to z^-$ in $H_0^1(\Omega)$. Since z_n is compactly supported, we can find a finite set $I_n \subset \mathbb{N}$ such that $\{\mathcal{I}\} \cup \{V_i\}_{i\in I_n}$ covers the support of z_n. Using the usual partition-of-unity argument, we obtain the partition

$$z_n = \varphi_n + \sum_{i\in I_n} \psi_{i,n} \quad \text{such that} \quad \varphi_n \in C_c^\infty(\mathcal{I})_+,\ \psi_{i,n} \in C_c^\infty(V_i)_+ \ \forall i \in I_n.$$

Further, $\langle \nu,\, \varphi_n\rangle = 0$ by (5.7a) and $\langle \nu,\, \psi_{i,n}\rangle = 0$ follows from (5.7d). This implies $\langle \nu,\, z^-\rangle = \lim_{n\to\infty}\langle \nu,\, z_n\rangle = 0$. Similarly, we can argue for z^+, which lives on $\mathcal{I} \cup \hat{\mathcal{I}} \cup \hat{\mathcal{B}} = \{\bar{y} > 0\} \cup \{w < 0\} \cup \{w > 0\}$. By using (5.7a), (5.7c) and (5.7d), this yields $\langle \nu, z\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} \le 0$. This finishes the proof of $\nu \in \hat{\mathcal{K}}(\hat{\mathcal{B}}, \hat{\mathcal{A}}_s)^\circ$.

5.6 Strong stationarity

Finally, we review the system of strong stationarity for (1.1). In [Mignot, 1976, Proposition 4.1] it has been shown that every local solution of (1.1) with $U_{\text{ad}} = L^2(\Omega)$ satisfies the optimality system (1.3) with $\mathcal{N}_{\mathbb{K}}^{\sharp}(\bar{y}, \bar{\lambda})$ replaced by by

$$\begin{aligned} \mathcal{N}_{\mathbb{K}}^{\text{strong}}(\bar{y}, \bar{\lambda}) := \{w \in H_0^1(\Omega) \mid w \geq 0 \text{ q.e. on } \mathcal{A} \text{ and } \langle w,\, \bar{\lambda}\rangle = 0\}^\circ \\ \times \{w \in H_0^1(\Omega) \mid w \geq 0 \text{ q.e. on } \mathcal{A} \text{ and } \langle w,\, \bar{\lambda}\rangle = 0\}. \end{aligned} \tag{5.8}$$

In view of (3.4) and (3.5), we have

$$\mathcal{N}_{\mathbb{K}}^{\text{strong}}(\bar{y}, \bar{\lambda}) = \mathcal{K}_K(\bar{y}, \bar{\lambda})^\circ \times \mathcal{K}_K(\bar{y}, \bar{\lambda})$$

and due to (2.1),

$$\mathcal{N}_{\mathbb{K}}^{\text{strong}}(\bar{y}, \bar{\lambda}) = \widehat{\mathcal{N}}_{\mathbb{K}}(\bar{y}, \bar{\lambda}) \tag{5.9}$$

follows. These equalities show that strong stationarity in the sense of Mignot for (1.1) coincides with the notion of strong stationarity (4.9)–(4.10) defined for abstract problems with complementarity constraints in Section 4.2. We also mention that the critical cone in the equation above can be evaluated via (3.4).

As already said, Mignot's technique needs $U_{\text{ad}} = L^2(\Omega)$ and this condition can be slightly weakened to the requirement that $\mathcal{T}_{U_{\text{ad}}}(\bar{u})$ is dense in $H^{-1}(\Omega)$. The case with box constraints was considered in [G. Wachsmuth, 2017]. In particular, under conditions on J and on the control bounds, which can be checked a-priori, one obtains that all local minimizers of (1.1) are strongly stationary. However, the counterexamples in [G. Wachsmuth, 2017, Section 6] show that this condition cannot be dropped and local minimizers of (1.1) are, in general, not strongly stationary if control bounds are present. Note that the minimizers in these counterexamples are still M-stationary in the sense of Section 5.3.

5.7 Almost everywhere versus quasi-everywhere

In this section, we consider the case that the multiplier $\bar{\lambda}$ of the obstacle problem enjoys the increased regularity $\bar{\lambda} \in L^2(\Omega)$. This holds, e.g., if $U_{\text{ad}} \subset L^{\max(2,s)}(\Omega)$ and $f \in L^{\max(2,s)}(\Omega)$ for some $s > d$, see [Kinderlehrer, Stampacchia, 1980, Theorem IV.2.3]. Now, since $\bar{\lambda}$ is a function, we can introduce the set

$$\hat{\mathcal{A}}_s := \{\bar{\lambda} < 0\}. \tag{5.10}$$

Note that, in difference to $\mathcal{A}_s$, $\hat{\mathcal{A}}_s$ is only defined up to a set of measure zero.

Under this increased regularity, one can find many contributions in the literature in which a system of strong stationarity is defined via using

$$\begin{aligned} \mathcal{N}_{\mathbb{K}}^{\text{ae-strong}}(\bar{y}, \bar{\lambda}) := \{w \in H_0^1(\Omega) \mid w \geq 0 \text{ a.e. on } \mathcal{A} \text{ and } w = 0 \text{ a.e. on } \hat{\mathcal{A}}_s\}^\circ \\ \times \{w \in H_0^1(\Omega) \mid w \geq 0 \text{ a.e. on } \mathcal{A} \text{ and } w = 0 \text{ a.e. on } \hat{\mathcal{A}}_s\} \end{aligned} \tag{5.11}$$

in (1.3). Note that this definition includes sign condition in the almost everywhere sense, whereas (5.8) includes similar sign conditions in the quasi-everywhere sense, and this is a crucial difference. It is easy to check that

$$\begin{aligned}&\{w \in H_0^1(\Omega) \mid w \geq 0 \text{ a.e. on } \mathcal{A} \text{ and } w = 0 \text{ a.e. on } \hat{\mathcal{A}}_s\}\\ &\qquad \supset \{w \in H_0^1(\Omega) \mid w \geq 0 \text{ q.e. on } \mathcal{A} \text{ and } w = 0 \text{ a.e. on } \hat{\mathcal{A}}_s\}\\ &\qquad = \{w \in H_0^1(\Omega) \mid w \geq 0 \text{ q.e. on } \mathcal{A} \text{ and } w = 0 \text{ q.e. on } \mathcal{A}_s\}\end{aligned}$$

and the inclusion is, in general, strict. Since we have the reverse inclusion for the polar sets, $\mathcal{N}_{\mathbb{K}}^{\text{ae-strong}}(\bar{y}, \bar{\lambda})$ and $\mathcal{N}_{\mathbb{K}}^{\text{strong}}(\bar{y}, \bar{\lambda})$ are, in general, not comparable.

Finally, we prove by means of a counterexample that (5.11) cannot be used to provide a necessary optimality condition for (1.1). We consider the one-dimensional problem

$$\begin{aligned}\text{minimize} \quad & |y(1/2) + 1|^2 + \frac{1}{2}\|u\|_{L^2(\Omega)}^2,\\ \text{such that} \quad & -\Delta y + \lambda = u + f,\\ & (y, \lambda) \in \mathbb{K}.\end{aligned}$$

where $\Omega = (0, 1)$ and

$$f(x) = 12\left(x - \frac{1}{2}\right)^2 - \frac{1}{2}.$$

It is easy to check that the global solution of this problem is given by

$$\bar{u} = \bar{\lambda} = 0, \qquad \bar{y}(x) = x\,(1 - x)\left(\frac{1}{2} - x\right)^2.$$

Hence, $\mathcal{A} = \{1/2\}$ and $\mathcal{A}_s = \hat{\mathcal{A}}_s = \emptyset$. Now, the system (1.3) reads

$$\begin{aligned}2\,\delta_{1/2} + \nu - \Delta p &= 0, & \mu &\in \mathcal{N}_{U_{\text{ad}}}(\bar{u}) = \{0\},\\ \mu - p &= 0, & (\nu, -p) &\in \mathcal{N}_{\mathbb{K}}^{\sharp}(\bar{y}, \bar{\lambda}),\end{aligned}$$

where $\delta_{1/2}$ is the Dirac measure at $1/2$. Note that this directly implies $p = 0$ and $\nu = -2\,\delta_{1/2}$. Finally, it can be checked that

$$(-2\,\delta_{1/2}, 0) \in \mathcal{N}_{\mathbb{K}}^{\text{strong}}(\bar{y}, \bar{\lambda}), \qquad (-2\,\delta_{1/2}, 0) \notin \mathcal{N}_{\mathbb{K}}^{\text{ae-strong}}(\bar{y}, \bar{\lambda}).$$

Hence, the system (1.3) with $\mathcal{N}_{\mathbb{K}}^{\text{ae-strong}}(\bar{y}, \bar{\lambda})$ instead of $\mathcal{N}_{\mathbb{K}}^{\sharp}(\bar{y}, \bar{\lambda})$ is not a necessary optimality system.

5.8 Comparison

Finally, we comment on the known relation between all of the introduced replacements for $\mathcal{N}_{\mathbb{K}}^{\sharp}(\bar{y}, \bar{\lambda})$, except the almost-everywhere variants from Section 5.7.

Theorem 5.6. Let $(\bar y, \bar\lambda) \in \mathbb{K}$ be given. Then, the inclusions

$$\mathcal{N}_{\mathbb{K}}^{\text{strong}}(\bar y, \bar\lambda) \subset \mathcal{N}_{\mathbb{K}}^{\text{M}}(\bar y, \bar\lambda) \subset \mathcal{N}_{\mathbb{K}}^{\text{C}}(\bar y, \bar\lambda) \subset \mathcal{N}_{\mathbb{K}}^{\text{weak}}(\bar y, \bar\lambda),$$
$$\mathcal{N}_{\mathbb{K}}^{\text{strong}}(\bar y, \bar\lambda) \subset \mathcal{N}_{\mathbb{K}}^{\text{M}}(\bar y, \bar\lambda) \subset \mathcal{N}_{\mathbb{K}}^{\text{lim}}(\bar y, \bar\lambda) \subset \mathcal{N}_{\mathbb{K}}^{\text{weak}}(\bar y, \bar\lambda)$$

hold.

Proof. The first chain of inclusions follows trivially from the definitions. The second chain follows from the identity $\mathcal{N}_{\mathbb{K}}^{\text{weak}}(\bar y, \bar\lambda) = \mathcal{N}_{\mathbb{K}}^{\text{Clarke}}(\bar y, \bar\lambda)$, see Theorem 5.2, the inclusion $\mathcal{N}_{\mathbb{K}}^{\text{lim}}(\bar y, \bar\lambda) \subset \mathcal{N}_{\mathbb{K}}^{\text{Clarke}}(\bar y, \bar\lambda)$, which follow from the definition, see Section 2, and Theorem 5.3.

In the one-dimensional case $d = 1$, we further have

$$\mathcal{N}_{\mathbb{K}}^{\text{lim}}(\bar y, \bar\lambda) = \mathcal{N}_{\mathbb{K}}^{\text{M}}(\bar y, \bar\lambda),$$

see Theorem 5.4. Hence, both chains in Theorem 5.6 can be combined to a single one. However, for $d \geq 2$ the situation is less clear, cf. Section 5.4.

Similar to the finite-dimensional case, one can check that all stationarity system coincides if the biactive set vanishes (in the sense of capacity).

Theorem 5.7. If $\operatorname{cap}(\mathcal{B}) = 0$, then all inclusions in Theorem 5.6 hold with equality. In case $\operatorname{cap}(\mathcal{B}) > 0$, we have

$$\mathcal{N}_{\mathbb{K}}^{\text{strong}}(\bar y, \bar\lambda) \subsetneq \mathcal{N}_{\mathbb{K}}^{\text{M}}(\bar y, \bar\lambda) \subsetneq \mathcal{N}_{\mathbb{K}}^{\text{C}}(\bar y, \bar\lambda) \subsetneq \mathcal{N}_{\mathbb{K}}^{\text{weak}}(\bar y, \bar\lambda).$$

If even $\operatorname{meas}(\mathcal{B}) > 0$ and $d \geq 2$, then

$$\mathcal{N}_{\mathbb{K}}^{\text{M}}(\bar y, \bar\lambda) \subsetneq \mathcal{N}_{\mathbb{K}}^{\text{lim}}(\bar y, \bar\lambda), \quad \mathcal{N}_{\mathbb{K}}^{\text{C}}(\bar y, \bar\lambda) \not\supset \mathcal{N}_{\mathbb{K}}^{\text{lim}}(\bar y, \bar\lambda).$$

Proof. First, we consider the case $\operatorname{cap}(\mathcal{B}) = 0$. It is sufficient to show

$$\mathcal{N}_{\mathbb{K}}^{\text{strong}}(\bar y, \bar\lambda) = \mathcal{N}_{\mathbb{K}}^{\text{weak}}(\bar y, \bar\lambda)$$

and this follows directly from the definition.

Now, let $\operatorname{cap}(\mathcal{B}) > 0$.

$\mathcal{N}_{\mathbb{K}}^{\text{strong}}(\bar y, \bar\lambda) \subsetneq \mathcal{N}_{\mathbb{K}}^{\text{M}}(\bar y, \bar\lambda)$: Since $\mathcal{I} \cup \mathcal{B}$ is quasi-open, there exists $p \in H_0^1(\Omega)_+$ with $\{p > 0\} = \mathcal{I} \cup \mathcal{B}$, see Lemma 3.7. Now, it is easy to check $(0, -p) \in \mathcal{N}_{\mathbb{K}}^{\text{M}}(\bar y, \bar\lambda)$ by choosing $\hat{\mathcal{I}} = \mathcal{B}$, $\hat{\mathcal{B}} = \hat{\mathcal{A}}_s = \emptyset$ and $(0, -p) \notin \mathcal{N}_{\mathbb{K}}^{\text{strong}}(\bar y, \bar\lambda)$ since $-p < 0$ q.e. on $\mathcal{B}$.

$\mathcal{N}_{\mathbb{K}}^{\text{M}}(\bar y, \bar\lambda) \subsetneq \mathcal{N}_{\mathbb{K}}^{\text{C}}(\bar y, \bar\lambda)$: As in the previous case, we choose $p \in H_0^1(\Omega)_+$ with $\{p > 0\} = \mathcal{I} \cup \mathcal{B}$. Moreover, Lemma 3.10 implies the existence of a non-zero $\nu \in H^{-1}(\Omega)_+$

with f-supp$(\nu) \subset \mathcal{B}$. Now, it is easy to check that $(\nu, -p) \in \mathcal{N}_{\mathbb{K}}^{\mathrm{C}}(\bar{y}, \bar{\lambda})$ and we claim $(\nu, -p) \notin \mathcal{N}_{\mathbb{K}}^{\mathrm{M}}(\bar{y}, \bar{\lambda})$. Indeed, $-p \in \hat{\mathcal{K}}(\hat{\mathcal{B}}, \hat{\mathcal{A}}_s)$ requires $\hat{\mathcal{I}} = \mathcal{B}$, $\hat{\mathcal{B}} = \hat{\mathcal{A}}_s = \emptyset$ (all up to zero capacity). But then, $\nu \in \hat{\mathcal{K}}(\emptyset, \emptyset)^\circ$ implies f-supp$(\nu) \subset \mathcal{A}_s$ which does not hold.

$\mathcal{N}_{\mathbb{K}}^{\mathrm{C}}(\bar{y}, \bar{\lambda}) \subsetneq \mathcal{N}_{\mathbb{K}}^{\mathrm{weak}}(\bar{y}, \bar{\lambda})$: We choose p and ν as in the previous case. Then, $(-\nu, -p) \in \mathcal{N}_{\mathbb{K}}^{\mathrm{weak}}(\bar{y}, \bar{\lambda})$ is clear. Now, if we use $\varphi = \chi_\Omega \in W^{1,\infty}(\Omega)$ in the definition of $\mathcal{N}_{\mathbb{K}}^{\mathrm{C}}(\bar{y}, \bar{\lambda})$, we have

$$\langle -\nu, -p\,\varphi\rangle = \int_\Omega p \,\mathrm{d}\nu > 0$$

since p does not vanish on the fine support of ν. Hence, $(-\nu, -p) \notin \mathcal{N}_{\mathbb{K}}^{\mathrm{C}}(\bar{y}, \bar{\lambda})$

Finally, let us assume that meas$(\mathcal{B}) > 0$. As before, we choose $p \in H_0^1(\Omega)_+$ with $\{p > 0\} = \mathcal{I} \cup \mathcal{B}$. We set $\nu = \chi_{\mathcal{B}}$. Then, $(-\nu, -p) \in \mathcal{N}_{\mathbb{K}}^{\mathrm{weak}}(\bar{y}, \bar{\lambda})$, hence, $(-\nu, -p) \in \mathcal{N}_{\mathbb{K}}^{\lim}(\bar{y}, \bar{\lambda})$ by (5.6). However, $(-\nu, -p) \notin \mathcal{N}_{\mathbb{K}}^{\mathrm{C}}(\bar{y}, \bar{\lambda})$ can be checked as above.

6 Conclusion

The theory for optimality conditions for the finite-dimensional problem (4.1) is well understood. The systems of weak and strong stationarity can be transferred to the optimal control of the obstacle problem by using some notions of capacity theory. The situation is different for the intermediate concepts of C-stationarity and M-stationarity. In particular, in dimensions $d \geq 2$, there is a discrepancy between the notion of M-stationarity as introduced in Section 5.3 and the stationarity concept involving the limiting normal cone, see Section 5.4. Finally, we mention two open questions.

- Do we have $\mathcal{N}_{\mathbb{K}}^{\lim}(\bar{y}, \bar{\lambda}) = \mathcal{N}_{\mathbb{K}}^{\mathrm{weak}}(\bar{y}, \bar{\lambda})$ in dimension $d \geq 2$?
- Is every local minimizer of (1.1) M-stationary in the sense of Section 5.3 if $d \geq 2$?

A first step towards answering the first question is the equivalence (5.6) and from the results of [G. Wachsmuth, 2017, Section 5], one could expect that the answer to the second question is affirmative.

Another open question concerning the fine support was raised after the proof of Theorem 5.3.

- Given a quasi-closed set $A \subset \Omega$. Does there exist $\xi \in H^{-1}(\Omega)_+$ with $A =$ f-supp(ξ)?

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