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The Multiplier-Penalty Method for Generalized Nash Equilibrium Problems in Banach Spaces*

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Abstract

This paper deals with generalized Nash equilibrium problems (GNEPs) in Banach spaces. We prove an existence result for normalized equilibria of jointly convex GNEPs and then propose an augmented Lagrangian-type algorithm for their computation. A thorough convergence analysis is conducted which considers the existence of subproblem solutions as well as feasibility and optimality of limit points. We then apply our investigations to a class of multiobjective optimal control problems which are governed by a linear partial differential equation and provide some numerical results to demonstrate the performance of the method.

1 Introduction

We consider the following generalized Nash equilibrium problem (GNEP). Let $N \in \mathbb{N}$ be the number of players, each in control of a variable $x^\nu \in X_\nu$, where X_ν is a (real) Banach space. We write $X := X_1 \times \ldots \times X_N$ for the strategy space of all players. In the following, $x^{-\nu}$ denotes the strategies of all players except the ν-th player, and $X_{-\nu}$ the corresponding strategy space. We use the notations $x = (x^\nu, x^{-\nu})$ and $X = X_\nu \times X_{-\nu}$ to emphasize the role of player ν's variable x^ν. Each player ν attempts to solve the optimization problem

$$\underset{x^\nu \in X_\nu}{\text{minimize}}\ \theta_\nu(x^\nu, x^{-\nu}) \quad \text{subject to} \quad (x^\nu, x^{-\nu}) \in \mathcal{F}. \tag{1}$$

Here, $\theta_\nu : X \to \mathbb{R}$ denotes the objective or utility function of player ν and $\mathcal{F} \subseteq X$ is a nonempty closed convex set. We will also assume that the objective functions $\theta_\nu(\cdot, x^{-\nu})$

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are convex and continuously differentiable for any given $x^{-\nu}$; in this setting, the GNEP (1) is usually called *jointly convex*. Note that if $\mathcal{F} = \mathcal{F}_1 \times \ldots \times \mathcal{F}_N$ for closed convex sets $\mathcal{F}_\nu \subseteq X_\nu$, then (1) reduces to the standard Nash equilibrium problem (NEP) where each player attempts to solve

$$\underset{x^\nu \in X_\nu}{\text{minimize}}\ \theta_\nu(x^\nu, x^{-\nu}) \quad \text{subject to} \quad x^\nu \in \mathcal{F}_\nu. \tag{2}$$

In the past years a substantial amount of research has been conducted for GNEPs in finite dimensions, particularly for the jointly convex case. For a broad overview of the finite-dimensional case we refer the reader to the survey papers [10, 12]. However, until now there is not much literature about infinite-dimensional GNEPs and their applications. In [6] Carlson extended the work of Rosen [30] to infinite dimensional games with strategies in Banach spaces. He provided conditions for existence and uniqueness of normalized Nash equilibriums.

Some work has been done for standard NEPs in certain specific problem settings [5, 27, 28, 29, 31, 32, 33]. For GNEPs, we are only aware of the papers [8, 13, 14], the latter two of which are confined to an optimal control setting. It is not difficult to see that GNEPs of this type can be reformulated to fit into our general framework (1). In the optimal control setting, each player has a control variable and the players' problems are coupled via the (joint) state variable which is the unique solution of a linear PDE. Working in the reduced formulation, we can simply define $\mathcal{F}$ as the set of all admissible controls where the corresponding state satisfies the given state constraints (if any), see also Section 6.

To the best of our knowledge, the only paper which considers "generic" jointly convex GNEPs in infinite dimensions is [8], where a relaxation method is presented. The aim of the present paper is to discuss some theoretical background on GNEPs in infinite dimensions and to provide an alternative algorithm. Our main approach is to apply an augmented Lagrangian (or multiplier-penalty) scheme to eliminate some or all of the constraints in (1) and therefore reduce the GNEP to a sequence of "easier" problems. This idea is not completely new: in [18], an augmented Lagrangian method was presented for finite-dimensional GNEPs. Furthermore, in [20] an augmented Lagrangian method for optimization problems in Banach spaces was considered. Motivated by the promising results of these two works, we want to combine them and extend the given approaches to infinite-dimensional GNEPs in a general setting.

This paper is organized as follows. In Section 2, we deal with some preliminary results like existence of solutions of the GNEP and state the KKT-conditions of the considered problem. Section 3 contains a detailed description of our proposed multiplier-penalty method. Section 4 is dedicated to the convergence analysis, where we cover solvability of the arising penalized subproblems as well as feasibility and Nash optimality of every weak limit point generated by the presented algorithm. Section 5 leads us to strong convergence of the primal iterates and weak-* convergence of the corresponding multiplier sequence under certain regularity assumptions. Since some well-known applications of jointly convex GNEPs are given in the optimal control setting, a description how these type of problems fit in our convergence analysis is given Section 6, where we also include some numerical results. Section 7 contains some final remarks.

Notation: Throughout the paper, X and Y are Banach spaces, and we denote strong and weak convergence by $\to$ and $\rightharpoonup$, respectively. Moreover, duality pairings are written as $\langle \cdot, \cdot \rangle$, scalar products in Hilbert spaces as $(\cdot, \cdot)$, and norms are denoted by $\| \cdot \|$ with an appropriate subscript to emphasize the corresponding space (e.g. $\| \cdot \|_X$). If S is a nonemtpy closed convex subset of a Hilbert space, we denote by P_S and d_S the projection and distance function to S, respectively. The partial derivative with respect to x^ν is denoted by D_{x^ν}.

2 Preliminaries

This section is dedicated to the existence of solutions of the jointly convex GNEP and its KKT conditions. Before we analyze these two topics we want to recall the definitions of generalized Nash equilibria and normalized equilibria first.

2.1 Equilibria and Normalized Equilibria

For the definition of the two types of equilibria, recall that $\mathcal{F}$ is the joint constraint set of all players. For a given point $x^{-\nu} \in X_{-\nu}$, we write

$$\mathcal{F}_\nu(x^{-\nu}) = \{x^\nu \in X_\nu : (x^\nu, x^{-\nu}) \in \mathcal{F}\}$$

for the feasible set of player ν's optimization problem. Note that this set might be empty for some (or many) $x^{-\nu}$. If we are dealing with a standard NEP, then $\mathcal{F}_\nu$ is independent of $x^{-\nu}$, see (2). Hence, in that case, $\mathcal{F}_\nu$ is always nonempty.

Definition 2.1. *Let $\bar{x} \in \mathcal{F}$ be a feasible point. We say that $\bar{x}$ is a*

(a) generalized Nash equilibrium *or simply a* solution *of the GNEP if, for every ν,*

$$\theta_\nu(\bar{x}^\nu, \bar{x}^{-\nu}) \le \theta_\nu(y^\nu, \bar{x}^{-\nu}) \quad \text{for all } y^\nu \in \mathcal{F}_\nu(\bar{x}^{-\nu}). \tag{3}$$

(b) normalized (Nash) equilibrium *if*

$$\sum_{\nu=1}^N \theta_\nu(\bar{x}^\nu, \bar{x}^{-\nu}) \le \sum_{\nu=1}^N \theta_\nu(y^\nu, \bar{x}^{-\nu}) \quad \text{for all } y \in \mathcal{F}. \tag{4}$$

Note that every normalized equilibrium is also a generalized Nash equilibrium, which can be seen by inserting points of the form $y := (y^\nu, \bar{x}^{-\nu})$ into (4). The converse however is not true in general. For NEPs, both concepts are equivalent.

The existence of Nash equilibria is a rather delicate topic, even when restricted to standard NEPs or finite-dimensional problems. Most existence results [1, 3, 10, 25] assume

(i) compactness of the set $\mathcal{F}$, and

(ii) appropriate continuity of the functions θ_ν.

In the infinite-dimensional setting, condition (i) effectively forces us to work in the weak topology of the underlying space; however, this introduces certain problems with condition (ii), since few functions are actually continuous in the weak topology. Note that (weak) lower semicontinuity of θ_ν with respect to x^ν is not enough for existence: consider, for instance, the NEP given by

$$\theta_1(x,y) := \begin{cases} x^2 & \text{if } y < 1 \\ -x & \text{if } y = 1 \end{cases}, \quad \theta_2(x,y) := \frac{1}{2}y^2 + (x-1)y, \quad \mathcal{F} := [0,1]^2,$$

where x, y are the respective player variables. Then both objective functions are continuous and convex with respect to the corresponding variable, but θ_1 is not continuous with respect to y. It is easily verified that this problem does not admit a Nash equilibrium.

On the other hand, the compactness of $\mathcal{F}$ is also hard to relax, even in seemingly "good" cases. For instance, the unconstrained NEP [10, Example 4.5] given by

$$\theta_1(x,y) := \frac{1}{2}x^2 - xy, \quad \theta_2(x,y) := \frac{1}{2}y^2 - (x+1)y,$$

where x, y again are the respective player variables, does not admit a Nash equilibrium, even though both objective functions are strongly convex, and uniformly so with respect to the rival's variable.

It follows from the above observations that a careful approach to the existence of Nash equilibria is necessary in our setting. To this end, we consider the existence of normalized equilibria, and define the Nikaido-Isoda (NI) function [26]

$$\Psi(x,y) := \sum_{\nu=1}^{N} \left[\theta_\nu(x^\nu, x^{-\nu}) - \theta_\nu(y^\nu, x^{-\nu})\right]. \tag{5}$$

It is evident that a point $\bar{x} \in \mathcal{F}$ is a normalized equilibrium if and only if

$$\Psi(\bar{x}, y) \le 0 \quad \forall y \in \mathcal{F}. \tag{6}$$

which is equivalent to $\bar{x}$ being a solution of the maximization problem

$$\max_y \ \Psi(\bar{x}, y) \quad \text{s.t.} \quad y \in \mathcal{F} \tag{7}$$

Problems of the type (6) are usually referred to as equilibrium problems [16, 17]. Taking into account the existence theory for equilibrium problems, we are prompted to make the following assumption.

Assumption 2.2. *The Nikaido-Isoda function* (5) *is weakly lower semicontinuous with respect to x.*

Before we discuss this assumption, let us first give a direct consequence which is an existence theorem for normalized equilibria.

Theorem 2.3. *Let Assumption 2.2 hold and assume that $\mathcal{F}$ is nonempty and weakly compact. Then the GNEP admits a normalized equilibrium.*

Proof. This follows from the Ky-Fan theorem, cf. [11, Thm. 1] or [17, Thm. 1.1]. $\square$

The assumption that Ψ is weakly lower semicontinuous (lsc) with respect to x arises naturally from the Ky-Fan theorem. However, in general, this is a nontrivial assumption due to the minus sign in (5). Clearly, a sufficient condition is the weak lower semicontinuity of the functions

$$x \mapsto \theta_\nu(x^\nu, x^{-\nu}) - \theta_\nu(y^\nu, x^{-\nu})$$

for all ν and fixed y^ν, which can be expected to hold in certain applications. It is also easy to verify that Assumption 2.2 is always satisfied in the optimal control framework from [13, 14]; some examples will be given in Section 6.

2.2 Cones, Convexity and the KKT Conditions

For a nonempty convex set S and a point x in some Banach space (e.g. X), we denote by

$$\mathcal{R}_S(x) = \{\alpha(s-x) \mid \alpha \geq 0,\ s \in S\}, \quad \mathcal{T}_S(x) = \overline{\mathcal{R}_S(x)}$$

the *radial cone* (or cone of feasible directions) and the *tangent cone* of S at x, respectively. Furthermore, we write

$$S^+ = \{\varphi \in X^* : \langle \varphi, s\rangle \geq 0\ \forall s \in S\}, \quad S^\circ = -S^+$$

for the dual and polar cones of S.

We now consider a special case of (1) where the constraint set $\mathcal{F}$ is given by

$$\mathcal{F} = \{x \in C : g(x) \in K\}. \tag{8}$$

Here, the function $g : X \to Y$ represents the joint constraints (i.e. the constraint that couples the players' individual strategies), Y is assumed to be a Banach space, and $K \subseteq Y$ is a nonempty closed convex cone. The set C denotes the players' individual constraints which are given by

$$C = C_1 \times \ldots \times C_N.$$

To make the feasible set $\mathcal{F}$ convex and the GNEP a *jointy convex* one, recall that the cone K induces the order relation

$$y \leq_K z :\Longleftrightarrow z - y \in K, \tag{9}$$

which allows us to extend various familiar concepts from finite-dimensional optimization theory to our setting. For instance, we say that g is concave if

$$g(\alpha x + (1-\alpha)y) \geq_K \alpha g(x) + (1-\alpha)g(y)$$

holds for all $x, y \in X$ and $\alpha \in [0, 1]$. Other notions which involve an order such as increasing, decreasing or convex functions are also defined in a straightforward way. For instance, the distance function $d_K : Y \to \mathbb{R}$ is decreasing since $z \geq_K y$ implies $z = y + k$, $k \in K$, and

$$d_K(z) = d_K(y + k) \leq \|y + k - (P_K(y) + k)\| = \|y - P_K(y)\| = d_K(y), \tag{10}$$

where the inequality uses the convexity of K. It turns out that concavity of g with respect to K is the appropriate condition to ensure the convexity of the set $\mathcal{F}$. This result along with some other useful observations is formulated in the following lemma.

Lemma 2.4. *Assume that $g : X \to Y$ is concave. If $m : Y \to \mathbb{R}$ is convex and decreasing, then $m \circ g$ is convex. In particular:*

(a) *The function $d_K \circ g : X \to \mathbb{R}$ is convex.*

(b) *If $\lambda \in K^\circ$, then $\langle \lambda, g \rangle : X \to \mathbb{R}$ is convex, where $\langle \lambda, g \rangle (x) := \langle \lambda, g(x) \rangle$.*

(c) *The set $\mathcal{F} = \{x \in C : g(x) \in K\}$ is convex.*

Proof. Let $x, y \in X$ and $x_\alpha = \alpha x + (1-\alpha)y$, $\alpha \in (0, 1)$. Then $g(x_\alpha) \geq_K \alpha g(x) + (1-\alpha)g(y)$ by the concavity of g. Applying m on both sides yields

$$m(g(x_\alpha)) \leq m(\alpha g(x) + (1 - \alpha)g(y)) \leq \alpha m(g(x)) + (1 - \alpha)m(g(y)),$$

where we used the monotonicity and the convexity of m. Hence, the real-valued mapping $m \circ g$ is convex in the usual sense. Assertion (a) now follows because d_K is decreasing (see above) and convex [2, Cor. 12.12]. Similarly, for (b), the function $y \mapsto \langle \lambda, y \rangle$ with $\lambda \in K^\circ$ is obviously a convex function, and it is decreasing because $\langle \lambda, k \rangle \leq 0$ for all $k \in K$. Finally, for (c), note that

$$M = \{x \in C : g(x) \in K\} = C \cap \{x \in X : d_K(g(x)) \leq 0\}.$$

The second set is a lower level set of the convex function $d_K \circ g$. Hence, $\mathcal{F}$ is convex. $\square$

We now discuss the KKT conditions of the GNEP.

Definition 2.5. *A tuple $(\bar{x}, \bar{\lambda}^1, \ldots, \bar{\lambda}^N) \in X \times (Y^*)^N$ is a KKT point of the GNEP if*

$$D_{x^\nu}\theta_\nu(\bar{x}) + (D_{x^\nu} g(\bar{x}))^* \, \bar{\lambda}^\nu \in \mathcal{T}_{C_\nu}(\bar{x}^\nu)^+,$$
$$\bar{x} \in \mathcal{F}, \quad \bar{\lambda}^\nu \in K^\circ, \quad \text{and} \quad \left\langle \bar{\lambda}^\nu, g(\bar{x}) \right\rangle = 0$$

hold for all ν.

The connection between the GNEP and its KKT conditions is well-known and essentially follows from the fact that the KKT system of the GNEP is just the concatenation of the KKT systems of each player. Since the player problems are convex, it follows that KKT points are always solutions of the GNEP. Moreover, if $\bar{x}$ is a solution of the GNEP and an appropriate constraint qualification is satisfied, then there are multipliers $\bar{\lambda}^1, \dots, \bar{\lambda}^N$ such that $(\bar{x}, \bar{\lambda}^1, \dots, \bar{\lambda}^N)$ is a KKT point of the GNEP. In this case, the joint constraint yields separate multipliers $\bar{\lambda}^\nu$ for each player.

For normalized equilibria (cf. Definition 2.1), it is possible to obtain a stronger notion of KKT points, cf. [6]. To this end, recall that $\bar{x}$ is a normalized equilibrium if and only if $\Psi(\bar{x}, y) \leq 0$ for all $y \in \mathcal{F}$ or, equivalently, if $\bar{x}$ solves the concave maximization problem (7). Assuming a suitable constraint qualification, which will be given later, for the set $\mathcal{F}$, this problem is equivalent to its KKT conditions, which are given by

$$-D_y \Psi(\bar{x}, \bar{x}) + g'(\bar{x})^* \bar{\lambda} \in \mathcal{T}_C(\bar{x})^+,$$
$$\bar{x} \in \mathcal{F}, \quad \bar{\lambda} \in K^\circ, \quad \text{and} \quad \langle \bar{\lambda}, g(\bar{x}) \rangle = 0,$$

where D_y is the derivative with respect to y. Recalling the definition of the NI function and the product form of the set C, we get $\mathcal{T}_C(\bar{x})^+ = \mathcal{T}_{C_1}(\bar{x}^1)^+ \times \ldots \times \mathcal{T}_{C_N}(\bar{x}^N)^+$ (see [24, Prop. 1.2]), and the first inclusion can be reformulated as

$$D_{x^\nu} \theta_\nu(\bar{x}) + D_{x^\nu} g(\bar{x})^* \bar{\lambda} \in \mathcal{T}_{C_\nu}(\bar{x}^\nu)^+$$

for all ν. In other words, $\bar{x}$ satisfies the KKT conditions from Definition 2.5 with $\bar{\lambda}^\nu := \bar{\lambda}$ for each ν, i.e. the multiplier is the same for every player.

3 The Multiplier-Penalty Method

The method which we present in this section aims to compute normalized equilibria of GNEPs whose constraint set has the form (8) with $g : X \to Y$ a concave operator. For the construction of the method, we assume that there is a (linear and continuous) embedding $e : Y \to H$ for some Hilbert space H, and that $K_H \subseteq H$ is a closed convex cone with $e^{-1}(K_H) = K$. Hence, we have

$$M = \{x \in C : g(x) \in K\} = \{x \in C : e(g(x)) \in K_H\}.$$

Moreover, we use the Moreau decomposition theorem, which can be stated as follows.

Lemma 3.1. *Every $y \in H$ can be uniquely written as $y = y_+ + y_-$ with $y_+ \in K_H$, $y_- \in K_H^\circ$, and $y_+ \perp y_-$. Moreover, we have $y_+ = P_{K_H}(y)$ and $y_- = P_{K_H^\circ}(y)$.*

In the following, $(\cdot)_-$ and $(\cdot)_+$ will always denote the projections from the Moreau decomposition. We now turn to the multiplier-penalty method for the GNEP (1). The main idea of the method is to replace the (supposedly difficult) GNEP by a sequence of standard NEPs which include the constraint g within a penalty term. For the formal description of the method, we define the Lagrangian of player ν as

$$L^\nu : X \times H \to \mathbb{R}, \quad L^\nu(x, \lambda) = \theta_\nu(x) + (\lambda, g(x)), \tag{11}$$

and the corresponding augmented Lagrangian as

$$L^\nu_\rho : X \times H \to \mathbb{R}, \quad L^\nu_\rho(x, \lambda) = \theta_\nu(x) + \frac{\rho}{2} \left\| \left(g(x) + \frac{\lambda}{\rho} \right)_- \right\|_H^2. \tag{12}$$

Recall that $\|(\cdot)_-\|$ is the distance to K_H by the Moreau decomposition. Moreover, we note that there are other variants of L_ρ in the literature. However, these differ from (12) only by an additive constant (with respect to x).

For the definition of our penalty updating scheme, we also define the utility function

$$V(x, \lambda, \rho) = \left\| g(x) - \left(g(x) + \frac{\lambda}{\rho} \right)_+ \right\|_H . \tag{13}$$

This enables us to formulate our algorithm as follows.

Algorithm 3.2. *(Multiplier-penalty method)*

(S.0) Choose $(x_0, \lambda_0) \in X \times H$*, a bounded set* $B \subseteq K_H^\circ$*, parameters* $\rho_0 > 0$*,* $\gamma > 1$*,* $\tau \in (0, 1)$*, and set* $k := 0$*.*

(S.1) If (x_k, λ_k) *satisfies a suitable stopping criterion: STOP.*

(S.2) Choose $w_k \in B$ *and compute an approximate KKT point (see Assumption 3.3)* x_{k+1} *of the NEP consisting of the minimization problems*

$$\min_{x^\nu} \; L^\nu_{\rho_k}(x^\nu, x^{-\nu}, w_k) \quad \text{s.t.} \quad x^\nu \in C_\nu. \tag{14}$$

(S.3) Update the multiplier estimate to

$$\lambda_{k+1} := (w_k + \rho_k g(x_{k+1}))_- .$$

(S.4) If $k = 0$ *or*

$$V(x_{k+1}, w_k, \rho_k) \leq \tau V(x_k, w_{k-1}, \rho_{k-1}) \tag{15}$$

holds, set $\rho_{k+1} := \rho_k$*. Otherwise, set* $\rho_{k+1} := \gamma \rho_k$*.*

(S.5) Set $k \leftarrow k + 1$ *and go to (S.1).*

Some comments are due. First, note that we consider the case $k = 0$ separately in Step 4, since w_{k-1} and ρ_{k-1} are not defined for $k = 0$. This treatment has no influence on our convergence theory.

Secondly, let us emphasize the importance of the sequence (w_k) in the above method. It is best to think of w_k as a safeguarded analogue of λ_k whose boundedness is enforced by requiring that $w_k \in B$ for all k. This simple fact will be crucial for our convergence analysis. Note that similar bounding schemes have been used, e.g., for augmented Lagrangian methods in nonlinear optimization [4], and that the resulting algorithm possesses strictly stronger convergence properties than the classical augmented Lagrangian method (which uses the possibly unbounded sequence $w_k := \lambda_k$), see also the example in [19].

We now consider the subproblems occurring in Algorithm 3.2, which we refer to as the *augmented NEPs*. Note that we have not specified what constitutes an "approximate KKT point" in Step 2. Before we make this more precise, let us introduce the notation

$$L^\nu_k(x^\nu, x^{-\nu}) := L^\nu_{\rho_k}(x^\nu, x^{-\nu}, w_k)$$

for the utility function of player ν in the augmented NEP at iteration k. Therefore, x_{k+1}^{ν} should be an approximate KKT point of $L_k^{\nu}(\cdot, x_{k+1}^{-\nu})$. This is reflected in the following assumption.

Assumption 3.3. *We have $x_{k+1} \in C$ for all k and there is a zero sequence $(\varepsilon_k) \subseteq X^* = X_1^* \times \ldots \times X_N^*$ such that*

$$D_{x^{\nu}} L_k^{\nu}(x_{k+1}) \in \mathcal{T}_{C_{\nu}}(x_{k+1}^{\nu})^+ + \varepsilon_k^{\nu} \tag{16}$$

for all ν and k.

The above is a fairly natural assumption which basically asserts that x_{k+1} is an approximate stationary point of the subproblem, and the degree of inexactness vanishes as $k \to \infty$. Note that we assumed that each iterate x_{k+1} satisfies the additional constraint $x \in C$ exactly. This assumption is not strictly necessary for our analysis, but it is nevertheless convenient and usually satisfied in practice since the set C is assumed to consist of "simple" constraints.

4 Convergence to Nash Equilibria

We now analyze the convergence properties of Algorithm 3.2. In finite-dimensional optimization, a standard way of stating convergence theorems is to assert optimality for any accumulation point of the sequence of iterates. Since we are dealing with possibly infinite-dimensional spaces, it is more natural to consider the case of *weak* limit points. The resulting convergence theorems obviously cover the case where the sequence (x_k) has a strong limit point, since any such point is also a weak limit point.

Throughout this section, we will make extensive use of Assumption 2.2 which asserts the weak lower semicontinuity of the Nikaido-Isoda function with respect to x. As we will see, this condition is not only useful for the existence of equilibria, cf. Theorem 2.3, but also implies certain convergence properties for our Algorithm 3.2.

Before we proceed, recall that $g : X \to Y$ is assumed to be concave with respect to K. In the context of our multiplier-penalty method, we used the embedding $e : Y \to H$ into the Hilbert space H and the closed convex cone $K_H \subseteq H$ with $e^{-1}(K_H) = K$. In this setting, it is quite easy to see that concavity of g with respect to K is equivalent to concavity of $e \circ g$ with respect to K_H. This should be kept in mind when applying results related to convexity such as Lemma 2.4.

Before we proceed, we give an auxiliary result.

Lemma 4.1. *The functions $h_k(x) := \|(g(x) + w_k/\rho_k)_-\|_H^2$ are convex, continuously differentiable, and weakly lower semicontinuous.*

Proof. By Lemma 2.4 it is not difficult to see that h_k is continuous and convex, hence weakly lower semicontinuous [2, Thm. 9.1]. The continuous differentiability follows from [2, Cor. 12.30]. $\square$

4.1 Existence of Subproblem Solutions

In every iteration of Algorithm 3.2, we have to solve the augmented NEP for the given values w_k and ρ_k. Since the existence of Nash equilibria is not trivial in general (see also the example in Section 2.1), we want to state a result regarding this question.

Lemma 4.2. *Let Assumption 2.2 be satisfied. If C is weakly compact, then the augmented NEPs (14) admit solutions for all k.*

Proof. Let $k \in \mathbb{N}$ and define $h_k(x)$ as in Lemma 4.1. Consider now the function

$$\Psi_k(x, y) := \Psi(x, y) + \frac{\rho_k}{2}\Big[h_k(x) - h_k(y)\Big],$$

where Ψ is the usual NI function. Then Ψ_k is weakly lsc with respect to x in view of Assumption 2.2 and Lemma 4.1. Hence, as in Theorem 2.3, there is a point $\hat{x} \in C$ with $\Psi_k(\hat{x}, y) \leq 0$ for all $y \in C$. We claim that $\hat{x}$ is a solution of the penalized NEP (14). To this end, let μ be an arbitrary player index and let $y^\mu \in C_\mu$. With $y := (y^\mu, \hat{x}^{-\mu}) \in C$ we obtain

$$\begin{aligned}
0 \geq \Psi_k(\hat{x}, y) &= \sum_{\nu=1}^{N} \Big[\theta_\nu(\hat{x}^\nu, \hat{x}^{-\nu}) - \theta_\nu(y^\nu, \hat{x}^{-\nu})\Big] + \frac{\rho_k}{2}\Big[h_k(\hat{x}) - h_k(y)\Big] \\
&= \theta_\mu(\hat{x}^\mu, \hat{x}^{-\mu}) - \theta_\mu(y^\mu, \hat{x}^{-\mu}) + \frac{\rho_k}{2}\Big[h_k(\hat{x}) - h_k(y)\Big] \\
&= L_k^\mu(\hat{x}) - L_k^\mu(y).
\end{aligned}$$

This completes the proof. □

Lemma 4.2 yields the existence of solutions of the subproblem for a set C that is weakly compact. If C is not weakly compact (e.g. unbounded), then the existence of penalized Nash equilibria becomes much more complicated. In theory, an appropriate form of coercivity should yield the existence of solutions of the subproblems, but this is a rather delicate topic due to the involved nature of NEPs and GNEPs, see the discussion in Section 2 and in [10].

4.2 Convergence to Nash Equilibria

The aim of this section is to show feasibility and optimality of every weak limit point of the sequence (x_k) generated by Algorithm 3.2. We start by addressing feasibility.

Lemma 4.3. *Every weak limit point of (x_k) is feasible.*

Proof. Recall that $\|g_-\|_H$ measures the distance of $g(x)$ to K_H. Exploiting the properties of a distance function we get that $\|g_-\|_H$ is convex and continuous, hence weakly lower semicontinuous. Let us first consider the case where (ρ_k) remains bounded. The penalty updating scheme (15) yields

$$\left\| g(x_{k+1}) - \left(g(x_{k+1}) + \frac{w_k}{\rho_k} \right)_+ \right\|_H \to 0.$$

But $(g(x_{k+1}) + w_k/\rho_k)_+ \in K_H$, therefore $\|g_-(x_{k+1})\|_H = d_K(g(x_{k+1})) \to 0$, and the result follows. We now assume that $\rho_k \to \infty$, and define $h_k(x)$ as in Lemma 4.1. Let $x_{k+1} \rightharpoonup_{\mathcal{K}} \bar{x}$ for some $\mathcal{K} \subseteq \mathbb{N}$ and assume that $\bar{x}$ is infeasible, i.e. $\|g_-(\bar{x})\|_H > 0$. Since $\mathcal{F}$ is nonempty, there is a point $y \in \mathcal{F}$, and we get

$$\liminf_{k\in\mathcal{K}} \Big[h_k(x_{k+1}) - h_k(y)\Big] = \liminf_{k\in\mathcal{K}} h_k(x_{k+1}) = \liminf_{k\in\mathcal{K}} \|g_-(x_{k+1})\|_H^2 \geq \|g_-(\bar{x})\|_H^2 > 0.$$

Hence, there is a constant $c_1 > 0$ such that $h_k(x_{k+1}) - h_k(y) \geq c_1$ for all $k \in \mathcal{K}$ sufficiently large. Since h_k is convex and continuously differentiable by Lemma 4.1, it follows that

$$\langle h_k'(x_{k+1}), y - x_{k+1}\rangle \leq h_k(y) - h_k(x_{k+1}) \leq -c_1 \tag{17}$$

for all $k \in \mathcal{K}$ sufficiently large. Now, let (ε_k) be the sequence from Assumption 3.3. Then

$$\begin{aligned}
\langle \varepsilon_k, y - x_{k+1}\rangle &\leq \sum_{\nu=1}^{N} \big\langle D_{x^\nu} L_k^\nu(x_{k+1}), y^\nu - x_{k+1}^\nu \big\rangle \\
&= \sum_{\nu=1}^{N} \Big[D_{x^\nu}\theta_\nu(x_{k+1})(y^\nu - x_{k+1}^\nu)\Big] + \frac{\rho_k}{2}\langle h_k'(x_{k+1}), y - x_{k+1}\rangle \\
&\leq \sum_{\nu=1}^{N} \Big[\theta_\nu(y^\nu, x_{k+1}^{-\nu}) - \theta_\nu(x_{k+1})\Big] + \frac{\rho_k}{2}\langle h_k'(x_{k+1}), y - x_{k+1}\rangle \\
&= \frac{\rho_k}{2}\langle h_k'(x_{k+1}), y - x_{k+1}\rangle - \Psi(x_{k+1}, y),
\end{aligned}$$

where Ψ is the NI-function from (5). By Assumption 2.2, Ψ is weakly lsc with respect to the first argument; hence, there is a constant $c_2 \in \mathbb{R}$ such that $\Psi(x_{k+1}, y) \geq c_2$ for all $k \in \mathcal{K}$. This together with (17) implies

$$\langle \varepsilon_k, y - x_{k+1}\rangle \leq -\frac{\rho_k c_1}{2} - c_2 \to -\infty$$

and therefore contradicts $\varepsilon_k \to 0$. □

The feasibility of the iterates or their (weak) limit points is obviously a crucial issue for the success of Algorithm 3.2, since the method is essentially a penalty-type method. Before proving the optimality of weak limit points, we first need a technical lemma which will also be of use later on.

Lemma 4.4. *We have* $\liminf_{k\to\infty} \langle \lambda_{k+1}, g(x_{k+1})\rangle \geq 0$.

Proof. Using the Moreau decomposition, we obtain

$$\begin{aligned}
(\lambda_{k+1}, g(x_{k+1})) &= \frac{1}{\rho_k}(\lambda_{k+1}, w_k + \rho_k g(x_{k+1})) - \frac{1}{\rho_k}(\lambda_{k+1}, w_k) \\
&= \frac{1}{\rho_k}\Big[\|\lambda_{k+1}\|_H^2 - (\lambda_{k+1}, w_k)\Big].
\end{aligned} \tag{18}$$

Now, if (ρ_k) is bounded, then (15) implies

$$\left\| \left(g(x_{k+1}) + \frac{w_k}{\rho_k} \right)_- - \frac{w_k}{\rho_k} \right\|_H = \left\| g(x_{k+1}) - \left(g(x_{k+1}) + \frac{w_k}{\rho_k} \right)_+ \right\|_H \to 0.$$

But the latter is obviously equal to $\|\lambda_{k+1} - w_k\|_H / \rho_k$. Therefore, $\|\lambda_{k+1} - w_k\|_H \to 0$, which implies the boundedness of (λ_{k+1}) in H as well as $\|\lambda_{k+1}\|_H^2 - (\lambda_{k+1}, w_k) = (\lambda_{k+1}, \lambda_{k+1} - w_k) \to 0$. Hence, the desired result follows from (18). We now assume that $\rho_k \to \infty$. Note that (18) is a quadratic function in λ. A simple calculation therefore shows that

$$(\lambda_{k+1}, g(x_{k+1})) \geq -\frac{1}{4\rho_k} \|w_k\|_H^2.$$

This completes the proof. □

Exploiting the feasibility from Lemma 4.3 and the result from Lemma 4.4 we are now able to prove that every weak limit point of (x_k) is a normalized equilibrium of the GNEP.

Theorem 4.5. *Every weak limit point of (x_k) is a normalized equilibrium of the GNEP.*

Proof. Let $x_{k+1} \rightharpoonup_{\mathcal{K}} \bar{x}$ for some $\mathcal{K} \subseteq \mathbb{N}$ (recall that $\bar{x}$ is feasible by Lemma 4.3), and let $y \in \mathcal{F}$ be any point. An easy calculation shows that $D_{x^\nu} L_k^\nu(x_{k+1}) = D_{x^\nu} L^\nu(x_{k+1}, \lambda_{k+1})$ for all k. Since $y^\nu \in C_\nu$ for all ν, Assumption 3.3 implies

$$\begin{aligned} \left\langle \varepsilon_k^\nu, y^\nu - x_{k+1}^\nu \right\rangle &\leq \left\langle D_{x^\nu} L_k^\nu(x_{k+1}), y^\nu - x_{k+1}^\nu \right\rangle \\ &\leq \theta_\nu(y^\nu, x_{k+1}^{-\nu}) - \theta_\nu(x_{k+1}) + \left\langle \lambda_{k+1}, D_{x^\nu} g(x_{k+1})(y^\nu - x_{k+1}^\nu) \right\rangle, \end{aligned}$$

where we used the convexity of θ_ν with respect to x^ν in the last estimate. Summing this inequality over all ν and using the convexity of $x \mapsto \langle \lambda_{k+1}, g(x) \rangle$ by Lemma 2.4 yields

$$\begin{aligned} \langle \varepsilon_k, y - x_{k+1} \rangle &\leq -\Psi(x_{k+1}, y) + \langle \lambda_{k+1}, g'(x_{k+1})(y - x_{k+1}) \rangle \\ &\leq -\Psi(x_{k+1}, y) + \langle \lambda_{k+1}, g(y) - g(x_{k+1}) \rangle \\ &\leq -\Psi(x_{k+1}, y) - \langle \lambda_{k+1}, g(x_{k+1}) \rangle. \end{aligned}$$

Taking the limit $k \to_{\mathcal{K}} \infty$ on both sides and using Lemma 4.4, $\varepsilon_k \to 0$ as well as the weak lower semicontinuity of Ψ with respect to the first argument, we see that $\Psi(\bar{x}, y) \leq 0$. Since $y \in \mathcal{F}$ was arbitrary, we conclude that $\bar{x}$ is a normalized equilibrium. □

5 Further Convergence Results

In this section we deal with further convergence results under stronger assumptions than those used in Section 4. After establishing an auxiliary result we prove two central results: (i) the strong convergence of the primal iterates to the unique normalized equilibrium, and (ii) the weak-* convergence of the multiplier sequence.

For the remainder of this section, let $F : X \to X^*$ be given by

$$F(x) = \begin{pmatrix} D_{x^1}\theta_1(x) & \cdots & D_{x^N}\theta_N(x) \end{pmatrix}.$$

It is well-known and easy to verify that the normalized equilibria of the GNEP can be characterized by means of the variational inequality

$$x \in \mathcal{F}, \quad \langle F(x), y - x\rangle \geq 0 \quad \forall y \in \mathcal{F}. \tag{19}$$

We refer the reader to [9] for a proof of this relationship in finite dimensions which directly extends to the infinite-dimensional case. Alternatively, one may simply observe that (19) is the first-order necessary condition of the concave maximization problem (7).

5.1 Strong Convergence of the Primal Iterates

Theorem 5.1. *Assume that X is reflexive and that F is strongly monotone on C, i.e. there is a $c > 0$ such that*

$$\langle F(x) - F(y), x - y\rangle \geq c\|x - y\|_X^2 \quad \forall x, y \in C. \tag{20}$$

Then there is a unique normalized equilibrium $\bar{x}$ of the GNEP. Moreover, if Assumption 3.3 holds, then $x_k \to \bar{x}$.

Proof. Existence and uniqueness of $\bar{x}$ for the equivalent variational inequality (19) follow from standard arguments, see, e.g., [21]. For the proof of convergence, we first show that (x_k) is bounded. By Assumption 3.3, we have $D_{x^\nu} L_k^\nu(x_{k+1}) \in \mathcal{T}_{C_\nu}(x_{k+1})^+ + \varepsilon_k^\nu$ with a zero sequence $(\varepsilon_k) \subseteq X^*$. Concatenating these relations for all ν and using the formula (12) of the augmented Lagrangian yields

$$F(x_{k+1}) + g'(x_{k+1})^*(w_k + \rho_k g(x_{k+1}))_- \in \mathcal{T}_C(x_{k+1})^+ + \varepsilon_k, \tag{21}$$

where $\mathcal{T}_C(x_{k+1})^+ = \mathcal{T}_{C_1}(x_{k+1}^1)^+ \times \ldots \times \mathcal{T}_{C_N}(x_{k+1}^N)^+$ (see [24, Prop. 1.2]). Writing $F_k(x) := F(x) + g'(x)^*(w_k + \rho_k g(x))_-$, we see that F_k is the sum of the strongly monotone function F and the gradient of the convex function $x \mapsto (\rho_k/2)\|(g(x) + w_k/\rho_k)_-\|_H^2$. Hence, F_k is strongly monotone for all k with the same modulus c as in (20). This yields

$$c\|x_{k+1} - \bar{x}\|_X^2 \leq \langle F_k(\bar{x}) - F_k(x_{k+1}), \bar{x} - x_{k+1}\rangle \leq \langle F_k(\bar{x}) - \varepsilon_k, \bar{x} - x_{k+1}\rangle .$$

Recall that d_{K_H} is monotonically decreasing by (10). This implies $\|(w_k + \rho_k g(\bar{x}))_-\|_H \leq \|(w_k)_-\|_H = \|w_k\|_H$ for all k. Hence, $(F_k(\bar{x}))$ is bounded, and we obtain the existence of a $c_1 > 0$ with $c\|x_{k+1} - \bar{x}\|_X^2 \leq c_1\|x_{k+1} - \bar{x}\|_X$. This yields the boundedness of (x_k).

We now prove the strong convergence of (x_k) to $\bar{x}$. Since (x_k) is bounded and X is reflexive, it follows from Theorem 4.5 that $x_k \rightharpoonup \bar{x}$. Now, using (20), it follows that

$$c\|x_{k+1} - \bar{x}\|_X^2 \leq \langle F(x_{k+1}) - F(\bar{x}), x_{k+1} - \bar{x}\rangle .$$

Since $x_{k+1} \rightharpoonup \bar{x}$, we see that $\langle F(\bar{x}), x_{k+1} - \bar{x}\rangle \to 0$. Hence, to conclude the proof, it suffices to show that $\limsup_{k\to\infty} \langle F(x_{k+1}), x_{k+1} - \bar{x}\rangle \leq 0$. Using (21) and the definition of λ_{k+1}, we see that

$$\langle F(x_{k+1}) + g'(x_{k+1})^*\lambda_{k+1}, \bar{x} - x_{k+1}\rangle \geq -\|\varepsilon_k\|_{X^*} \cdot \|\bar{x} - x_{k+1}\|_X.$$

Therefore, it suffices to show $\limsup_{k\to\infty} r_k \leq 0$, where $r_k := \langle g'(x_{k+1})^*\lambda_{k+1}, \bar{x} - x_{k+1}\rangle$. By Lemma 2.4, $x \mapsto \langle \lambda_{k+1}, g(x)\rangle$ is convex. This yields $r_k \leq \langle \lambda_{k+1}, g(\bar{x}) - g(x_{k+1})\rangle$ and, hence, $r_k \leq -\langle \lambda_{k+1}, g(x_{k+1})\rangle$. Therefore, the result follows from Lemma 4.4. □

5.2 Convergence of the Multipliers

Having proved the strong convergence of the primal iterates, in the following theorem we want to show convergence of the multiplier sequence. Recall that the radial cones $\mathcal{R}_C$, $\mathcal{R}_K$ to C and K are defined in Section 2.2.

Theorem 5.2. *Let Assumption 3.3 be satisfied. If $x_k \to \bar{x}$ and the Zowe-Kurcyusz regularity condition*

$$g'(\bar{x})\mathcal{R}_C(\bar{x}) - \mathcal{R}_K(g(\bar{x})) = Y, \tag{22}$$

is satisfied, then (λ_k) is bounded in Y^*. Furthermore, every weak- limit point of (λ_k) is a Lagrange multiplier corresponding to $\bar{x}$.*

Proof. By Assumption 3.3 and Lemma 4.4, we have

$$\begin{aligned} F(x_{k+1}) + g'(x_{k+1})^*\lambda_{k+1} \in \mathcal{T}_C(x_{k+1})^+ + \varepsilon_k \\ \text{and} \quad \liminf_{k\to\infty} \langle \lambda_{k+1}, g(x_{k+1}) \rangle \geq 0 \end{aligned} \tag{23}$$

with a zero sequence $(\varepsilon_k) \subseteq X^*$ and $\lambda_{k+1} \in K^\circ$. Since $\bar{x}$ is feasible (Lemma 4.3), this implies the second statement. We now show the boundedness of (λ_k) in Y^*. By [35, Thm. 2.1], there is an $r > 0$ such that

$$B_r^Y \subseteq g'(\bar{x}) \left[(C - \bar{x}) \cap B_1^X\right] - (K - g(\bar{x})) \cap B_1^Y,$$

where B_r^X and B_r^Y are the closed r-balls around zero in X and Y, respectively. Since $x_k \to \bar{x}$, we can choose choose $k_0 \in \mathbb{N}$ such that

$$\|g(x_k) - g(\bar{x})\|_Y \leq \frac{r}{4} \quad \text{and} \quad \|g'(x_k) - g'(\bar{x})\|_{\mathcal{L}(X,Y)} \leq \frac{r}{4}$$

for all $k \geq k_0$. Now, let $u \in B_r^Y$. It follows that $-u = g'(\bar{x})w - z$ with $\|w\|_X, \|z\|_Y \leq 1$, and $w = w^1 - \bar{x}$, $z = z^1 - g(\bar{x})$ for some $w^1 \in C$, $z^1 \in K$. Furthermore,

$$\langle \lambda_k, z \rangle = \left\langle \lambda_k, z^1 - g(\bar{x}) \right\rangle \leq \langle \lambda_k, -g(\bar{x}) \rangle \leq \langle \lambda_k, -g(x_k) \rangle + \frac{r}{4}\|\lambda_k\|_{Y^*}. \tag{24}$$

Moreover, by (23), $\varphi_k := F(x_k) + g'(x_k)^*\lambda_k - \varepsilon_{k-1} \in \mathcal{R}_C(x_k)^+$ for all $k \geq 2$. Hence,

$$\begin{aligned} \langle \lambda_k, g'(\bar{x})w \rangle = \langle g'(\bar{x})^*\lambda_k, w \rangle &\geq \langle g'(x_k)^*\lambda_k, w \rangle - \frac{r}{4}\|\lambda_k\|_{Y^*} \\ &= \langle \varepsilon_{k-1} + \varphi_k - F(x_k), w \rangle - \frac{r}{4}\|\lambda_k\|_{Y^*}. \end{aligned} \tag{25}$$

We now use the fact that both $\langle \lambda_k, g(x_k) \rangle$ and $\langle \varepsilon_{k-1} + \varphi_k - F(x_k), w \rangle$ are bounded from below independently of w, which is an easy consequence of (23) and $\varepsilon_k \to 0$. Putting together (24) and (25), we obtain

$$\langle \lambda_k, u \rangle = \langle \lambda_k, z \rangle - \langle \lambda_k, g'(\bar{x})w \rangle \leq \frac{r}{2}\|\lambda_k\|_{Y^*} + c$$

for some constant $c > 0$. This implies

$$\|\lambda_k\|_{Y^*} = \sup_{\|u\| \leq r} \left\langle \lambda_k, \frac{1}{r}u \right\rangle \leq \frac{1}{r}\left(c + \frac{r}{2}\|\lambda_k\|_{Y^*}\right)$$

and, hence, $\|\lambda_k\|_{Y^*} \leq 2c/r$. ∎

Note that the above result obviously remains true if we assume that $x_k \to_{\mathcal{K}} \bar{x}$ on a subset $\mathcal{K} \subseteq \mathbb{N}$. In this case, we get the boundedness of $(\lambda_k)_{\mathcal{K}}$ in Y^* and the same assertion about weak-* limit points of this subsequence.

6 Applications

In the following we give some applications and numerical examples for Algorithm 3.2. Optimal control problems state a suitable problem class for infinite-dimensional generalized Nash equilibrium problems. Therefore, we have chosen two examples of this problem class from the literature. The first example is a state constrained elliptic optimal control problem. First we give a detailed overview of this example and analyse in detail why this type of problem is suitable for our convergence analysis. After that, we examine another example that is control constrained only.

All implementations in this section were done in FEniCS [22] using the DOLFIN [23] Python interface and the domain $\Omega = (0,1)^2$. Unless stated otherwise, we choose $\rho_0 = 1.0$, $\tau = 0.1$ and $\gamma = 10$ as the parameters for the algorithm.

Let us introduce a slight change of notation for this section. In the optimal control setting the players' strategies $x^\nu \in X_\nu$ are called the controls $u^\nu \in L^2(\Omega)$. The so called state $y \in Y$ is in general the solution of a PDE constraint that is dependent of the players controls $u = (u^\nu, u^{-\nu})$. Here Y depends on the kind of partial differential equation. Each player's cost functional in this context is denoted by $J_\nu(y, u^\nu)$.

6.1 State-Constrained Elliptic Optimal Control Problems

Let us start with a multiobjective optimal control problem including tracking type cost functionals and elliptic PDE constraints as well as state constraints. Problems of this type have also been investigated in detail in [13] where a Moreau-Yosida regularization has been used to handle the state constraints. Each player ν is equipped with a cost functional $J_\nu(y, u)$, where the state y is dependent on the decisions $u^{-\nu} \in L^2(\Omega)^{N-1}$ of the other competitors. Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded domain with $C^{1,1}$-boundary Γ. Then player ν wants to minimize

$$\frac{1}{2} \|y - y_d^\nu\|^2_{L^2(\Omega)} + \frac{\alpha_\nu}{2} \|u^\nu\|^2_{L^2(\Omega)} \tag{26}$$

over all $(y, u^\nu) \in (H_0^1(\Omega) \cap C(\bar{\Omega})) \times L^2(\Omega)$ subject to the partial differential equation and pointwise control and state constraints

$$Ay = \sum_{i=1}^{N} \chi_{\Omega_\nu} u^\nu, \quad u^\nu \in U_{\text{ad}}^\nu, \quad \text{and} \quad y \geq \psi \quad \text{a.e. in } \Omega, \tag{27}$$

where A is a suitable elliptic differential operator (e.g. $A = -\Delta$) and $U_{\text{ad}}^\nu \subseteq L^2(\Omega)$ is a nonempty closed convex set. The given data satisfy $y_d^\nu \in L^2(\Omega)$, $\alpha > 0$, and

$\psi \in C(\bar{\Omega})$. Moreover, $\chi_{\Omega_\nu} : \mathbb{R}^d \to \{0,1\}$ denotes the characteristic function of a suitable player-specific domain $\Omega_\nu \subseteq \Omega$. The sets U_{ad}^ν are given by

$$U_{\text{ad}}^\nu = \{u^\nu \in L^2(\Omega) : \; u_a^\nu(x) \le u^\nu(x) \le u_b^\nu(x) \text{ a.e. in } \Omega\}$$

with $u_a^\nu, u_b^\nu \in L^2(\Omega)$ and $u_a^\nu \le u_b^\nu$ for all ν, and we define $U_{\text{ad}} := U_{\text{ad}}^1 \times \cdots \times U_{\text{ad}}^N$. Obviously, U_{ad} and $U_{\text{ad}}^1, \ldots, U_{\text{ad}}^N$ are closed, bounded, and convex. Using the control-to-state mapping

$$S : u \mapsto y, \; L^2(\Omega) \to H^1(\Omega) \cap C(\bar{\Omega}), \qquad Su = \sum_{i=1}^N S_\nu u^\nu$$

we get the reduced formulation of the optimal control problem that coincides with the definition of a jointly convex GNEP:

$$\begin{aligned} \min_{u^\nu} \quad & J_\nu(u) := \frac{1}{2} \left\| Su - y_d^\nu \right\|_{L^2(\Omega)}^2 + \frac{\alpha_\nu}{2} \left\| u^\nu \right\|_{L^2(\Omega)}^2 , \\ \text{s.t.} \quad & u^\nu \in U_{\text{ad}}^\nu, \quad Su \ge \psi \text{ a.e. in } \Omega. \end{aligned} \tag{28}$$

In the notation of our abstract setting (1), (8), we have $C = U_{\text{ad}}$, $C_\nu = U_{\text{ad}}^\nu$ for every ν, $g(u) = Su - \psi$, and K is the nonnegative cone in $C(\bar{\Omega})$. The feasible set $\mathcal{F}$ as defined in (8) takes on the form $\mathcal{F} = \{u \in U_{\text{ad}} : Su \ge \psi\}$ and is closed, bounded, and convex. The Nikaido-Isoda function of the GNEP is given by

$$\Psi(u, w) = \sum_{\nu=1}^N \Big[J_\nu(u^\nu, u^{-\nu}) - J_\nu(w^\nu, u^{-\nu}) \Big], \tag{29}$$

where $w = (w^\nu, w^{-\nu})$ and $u, w \in L^2(\Omega)$. The next lemma gives us the weak lower semicontinuity of the corresponding Nikaido-Isoda function, i.e. Assumption 2.2 is satisfied.

Lemma 6.1. *The Nikaido-Isoda function (29) is weakly lower semicontinuous with respect to u.*

Proof. The result follows from the weak lower semicontinuity of the norm and the compactness of the solution operator S, i.e. weakly convergent sequences $u_k \rightharpoonup u$ are mapped onto strongly convergent sequences $Su_k \to Su$. □

Lemma 6.1 has a number of important consequences. First, the weak lower semicontinuity of Ψ together with the weak compactness of $\mathcal{F}$ implies the existence of a normalized Nash equilibrium by Theorem 2.3. Moreover, it follows that the augmented Lagrangian subproblems generated by Algorithm 3.2 always admit solutions (Lemma 4.2) and every weak limit point of the sequence of controls (u_k) is a normalized Nash equilibrium of the GNEP (Theorem 4.5).

Strong Monotonicity of the Mapping F

We now show that the mapping F induced by the GNEP (28) is strongly monotone in the sense of Theorem 5.1. This function is given by

$$F(u) = \begin{pmatrix} D_{u_1}J_1(u) & \cdots & D_{u_N}J_N(u) \end{pmatrix}.$$

Recall that strong monotonicity of F implies the strong convergence of the whole sequence (u_k) to the unique normalized Nash equilibrium $\bar{u}$.

Lemma 6.2. *The operator F is strongly monotone.*

Proof. Splitting the cost functional J_ν into two parts $J_\nu(u) = J_\nu^1(u) + J_\nu^2(u)$ with

$$J_\nu^1(u) := \|Su - y_d^\nu\|^2_{L^2(\Omega)}, \quad J_\nu^2(u) := \frac{\alpha_\nu}{2}\|u^\nu\|^2_{L^2(\Omega)}$$

yields $F(u) = F_1(u) + F_2(u)$ with $F_1(u) = (D_{u_1}J_1^1(u), \ldots, D_{u_N}J_N^1(u))$ and F_2 defined similarly. For the first part, we use $Su = \sum_{\nu=1}^N S_\nu u^\nu$ and obtain

$$\begin{aligned}(F_1(u) - F_1(u'), u - u') &= \sum_{\nu=1}^N (S_\nu^*(Su - y_d^\nu) - S_\nu^*(Su' - y_d^\nu), u^\nu - u^{\nu'}) \\ &= \sum_{\nu=1}^N (S(u-u'), S_\nu(u^\nu - u^{\nu'})) = \|S(u-u')\|^2_{L^2(\Omega)} \geq 0.\end{aligned}$$

We now analyze F_2 and set $\alpha := \min\{\alpha_1, \ldots, \alpha_N\} > 0$. Then

$$(F_2(u) - F_2(u'), u - u') = \sum_{\nu=1}^N (\alpha_\nu(u^\nu - u^{\nu'}), u^\nu - u^{\nu'}) \geq \alpha\|u - u'\|^2_{L^2(\Omega)},$$

and the proof is complete. □

Let us remark here that the Tikhonov terms in the objective functions are of great importance since they yield the strong monotonicity of the operator F. In particular, $F_1(u)$ is only monotone and *not* strongly monotone since there is no constant $c > 0$ with

$$\|Su\|_{H^1(\Omega)\cap C(\bar{\Omega})} \geq c\,\|u\|_{L^2(\Omega)} \quad \text{for all } u \in L^2(\Omega).$$

For instance, if $u_k \rightharpoonup 0$ and $\|u_k\|_{L^2(\Omega)} = 1$, then $Su_k \to 0$ by the compactness of S, yielding a contradiction.

Existence and Convergence of Multipliers

Note that the GNEP (28) admits a normalized equilibrium by Lemma 6.1 and Theorem 2.3. Denoting by $\bar{u}$ such an equilibrium, we obtain a corresponding optimal state $\bar{y} = S\bar{u}$. Using a suitable constraint qualification, we can furthermore establish the existence of a Lagrange multiplier $\bar{\lambda}$. To this end, we use the following Slater condition.

Assumption 6.3. *We assume that there are $\hat{u} \in U_{\mathrm{ad}}$ and $\sigma > 0$ such that*

$$S\hat{u}(x) \geq \psi(x) + \sigma \quad \forall x \in \bar{\Omega}. \tag{30}$$

The above assumption essentially boils down to $S\hat{u} - \psi$ lying in the interior of the nonnegative cone of Y. Note that $Y = H^1(\Omega) \cap C(\bar{\Omega})$ and, therefore, the nonnegative cone has a nonempty interior (as opposed to spaces such as $L^2(\Omega)$). Furthermore, since S is linear, it is easy to see that Assumption 6.3 is equivalent to the *linearized* Slater condition

$$\exists u \in U_{\mathrm{ad}}: \; S\bar{u}(x) + S(u - \bar{u})(x) \geq \psi(x) + \sigma,$$

e.g. by taking $u := \hat{u}$ with $\hat{u}$ as in (30). The linearized Slater condition in turn implies the Zowe-Kurcyusz regularity condition which we used in Section 5 (see [34, p. 332]).

The above discussion implies two things: first, the optimal control and state $(\bar{u}, \bar{y})$ admit a Lagrange multiplier $\bar{\lambda} \in C(\bar{\Omega})^*$ such that the first-order necessary conditions

$$\begin{aligned}
A\bar{y} &= \sum_{i=1}^{N} \chi_{\Omega_\nu} \bar{u}^\nu, && (31a)\\
A^* \bar{p}^\nu &= \bar{y} - y_d^\nu + \bar{\lambda}, && (31b)\\
(\bar{p}^\nu + \alpha_\nu \bar{u}^\nu, z - \bar{u}^\nu) &\geq 0 \qquad \forall z \in U_{\mathrm{ad}}^\nu, && (31c)\\
\langle \bar{\lambda}, \psi - \bar{y} \rangle_{\mathcal{M}(\bar{\Omega}), C(\bar{\Omega})} &= 0, \qquad \bar{\lambda} \leq 0, && (31d)
\end{aligned}$$

are satisfied for all ν, cf. [7] and the discussion in Section 2.2. Here, $\bar{p}^\nu \in W^{1,s}$, $1 < s < N/(N-1)$, is the adjoint state of player ν, and the inequality $\bar{\lambda} \leq 0$ has to be understood as $\langle \bar{\lambda}, \varphi \rangle_{\mathcal{M}(\bar{\Omega}), C(\bar{\Omega})} \leq 0$ for all $\varphi \in C(\bar{\Omega})$ with $\varphi \geq 0$. In other words, $\bar{\lambda}$ lies in the polar of the nonnegative cone of $C(\bar{\Omega})$.

The second implication of the Slater condition (or, equivalently, of the Zowe-Kurcyusz condition) is that the assertions of Theorem 5.2 hold, i.e. the multiplier sequence (λ_k) generated by Algorithm 3.2 is bounded in $C(\bar{\Omega})^*$ and each of its weak-* limit points is a Lagrange multiplier satisfying the optimality system (31).

Now, let us briefly consider the subproblems which occur in every iteration of the algorithm. We know that, by Lemmas 6.1 and 4.2, these problems always admit a Nash equilibrium $\bar{u}_k$. Since the problem is convex and control constrained only, first-order necessary optimality conditions can be established without any further regularity assumptions. Setting $\bar{y}_k := S\bar{u}_k$, there exist unique adjoint states $\bar{p}_k^\nu \in H^1(\Omega)$ which satisfy (compare with (31)) the system

$$\begin{aligned}
A\bar{y}_k &= \sum_{i=1}^{N} \chi_{\Omega_\nu} \bar{u}_k^\nu, && (32a)\\
A^* \bar{p}_k^\nu &= \bar{y}_k - y_d^\nu + \bar{\lambda}_k, && (32b)\\
(\bar{p}_k^\nu + \alpha_\nu \bar{u}_k^\nu, z - \bar{u}_k^\nu) &\geq 0 \qquad \forall z \in U_{\mathrm{ad}}^\nu, && (32c)\\
\bar{\lambda}_k &= (\lambda_k + \rho_k(S\bar{u}_k - \psi))_- && (32d)
\end{aligned}$$

for all ν.

Summary of the Convergence Properties

We can associate with each iterate u_k and its state $y_k = Su_k$ the adjoint states $p_k^\nu = G^*(y_k - y_d^\nu + \lambda_k)$, where $G := ES$ and E is the canonical embedding from $H^1(\Omega) \cap C(\bar{\Omega})$ into $L^2(\Omega)$. Using standard arguments (e.g. [15, Lemma 2.6]), it is easy to show that $p_k^\nu \rightharpoonup \bar{p}^\nu$ in $L^2(\Omega)$ for all i.

Concluding, as $k \to \infty$ we have for the sequence $(y_k, u_k^\nu, u_k^{-\nu}, p_k^\nu, \lambda_k)$ generated by Algorithm 3.2 that

$$\begin{aligned}
(y_k, u_k) &\to (\bar{y}, \bar{u}) && \text{in } (H^1(\Omega) \cap C(\bar{\Omega})) \times L^2(\Omega), \\
p_k^\nu &\rightharpoonup \bar{p}_\nu && \text{in } L^2(\Omega), \\
\lambda_k &\overset{*}{\rightharpoonup} \bar{\lambda} && \text{in } C(\bar{\Omega})^*.
\end{aligned}$$

Numerical Results

In the following let us report about numerical results. As a test problem, we chose the four-player game presented in [13], which is a special instance of the problem presented above, where $\Omega_i = \Omega$ for all i and $f \equiv 1$. The vector of Tikhonov-parameters is given by $\alpha = (2.8859, 4.3374, 2.5921, 3.9481)$, and the control constraints are defined by $u_a^\nu \equiv -12$, $u_b^\nu \equiv 12$ for all ν. The state has to fulfill the state constraint for

$$\psi(x_1, x_2) = \cos(5\sqrt{(x_1 - 0.5)^2 + (x_2 - 0,5)^2} + 0.1.$$

Defining

$$\xi_\nu(x_1, x_2) := 10^3 \max(0, 4(0.25 - \max(|x_1 - z_\nu^1|, |x_2 - z_\nu^2|)))$$

with $z^1 := (0.25, 0.75, 0.25, 0.75)$ and $z^2 := (0.25, 0.25, 0.75, 0.75)$, we set

$$\begin{aligned}
y_d^1 &:= \xi_1 - \xi_4, & y_d^2 &:= \xi_2 - \xi_3, \\
y_d^3 &:= \xi_3 - \xi_2, & y_d^4 &:= \xi_4 - \xi_1.
\end{aligned}$$

The algorithm was stopped as soon as the quantities $\|(\psi - y_k)_+\|_{C(\bar{\Omega})}$ and $|(\lambda_k, y_k - \psi)|$ drop below 10^{-4}. The subproblems arising within the computation are solved by applying an active set method to the corresponding KKT conditions (32).

Table 6.1 denotes some iteration numbers for different discretizations as well as the maximum penalty parameter $\rho_{\max}$ reached during the given iterations. Note that the inner iterations are accumulated over the whole outer iterations.

n	16	32	64	128	256
outer it.	10	11	11	11	11
inner it.	25	35	36	40	45
$\rho_{\max}$	10^7	10^9	10^{10}	10^{10}	10^{10}

It is worth noting that the outer iteration numbers stay approximately the same as n increases. Moreover, the same holds for the final penalty parameter $\rho_{\max}$, which is equal to 10^{10} for $n \geq 64$. This observation suggests that our algorithm works quite well for the given optimal control problem.

The Figures 1, 2 and 3 show the numerical solution of Example 1. All figures depict results gained for a triangular mesh with $n = 128$ grid points.

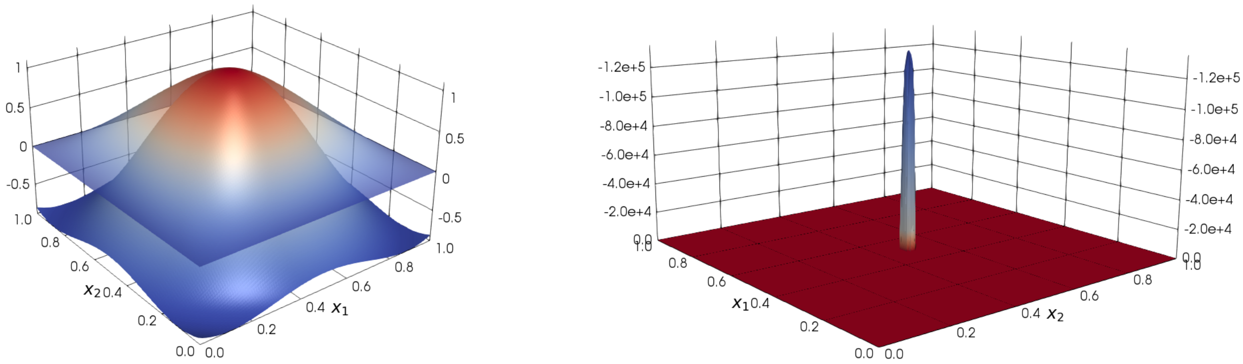

Figure 1: (Example 1) Left: Computed discrete optimal state y_h (transparent, up) and state constraint ψ, Right: computed Lagrange multiplier λ_h

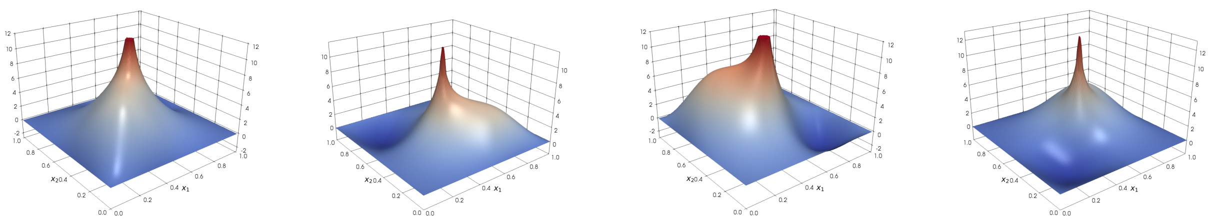

Figure 2: (Example 1) Computed optimal control $\bar{u}_h = (\bar{u}_h^1, \bar{u}_h^2, \bar{u}_h^3, \bar{u}_h^4)$.

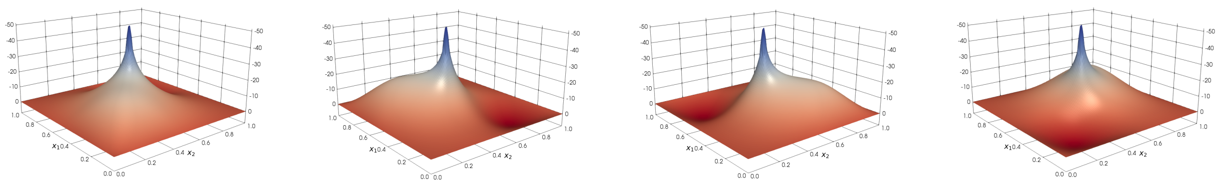

Figure 3: (Example 1) Computed adjoint state $\bar{p}_h = (\bar{p}_h^1, \bar{p}_h^2, \bar{p}_h^3, \bar{p}_h^4)$.

6.2 Control-Constrained Optimal Control Problems

The following example does not include constraints on the state. However this example is still of interest, since it has a known analytic solution, allowing us to do error estimates on our computed solution. Let $N = 2$ be the number of players. Every player wants to minimize the tracking-type functional (26) over all $(y, u^\nu) \in (H^1(\Omega) \cap C(\bar{\Omega})) \times L^2(\Omega)$ subject to the PDE and control constraints

$$-\Delta y = \sum_{\nu=1}^{N} \chi_{\Omega_\nu} u^\nu + f \quad \text{and} \quad u^\nu \in U_{\text{ad}}^\nu,$$

where $U_{\text{ad}}^\nu = \{u^\nu \in L^2(\Omega) : \; u_a^\nu \leq u^\nu \leq u_b^\nu \text{ a.e. in } \Omega\}$ as before. Choosing $C = L^2(\Omega)$ and setting $g(u) = (g_1(u), g_2(u), \ldots, g_N(u))$ with

$$g_i(u) = \begin{pmatrix} u^\nu - u_a^\nu \\ u_b^\nu - u^\nu \end{pmatrix}, \quad \nu = 1, \ldots, N,$$

our set of constraints from (8) is given by $\mathcal{F} = \{u \in L^2(\Omega) : g(u) \geq 0\}$. Here, one player's feasible set does not depend on the rival players' strategies, so we have $\mathcal{F}_\nu(u^{-\nu}) = U_{\text{ad}}^\nu$ for all ν, and the problem is a standard NEP. Let $\lambda_a^\nu, \lambda_b^\nu$ denote the multipliers corresponding to the lower and upper control constraints. Then the complete multiplier vector λ is given by $\lambda = (\lambda_1, \ldots, \lambda_N)$ with $\lambda_i = (\lambda_a^\nu, \lambda_b^\nu)$ for all ν.

In this example we apply our algorithm by augmenting the given control constraints. Special care needs to be taken because the set C is not bounded and therefore not weakly compact as assumed in Lemma 4.2. However, because of the special structure of the cost functional, the augmented NEP can be reduced to a single control problem, yielding the existence of a unique normalized Nash equilibrium, cf. [14, Prop. 3.10].

The test problem we chose was first presented in [5]. Here, we state a reformulated version from [8]. The Tikhonov-parameters are given by $\alpha_1 = \alpha_2 = 1$. Moreover, we define the subsets $B_\nu \subseteq \Omega$ by

$$B_1 := (0,1) \times (0, 0.5), \qquad B_2 := (0,1) \times (0.5, 1)$$

and the control constraints $u_a^\nu := a_\nu \chi_{B_\nu}$, $u_b^\nu := b_\nu \chi_{B_\nu}$, where $a_\nu := -0.5$ and $b_\nu := 0.5$ for all ν. Finally, we set

$$y_d^1(x) := y(x) + 8\pi^2 y(2x), \quad y_d^2(x) := y(x) + 18\pi^2 y(3x)$$

and $f := -\Delta y - u_1 - u_2$. The exact solution of the resulting problem is given by

$$y(x) := \sin(\pi x_1)\sin(\pi x_2), \quad u_1(x) := \chi_{B_1} P_{[a_1,b_1]}\, y(2x), \quad u_2(x) := \chi_{B_2} P_{[a_2,b_2]}\, y(3x).$$

The algorithm was stopped as soon as

$$\sum_{\nu=1}^{2} \Big[\|(u_a^\nu - u^\nu)_+\|_{C(\bar\Omega)} + |(\lambda_a^\nu, u^\nu - u_a^\nu)| + \|(u^\nu - u_b^\nu)_+\|_{C(\bar\Omega)} + |(\lambda_b^\nu, u_b^\nu - u^\nu)| \Big] \leq 10^{-7}$$

was satisfied. Once again, the subproblems occurring within the algorithm were solved by applying an active set method.

Numerical Results

Figure 4 depicts the obtained results for the optimal control u_h^1, u_h^2 of the two players using $n = 128$ gridpoints.

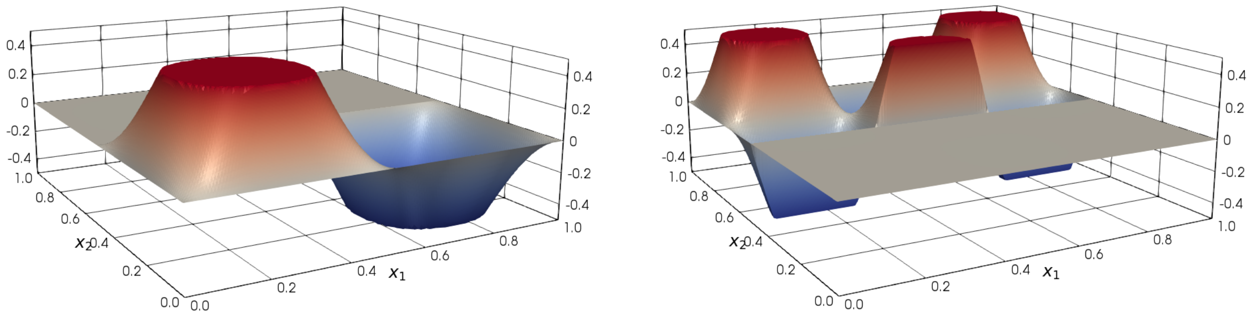

Figure 4: (Example 2) Computed discrete optimal control u_h^1 (left) and u_h^2 (right).

The iteration numbers of outer and inner iterations as well as the final value of the penalty parameter $\rho_{\max}$ are shown in the table below.

n	16	32	64	128	256	512
outer it.	7	7	7	7	7	7
inner it.	19	12	21	12	13	31
$\rho_{\max}$	10	10	10	10	10	10

Note that the outer iteration numbers and the final penalty parameter remain constant with increasing n. Finally, let us report about the behavior of the discretized errors with increasing dimension. The following table illustrates the corresponding L^2-norms for different numbers of gridpoints.

n	16	32	64	128	256
$\|\bar{u}_1 - u_h^1\|_{L^2(\Omega)}$	1.72e-3	4.29e-4	1.16e-4	2.89e-5	7.24e-6
$\|\bar{u}_2 - u_h^2\|_{L^2(\Omega)}$	3.84e-3	1.04e-3	2.59e-4	6.50e-5	1.63e-5
$\|\bar{y} - y_h\|_{L^2(\Omega)}$	1.60e-3	4.01e-4	1.00e-4	2.50e-5	6.27e-6

Considering the consistent number of outer and also inner iterations and the quite good approximation of the Nash equilibrium, we can conclude that our algorithm works fine for this kind of problems.

7 Final Remarks

In this paper, we have introduced an augmented Lagrangian method for jointly convex GNEPs in Banach spaces. Under relatively weak assumptions, we obtain feasibility and optimality of every weak limit point of the generated sequence, and under additional regularity assumptions, we show strong convergence of the primal sequence and weak-*

convergence of the multiplier sequence. The fact that we stated the problem in a quite general setting allows us to consider a broad range of applications to our setting, including the important field of multiobjective optimal control problems. Our numerical tests point out that our method works quite well for this kind of problems since it possesses good convergence properties and yields relatively high accuracy.

References

[1] K. J. Arrow and G. Debreu. Existence of an equilibrium for a competitive economy. *Econometrica*, 22:265–290, 1954.

[2] H. H. Bauschke and P. L. Combettes. *Convex Analysis and Monotone Operator Theory in Hilbert Spaces.* Springer, New York, 2011.

[3] M. R. Baye, G. Q. Tian, and J. Zhou. Characterizations of the existence of equilibria in games with discontinuous and nonquasiconcave payoffs. *Rev. Econom. Stud.*, 60(4):935–948, 1993.

[4] E. G. Birgin and J. M. Martínez. *Practical Augmented Lagrangian Methods for Constrained Optimization.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2014.

[5] A. Borzì and C. Kanzow. Formulation and numerical solution of Nash equilibrium multiobjective elliptic control problems. *SIAM J. Control Optim.*, 51(1):718–744, 2013.

[6] D. A. Carlson. Uniqueness of normalized Nash equilibrium for a class of games with strategies in Banach spaces. pages 333–348, 2002.

[7] E. Casas. Control of an elliptic problem with pointwise state constraints. *SIAM J. Control Optim.*, 24(6):1309–1318, 1986.

[8] A. Dreves and J. Gwinner. Jointly convex generalized Nash equilibria and elliptic multiobjective optimal control. *J. Optim. Theory Appl.*, 168(3):1065–1086, 2016.

[9] F. Facchinei, A. Fischer, and V. Piccialli. On generalized Nash games and variational inequalities. *Oper. Res. Lett.*, 35(2):159–164, 2007.

[10] F. Facchinei and C. Kanzow. Generalized Nash equilibrium problems. *Ann. Oper. Res.*, 175:177–211, 2010.

[11] K. Fan. A minimax inequality and applications. *Inequalities, III (Proc. Third Sympos., Univ. California, Los Angeles, Calif., 1969; dedicated to the memory of Theodore S. Motzkin)*, pages 103–113, 1972.

[12] A. Fischer, M. Herrich, and K. Schönefeld. Generalized nash equilibrium problems-recent advances and challenges. *Pesquisa Operacional*, 34(3):521–558, 2014.

[13] M. Hintermüller and T. Surowiec. A PDE-constrained generalized Nash equilibrium problem with pointwise control and state constraints. *Pac. J. Optim.*, 9(2):251–273, 2013.

[14] M. Hintermüller, T. Surowiec, and A. Kämmler. Generalized Nash equilibrium problems in Banach spaces: theory, Nikaido-Isoda-based path-following methods, and applications. *SIAM J. Optim.*, 25(3):1826–1856, 2015.

[15] M. Hinze and C. Meyer. Variational discretization of Lavrentiev-regularized state constrained elliptic optimal control problems. *Comput. Optim. Appl.*, 46(3):487–510, 2010.

[16] A. N. Iusem, G. Kassay, and W. Sosa. On certain conditions for the existence of solutions of equilibrium problems. *Math. Program.*, 116(1-2, Ser. B):259–273, 2009.

[17] A. N. Iusem and W. Sosa. New existence results for equilibrium problems. *Nonlinear Anal.*, 52(2):621–635, 2003.

[18] C. Kanzow and D. Steck. Augmented Lagrangian methods for the solution of generalized Nash equilibrium problems. *SIAM J. Optim.*, 26(4):2034–2058, 2016.

[19] C. Kanzow and D. Steck. An example comparing the standard and modified augmented Lagrangian methods. *Technical Report*, Institute of Mathematics, University of Würzburg, February 2017.

[20] C. Kanzow, D. Steck, and D. Wachsmuth. An augmented Lagrangian method for optimization problems in Banach spaces. *Technical Report*, Institute of Mathematics, University of Würzburg, December 2016.

[21] D. Kinderlehrer and G. Stampacchia. *An introduction to variational inequalities and their applications*, volume 31 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2000. Reprint of the 1980 original.

[22] A. Logg, K.-A. Mardal, G. N. Wells, et al. *Automated Solution of Differential Equations by the Finite Element Method*. Springer, 2012.

[23] A. Logg and G. N. Wells. Dolfin: Automated finite element computing. *ACM Transactions on Mathematical Software*, 37(2), 2010.

[24] B. S. Mordukhovich. *Variational analysis and generalized differentiation. I*, volume 330 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2006. Basic theory.

[25] J. Morgan and V. Scalzo. Existence of equilibria in discontinuous abstract economies. *Preprint 53-2004*, Dipartimento di Matematica e Applicazioni R. Caccioppoli, Napoli, 2004.

[26] H. Nikaido and K. Isoda. Note on non-cooperative convex games. *Pacific J. Math.*, 5:807–815, 1955.

[27] A. M. Ramos, R. Glowinski, and J. Periaux. Nash equilibria for the multiobjective control of linear partial differential equations. *J. Optim. Theory Appl.*, 112(3):457–498, 2002.

[28] A. M. Ramos, R. Glowinski, and J. Periaux. Pointwise control of the Burgers equation and related Nash equilibrium problems: computational approach. *J. Optim. Theory Appl.*, 112(3):499–516, 2002.

[29] A. M. Ramos and T. Roubíček. Nash equilibria in noncooperative predator-prey games. *Appl. Math. Optim.*, 56(2):211–241, 2007.

[30] J. B. Rosen. Existence and uniqueness of equilibrium points for concave n-person games. *Econometrica*, 33:520–534, 1965.

[31] T. Roubíček. Noncooperative games with elliptic systems. In *Optimal control of partial differential equations (Chemnitz, 1998)*, volume 133 of *Internat. Ser. Numer. Math.*, pages 245–255. Birkhäuser, Basel, 1999.

[32] T. Roubíček. On noncooperative nonlinear differential games. *Kybernetika (Prague)*, 35(4):487–498, 1999.

[33] Z. Tang, J.-A. Désidéri, and J. Périaux. Multicriterion aerodynamic shape design optimization and inverse problems using control theory and Nash games. *J. Optim. Theory Appl.*, 135(3):599–622, 2007.

[34] F. Tröltzsch. *Optimal Control of Partial Differential Equations*. American Mathematical Society, Providence, RI, 2010.

[35] J. Zowe and S. Kurcyusz. Regularity and stability for the mathematical programming problem in Banach spaces. *Appl. Math. Optim.*, 5(1):49–62, 1979.