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Constantin Christof, Gerd Wachsmuth



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Edited by
SPP1962 at Weierstrass Institute for Applied Analysis and Stochastics (WIAS)
Leibniz Institute in the Forschungsverbund Berlin e.V.
Mohrenstraße 39, 10117 Berlin, Germany
E-Mail: spp1962@wias-berlin.de

World Wide Web: <http://spp1962.wias-berlin.de/>

No-Gap Second-Order Conditions via a Directional Curvature Functional*

Constantin Christof[†] Gerd Wachsmuth[‡] 

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This paper is concerned with necessary and sufficient second-order conditions for finite-dimensional and infinite-dimensional constrained optimization problems. Using a suitably defined directional curvature functional for the admissible set, we derive no-gap second-order optimality conditions in an abstract functional analytic setting. Our theory not only covers those cases where the classical assumptions of polyhedricity or second-order regularity are satisfied but also allows to study problems in the absence of these requirements. As a tangible example, we consider no-gap second-order conditions for bang-bang optimal control problems.

Keywords: second-order condition, bang-bang control, polyhedricity, second-order regularity, no-gap optimality condition

MSC: [49J53](#), [49K20](#), [49K27](#), [49K30](#)

1 Introduction

The aim of this paper is to develop a theoretical framework for necessary and sufficient second-order optimality conditions (henceforth referred to as SNC and SSC, respectively). The main feature of our approach is that we use a suitably defined “directional” curvature functional to take into account the influence of the admissible set. Our definition of curvature allows us to derive no-gap second-order conditions that provide more flexibility than classical results. In particular, our approach makes it possible to exploit additional information about the gradient of the objective functional. Such information

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[†]Technische Universität Dortmund, Faculty of Mathematics, LS X, 44227 Dortmund, Germany

[‡]Technische Universität Chemnitz, Faculty of Mathematics, Professorship Numerical Mathematics (Partial Differential Equations), 09107 Chemnitz, Germany, http://www.tu-chemnitz.de/mathematik/part_dgl/people/wachsmuth/, gerd.wachsmuth@mathematik.tu-chemnitz.de

is, e.g., often available in the optimal control of partial differential equations (PDEs) where the gradient of the objective is typically characterized by an adjoint equation and, as a consequence, enjoys additional regularity properties.

The analysis found in this paper originated from the idea to extend the results of [Casas et al., 2017b]. In this paper, the authors derived an SSC for a class of bang-bang optimal control problems that does not fit into the classical setting of polyhedricity and second-order regularity, cf. [Bonnans and Shapiro, 2000]. Our results turned out to be of relevance for other problems as well and ultimately gave rise to the abstract framework of Section 4 that not only covers large parts of the classical SNC and SSC theory but also allows to study situations where the admissible set exhibits a singular or degenerate curvature behavior, cf. Section 6. We hope that with the subsequent analysis we can, on the one hand, offer an alternative view on well-known SNC and SSC results and, on the other hand, also provide some new ideas for the study of problems that do not satisfy the classical assumptions of polyhedricity and second-order regularity.

Let us give some references: There are several contributions addressing second-order optimality conditions for optimization problems posed in infinite-dimensional spaces. We mention exemplarily [Maurer and Zowe, 1979; Bonnans et al., 1999; Bonnans and Zidani, 1999; Bonnans and Shapiro, 2000; Casas and Tröltzsch, 2012; 2015]. No-gap optimality conditions in the infinite-dimensional case can be found, e.g., in [Bonnans and Zidani, 1999, Theorems 2.7, 2.10], [Casas and Tröltzsch, 2012, Theorems 2.2, 2.3], and [G. Wachsmuth, 2016, Theorem 5.7]. Note that the latter results all rely on the concept of polyhedricity (in contrast to our Theorem 4.5).

Before we begin with our analysis, we give a short overview over the contents and the structure of this paper:

In Section 2, we clarify the notation, make our assumptions precise and recall several definitions that are needed for our investigation.

In Section 3, we define the directional curvature functional that is at the heart of our SNC and SSC analysis. Here, we also discuss basic properties of the curvature functional as, e.g., positivity and lower semicontinuity, that are used in the remainder of the paper.

Section 4 addresses SNC and SSC for constrained optimization problems on an abstract functional analytic level. The main results of this section (and of the paper as a whole) are Theorems 4.3 to 4.5. These theorems illustrate the advantages of working with the directional curvature functional and demonstrate that our approach allows for a very short and elegant derivation of the second-order theory.

In Section 5, we demonstrate that the framework of Section 4 indeed covers the classical SNC and SSC theory for optimization problems with polyhedric or second-order regular sets. Here, we further comment on how the assumptions (NDC) and (MRC) appearing in our second-order conditions can be verified in practice and interpreted in the context of generalized Legendre forms and Tikhonov regularization. Section 5 also includes two tangible examples that demonstrate the usefulness of the theorems in Section 4.

Section 6 is devoted to problems that are covered by our analysis but do not fall under the scope of the classical SNC- and SSC-framework. The first example that we consider here is a finite-dimensional optimization problem whose admissible set exhibits a singular curvature behavior. In Section 6.2, we then study no-gap second-order conditions for

bang-bang optimal control problems in the measure space $\mathcal{M}(\Omega)$. The results obtained in this subsection extend those of [Casas et al., 2017b], see [Theorems 6.4](#) and [6.12](#).

[Section 7](#) summarizes our findings and gives some pointers to further research.

2 Notation, Preliminaries and Basic Concepts

Throughout this paper, we always consider the following situation.

Assumption 2.1 (Standing Assumptions and Notation).

- (i) X is the (topological) dual of a separable Banach space Y ,
- (ii) $\iota : Y \rightarrow X^*$ denotes the canonical embedding of Y into the dual X^* of X ,
- (iii) the admissible set C is a closed, non-empty subset of X .

Note that, under the above assumptions, X is necessarily a Banach space with a weak- \star sequentially compact unit ball, cf. the Banach–Alaoglu theorem. We remark that the overwhelming majority of our results also holds when the space Y is assumed to be reflexive instead of separable. We restrict our analysis to the above setting since it allows to study more interesting practical examples, see [Section 6.2](#). For our analysis, we need the following classical concepts (cf. [Bonnans and Shapiro, 2000, Section 2.2.4]).

Definition 2.2. Let $x \in C$ be given. We define the radial cone, the strong outer tangent cone and the weak- \star outer tangent cone to C at x , respectively, by

$$\begin{aligned}\mathcal{R}_C(x) &:= \{h \in X \mid \exists T > 0 \forall t \in [0, T], x + th \in C\}, \\ \mathcal{T}_C(x) &:= \left\{ h \in X \mid \exists t_k \searrow 0, \exists x_k \in C \text{ such that } \frac{x_k - x}{t_k} \rightarrow h \right\}, \\ \mathcal{T}_C^*(x) &:= \left\{ h \in X \mid \exists t_k \searrow 0, \exists x_k \in C \text{ such that } \frac{x_k - x}{t_k} \overset{\star}{\rightarrow} h \right\}.\end{aligned}$$

Moreover, we define the weak- \star (outer) normal cone to x by

$$\mathcal{N}_C^*(x) := \{x^* \in \iota(Y) \mid \forall h \in \mathcal{T}_C^*(x) : \langle x^*, h \rangle \leq 0\}.$$

Finally, given a $\varphi \in -\mathcal{N}_C^*(x)$, we define the weak- \star critical cone by

$$\mathcal{K}_C^*(x, \varphi) := \mathcal{T}_C^*(x) \cap \varphi^\perp.$$

We emphasize that $\mathcal{N}_C^*(x)$ is defined to be a subset of $\iota(Y)$ and may thus be identified with a subset of the predual space Y . Note that all “cones” in the above are indeed cones in the mathematical sense, i.e., $h \in \mathcal{R}_C(x)$ implies $\alpha h \in \mathcal{R}_C(x)$ for all $\alpha \geq 0$ etc. We point out that $\mathcal{R}_C(x) \subset \mathcal{T}_C(x) \subset \mathcal{T}_C^*(x)$. If C is convex, then [Bonnans and Shapiro, 2000, Proposition 2.55]

$$\mathcal{R}_C(x) = \mathbb{R}^+(C - x), \quad \mathcal{T}_C(x) = \text{cl}(\mathcal{R}_C(x)) \quad \forall x \in C.$$

If, in addition, X is reflexive, then we have $\mathcal{T}_C^*(x) = \mathcal{T}_C(x)$ by Mazur’s lemma. We remark that $\mathcal{T}_C^*(x)$ is in general not closed since the weak- \star topology is not sequential on infinite-dimensional spaces, cf. [Mehlitz and G. Wachsmuth, 2017, Example 5.9].

3 The Directional Curvature Functional

The basic idea of our SNC and SSC approach is to not discuss the curvature properties of the admissible set C separately, i.e., independently of the optimization problem at hand, but to develop a second-order analysis that takes into account the gradient of the objective. To accomplish the latter, we introduce the directional curvature functional.

Definition 3.1. *Let $x \in C$ and $\varphi \in -\mathcal{N}_C^*(x)$ be given. The weak- \star directional curvature functional $Q_C^{x,\varphi}: \mathcal{K}_C^*(x, \varphi) \rightarrow [-\infty, \infty]$ associated with the triple (x, φ, C) is defined by*

$$Q_C^{x,\varphi}(h) := \inf \left\{ \liminf_{k \rightarrow \infty} \langle \varphi, r_k \rangle \left| \begin{array}{l} \{r_k\} \subset X, \{t_k\} \subset \mathbb{R}^+ : t_k \searrow 0, t_k r_k \xrightarrow{\star} 0, \\ x + t_k h + \frac{1}{2} t_k^2 r_k \in C \end{array} \right. \right\}. \quad (1)$$

Note that $Q_C^{x,\varphi}(\cdot)$ is precisely that function whose epigraph coincides with the weak- \star limes superior (in the sense of Kuratowski) of the epigraphs of the second-order difference quotients associated with the characteristic function of the set C , the point x , and the element $\varphi \in -\mathcal{N}_C^*(x)$, cf. [Do, 1992, Sections 1, 4]. The functional $Q_C^{x,\varphi}(\cdot)$ may thus be identified with a second-order weak- \star lower epiderivative.

Definition 3.1 can be motivated as follows: Consider an optimization problem of the form $\min J(x) := \langle \varphi, x \rangle$ s.t. $x \in C$, where φ satisfies $\varphi \in -\mathcal{N}_C^*(\bar{x})$ for some $\bar{x} \in C$, i.e., $\langle \varphi, h \rangle \geq 0$ for all $h \in \mathcal{T}_C^*(\bar{x})$. Then \bar{x} is a critical point and we have to study perturbations into the critical directions $h \in \mathcal{K}_C^*(\bar{x}, \varphi)$ to decide whether \bar{x} is a local minimizer or not. If we fix a critical direction $h \in \mathcal{K}_C^*(\bar{x}, \varphi)$, then the definition of $\mathcal{T}_C^*(\bar{x})$ yields the existence of sequences $\{r_k\} \subset X$, $\{t_k\} \subset \mathbb{R}^+$ with $t_k \searrow 0$, $t_k r_k \xrightarrow{\star} 0$ and $x_k := \bar{x} + t_k h + \frac{1}{2} t_k^2 r_k \in C$, and we may calculate that $J(x_k) - J(\bar{x}) = \frac{1}{2} t_k^2 \langle \varphi, r_k \rangle$. This identity suggests that the limiting behavior of the dual pairing $\langle \varphi, r_k \rangle$ between the gradient φ of the objective J and the second-order correction r_k is a decisive factor in the study of the optimality of the critical point \bar{x} . Since the directional curvature functional $Q_C^{\bar{x},\varphi}(\cdot)$ allows to estimate the limes inferior of precisely that quantity for all possible sequences $\{r_k\}$ and $\{t_k\}$, it is only natural to consider it an adequate tool for the derivation of optimality conditions. Note that, instead of looking at the accumulation points of the corrections r_k , which is the idea of second-order tangent sets, cf. **Definition 5.2**, we only study accumulation points of the scalar sequences $\langle \varphi, r_k \rangle$ when working with the functional $Q_C^{\bar{x},\varphi}(\cdot)$. Thus, it is possible to obtain information even when the second-order corrections r_k diverge or cannot be analyzed properly. Before turning our attention to SNC and SSC, in what follows, we first state some preliminary results on the properties of the directional curvature functional that are needed for our investigation.

Lemma 3.2. *Let $x \in C$ be given. Then the following assertions hold.*

- (i) *For all $\varphi \in -\mathcal{N}_C^*(x)$, $h \in \mathcal{K}_C^*(x, \varphi)$ and $\alpha > 0$, we have $Q_C^{x,\varphi}(\alpha h) = \alpha^2 Q_C^{x,\varphi}(h)$.*
- (ii) *If C is convex, then $Q_C^{x,\varphi}(h) \geq 0$ for all $\varphi \in -\mathcal{N}_C^*(x)$ and $h \in \mathcal{K}_C^*(x, \varphi)$.*

Proof. Assertion (i) can be checked by a simple scaling argument. To prove (ii), suppose that C is convex, let $h \in \mathcal{K}_C^*(x, \varphi)$ and $\varphi \in -\mathcal{N}_C^*(x)$ be arbitrary but fixed, and let $\{r_k\}$,

$\{t_k\}$ be sequences as in the definition of $Q_C^{x,\varphi}(h)$. Then it holds $\frac{2}{t_k}h + r_k \in \mathcal{R}_C(x) \subset \mathcal{T}_C^*(x)$ due to the convexity of C and, consequently, $\langle \varphi, r_k \rangle = \langle \varphi, \frac{2}{t_k}h + r_k \rangle \geq 0$. Taking the limes inferior for $k \rightarrow \infty$ and the infimum over all $\{r_k\}, \{t_k\}$ now yields the claim. \square

In addition to [Lemma 3.2](#), we have the following weak- \star lower semicontinuity result.

Lemma 3.3. *Let $x \in C$ and $\varphi \in -\mathcal{N}_C^*(x)$ be given. Let $\{h_n\} \subset \mathcal{K}_C^*(x, \varphi)$ be a sequence such that $h_n \xrightarrow{\star} h$ holds for some $h \in X$ and such that there exist sequences $\{r_{n,k}\} \subset X$ and $\{t_{n,k}\} \subset \mathbb{R}^+$ and a constant $M > 0$ with*

$$\begin{aligned} t_{n,k} \searrow 0, \quad t_{n,k} r_{n,k} \xrightarrow{\star} 0, \quad \langle \varphi, r_{n,k} \rangle \rightarrow Q_C^{x,\varphi}(h_n) \quad \text{for all } n \text{ as } k \rightarrow \infty \text{ and} \\ \|t_{n,k} r_{n,k}\|_X \leq M, \quad x + t_{n,k} h_n + \frac{1}{2} t_{n,k}^2 r_{n,k} \in C \quad \text{for all } n, k. \end{aligned}$$

Then, h is an element of the critical cone $\mathcal{K}_C^*(x, \varphi)$ and it holds

$$Q_C^{x,\varphi}(h) \leq \liminf_{n \rightarrow \infty} Q_C^{x,\varphi}(h_n). \quad (2)$$

Proof. Consider a countable dense subset $\{y_i\}$ of Y and choose a sequence k_n such that

$$t_{n,k_n} \leq \frac{1}{n}, \quad \langle \varphi, r_{n,k_n} \rangle \leq Q_C^{x,\varphi}(h_n) + \frac{1}{n} \quad \text{and} \quad |\langle y_i, t_{n,k_n} r_{n,k_n} \rangle| \leq \frac{1}{n} \quad \forall i \leq n \quad (3)$$

holds for all $n \in \mathbb{N}$. Then, our assumptions on $r_{n,k}$ imply $\|t_{n,k_n} r_{n,k_n}\|_X \leq M$, and we may deduce from (3) that the sequences $t_n := t_{n,k_n}$ and $r_n := r_{n,k_n} + \frac{2}{t_n}(h_n - h)$ satisfy

$$t_n \searrow 0, \quad t_n r_n \xrightarrow{\star} 0, \quad x + t_n h + \frac{1}{2} t_n^2 r_n \in C \quad \text{and} \quad \liminf_{n \rightarrow \infty} \langle \varphi, r_n \rangle \leq \liminf_{n \rightarrow \infty} Q_C^{x,\varphi}(h_n).$$

The above yields $h \in \mathcal{T}_C^*(x)$ and, since we trivially have $h \in \varphi^\perp$, $h \in \mathcal{K}_C^*(x, \varphi)$. Moreover, we obtain (2) from the definition of $Q_C^{x,\varphi}(h)$. This proves the claim. \square

We point out that [Lemma 3.3](#) is in particular applicable if for every $h \in \mathcal{K}_C^*(x, \varphi)$ we can find sequences $\{r_k\} \subset X$ and $\{t_k\} \subset \mathbb{R}^+$ that realize the infimum in (1) with strong convergence $t_k r_k \rightarrow 0$. Since sets C with the latter property prove to be useful also in different contexts, we introduce the following concept.

Definition 3.4 (Mosco Regularity Condition (MRC)). *We say that C is Mosco regular in $(x, \varphi) \in C \times -\mathcal{N}_C^*(x)$ if*

$$\begin{aligned} \forall h \in \mathcal{K}_C^*(x, \varphi) \exists \{r_k\} \subset X, \{t_k\} \subset \mathbb{R}^+ : \\ t_k \searrow 0, \quad t_k r_k \rightarrow 0, \quad x + t_k h + \frac{1}{2} t_k^2 r_k \in C, \quad Q_C^{x,\varphi}(h) = \lim_{k \rightarrow \infty} \langle \varphi, r_k \rangle. \end{aligned} \quad (\text{MRC})$$

Remark 3.5. *It is easy to see that (MRC) holds in $(x, \varphi) \in C \times -\mathcal{N}_C^*(x)$ if and only if*

$$Q_C^{x,\varphi}(h) = \inf \left\{ \liminf_{k \rightarrow \infty} \langle \varphi, r_k \rangle \mid \left. \begin{array}{l} \{r_k\} \subset X, \{t_k\} \subset \mathbb{R}^+ : t_k \searrow 0, t_k r_k \rightarrow 0, \\ x + t_k h + \frac{1}{2} t_k^2 r_k \in C \end{array} \right\} \right.$$

for all $h \in \mathcal{K}_C^*(x, \varphi)$, i.e., if and only if the functional $Q_C^{x, \varphi}(\cdot)$ remains unchanged when we replace the weak- \star convergence of $t_k r_k$ with strong convergence. In the context of Kuratowski limits, the latter means that the weak- \star limes superior and the strong limes superior of the epigraphs of the second-order difference quotients associated with the characteristic function of the set C , the point x , and the element $\varphi \in -\mathcal{N}_C^*(x)$ coincide. We again refer to [Do, 1992] for details.

We will see in the following section that the condition (MRC) is also of significance for the study of second-order optimality conditions as it allows to weaken the regularity assumptions on the objective needed for the derivation of SNC. Note that (MRC) is always satisfied when X is finite-dimensional. Further conditions ensuring (MRC) can be found in Lemmas 5.3 and 5.7.

4 Necessary and Sufficient Second-Order Conditions

Having introduced the directional curvature functional $Q_C^{x, \varphi}(\cdot)$, we now turn our attention to SNC and SSC for minimization problems of the form

$$\text{Minimize } J(x), \text{ such that } x \in C. \quad (\text{P})$$

In the remainder of this paper, when discussing optimality conditions for a problem of the type (P), we always require that (in addition to our standing Assumption 2.1) the following assumption holds.

Assumption 4.1.

- (i) \bar{x} is a fixed element of the set C (the minimizer/candidate for a minimizer),
- (ii) $J : C \rightarrow \mathbb{R}$ is a function such that there exist a $J'(\bar{x}) \in \iota(Y)$ and a bounded bilinear $J''(\bar{x}) : X \times X \rightarrow \mathbb{R}$ with

$$\lim_{k \rightarrow \infty} \frac{J(\bar{x} + t_k h_k) - J(\bar{x}) - t_k J'(\bar{x}) h_k - \frac{1}{2} t_k^2 J''(\bar{x}) h_k^2}{t_k^2} = 0 \quad (4)$$

for all $\{h_k\} \subset X$, $\{t_k\} \subset \mathbb{R}^+$ satisfying $t_k \searrow 0$, $h_k \xrightarrow{*} h \in X$ and $\bar{x} + t_k h_k \in C$.

Note that we use the abbreviations $J'(\bar{x}) h := \langle J'(\bar{x}), h \rangle$ and $J''(\bar{x}) h^2 := J''(\bar{x})(h, h)$ for all $h \in X$ in (4), and that (4) is automatically satisfied if J admits a second-order Taylor expansion of the form

$$J(\bar{x} + h) - J(\bar{x}) - J'(\bar{x}) h - \frac{1}{2} J''(\bar{x}) h^2 = o(\|h\|_X^2) \quad \text{as } \|h\|_X \rightarrow 0. \quad (5)$$

We begin our investigation by stating necessary conditions of first order.

Theorem 4.2 (First-Order Necessary Condition). *Suppose that \bar{x} is a local minimizer of (P), i.e., assume that there is an $\varepsilon > 0$ with*

$$J(x) \geq J(\bar{x}) \quad \forall x \in C \cap B_\varepsilon^X(\bar{x}),$$

where $B_\varepsilon^X(\bar{x})$ denotes the closed ball of radius ε in X around \bar{x} . Then, $J'(\bar{x}) \in -\mathcal{N}_C^*(\bar{x})$.

Proof. Let $h \in \mathcal{T}_C^*(\bar{x})$ be given. By definition, there are sequences $\{x_k\} \subset C$, $\{t_k\} \subset \mathbb{R}^+$ with $t_k \searrow 0$ and $(x_k - \bar{x})/t_k \xrightarrow{*} h$. We set $h_k := (x_k - \bar{x})/t_k$. Then for large enough k , we have (due to (4) and the boundedness of weakly- \star convergent sequences)

$$0 \leq \frac{J(x_k) - J(\bar{x})}{t_k} = \frac{J(\bar{x} + t_k h_k) - J(\bar{x})}{t_k} = J'(\bar{x}) h_k + \mathcal{O}(t_k) \rightarrow J'(\bar{x}) h.$$

The above and our assumption $J'(\bar{x}) \in \iota(Y)$ yield $J'(\bar{x}) \in -\mathcal{N}_C^*(\bar{x})$ as claimed. \square

Using **Theorem 4.2**, we can provide second-order optimality conditions for (P).

Theorem 4.3 (SNC Involving the Directional Curvature Functional). *Suppose that \bar{x} is a local minimizer of (P) such that*

$$J(x) \geq J(\bar{x}) + \frac{c}{2} \|x - \bar{x}\|_X^2 \quad \forall x \in C \cap B_\varepsilon^X(\bar{x}) \quad (6)$$

holds for some $c \geq 0$ and some $\varepsilon > 0$. Assume further that one of the following conditions is satisfied.

- (i) The map $h \mapsto J''(\bar{x}) h^2$ is weak- \star upper semicontinuous.
- (ii) The admissible set C satisfies (MRC) in $(\bar{x}, J'(\bar{x})) \in C \times -\mathcal{N}_C^*(\bar{x})$.

Then

$$Q_C^{\bar{x}, J'(\bar{x})}(h) + J''(\bar{x}) h^2 \geq c \|h\|_X^2 \quad \forall h \in \mathcal{K}_C^*(\bar{x}, J'(\bar{x})). \quad (7)$$

Proof. We first consider the case with (i): From the definition of $Q_C^{\bar{x}, J'(\bar{x})}(\cdot)$, it follows that for every $h \in \mathcal{K}_C^*(\bar{x}, J'(\bar{x}))$ and every $\delta > 0$ we can find sequences $\{r_k\} \subset X$ and $\{t_k\} \subset \mathbb{R}^+$ such that $t_k \searrow 0$, $t_k r_k \xrightarrow{*} 0$, $x_k := \bar{x} + t_k h + \frac{1}{2} t_k^2 r_k \in C$ and

$$\lim_{k \rightarrow \infty} \langle J'(\bar{x}), r_k \rangle \leq Q_C^{\bar{x}, J'(\bar{x})}(h) + \delta. \quad (8)$$

Since $x_k \rightarrow \bar{x}$ strongly in X , (6) entails $J(x_k) \geq J(\bar{x}) + \frac{c}{2} \|x_k - \bar{x}\|_X^2$ for large enough k . Hence, we may use (4), $J'(\bar{x})h = 0$ and the weak- \star lower semicontinuity of the norm $\|\cdot\|_X$ to obtain

$$\begin{aligned} 0 &= \lim_{k \rightarrow \infty} \frac{J(x_k) - J(\bar{x}) - t_k J'(\bar{x}) \left(h + \frac{1}{2} t_k r_k \right) - \frac{1}{2} t_k^2 J''(\bar{x}) \left(h + \frac{1}{2} t_k r_k \right)^2}{t_k^2} \\ &\geq \limsup_{k \rightarrow \infty} \frac{\frac{c}{2} t_k^2 \|h + \frac{1}{2} t_k r_k\|_X^2 - t_k J'(\bar{x}) \left(\frac{1}{2} t_k r_k \right) - \frac{1}{2} t_k^2 J''(\bar{x}) \left(h + \frac{1}{2} t_k r_k \right)^2}{t_k^2} \\ &\geq \frac{c}{2} \|h\|_X^2 - \frac{1}{2} \liminf_{k \rightarrow \infty} \langle J'(\bar{x}), r_k \rangle - \frac{1}{2} \limsup_{k \rightarrow \infty} (J''(\bar{x}) \left(h + \frac{1}{2} t_k r_k \right)^2). \end{aligned} \quad (9)$$

From (8), (9) and the weak- \star upper semicontinuity of $h \mapsto J''(\bar{x}) h^2$, it follows

$$c \|h\|_X^2 \leq Q_C^{\bar{x}, J'(\bar{x})}(h) + \delta + J''(\bar{x}) h^2.$$

Passing to the limit $\delta \searrow 0$ in this inequality yields the claim in the first case.

It remains to prove (7) under assumption (ii). To this end, we note that, if (MRC) holds in $(\bar{x}, J'(\bar{x})) \in C \times -\mathcal{N}_C^*(\bar{x})$, then for every $h \in \mathcal{K}_C^*(\bar{x}, J'(\bar{x}))$ we can find sequences $\{r_k\} \subset X$, $\{t_k\} \subset \mathbb{R}^+$ such that $t_k \searrow 0$, $t_k r_k \rightarrow 0$, $x_k := \bar{x} + t_k h + \frac{1}{2} t_k^2 r_k \in C$ and

$$\lim_{k \rightarrow \infty} \langle J'(\bar{x}), r_k \rangle = Q_C^{\bar{x}, J'(\bar{x})}(h).$$

Now, the second-order condition (7) follows analogously to case (i). \square

Theorem 4.4 (SSC Involving the Directional Curvature Functional). *Assume that the map $h \mapsto J''(\bar{x}) h^2$ is weak- \star lower semicontinuous, that $J'(\bar{x}) \in -\mathcal{N}_C^*(\bar{x})$ and that*

$$Q_C^{\bar{x}, J'(\bar{x})}(h) + J''(\bar{x}) h^2 > 0 \quad \forall h \in \mathcal{K}_C^*(\bar{x}, J'(\bar{x})) \setminus \{0\}. \quad (10)$$

Suppose further that

$$\begin{aligned} & \text{for all } \{h_k\} \subset X, \{t_k\} \subset \mathbb{R}^+ \text{ with } \bar{x} + t_k h_k \in C, h_k \xrightarrow{\star} 0, t_k \searrow 0 \\ & \text{and } \|h_k\|_X = 1, \text{ it is true that} \end{aligned} \quad (\text{NDC})$$

$$\liminf_{k \rightarrow \infty} (\langle J'(\bar{x}), h_k/t_k \rangle + \frac{1}{2} J''(\bar{x}) h_k^2) > 0.$$

Then \bar{x} satisfies the growth condition (6) with some constants $c > 0$ and $\varepsilon > 0$.

Proof. We argue by contradiction. Assume that there are no $c > 0$, $\varepsilon > 0$ such that (6) holds. Then there are sequences $\{x_k\} \subset C$ and $\{c_k\} \subset \mathbb{R}^+$ such that

$$c_k \searrow 0, \quad \|x_k - \bar{x}\|_X \rightarrow 0, \quad \text{and} \quad J(x_k) < J(\bar{x}) + \frac{c_k}{2} \|x_k - \bar{x}\|_X^2.$$

Define $t_k := \|x_k - \bar{x}\|_X$ and $h_k := (x_k - \bar{x})/t_k$. Then $\|h_k\|_X = 1$ for all k and we may extract a subsequence (not relabeled) such that $h_k \xrightarrow{\star} h \in \mathcal{T}_C^*(\bar{x})$. From (4), it follows

$$\begin{aligned} 0 &= \lim_{k \rightarrow \infty} \frac{J(\bar{x} + t_k h_k) - J(\bar{x}) - t_k J'(\bar{x}) h_k - \frac{1}{2} t_k^2 J''(\bar{x}) h_k^2}{t_k^2} \\ &\leq \liminf_{k \rightarrow \infty} \frac{\frac{c_k}{2} \|x_k - \bar{x}\|_X^2 - t_k J'(\bar{x}) h_k - \frac{1}{2} t_k^2 J''(\bar{x}) h_k^2}{t_k^2} \\ &= \liminf_{k \rightarrow \infty} \frac{-J'(\bar{x}) h_k - \frac{1}{2} t_k J''(\bar{x}) h_k^2}{t_k} \\ &\leq -\limsup_{k \rightarrow \infty} \frac{J'(\bar{x}) h_k}{t_k} - \liminf_{k \rightarrow \infty} \frac{\frac{1}{2} t_k J''(\bar{x}) h_k^2}{t_k} \leq -\limsup_{k \rightarrow \infty} \frac{J'(\bar{x}) h_k}{t_k} - \frac{1}{2} J''(\bar{x}) h^2. \end{aligned} \quad (11)$$

Hence, $\limsup_{k \rightarrow \infty} J'(\bar{x}) h_k \leq 0$. Together with $J'(\bar{x}) \in -\mathcal{N}_C^*(\bar{x})$ we find $J'(\bar{x}) h = 0$. Using $h \in \mathcal{T}_C^*(\bar{x})$, this yields $h \in \mathcal{K}_C^*(\bar{x}, J'(\bar{x}))$. Setting $r_k := 2(h_k - h)/t_k$ and using (11), $J'(\bar{x}) h = 0$ and Definition 3.1, we obtain

$$J''(\bar{x}) h^2 \leq -2 \limsup_{k \rightarrow \infty} \frac{J'(\bar{x}) h_k}{t_k} = -\limsup_{k \rightarrow \infty} J'(\bar{x}) r_k \leq -\liminf_{k \rightarrow \infty} J'(\bar{x}) r_k \leq -Q_C^{\bar{x}, J'(\bar{x})}(h).$$

From (10), we may now deduce that h is zero. This is a contradiction with (NDC), see the properties of the sequences $\{h_k\}$, $\{t_k\}$ and (11). \square

The acronym **(NDC)** stands for “non-degeneracy condition”. Comments on **(NDC)** are provided in **Section 5.1**. By combining the previous two theorems, we arrive at our main theorem on no-gap second-order conditions.

Theorem 4.5 (No-Gap Second-Order Optimality Condition). *Assume that the map $h \mapsto J''(\bar{x})h^2$ is weak- \star lower semicontinuous, that $J'(\bar{x}) \in -\mathcal{N}_C^*(\bar{x})$, that **(NDC)** holds, and that one of the conditions **(i)** and **(ii)** in **Theorem 4.3** is satisfied. Then, the condition*

$$Q_C^{\bar{x}, J'(\bar{x})}(h) + J''(\bar{x})h^2 > 0 \quad \forall h \in \mathcal{K}_C^*(\bar{x}, J'(\bar{x})) \setminus \{0\}$$

is equivalent to the quadratic growth condition **(6)** with constants $c > 0$ and $\varepsilon > 0$.

Some remarks regarding **Theorems 4.3** to **4.5** are in order.

Remark 4.6.

- (i) Note that **Theorem 4.3** yields that there are two ways to obtain the second-order necessary condition **(7)** in the situation of **Assumption 4.1**: We can assume either that the second derivative of J at \bar{x} has additional (semi)continuity properties (this is case **(i)**) or that the set C is sufficiently well-behaved at \bar{x} (this is case **(ii)**). The observation that one has the choice to impose additional assumptions either on the appearing sets or the appearing functions can be made frequently when working with the directional curvature functional.
- (ii) We point out that the regularity condition **(MRC)** is not helpful in the derivation of the SSC in **Theorem 4.4**. At the heart of the proof of **Theorem 4.4** is, after all, the contradiction argument which only provides a weak- \star convergent subsequence.
- (iii) It is easy to see that the differentiability assumptions on J in **Assumption 4.1** can be weakened when one is interested in only one of the results in **Theorems 4.3** and **4.4**. To derive the necessary condition **(7)** in case **(ii)** of **Theorem 4.3**, for example, it suffices to assume that

$$h \in X, \{r_k\} \subset X, \{t_k\} \subset \mathbb{R}^+, t_k \searrow 0, t_k r_k \rightarrow 0 \text{ in } X \\ \Rightarrow \max \left(0, \frac{J(\bar{x} + t_k h + \frac{1}{2} t_k^2 r_k) - J(\bar{x}) - t_k J'(\bar{x})(h + \frac{1}{2} t_k r_k) - \frac{1}{2} t_k^2 J''(\bar{x})h^2}{t_k^2} \right) \rightarrow 0$$

holds for some bounded, linear mapping $J'(\bar{x}) : X \rightarrow \mathbb{R}$ and some bilinear form $J''(\bar{x}) : X \times X \rightarrow \mathbb{R}$. We refrain from stating the minimal differentiability properties in each of the **Theorems 4.3** to **4.5** since this would just obscure the basic ideas of our analysis.

- (iv) We point out that it is possible to modify the proofs of **Theorems 4.3** and **4.4** to obtain second-order conditions in the setting of two-norms discrepancy, i.e., in the situation where the objective J only satisfies a second-order Taylor expansion à la **(5)** w.r.t. some norm $\|\cdot\|_Z$ that is stronger than $\|\cdot\|_X$. We leave it to the interested reader to work out this easy extension of our analysis in detail.
- (v) In the finite-dimensional setting, results similar to **Theorems 4.3** to **4.5** have been obtained in [**Rockafellar and Wets, 1998**, Theorem 13.24].

5 How to Apply and Interpret the Results of Section 4 in the Context of the Classical Theory

In this section, we comment on the verification and interpretation of the conditions (NDC) and (MRC) appearing in our second-order conditions, see Section 5.1, Lemma 5.3, and Lemma 5.7. Further, we address the computation of the directional curvature functional for polyhedral and second-order regular sets, see Section 5.2. Finally, we compare our theorems of Section 4 with classical results. In Section 5.3, we use the observations of Sections 5.1 and 5.2 to state two corollaries of Theorem 4.5 that reproduce (and slightly extend) classical results found, e.g., in [Bonnans and Shapiro, 2000] and [Casas and Tröltzsch, 2015]. We conclude this section with two examples that illustrate the usefulness of Theorems 4.3 to 4.5.

5.1 Remarks on the Non-Degeneracy Condition (NDC)

The condition (NDC) appearing in our SSC can be interpreted as a generalized Legendre condition, cf. [Bonnans and Shapiro, 2000, Section 3.3.2] and Lemma 5.1 (ii). In contrast to a classical Legendre condition, our requirement (NDC) is a condition on the interplay between the curvature of C in \bar{x} in the direction $J'(\bar{x})$ and the properties of $J''(\bar{x})$. In practice, (NDC) can be ensured, e.g., by assuming ellipticity of the second derivative $J''(\bar{x})$ or by assuming that the admissible set C has “positive” curvature at \bar{x} (or some combination of the both). Some sufficient criteria can be found in the following lemma.

Lemma 5.1. *Each of the following conditions is sufficient for (NDC).*

- (i) C is a convex subset of X , $J'(\bar{x}) \in -\mathcal{N}_C^*(\bar{x})$, and $J''(\bar{x})h^2 \geq r > 0$ for all $h \in X$ with $\|h\|_X = 1$.
- (ii) C is a convex subset of X , $J'(\bar{x}) \in -\mathcal{N}_C^*(\bar{x})$, and $J''(\bar{x})$ is a Legendre form in the sense that $h \mapsto J''(\bar{x})h^2$ is weak- \star lower semicontinuous and

$$h_k \xrightarrow{\star} h \text{ and } J''(\bar{x})h_k^2 \rightarrow J''(\bar{x})h^2 \quad \Rightarrow \quad h_k \rightarrow h$$

holds for all sequences $\{h_k\} \subset X$.

- (iii) There exist $c, \varepsilon > 0$ with $J'(\bar{x})(x - \bar{x}) \geq \frac{c}{2} \|x - \bar{x}\|^2$ for all $x \in B_\varepsilon^X(\bar{x}) \cap C$ and the map $h \mapsto J''(\bar{x})h^2$ is weak- \star lower semicontinuous.

- (iv) X is finite-dimensional.

Proof. Case (iv) is trivial. To prove (NDC) in the cases (i) to (iii), we have to show that for all sequences $\{t_k\}, \{h_k\}$ satisfying $t_k \searrow 0$, $h_k \xrightarrow{\star} 0$ for $k \rightarrow \infty$ and $\|h_k\|_X = 1$, $\bar{x} + t_k h_k \in C$ for all k , it holds $\liminf_{k \rightarrow \infty} (\langle J'(\bar{x}), h_k/t_k \rangle + \frac{1}{2} J''(\bar{x})h_k^2) > 0$. Therefore, we assume that sequences $\{t_k\}, \{h_k\}$ with the above properties are given.

We first discuss the case that C is convex. For such a C , it holds $h_k \in \mathcal{R}_C(\bar{x})$ and, as a consequence, $J'(\bar{x})h_k \geq 0$, so it suffices to prove $\liminf_{k \rightarrow \infty} J''(\bar{x})h_k^2 > 0$ to obtain the claim. The latter is trivially true under assumption (i). Similarly, in case (ii), we

have $\liminf_{k \rightarrow \infty} J''(\bar{x}) h_k^2 \geq 0$ by lower semicontinuity. Moreover, if we had $J''(\bar{x}) h_k^2 \rightarrow 0$ (along a subsequence), we would obtain the contradiction $h_k \rightarrow 0$ (along a subsequence). Hence, $\liminf_{k \rightarrow \infty} J''(\bar{x}) h_k^2 > 0$ and (NDC) is proved for (ii).

In case (iii), we have

$$\langle J'(\bar{x}), h_k/t_k \rangle = t_k^{-2} \langle J'(\bar{x}), t_k h_k \rangle \geq \frac{c}{2} t_k^{-2} \|t_k h_k\|^2 = \frac{c}{2}$$

for k large enough. Together with $\liminf_{k \rightarrow \infty} J''(\bar{x}) h_k^2 \geq 0$, the above yields the desired inequality $\liminf_{k \rightarrow \infty} (\langle J'(\bar{x}), h_k/t_k \rangle + \frac{1}{2} J''(\bar{x}) h_k^2) > 0$. This completes the proof. \square

Note that in case (iii) of Lemma 5.1, the second derivative $J''(\bar{x})$ is allowed to be negative definite. This effect occurs whenever the curvature of the set C at \bar{x} is such that it can compensate for negative curvature of the objective J . In Section 6.2, we will see an example, where the latter actually happens. We further point out that, under the assumptions of Lemma 5.1 (iii), the directional curvature functional is coercive in the sense that

$$Q_C^{\bar{x}, J'(\bar{x})}(h) \geq c \|h\|_X^2 \quad \forall h \in \mathcal{K}_C^*(\bar{x}, J'(\bar{x})), \quad (12)$$

where c is the constant in Lemma 5.1 (iii). Indeed, for $h \in \mathcal{K}_C^*(\bar{x}, J'(\bar{x}))$, $t_k \searrow 0$, $t_k r_k \xrightarrow{*} 0$ and $x_k := \bar{x} + t_k h + \frac{1}{2} t_k^2 r_k \in C$, we have

$$\begin{aligned} \liminf_{k \rightarrow \infty} J'(\bar{x}) r_k &= \liminf_{k \rightarrow \infty} J'(\bar{x}) \frac{x_k - \bar{x}}{\frac{1}{2} t_k^2} \\ &\geq \liminf_{k \rightarrow \infty} c \frac{\|x_k - \bar{x}\|_X^2}{t_k^2} = \liminf_{k \rightarrow \infty} c \|h + \frac{1}{2} t_k r_k\|_X^2 \geq c \|h\|_X^2. \end{aligned}$$

The above gives a lower estimate for the directional curvature functional that is often also interesting for its own sake. We will get back to this topic in Section 6.2.

5.2 Directional Curvature, Polyhedricity and Second-Order Regularity

In this section, we discuss how the notion of directional curvature is related to the classical concepts of polyhedricity and second-order regularity, and how these properties can be used to calculate the functional $Q_C^{x, \varphi}(\cdot)$ for a given tuple $(x, \varphi) \in C \times -\mathcal{N}_C^*(x)$. Recall the following definitions, cf. [Bonnans and Shapiro, 2000, Section 3.2.1, Definitions 3.51, 3.85], [Mignot, 1976], and [G. Wachsmuth, 2016, Lemma 4.1].

Definition 5.2 (Polyhedricity and Second-Order Regularity).

(i) The set C is said to be polyhedric at $x \in C$ if C is convex and

$$\mathcal{T}_C(x) \cap \varphi^\perp = \text{cl}(\mathcal{R}_C(x) \cap \varphi^\perp) \quad \forall \varphi \in X^*.$$

(ii) The strong (outer) second-order tangent set to a tuple $(x, h) \in C \times \mathcal{T}_C(x)$ is given by

$$\mathcal{T}_C^2(x, h) := \left\{ r \in X \mid \exists t_n \searrow 0, \text{dist}(x + t_n h + \frac{1}{2} t_n^2 r, C) = o(t_n^2) \right\}.$$

(iii) The set C is called (outer) second-order regular at $x \in C$ if for all $h \in \mathcal{T}_C(x)$ and all $x_n \in C$ of the form $x_n := x + t_n h + \frac{1}{2} t_n^2 r_n$ with $t_n \searrow 0$ and $t_n r_n \rightarrow 0$ it holds

$$\lim_{n \rightarrow \infty} \text{dist} \left(r_n, \mathcal{T}_C^2(x, h) \right) = 0.$$

First, we show that the boundary of polyhedric sets is not curved.

Lemma 5.3 (Curvature of Polyhedric Sets). *Assume that X is reflexive and that C is polyhedric at $x \in C$. Then (MRC) is satisfied in (x, φ) for all $\varphi \in -\mathcal{N}_C^*(x)$ and*

$$Q_C^{x, \varphi}(h) = 0 \quad \forall h \in \mathcal{K}_C^*(x, \varphi).$$

Proof. From the reflexivity, Mazur's lemma and the convexity and closedness of C , it follows $\mathcal{T}_C^*(x) = \mathcal{T}_C(x)$. Consequently, $\mathcal{K}_C^*(x, \varphi) = \mathcal{T}_C(x) \cap \varphi^\perp$ for every $\varphi \in -\mathcal{N}_C^*(x)$. For $h \in \mathcal{R}_C(x) \cap \varphi^\perp$ the choice $t_k = 1/k$, $k \in \mathbb{N}$ sufficiently large, and $r_k = 0$ shows $Q_C^{x, \varphi}(h) = 0$, see also [Lemma 3.2 \(ii\)](#). Now, let $h \in \mathcal{K}_C^*(x, \varphi) = \mathcal{T}_C(x) \cap \varphi^\perp$ be given. Owing to the polyhedricity of C at x , there exists a sequence $\{h_k\} \subset \mathcal{R}_C(x) \cap \varphi^\perp$ with $h_k \rightarrow h$ in X . Now, we can apply [Lemma 3.2 \(ii\)](#) and [Lemma 3.3](#) (with $M = 0$) to obtain

$$0 \leq Q_C^{x, \varphi}(h) \leq \liminf_{k \rightarrow \infty} Q_C^{x, \varphi}(h_k) = 0.$$

The recovery sequence in (MRC) can be constructed straightforwardly from $\{h_k\}$. \square

[Lemma 5.3](#) shows that, for a polyhedric set C , the directional curvature functional is always identical zero. This is, of course, exactly what one would expect (cf. also with [\[Do, 1992, Example 2.10\]](#) in this context). The situation is different when C possesses curvature in the sense of second-order regularity – a concept that is promoted and extensively used in [\[Bonnans and Shapiro, 2000\]](#). The curvature of second-order regular sets is addressed in the next lemma.

Lemma 5.4 (Curvature of Second-Order Regular Sets). *Assume that C is outer second-order regular at $x \in C$, that $\varphi \in -\mathcal{N}_C^*(x)$, and that (MRC) is satisfied in (x, φ) . Then*

$$Q_C^{x, \varphi}(h) = \inf_{r \in \mathcal{T}_C^2(x, h)} \langle \varphi, r \rangle \quad \forall h \in \mathcal{K}_C^*(x, \varphi).$$

Proof. Let $h \in \mathcal{K}_C^*(x, \varphi)$ be given. Then (MRC) yields that we can find sequences $\{r_k\} \subset X$, $\{t_k\} \subset \mathbb{R}^+$ such that $t_k \searrow 0$, $t_k r_k \rightarrow 0$, $x + t_k h + \frac{1}{2} t_k^2 r_k \in C$ and

$$Q_C^{x, \varphi}(h) = \lim_{k \rightarrow \infty} \langle \varphi, r_k \rangle.$$

Note that the above implies in particular that $h \in \mathcal{T}_C(x)$. From the outer second-order regularity of C in x , we now obtain

$$\lim_{k \rightarrow \infty} \text{dist} \left(r_k, \mathcal{T}_C^2(x, h) \right) = 0,$$

i.e., there exists a sequence $\{\tilde{r}_k\} \subset \mathcal{T}_C^2(x, h)$ with

$$Q_C^{x, \varphi}(h) = \lim_{k \rightarrow \infty} \langle \varphi, r_k \rangle = \lim_{k \rightarrow \infty} \langle \varphi, \tilde{r}_k \rangle \geq \inf_{r \in \mathcal{T}_C^2(x, h)} \langle \varphi, r \rangle. \quad (13)$$

If, on the other hand, $r \in \mathcal{T}_C^2(x, h)$ is arbitrary but fixed, then we know that there are sequences $\{t_k\} \subset \mathbb{R}^+$, $\{s_k\} \subset X$ with

$$t_k \searrow 0, \quad x + t_k h + \frac{1}{2} t_k^2 (r + s_k) \in C, \quad s_k \rightarrow 0,$$

and we obtain from the definition of $Q_C^{x, \varphi}(h)$ that

$$\begin{aligned} Q_C^{x, \varphi}(h) &\leq \inf \left\{ \liminf_{k \rightarrow \infty} \langle \varphi, r_k \rangle \mid \begin{array}{l} \{r_k\} \subset X, \{t_k\} \subset \mathbb{R}^+ : t_k \searrow 0, t_k r_k \rightarrow 0, \\ x + t_k h + \frac{1}{2} t_k^2 r_k \in C \end{array} \right\} \\ &\leq \liminf_{k \rightarrow \infty} \langle \varphi, r + s_k \rangle = \langle \varphi, r \rangle. \end{aligned}$$

This yields

$$Q_C^{x, \varphi}(h) \leq \inf_{r \in \mathcal{T}_C^2(x, h)} \langle \varphi, r \rangle$$

which, together with (13), proves the claim. \square

We would like to point out that [Lemma 5.3](#) cannot be obtained as a corollary of [Lemma 5.4](#). The reason for this is that polyhedral sets do not necessarily have to be second-order regular. In fact, we have the following result.

Lemma 5.5 (Necessary Condition for Second-Order Regularity). *Assume that C is outer second-order regular at $x \in C$. Then, for all $h \in \mathcal{T}_C(x)$, it holds $\mathcal{T}_C^2(x, h) \neq \emptyset$ and there exists a positive sequence $t_k \searrow 0$ such that $\text{dist}(x + t_k h, C) = \mathcal{O}(t_k^2)$ as $k \rightarrow \infty$.*

Proof. Let $h \in \mathcal{T}_C(x)$ be given. Then, by the definition of $\mathcal{T}_C(x)$, there exist sequences $\{t_k\} \subset \mathbb{R}^+$ and $\{x_k\} \subset C$ with $t_k \searrow 0$ and $(x_k - x)/t_k \rightarrow h$. Set $r_k := 2t_k^{-2}(x_k - x - t_k h)$. Then r_k satisfies $x_k = x + t_k h + \frac{1}{2} t_k^2 r_k$ and $t_k r_k \rightarrow 0$, and the second-order regularity of C at x yields $\text{dist}(r_k, \mathcal{T}_C^2(x, h)) \rightarrow 0$. This proves $\mathcal{T}_C^2(x, h) \neq \emptyset$.

Now, let $r \in \mathcal{T}_C^2(x, h)$ be given. By definition, there is a positive sequence $t_k \searrow 0$ such that $\text{dist}(x + t_k h + \frac{1}{2} t_k^2 r, C) = o(t_k^2)$. Using the latter and the triangle inequality, we obtain

$$\text{dist}(x + t_k h, C) \leq \text{dist}\left(x + t_k h + \frac{1}{2} t_k^2 r, C\right) + \frac{1}{2} t_k^2 \|r\|_X = o(t_k^2) + \mathcal{O}(t_k^2) = \mathcal{O}(t_k^2).$$

This finishes the proof. \square

Using [Lemma 5.5](#), we can prove that even the most elementary examples of infinite-dimensional polyhedral sets can lack the property of second-order regularity.

Example 5.6. *Let $(0, 1)$ be equipped with Lebesgue's measure. Define $X := L^2(0, 1)$ and $C := \{v \in L^2(0, 1) : v \geq 0\}$, and let x be the unique element of X with $x(\xi) = 1$ for a.a. $\xi \in (0, 1)$. Then, C is polyhedral at x and it holds $\mathcal{T}_C(x) = L^2(0, 1)$. Consider now an arbitrary but fixed $\alpha \in (-1/2, -1/4)$ and let $h \in L^2(0, 1)$ be defined via $h(\xi) = -\xi^\alpha$ for a.a. $\xi \in (0, 1)$. Then, for all sequences $t_k \searrow 0$, it holds*

$$\text{dist}(x + t_k h, C) = \left(\int_0^{t_k^{-1/\alpha}} (1 - t_k \xi^\alpha)^2 d\xi \right)^{1/2} = c_\alpha^{1/2} t_k^{-1/(2\alpha)} \neq \mathcal{O}(t_k^2),$$

where $c_\alpha = \frac{2\alpha^2}{(\alpha+1)(2\alpha+1)} > 0$. Hence, $\mathcal{T}_C^2(x, h) = \emptyset$ and, consequently, C cannot be outer second-order regular at x by [Lemma 5.5](#). Using similar arguments, we can show that C fails to be outer second-order regular at all $x \in C \setminus \{0\}$.

Note that, due to the effects appearing in [Example 5.6](#), the concept of second-order regularity is typically not suited for the analysis of optimal control problems with pointwise control or state constraints. It is, however, quite useful when the optimization problem at hand is finite-dimensional or involves constraints of the form $G(x) \in K$, where $K \subset \mathbb{R}^n$ is a closed, convex, non-empty set, cf. [Example 5.13](#). To simplify the derivation of second-order optimality conditions for problems of the latter type, we provide a calculus rule for the curvature of preimages.

Lemma 5.7. *Let Z be a Banach space. Assume that $C = G^{-1}(K)$ holds for some twice continuously Fréchet differentiable function $G : X \rightarrow Z$ and some closed, convex, non-empty set $K \subset Z$. Suppose further that a tuple $(x, \varphi) \in C \times -\mathcal{N}_C^*(x)$ is given such that the Zowe-Kurcyusz constraint qualification*

$$G'(x)X - \mathcal{R}_K(G(x)) = Z \quad (\text{ZKCQ})$$

is satisfied in x , such that K is second-order regular in $G(x)$, and such that the maps $h \mapsto G'(x)h$ and $h \mapsto G''(x)h^2$ are weak- \star -to-strong continuous. Then, C satisfies [\(MRC\)](#) in (x, φ) , it holds $\mathcal{K}_C^*(x, \varphi) = \varphi^\perp \cap G'(x)^{-1}\mathcal{T}_K(G(x))$, and for every $h \in \mathcal{K}_C^*(x, \varphi)$ it is true that

$$Q_C^{x, \varphi}(h) = \inf_{r \in G'(x)^{-1}(\mathcal{T}_K^2(G(x), G'(x)h) - G''(x)h^2)} \langle \varphi, r \rangle. \quad (14)$$

Proof. The proof of [Lemma 5.7](#) follows the lines of that of [[Bonnans and Shapiro, 2000](#), Proposition 3.88]: Let $h \in \mathcal{K}_C^*(x, \varphi)$ be fixed, and let $\{r_k\} \subset X$, $\{t_k\} \subset \mathbb{R}^+$ be sequences satisfying $t_k \searrow 0$, $t_k r_k \xrightarrow{\star} 0$ and $x + t_k h + \frac{1}{2} t_k^2 r_k \in C$, i.e., $G(x + t_k h + \frac{1}{2} t_k^2 r_k) \in K$. Then, by a Taylor expansion of G , cf. [[Cartan, 1967](#), Theorem 5.6.3], and using our assumptions on $G''(x)$, we obtain

$$\begin{aligned} G(x + t_k h + \frac{1}{2} t_k^2 r_k) &= G(x) + t_k G'(x)h + \frac{1}{2} t_k^2 (G'(x)r_k + G''(x)(h + \frac{1}{2} t_k r_k)^2) + o(t_k^2) \\ &= G(x) + t_k G'(x)h + \frac{1}{2} t_k^2 (G'(x)r_k + G''(x)h^2 + \varphi_k) \in K \end{aligned} \quad (15)$$

with some φ_k satisfying $\varphi_k \rightarrow 0$ in Z . Since $G'(x)$ is weak- \star -to-strong continuous, [\(15\)](#) implies $G'(x)h \in \mathcal{T}_K(G(x))$ and, by [[Bonnans and Shapiro, 2000](#), Corollary 2.91], $h \in \mathcal{T}_C(x) = G'(x)^{-1}\mathcal{T}_K(G(x))$. This yields $\mathcal{K}_C^*(x, \varphi) = \varphi^\perp \cap G'(x)^{-1}\mathcal{T}_K(G(x))$ and shows that it makes sense to talk about the second-order tangent sets $\mathcal{T}_C^2(x, h)$ and $\mathcal{T}_K^2(G(x), G'(x)h)$ in the following. Next, we will prove that $\text{dist}(r_k, \mathcal{T}_C^2(x, h)) \rightarrow 0$. We start by observing that

$$\begin{aligned} 0 \leq D_k &:= \text{dist} \left(G'(x)r_k, \mathcal{T}_K^2(G(x), G'(x)h) - G''(x)h^2 \right) \\ &\leq \text{dist} \left(G'(x)r_k + G''(x)h^2 + \varphi_k, \mathcal{T}_K^2(G(x), G'(x)h) \right) + \|\varphi_k\|_Z \rightarrow 0, \end{aligned}$$

where we used (15), $t_k G'(x) r_k \rightarrow 0$ and the outer second-order regularity of K at $G(x)$.

Now, let $\{\eta_k\} \subset Z$ be a sequence with $G'(x)r_k + G''(x)h^2 + \eta_k \in \mathcal{T}_K^2(G(x), G'(x)h)$ and $\|\eta_k\|_Z \leq D_k + 1/k$. From [Zowe and Kurcyusz, 1979, Theorem 2.1] and [Bonnans and Shapiro, 2000, Proposition 2.95], we obtain that there exists a $\rho > 0$ with

$$B_\rho^Z(0) \subset G'(x)B_1^X(0) - (K - G(x)) \cap B_1^Z(0).$$

In particular, we may find sequences $\mu_k \in X$ and $\lambda_k \in \mathcal{R}_K(G(x))$ such that

$$\eta_k = G'(x)\mu_k - \lambda_k \quad \text{and} \quad \|\mu_k\|_X \leq \rho^{-1}\|\eta_k\|_Z. \quad (16)$$

From (16) and the inclusions

$$\mathcal{R}_K(G(x)) \subset \mathcal{T}_{\mathcal{T}_K(G(x))}(h), \quad (17a)$$

$$\mathcal{T}_K^2(G(x), G'(x)h) + \mathcal{T}_{\mathcal{T}_K(G(x))}(G'(x)h) \subset \mathcal{T}_K^2(G(x), G'(x)h), \quad (17b)$$

which follow from the fact that $\mathcal{T}_K(G(x))$ is a closed convex cone and [Bonnans and Shapiro, 2000, Proposition 3.34], we obtain

$$G'(x)(r_k + \mu_k) \in \mathcal{T}_K^2(G(x), G'(x)h) + \lambda_k - G''(x)h^2 \subset \mathcal{T}_K^2(G(x), G'(x)h) - G''(x)h^2.$$

Using the identity

$$\mathcal{T}_C^2(x, h) = G'(x)^{-1} \left(\mathcal{T}_K^2(G(x), G'(x)h) - G''(x)(h, h) \right) \quad \forall h \in \mathcal{T}_C(x) \quad (18)$$

that is found, e.g., in [Bonnans and Shapiro, 2000, Proposition 3.33], we infer

$$r_k + \mu_k \in G'(x)^{-1} \left(\mathcal{T}_K^2(G(x), G'(x)h) - G''(x)h^2 \right) = \mathcal{T}_C^2(x, h).$$

The above implies that we indeed have

$$\text{dist}(r_k, \mathcal{T}_C^2(x, h)) \leq \|\mu_k\|_X \leq \rho^{-1}\|\eta_k\|_Z \leq \rho^{-1}(D_k + 1/k) \rightarrow 0.$$

Arguing as in the first part of the proof of Lemma 5.4, we now obtain

$$\liminf_{k \rightarrow \infty} \langle \varphi, r_k \rangle \geq \inf_{r \in \mathcal{T}_C^2(x, h)} \langle \varphi, r \rangle,$$

and, as a consequence,

$$Q_C^{x, \varphi}(h) \geq \inf_{r \in \mathcal{T}_C^2(x, h)} \langle \varphi, r \rangle.$$

On the other hand, an argumentation analogous to that employed in the second part of the proof of Lemma 5.4 yields

$$\begin{aligned} Q_C^{x, \varphi}(h) &\leq \inf \left\{ \liminf_{k \rightarrow \infty} \langle \varphi, r_k \rangle \left| \begin{array}{l} \{r_k\} \subset X, \{t_k\} \subset \mathbb{R}^+ : t_k \searrow 0, t_k r_k \rightarrow 0, \\ x + t_k h + \frac{1}{2} t_k^2 r_k \in C \end{array} \right. \right\} \\ &\leq \inf_{r \in \mathcal{T}_C^2(x, h)} \langle \varphi, r \rangle \leq Q_C^{x, \varphi}(h). \end{aligned}$$

Hence, equality holds everywhere, (14) is valid (cf. (18)) and (MRC) is satisfied in (x, φ) by the observation in Remark 3.5. This completes the proof. \square

Under stronger assumptions on x and G , the right-hand side of (14) is directly related to the directional curvature functional of K .

Lemma 5.8. *In the situation of Lemma 5.7, assume that Z is the dual of a separable Banach space and that there exists a $\lambda \in \mathcal{N}_K^*(G(x))$ with $\varphi + G'(x)^*\lambda = 0$ such that (MRC) of K holds in $(G(x), -\lambda)$ and*

$$Z = G'(x)X - \mathcal{R}_K(G(x)) \cap \lambda^\perp. \quad (19)$$

Then,

$$Q_C^{x,\varphi}(h) = Q_K^{G(x),-\lambda}(G'(x)h) + \langle \lambda, G''(x)h^2 \rangle \quad \forall h \in \mathcal{K}_C^*(x, \varphi). \quad (20)$$

Proof. Let $h \in \mathcal{K}_C^*(x, \varphi)$ be arbitrary but fixed. From (17), (19) and [Zowe and Kurcyusz, 1979, Theorem 2.1], it follows (analogously to the proof of Lemma 5.7) that for every $w \in \mathcal{T}_K^2(G(x), G'(x)h) - G''(x)h^2$ there exist an $r \in X$ and an $\eta \in \mathcal{R}_K(G(x)) \cap \lambda^\perp$ with

$$w = G'(x)r - \eta \quad \text{and} \quad G'(x)r = w + \eta \in \mathcal{T}_K^2(G(x), G'(x)h) - G''(x)h^2.$$

Note that an r with the latter properties necessarily satisfies $\langle -\lambda, w \rangle = \langle -\lambda, G'(x)r \rangle$ and $G'(x)r \in G'(x)X \cap (\mathcal{T}_K^2(G(x), G'(x)h) - G''(x)h^2)$. Consequently, we may deduce

$$\inf_{w \in \mathcal{T}_K^2(G(x), G'(x)h) - G''(x)h^2} \langle -\lambda, w \rangle \geq \inf_{w \in G'(x)X \cap (\mathcal{T}_K^2(G(x), G'(x)h) - G''(x)h^2)} \langle -\lambda, w \rangle. \quad (21)$$

On the other hand, we trivially have

$$\inf_{w \in \mathcal{T}_K^2(G(x), G'(x)h) - G''(x)h^2} \langle -\lambda, w \rangle \leq \inf_{w \in G'(x)X \cap (\mathcal{T}_K^2(G(x), G'(x)h) - G''(x)h^2)} \langle -\lambda, w \rangle,$$

so equality has to hold in (21). Using this equality, (18), Lemmas 5.4 and 5.7, the identity $\varphi + G'(x)^*\lambda = 0$ and a straightforward calculation, we obtain

$$\begin{aligned} Q_C^{x,\varphi}(h) &= \inf_{r \in \mathcal{T}_C^2(x, h)} \langle \varphi, r \rangle = \inf_{r \in G'(x)^{-1}(\mathcal{T}_K^2(G(x), G'(x)h) - G''(x)h^2)} \langle -\lambda, G'(x)r \rangle \\ &= \inf_{w \in G'(x)X \cap (\mathcal{T}_K^2(G(x), G'(x)h) - G''(x)h^2)} \langle -\lambda, w \rangle = \inf_{w \in \mathcal{T}_K^2(G(x), G'(x)h) - G''(x)h^2} \langle -\lambda, w \rangle \\ &= \inf_{w \in \mathcal{T}_K^2(G(x), G'(x)h)} \langle -\lambda, w \rangle + \langle \lambda, G''(x)h^2 \rangle = Q_K^{G(x),-\lambda}(G'(x)h) + \langle \lambda, G''(x)h^2 \rangle. \end{aligned}$$

This proves the claim. \square

Several things are noteworthy regarding Lemma 5.8 and its assumptions.

Remark 5.9.

(i) The pull-back formula (20) is, in fact, valid in a setting that is far more general than the one considered in Lemmas 5.7 and 5.8. It holds, e.g., also for polyhedral sets K provided the strengthened Zowe-Kurcyusz condition (19) is satisfied, cf. [G. Wachsmuth, 2016, Theorem 5.7]. It is further remarkable that the estimate

$$Q_C^{x,\varphi}(h) \geq Q_K^{G(x),-\lambda}(G'(x)h) + \langle \lambda, G''(x)h^2 \rangle \quad \forall h \in \mathcal{K}_C^*(x, \varphi),$$

which yields an SSC for the problem (P), can often be proved without any constraint qualifications at all. To avoid overloading this paper, we leave a detailed discussion of the latter topics for future research.

- (ii) A possible interpretation of the formula (20) is that the (directional) curvature of the set C has its origin in the nonlinearity of G or in the curvedness of K .
- (iii) The condition (19) is well-known and appears, e.g., also in the study of the uniqueness of Lagrange multipliers. It is precisely the ordinary Zowe-Kurcyusz constraint qualification for the set $\tilde{K} := \{u \in K : (u - G(x)) \in \lambda^\perp\}$. We refer to [Shapiro, 1997, Theorem 2.2] for details on this topic.

5.3 Two Corollaries of Theorem 4.5 and Some Tangible Examples

If we combine the findings of Sections 5.1 and 5.2 with the analysis of Section 4, then we arrive, e.g., at the following two results.

Theorem 5.10 (No-Gap Second-Order Condition for Polyhedral Sets). *Suppose that X is reflexive and that C is polyhedral at \bar{x} . Assume that $J'(\bar{x}) \in -\mathcal{N}_C^*(\bar{x})$ holds and that $J''(\bar{x})$ is a Legendre form in the sense of Lemma 5.1 (ii). Then, the condition*

$$J''(\bar{x})h^2 > 0 \quad \forall h \in \mathcal{K}_C^*(\bar{x}, J'(\bar{x})) \setminus \{0\}$$

is equivalent to the quadratic growth condition (6) with constants $c > 0$ and $\varepsilon > 0$.

Proof. Use Lemmas 5.1 and 5.3 in Theorem 4.5. □

Theorem 5.11 (No-Gap Second-Order Condition for Second-Order Regular Sets). *Let Z be a Banach space. Assume that $C = G^{-1}(K)$ holds for some twice continuously Fréchet differentiable function $G : X \rightarrow Z$ and some closed, convex, non-empty set $K \subset Z$. Assume further that $J'(\bar{x}) \in -\mathcal{N}_C^*(\bar{x})$ holds, that $J''(\bar{x})$ is weak- \star lower semi-continuous, that (NDC) holds, that the maps $h \mapsto G'(\bar{x})h$ and $h \mapsto G''(\bar{x})h^2$ are weak- \star -to-strong continuous, that K is second-order regular in $G(\bar{x})$, and that the constraint qualification*

$$G'(\bar{x})X - \mathcal{R}_K(G(\bar{x})) = Z$$

is satisfied. Then, the condition

$$J''(\bar{x})h^2 + \inf_{r \in G'(\bar{x})^{-1}(\mathcal{T}_K^2(G(\bar{x}), G'(\bar{x})h) - G''(\bar{x})h^2)} \langle J'(\bar{x}), r \rangle > 0 \quad (22)$$

$$\forall h \in J'(\bar{x})^\perp \cap G'(\bar{x})^{-1}\mathcal{T}_K(G(\bar{x})) \setminus \{0\}$$

is equivalent to the quadratic growth condition (6) with constants $c > 0$ and $\varepsilon > 0$. If, moreover, we know that Z is the dual of a separable Banach space and that there exists a Lagrange multiplier $\lambda \in \mathcal{N}_K^*(G(\bar{x}))$ satisfying $J'(\bar{x}) + G'(\bar{x})^*\lambda = 0$ such that (MRC) holds in $(G(\bar{x}), -\lambda)$ and such that

$$Z = G'(\bar{x})X - \mathcal{R}_K(G(\bar{x})) \cap \lambda^\perp,$$

then (22) is equivalent to

$$\partial_{xx}L(x, \lambda)h^2 + Q_K^{G(\bar{x}), -\lambda}(G'(\bar{x})h) > 0 \quad \forall h \in J'(\bar{x})^\perp \cap G'(\bar{x})^{-1}\mathcal{T}_K(G(\bar{x})) \setminus \{0\},$$

where $L(x, \lambda) := J(x) + \langle \lambda, G(x) \rangle$ is the Lagrangian associated with (P).

Proof. Use Lemmas 5.4, 5.7 and 5.8 in Theorem 4.5. \square

We remark that no-gap second-order conditions for specific classes of optimization problems with polyhedral admissible sets (in particular, optimal control problems with box constraints) can be found frequently in the literature. We only mention [Bonnans, 1998, Theorem 2.7], [Casas and Tröltzsch, 2012, Theorems 2.2, 2.3], and [Casas and Tröltzsch, 2015, Theorem 4.13] as examples here. Theorem 5.10 reproduces these results on an abstract level.

Second-order conditions similar to those in Theorem 5.11, on the other hand, have been studied extensively in [Bonnans and Shapiro, 2000] in various formats and settings, see ibidem Theorems 3.45, 3.83, 3.86, 3.109, 3.137, 3.145, 3.148, 3.155 and Proposition 3.46. It should be noted that the no-gap conditions derived in [Bonnans and Shapiro, 2000, Section 3.3.3] all require X to be finite-dimensional and, in addition, all need further assumptions on, e.g., the second-order tangent set $\mathcal{T}_K^2(G(\bar{x}), G'(\bar{x})h)$. Such assumptions are not needed for the derivation of our second-order condition (22), but may be required for reformulations of (22) as we have seen in the second part of Theorem 5.11.

We conclude this section with two simple examples that demonstrate the usefulness of Theorems 5.10 and 5.11.

Example 5.12 (A Simple Optimal Control Problem with Control Constraints). *Consider a minimization problem of the form*

$$\begin{aligned} \text{Minimize} \quad & j(y) + \frac{\gamma}{2} \int_{\Omega} u^2 \, d\mathcal{L}^d \\ \text{such that} \quad & u \in L^2(\Omega), \quad -1 \leq u \leq 1 \text{ a.e. in } \Omega, \quad S(u) = y, \end{aligned} \tag{23}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain, $j : L^\infty(\Omega) \rightarrow \mathbb{R}$ is twice continuously differentiable, $\gamma > 0$ is a Tikhonov parameter, \mathcal{L}^d is the Lebesgue measure, and $S : L^2(\Omega) \rightarrow L^\infty(\Omega)$ is (for simplicity) linear and compact. In this situation, the space $X := L^2(\Omega)$ is obviously reflexive, the set $C := L^2(\Omega, [-1, 1])$ is closed, non-empty and polyhedral at every point, and the reduced objective $J(u) := j(Su) + \frac{\gamma}{2} \int_{\Omega} u^2 \, d\mathcal{L}^d$ is a C^2 -function with

$$\begin{aligned} J'(u)h &= j'(Su)(Sh) + \gamma(u, h)_{L^2(\Omega)} & \forall u, h \in L^2(\Omega), \\ J''(u)(h_1, h_2) &= j''(Su)(Sh_1, Sh_2) + \gamma(h_1, h_2)_{L^2(\Omega)} & \forall u, h_1, h_2 \in L^2(\Omega). \end{aligned}$$

Note that the map $h \mapsto J''(u)h^2$ is weakly lower semicontinuous for all $u \in L^2(\Omega)$, and that $J''(u)$ is a Legendre form for all $u \in L^2(\Omega)$ since

$$\begin{aligned} h_k \rightharpoonup h \text{ and } J''(u)h_k^2 \rightarrow J''(u)h^2 & \Rightarrow h_k \rightharpoonup h \text{ and } \|h_k\|_{L^2(\Omega)}^2 \rightarrow \|h\|_{L^2(\Omega)}^2 \\ & \Rightarrow h_k \rightarrow h. \end{aligned} \tag{24}$$

Consequently, [Theorem 5.10](#) is applicable in case of problem [\(23\)](#), and we may deduce that, given some $\bar{u} \in C$ with $\bar{p} + \gamma\bar{u} \in -\mathcal{N}_C^*(\bar{u})$, where $\bar{p} = S^*(j'(S\bar{u}))$ is the adjoint state, the condition

$$j''(S\bar{u})(Sh, Sh) + \gamma\|h\|_{L^2(\Omega)}^2 > 0 \quad \forall h \in \mathcal{K}_C^*(\bar{u}, \bar{p} + \gamma\bar{u}) \setminus \{0\}$$

is equivalent to the quadratic growth condition

$$J(u) \geq J(\bar{u}) + \frac{c}{2} \|u - \bar{u}\|_{L^2(\Omega)}^2 \quad \forall u \in C \cap B_\varepsilon^{L^2(\Omega)}(\bar{u})$$

with $c > 0$ and some $\varepsilon > 0$. Note that the Tikhonov regularization is of particular importance in the above setting: The condition $\gamma > 0$ ensures (in combination with the compactness of S) that the derivative $J''(u)$ is a Legendre form for all $u \in L^2(\Omega)$ and thus guarantees [\(NDC\)](#). We will see in [Section 6.2](#) that the situation changes drastically when the regularization parameter γ equals zero.

Example 5.13 (A Simple Optimal Control Problem with a State Constraint). Consider a minimization problem of the form

$$\begin{aligned} \text{Minimize} \quad & j(y) + \frac{\gamma}{2} \int_{\Omega} u^2 \, d\mathcal{L}^d \\ \text{such that} \quad & u \in L^2(\Omega), \quad Su = y, \quad Tu \in B_1^H(0), \end{aligned}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain, $j : L^\infty(\Omega) \rightarrow \mathbb{R}$ is twice continuously differentiable, $\gamma > 0$ is a Tikhonov parameter, \mathcal{L}^d is the Lebesgue measure, H is some Hilbert space, and $S : L^2(\Omega) \rightarrow L^\infty(\Omega)$, $T : L^2(\Omega) \rightarrow H$ are (for simplicity) linear and compact. Define $X := L^2(\Omega)$, $Z := \mathbb{R}$, $G(u) := \|Tu\|_H^2 - 1$, $K := (-\infty, 0]$, $C := G^{-1}(K)$, and $J(u) := j(Su) + \frac{\gamma}{2} \int_{\Omega} u^2 \, d\mathcal{L}^d$. Then X and Z are Hilbert spaces, C is non-empty, convex and closed, K is non-empty, convex and closed, G is twice continuously differentiable with $G'(u)h = 2(Tu, Th)_H$ and $G''(u)(h_1, h_2) = 2(Th_1, Th_2)_H$ for all $u \in L^2(\Omega)$, J is twice continuously differentiable with the same derivatives as in [Example 5.12](#), the map $h \mapsto J''(u)h^2$ is weakly lower semicontinuous for all $u \in L^2(\Omega)$, $J''(u)$ is a Legendre form for all $u \in L^2(\Omega)$ (cf. [Example 5.12](#)), [\(ZKCCQ\)](#) is satisfied in every $u \in C$ (just use a distinction of cases), and K is second-order regular at every $z \in K$ with

$$\mathcal{T}_K(z) = \begin{cases} \mathbb{R} & \text{if } z \in (-\infty, 0), \\ (-\infty, 0] & \text{if } z = 0, \end{cases} \quad \mathcal{T}_K^2(z, h) = \begin{cases} \mathbb{R} & \text{if } z \in (-\infty, 0), \, h \in \mathbb{R}, \\ \mathbb{R} & \text{if } z = 0, \, h \in (-\infty, 0), \\ (-\infty, 0] & \text{if } z = 0, \, h = 0. \end{cases}$$

Now, let $\bar{u} \in C$ be given, such that $\bar{p} + \gamma\bar{u} \in -\mathcal{N}_C^*(\bar{u})$, where $\bar{p} = S^*(j'(S\bar{u}))$ is the adjoint state. Using the above observations and [\(22\)](#), we obtain that the condition

$$\begin{aligned} j''(S\bar{u})(Sh, Sh) + \gamma\|h\|_{L^2(\Omega)}^2 + \inf_{r \in G'(\bar{u})^{-1}(\mathcal{T}_K^2(G(\bar{u}), G'(\bar{u})h) - 2\|Th\|_H^2)} \langle \bar{p} + \gamma\bar{u}, r \rangle > 0 \\ \forall h \in (\bar{p} + \gamma\bar{u})^\perp \cap G'(\bar{u})^{-1}\mathcal{T}_K(G(\bar{u})) \setminus \{0\} \end{aligned} \quad (25)$$

is equivalent to the quadratic growth condition (6) with constants $c > 0$ and $\varepsilon > 0$. Note that $G'(\bar{u}) : L^2(\Omega) \rightarrow \mathbb{R}$ is surjective if $\|T\bar{u}\|_H > 0$. Consequently, if $0 < \|T\bar{u}\| \leq 1$, we may use the second part of [Theorem 5.11](#) to simplify (25). This yields the condition

$$j''(S\bar{u})(Sh, Sh) + \gamma\|h\|_{L^2(\Omega)}^2 > 0 \quad \forall h \in (\bar{p} + \gamma\bar{u})^\perp \setminus \{0\}$$

for the case $0 < \|T\bar{u}\| < 1$ and the condition

$$j''(S\bar{u})(Sh, Sh) + \gamma\|h\|_{L^2(\Omega)}^2 + 2\lambda\|Th\|_H^2 > 0 \\ \forall h \in (\bar{p} + \gamma\bar{u})^\perp \cap \left\{ h \in L^2(\Omega) \setminus \{0\} : (Tu, Th)_H \leq 0 \right\}$$

for the case $\|T\bar{u}\|_H = 1$. Here, $\lambda \geq 0$ is the (in this case necessarily unique) Lagrange multiplier associated with \bar{u} .

6 Advantages of our Approach

Having demonstrated that the framework of [Section 4](#) indeed allows to reproduce classical results for minimization problems with polyhedral and second-order regular sets, we now turn our attention to the benefits offered by our approach in comparison with the classical theory. The main advantages of our method are the following.

- (i) Our approach splits the task of proving no-gap second-order optimality conditions for problems of the type [\(P\)](#) into subproblems that can be tackled independently from each other (namely, verifying [\(NDC\)](#), checking the differentiability of J , and computing the directional curvature functional $Q_C^{x,\varphi}(\cdot)$). We can further state our second-order conditions without imposing any preliminary assumptions (as, e.g., polyhedricity or second-order regularity) on the admissible set C , cf. [Theorem 4.5](#). All of this makes our method more flexible than the classical “all-at-once” approach.
- (ii) Our results can also be employed in situations where the admissible set exhibits a singular or degenerate curvature behavior, cf. the examples in [Sections 6.1](#) and [6.2](#).
- (iii) Our approach does not require a detailed analysis of the curvature of the set C . To obtain the second-order condition in [Theorem 4.5](#), we only have to study the behavior of the quantity $\langle J'(\bar{x}), r_k \rangle$ that appears in the definition of the functional $Q_C^{\bar{x}, J'(\bar{x})}(\cdot)$, i.e., we only have to analyze how the derivative $J'(\bar{x})$ acts on the second-order corrections r_k and not how the r_k behave in detail. This is a major difference to the concept of second-order regularity, cf. [Definition 5.2 \(iii\)](#), and often very advantageous since it allows to exploit additional information about the gradient of the objective. We will see this effect in [Section 6.2](#) below.

In the following, we demonstrate by means of two tangible examples that the above points are not only of academic interest but also of relevance in practice. We begin with a simple finite-dimensional optimization problem whose admissible set exhibits a singular curvature behavior.

6.1 Singular Curvature in Finite Dimensions

Consider a two-dimensional optimization problem of the form

$$\text{Minimize } J(x), \text{ such that } x \in C = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \geq |x_1|^\alpha\} \quad (26)$$

with a twice continuously differentiable objective $J : \mathbb{R}^2 \rightarrow \mathbb{R}$ and some $\alpha \in (1, 2)$. Set $\bar{x} := (0, 0)$ and suppose that \bar{x} is a critical point of (26) with a non-vanishing gradient, i.e., $J'(\bar{x}) = (0, \beta) \in -\mathcal{N}_C^*(\bar{x}) = \{0\} \times [0, \infty)$ for some $\beta > 0$. Then, for every critical direction $h \in \mathcal{K}_C^*(\bar{x}, J'(\bar{x})) = \{(h_1, h_2) \in \mathbb{R}^2 \mid h_2 = 0\}$ and all sequences $\{t_k\} \subset \mathbb{R}^+$ and $\{r_k\} \subset \mathbb{R}^2$ satisfying $t_k \searrow 0$, $t_k r_k \rightarrow 0$ and $\bar{x} + t_k h + \frac{1}{2} t_k^2 r_k \in C$, it holds

$$\begin{aligned} & \liminf_{k \rightarrow \infty} \langle J'(\bar{x}), r_k \rangle \\ &= 2 \liminf_{k \rightarrow \infty} J'(\bar{x})^\top \frac{\bar{x} + t_k h + \frac{1}{2} t_k^2 r_k}{t_k^2} = 2\beta \liminf_{k \rightarrow \infty} \frac{(\bar{x} + t_k h + \frac{1}{2} t_k^2 r_k)_2}{t_k^2} \\ &\geq 2\beta \liminf_{k \rightarrow \infty} \frac{|(\bar{x} + t_k h + \frac{1}{2} t_k^2 r_k)_1|^\alpha}{t_k^2} = 2\beta \liminf_{k \rightarrow \infty} |(h + \frac{1}{2} t_k r_k)_1|^\alpha t_k^{\alpha-2}. \end{aligned} \quad (27)$$

The above implies

$$Q_C^{\bar{x}, J'(\bar{x})}(h) = +\infty \quad \forall h \in \mathcal{K}_C^*(\bar{x}, J'(\bar{x})) \setminus \{0\} = \{h \in \mathbb{R}^2 \mid h_1 \neq 0, h_2 = 0\}. \quad (28)$$

Using (28), [Theorem 4.5](#) and the fact that the conditions [\(MRC\)](#) and [\(NDC\)](#) are trivially satisfied for (26), we obtain (analogous to [[Bonnans and Shapiro, 2000](#), Example 3.84]) the following result.

Theorem 6.1. *If $\bar{x} = (0, 0)$ is a critical point of (26) with $J'(\bar{x}) \neq 0$, then \bar{x} is a local minimizer of (26) and there exist parameters $c > 0$ and $\varepsilon > 0$ such that the quadratic growth condition [\(6\)](#) is satisfied.*

Note that the second derivative $J''(\bar{x})$ is not important for local optimality of \bar{x} . The reason for this is that the curvature of the boundary ∂C is singular at the origin and can thus compensate for any negative curvature that the objective J might have at \bar{x} . It should be noted further that the set C in (26) is neither polyhedric (trivially) nor second-order regular at \bar{x} (since $\mathcal{T}_C^2(\bar{x}, h) = \emptyset$ for all $h \in \mathcal{K}_C^*(\bar{x}, J'(\bar{x})) \setminus \{0\}$, cf. [Lemma 5.5](#), and the estimate (27)). This demonstrates that (26) does not fall under the setting of [Theorems 5.10](#) and [5.11](#) and is indeed not covered by what is typically seen as the classical second-order theory.

We remark that, given an optimization problem of the form

$$\text{Minimize } J(x), \text{ such that } x \in C = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \leq |x_1|^\alpha\} \quad (29)$$

with $J \in C^2(\mathbb{R}^2)$, $J'(\bar{x}) = (0, -\beta) \in -\mathcal{N}_C^*(\bar{x}) = \{0\} \times (-\infty, 0]$, $\beta > 0$, $\bar{x} := (0, 0)$, we can use exactly the same arguments as for (26) to prove

$$Q_C^{\bar{x}, J'(\bar{x})}(h) = -\infty \quad \forall h \in \mathcal{K}_C^*(\bar{x}, J'(\bar{x})) = \{h \in \mathbb{R}^2 \mid h_2 = 0\}.$$

The above yields in combination with [Theorems 4.2](#) and [4.3](#) that $\bar{x} = (0, 0)$ can never be a local minimizer of (29) unless the derivative $J'(\bar{x})$ is identical zero.

6.2 No-Gap Second-Order Conditions for Bang-Bang Problems

In this section, we demonstrate that the analysis of [Section 4](#) is not only relevant for finite-dimensional toy problems à la [\(26\)](#) and [\(29\)](#), but also applicable in more complicated situations. In what follows, we will use it to derive no-gap second-order conditions for bang-bang optimal control problems. As a motivation, let us consider the optimization problem [\(23\)](#) in [Example 5.12](#) with $\gamma = 0$, i.e., the problem

$$\begin{aligned} & \text{Minimize } j(y) \\ & \text{such that } u \in L^2(\Omega), \quad -1 \leq u \leq 1 \text{ a.e. in } \Omega, \quad S(u) = y. \end{aligned} \tag{30}$$

From [Theorem 4.2](#), we obtain that every minimizer \bar{u} of [\(30\)](#) satisfies $\bar{p} \in -\mathcal{N}_C^*(\bar{u})$, where $\bar{p} = S^*(j'(S\bar{u}))$ is the adjoint state. In particular, this implies that a minimizer \bar{u} with $\mathcal{L}^d(\{\bar{p} = 0\}) = 0$ can only take the values ± 1 a.e. in Ω . Such a solution \bar{u} is called bang-bang.

The major problem that arises when SSC for a bang-bang control \bar{u} are considered is the verification of the non-degeneracy condition (NDC). Recall that in [Example 5.12](#) the latter is satisfied since the Tikhonov regularization causes the second derivative of the reduced objective to be a Legendre form in $L^2(\Omega)$, cf. [\(24\)](#). For [\(30\)](#) such an argumentation is obviously not possible and this is not an artificial problem: It can be shown that quadratic growth in $L^2(\Omega)$ is in general not possible for a bang-bang solution \bar{u} of [\(30\)](#), i.e., the growth condition [\(6\)](#) typically does not hold with $X = L^2(\Omega)$ and $c, \varepsilon > 0$, cf. [[Casas, 2012](#), end of Section 2].

Hence, [Theorem 4.5](#) cannot be applicable when we work with the space $X = L^2(\Omega)$. Note that, if we calculate the critical cone $\mathcal{K}_C^*(\bar{u}, \bar{p})$ for a bang-bang control \bar{u} in the L^2 -setting, then we end up with $\mathcal{K}_C^*(\bar{u}, \bar{p}) = \mathcal{T}_C(\bar{u}) \cap \bar{p}^\perp = \{0\}$ so that the conditions $j''(S\bar{u})(Sh, Sh) > 0 \forall h \in \mathcal{K}_C^*(\bar{u}, \bar{p}) \setminus \{0\}$ and $j''(S\bar{u})(Sh, Sh) \geq 0 \forall h \in \mathcal{K}_C^*(\bar{u}, \bar{p})$, which are the natural candidates for the SSC and SNC, respectively, are both void. This also indicates that it is not useful to discuss [\(30\)](#) as a problem in $X = L^2(\Omega)$.

The above discussion shows that, if we want to derive no-gap second-order conditions for a bang-bang optimal control problem of the type [\(30\)](#), then we have to work with a space X that is different from $L^2(\Omega)$. As it turns out, the right choice is the measure space $X = \mathcal{M}(\Omega)$. Therefore, we consider the following setting.

Assumption 6.2 (Standing Assumptions and Notation for the Bang-Bang Setting). *We suppose that $\Omega \subset \mathbb{R}^d$, $d \geq 1$, is a domain with a Lipschitz boundary, cf. [[Grisvard, 1985](#), Definition 1.2.1.1]. We define the space $Y := C_0(\Omega) = \text{cl}_{\|\cdot\|_\infty}(C_c(\Omega))$ endowed with the usual supremum norm. Its dual space can be identified with $X := \mathcal{M}(\Omega)$ which is the space of signed finite Radon measures on Ω endowed with the norm $\|\mu\|_{\mathcal{M}(\Omega)} := |\mu|(\Omega)$, cf. [[Ambrosio, 2000](#), Theorem 1.54]. The space $L^1(\Omega)$ is identified with a closed subspace of $\mathcal{M}(\Omega)$ via the isometric embedding $x \mapsto x\mathcal{L}^d$, where \mathcal{L}^d is Lebesgue's measure. Finally, $C := L^\infty(\Omega, [-1, 1]) = \{x \in L^\infty(\Omega) : -1 \leq x \leq 1 \text{ a.e. in } \Omega\} \subset \mathcal{M}(\Omega)$.*

The reason for using the space $X = \mathcal{M}(\Omega)$ is the following observation, cf. [[Casas et al., 2017b](#), Proposition 2.7].

Lemma 6.3. *Let $\bar{x} \in C$ and $\bar{\varphi} \in -\mathcal{N}_C^*(\bar{x})$ be given. Define*

$$K(\bar{\varphi}) := \frac{1}{4} \liminf_{s \searrow 0} \left(\frac{s}{\mathcal{L}^d(\{|\bar{\varphi}| \leq s\})} \right) \in [0, +\infty]. \quad (31)$$

Then, there exists a family of constants $\{c_\varepsilon\}$ satisfying $c_\varepsilon \searrow 0$ as $\varepsilon \searrow 0$ such that

$$\langle \bar{\varphi}, x - \bar{x} \rangle \geq \left(\frac{1}{2} K(\bar{\varphi}) - c_\varepsilon \right) \|x - \bar{x}\|_X^2 \quad \forall x \in C \cap B_\varepsilon^X(\bar{x}) \quad \forall \varepsilon > 0. \quad (32)$$

Proof. We adapt the proof of [Casas et al., 2017b, Proposition 2.7]. Define $K := K(\bar{\varphi})$. If $K = 0$, then the claim is trivially true. If $K > 0$, then it necessarily holds that $\mathcal{L}^d(\{\bar{\varphi} = 0\}) = 0$ and we obtain from $\bar{\varphi} \in -\mathcal{N}_C^*(\bar{x})$ that \bar{x} is bang-bang with $\bar{x} = -\text{sgn } \bar{\varphi}$ a.e. in Ω . Using the latter and $\mathcal{N}_C^*(\bar{x}) \subset C_0(\Omega)$, we may calculate that for all $h \in X$ and all $t > 0$ with $\bar{x} + th \in C$ and $\|h\|_X = \|h\|_{L^1(\Omega)} = 1$, it holds

$$\begin{aligned} \int_\Omega \bar{\varphi} \frac{h}{t} d\mathcal{L}^d &= \int_\Omega |\bar{\varphi}| \frac{|h|}{t} d\mathcal{L}^d \geq \int_{\{|\bar{\varphi}| > Kt\}} |\bar{\varphi}| \frac{|h|}{t} d\mathcal{L}^d \geq \int_\Omega K|h| d\mathcal{L}^d - \int_{\{|\bar{\varphi}| \leq Kt\}} K|h| d\mathcal{L}^d \\ &\geq K - K\|h\|_{L^\infty(\Omega)} \mathcal{L}^d(\{|\bar{\varphi}| \leq Kt\}). \end{aligned}$$

By using $t := \|x - \bar{x}\|_X$, $h := (x - \bar{x})/\|x - \bar{x}\|_X$ and $\|h\|_{L^\infty(\Omega)} \leq 2/t$, this implies

$$\langle \bar{\varphi}, x - \bar{x} \rangle \geq \left(K - 2K^2 \sup_{0 < t \leq \varepsilon} \frac{\mathcal{L}^d(\{|\bar{\varphi}| \leq Kt\})}{Kt} \right) \|x - \bar{x}\|_X^2$$

for all $x \in C \cap B_\varepsilon^X(\bar{x}) \setminus \{\bar{x}\}$ and all $\varepsilon > 0$. Note that the coefficient on the right-hand side of the last estimate satisfies

$$\lim_{\varepsilon \searrow 0} \left(K - 2K^2 \sup_{0 < t \leq \varepsilon} \frac{\mathcal{L}^d(\{|\bar{\varphi}| \leq Kt\})}{Kt} \right) = K - 2K^2 \left(\limsup_{t \searrow 0} \frac{\mathcal{L}^d(\{|\bar{\varphi}| \leq t\})}{t} \right) = \frac{1}{2}K.$$

This proves the claim. \square

Lemma 6.3 shows that **Lemma 5.1 (iii)** is applicable when we consider a bang-bang solution \bar{x} whose gradient $\bar{\varphi} := J'(\bar{x})$ satisfies $K(\bar{\varphi}) > 0$. This allows us to verify **(NDC)** and to obtain the following result from **Theorem 4.5**.

Theorem 6.4 (No-Gap Second-Order Condition for Bang-Bang Problems). *Consider an optimization problem of the form (P) with C, X etc. as in Assumption 6.2. Assume that $\bar{x} \in C$ is fixed, that J satisfies the conditions in Assumption 4.1, that the map $X \ni h \mapsto J''(\bar{x})h^2 \in \mathbb{R}$ is weak- \star continuous, that $\bar{\varphi} := J'(\bar{x}) \in -\mathcal{N}_C^*(\bar{x})$, and that the constant $K(\bar{\varphi})$ in (31) is positive. Then, the condition*

$$Q_{C, \bar{\varphi}}^{\bar{x}}(h) + J''(\bar{x})h^2 > 0 \quad \forall h \in \mathcal{K}_C^*(\bar{x}, \bar{\varphi}) \setminus \{0\} \quad (33)$$

is equivalent to the quadratic growth condition

$$J(x) \geq J(\bar{x}) + \frac{c}{2} \|x - \bar{x}\|_{L^1(\Omega)}^2 \quad \forall x \in C \cap B_\varepsilon^X(\bar{x}) \quad (34)$$

with constants $c > 0$ and $\varepsilon > 0$.

Proof. We only need to check that [Theorem 4.5](#) is applicable. The setting in [Assumption 6.2](#) clearly fits into that of [Assumption 2.1](#), and [Assumption 4.1](#) is trivially satisfied. From $K(\bar{\varphi}) > 0$ and [Lemma 6.3](#), we further obtain that there exist constants $c, \varepsilon > 0$ with $J'(\bar{x})(x - \bar{x}) \geq \frac{c}{2} \|x - \bar{x}\|^2$ for all $x \in B_\varepsilon^X(\bar{x}) \cap C$. This yields, in combination with the weak- \star continuity of $h \mapsto J''(\bar{x})h^2$ and [Lemma 5.1 \(iii\)](#), that [\(NDC\)](#) holds. Note that the weak- \star continuity of $h \mapsto J''(\bar{x})h^2$ also implies [\(i\)](#) in [Theorem 4.3](#). This shows that [Theorem 4.5](#) is applicable and proves the claim. \square

Some remarks concerning [Lemma 6.3](#) and [Theorem 6.4](#) are in order.

Remark 6.5.

- (i) [Theorem 6.4](#) provides no-gap second-order conditions even in the case that we cannot characterize the directional curvature functional $Q_C^{\bar{x}, \bar{\varphi}}(\cdot)$ precisely.
- (ii) Recall that the condition in [Lemma 5.1 \(iii\)](#) not only implies [\(NDC\)](#) but also yields a coercivity estimate for the functional $Q_C^{\bar{x}, \bar{\varphi}}(\cdot)$, see [\(12\)](#). Using this estimate and [\(32\)](#), we obtain that, in the situation of [Theorem 6.4](#),

$$Q_C^{\bar{x}, \bar{\varphi}}(h) \geq K(\bar{\varphi}) \|h\|_X^2 \quad \forall h \in \mathcal{K}_C^*(\bar{x}, \bar{\varphi}). \quad (35)$$

Here, $K(\bar{\varphi}) > 0$ is again defined by [\(31\)](#). We point out that [\(35\)](#) implies that the set $C = L^\infty(\Omega, [-1, 1])$ possesses positive curvature as a subset of the space $\mathcal{M}(\Omega)$. This is not true if C is considered as a subset of the space $L^2(\Omega)$ as we have seen in [Example 5.12](#) (in $L^2(\Omega)$, C is polyhedral and the curvature functional is zero).

- (iii) From [\(35\)](#), it follows that $J''(\bar{x})h^2 > -K(\bar{\varphi}) \|h\|_X^2$ for all $h \in \mathcal{K}_C^*(\bar{x}, \bar{\varphi}) \setminus \{0\}$ is a sufficient condition for quadratic growth in the situation of [Theorem 6.4](#), cf. [\(33\)](#). We point out that this SSC is sharper than that found in [[Casas et al., 2017b](#), Corollary 2.15]. In this contribution the authors work with the slightly more restrictive “global” level set assumption

$$\tilde{K} \leq \frac{s}{4 \mathcal{L}^d(\{|\bar{\varphi}| \leq s\})} \quad \forall s > 0$$

for some $\tilde{K} > 0$. Note that such a \tilde{K} necessarily satisfies $\tilde{K} \leq K(\bar{\varphi})$. We further point out that the SSC in [[Casas et al., 2017b](#), Corollary 2.15] can be improved to

$$\exists \varepsilon > 0 : \quad J''(\bar{x})h^2 \geq -(\tilde{K} - \varepsilon) \|h\|_X^2 \quad \forall h \in \mathcal{K}_C^*(\bar{x}, \bar{\varphi}),$$

cf. [[Casas et al., 2017a](#), Theorem 2.4].

- (iv) We expect that the SSC $J''(\bar{x})h^2 > -K(\bar{\varphi}) \|h\|_X^2 \quad \forall h \in \mathcal{K}_C^*(\bar{x}, \bar{\varphi}) \setminus \{0\}$ can also be formulated as an inequality on the so-called extended critical cone introduced in [[Casas, 2012](#)], cf. [[Casas et al., 2017b](#), Theorem 2.14] and [[Casas et al., 2017a](#), Theorem 2.4]. We do not pursue this approach here.

The next step is the calculation of the curvature functional $Q_C^{\bar{x}, \bar{\varphi}}(\cdot)$ for a bang-bang solution \bar{x} in order to obtain a no-gap optimality condition that is more explicit than (33). Hence, we have to compute the (directional) curvature of the set $C := L^\infty(\Omega, [-1, 1])$ as a subset of the space $\mathcal{M}(\Omega)$. In the remainder of this section, we will consider a bang-bang solution \bar{x} whose gradient $\bar{\varphi} := J'(\bar{x}) \in -\mathcal{N}_C^*(\bar{x})$ is in $C^1(\Omega)$. Let us first fix our assumptions on the \bar{x} under consideration.

Assumption 6.6 (Assumptions and Notation for the Calculation of $Q_C^{\bar{x}, \bar{\varphi}}(\cdot)$).

In addition to [Assumption 6.2](#), we suppose that $\bar{x} \in C$ and $\bar{\varphi} \in -\mathcal{N}_C^*(\bar{x})$ are given. We require $\bar{\varphi} \in \iota(C_0(\Omega) \cap C^1(\Omega))$ and define $\mathcal{Z} := \{z \in \Omega : \bar{\varphi}(z) = 0\}$. We assume $\mathcal{Z} \subset \{z \in \Omega : |\nabla \bar{\varphi}(z)| \neq 0\}$. Here and in the sequel, $|\nabla \bar{\varphi}(z)|$ denotes the Euclidean norm of $\nabla \bar{\varphi}(z) \in \mathbb{R}^d$.

Finally, we denote by \mathcal{H}^{d-1} the $(d-1)$ -dimensional Hausdorff measure, which is scaled as in [[Evans and Gariepy, 2015, Definition 2.1](#)].

In the above situation, the set \mathcal{Z} is a $(d-1)$ -dimensional C^1 -submanifold of \mathbb{R}^d due to the implicit function theorem, cf. [[Holm et al., 2009, Theorem 2.32](#)]. This implies in particular that $\mathcal{L}^d(\mathcal{Z}) = \mathcal{L}^d(\{\bar{\varphi} = 0\}) = 0$ and that \bar{x} is indeed bang-bang with $\bar{x} = -\text{sgn } \bar{\varphi}$ a.e. in Ω . To calculate $Q_C^{\bar{x}, \bar{\varphi}}(\cdot)$, we need the following directional Taylor-like expansion of the $L^1(\Omega)$ -norm.

Lemma 6.7 ([[Christof and Meyer, 2016, Corollary 5.10](#)]). Given [Assumption 6.6](#), for all $v \in C_c(\Omega) \cap H^1(\Omega)$ and all sequences $t_k \in (0, \infty)$ with $t_k \searrow 0$, it is true that

$$\int_{\Omega} |-\bar{\varphi} + t_k v| d\mathcal{L}^d = \int_{\Omega} |\bar{\varphi}| d\mathcal{L}^d + t_k \int_{\Omega} \bar{x} v d\mathcal{L}^d + t_k^2 \int_{\mathcal{Z}} \frac{v^2}{|\nabla \bar{\varphi}|} d\mathcal{H}^{d-1} + o(t_k^2).$$

The next lemma provides a link between the curvature of C and $\nabla \bar{\varphi}$.

Lemma 6.8. For all $v \in C_c(\Omega)$, all $h \in \mathcal{K}_C^*(\bar{x}, \bar{\varphi})$, and all $\alpha > 0$, it holds

$$\frac{\alpha^2}{2} Q_C^{\bar{x}, \bar{\varphi}}(h) - \alpha \langle h, v \rangle + \int_{\mathcal{Z}} \frac{v^2}{|\nabla \bar{\varphi}|} d\mathcal{H}^{d-1} \geq 0. \quad (36)$$

Proof. Let $\alpha > 0$ and $h \in \mathcal{K}_C^*(\bar{x}, \bar{\varphi})$ be given. From the cone property of $\mathcal{K}_C^*(\bar{x}, \bar{\varphi})$, it follows $\alpha h \in \mathcal{K}_C^*(\bar{x}, \bar{\varphi})$. This implies that there exist sequences $\{r_k\} \subset X$, $\{t_k\} \subset \mathbb{R}^+$ with $t_k \searrow 0$, $t_k r_k \xrightarrow{*} 0$ and $\bar{x} + t_k \alpha h + \frac{1}{2} t_k^2 r_k \in C$. Fix such sequences $\{r_k\}$, $\{t_k\}$ and define $h_k := \alpha h + \frac{1}{2} t_k r_k$. Then, it holds $h_k \in \mathcal{R}_C(\bar{x}) \subset L^\infty(\Omega)$ and $h_k \xrightarrow{*} \alpha h$ in X , and [Lemma 6.7](#) yields

$$\begin{aligned} & t_k^2 \int_{\mathcal{Z}} \frac{v^2}{|\nabla \bar{\varphi}|} d\mathcal{H}^{d-1} + o(t_k^2) \\ &= \left(\sup_{q \in L^\infty(\Omega, [-1, 1])} \int_{\Omega} (-\bar{\varphi} + t_k v) q d\mathcal{L}^d \right) - \int_{\Omega} |\bar{\varphi}| d\mathcal{L}^d - t_k \int_{\Omega} \bar{x} v d\mathcal{L}^d \\ &= \left(\sup_{p \in L^\infty(\Omega): \bar{x} + t_k p \in L^\infty(\Omega, [-1, 1])} \int_{\Omega} (-\bar{\varphi} + t_k v)(\bar{x} + t_k p) d\mathcal{L}^d \right) - \int_{\Omega} |\bar{\varphi}| d\mathcal{L}^d - t_k \int_{\Omega} \bar{x} v d\mathcal{L}^d \end{aligned}$$

$$\begin{aligned}
&= \left(\sup_{p \in L^\infty(\Omega): \bar{x} + t_k p \in L^\infty(\Omega, [-1, 1])} \int_{\Omega} (-\bar{\varphi} + t_k v)(t_k p) d\mathcal{L}^d \right) \\
&\geq \int_{\Omega} (-\bar{\varphi} + t_k v)(t_k h_k) d\mathcal{L}^d \quad \forall v \in C_c(\Omega) \cap H^1(\Omega). \tag{37}
\end{aligned}$$

If we divide (37) by t_k^2 and let $k \rightarrow \infty$, then we obtain (with $\langle \bar{\varphi}, h_k \rangle = \langle \bar{\varphi}, \frac{1}{2} t_k r_k \rangle$)

$$\frac{1}{2} \liminf_{k \rightarrow \infty} \langle \bar{\varphi}, r_k \rangle - \langle \alpha h, v \rangle + \int_{\mathcal{Z}} \frac{v^2}{|\nabla \bar{\varphi}|} d\mathcal{H}^{d-1} \geq 0 \quad \forall v \in C_c(\Omega) \cap H^1(\Omega).$$

Taking the infimum over all sequences $\{r_k\}, \{t_k\}$, using the positive homogeneity of the functional $Q_C^{\bar{x}, \bar{\varphi}}(\cdot)$, and employing a density argument, we now arrive at

$$\frac{\alpha^2}{2} Q_C^{\bar{x}, \bar{\varphi}}(h) - \langle \alpha h, v \rangle + \int_{\mathcal{Z}} \frac{v^2}{|\nabla \bar{\varphi}|} d\mathcal{H}^{d-1} \geq 0 \quad \forall v \in C_c(\Omega).$$

This is the desired estimate. \square

We are now in the position to prove a lower bound for the curvature of C .

Proposition 6.9. *For all $h \in \mathcal{K}_C^*(\bar{x}, \bar{\varphi})$ with $Q_C^{\bar{x}, \bar{\varphi}}(h) < \infty$ there exists a g such that*

$$\begin{aligned}
h &= g \mathcal{H}^{d-1}|_{\mathcal{Z}}, \quad g \in L^1(\mathcal{Z}, \mathcal{H}^{d-1}) \cap L^2(\mathcal{Z}, |\nabla \bar{\varphi}| \mathcal{H}^{d-1}), \\
\frac{1}{2} \int_{\mathcal{Z}} g^2 |\nabla \bar{\varphi}| d\mathcal{H}^{d-1} &\leq Q_C^{\bar{x}, \bar{\varphi}}(h).
\end{aligned}$$

Proof. Let $h \in \mathcal{K}_C^*(\bar{x}, \bar{\varphi}) \subset \mathcal{M}(\Omega)$ with $Q_C^{\bar{x}, \bar{\varphi}}(h) < \infty$ be given. Then, for all $v \in C_c(\Omega)$ with $v = 0$ on \mathcal{Z} , we obtain from (36) that

$$\frac{\alpha}{2} Q_C^{\bar{x}, \bar{\varphi}}(h) \geq \langle h, v \rangle \quad \forall \alpha > 0,$$

i.e., $\langle h, v \rangle = 0$. This implies that the map

$$\tilde{h} : C_c(\mathcal{Z}) \rightarrow \mathbb{R}, \quad \tilde{v} \mapsto \langle h, v \rangle, \quad v \in C_c(\Omega), \quad v|_{\mathcal{Z}} = \tilde{v}$$

is well-defined as it is independent of the extension v of \tilde{v} appearing in its definition. Note that, given a $\tilde{v} \in C_c(\mathcal{Z})$, we can always find a $v \in C_c(\Omega)$ with $v|_{\mathcal{Z}} = \tilde{v}$, cf. the submanifold property of \mathcal{Z} . From (36) with

$$\alpha = \left(\int_{\mathcal{Z}} \frac{\tilde{v}^2}{|\nabla \bar{\varphi}|} d\mathcal{H}^{d-1} \right)^{1/2} \beta, \quad \beta > 0 \text{ arbitrary but fixed,}$$

it now follows

$$\left(\frac{\beta}{2} Q_C^{\bar{x}, \bar{\varphi}}(h) + \frac{1}{\beta} \right) \left(\int_{\mathcal{Z}} \frac{\tilde{v}^2}{|\nabla \bar{\varphi}|} d\mathcal{H}^{d-1} \right)^{1/2} \geq \langle \tilde{h}, \tilde{v} \rangle \quad \forall \tilde{v} \in C_c(\mathcal{Z}).$$

Using Hahn-Banach, we obtain that \tilde{h} can be extended from $C_c(\mathcal{Z})$ to the weighted Lebesgue-space $L^2(\mathcal{Z}, \mathcal{H}^{d-1}/|\nabla\bar{\varphi}|)$, i.e., there exists an $f \in L^2(\mathcal{Z}, \mathcal{H}^{d-1}/|\nabla\bar{\varphi}|)$ with

$$\left(\frac{\beta}{2}Q_C^{\bar{x},\bar{\varphi}}(h) + \frac{1}{\beta}\right) \left(\int_{\mathcal{Z}} \frac{\tilde{v}^2}{|\nabla\bar{\varphi}|} d\mathcal{H}^{d-1}\right)^{1/2} \geq \langle \tilde{h}, \tilde{v} \rangle = \int_{\mathcal{Z}} \frac{\tilde{v}f}{|\nabla\bar{\varphi}|} d\mathcal{H}^{d-1} \quad \forall \tilde{v} \in C_c(\mathcal{Z}).$$

Note that f is independent of β due to the density of $C_c(\mathcal{Z})$ in $L^2(\mathcal{Z}, \mathcal{H}^{d-1}/|\nabla\bar{\varphi}|)$. We thus arrive at

$$\langle h, v \rangle = \langle \tilde{h}, v|_{\mathcal{Z}} \rangle = \int_{\mathcal{Z}} \frac{vf}{|\nabla\bar{\varphi}|} d\mathcal{H}^{d-1} \quad \forall v \in C_c(\Omega)$$

with

$$\left(\int_{\mathcal{Z}} \frac{f^2}{|\nabla\bar{\varphi}|} d\mathcal{H}^{d-1}\right)^{1/2} \leq \left(\frac{\beta}{2}Q_C^{\bar{x},\bar{\varphi}}(h) + \frac{1}{\beta}\right) \quad \forall \beta > 0.$$

Choosing $\beta = (2/Q_C^{\bar{x},\bar{\varphi}}(h))^{1/2}$ for $Q_C^{\bar{x},\bar{\varphi}}(h) > 0$ and β arbitrarily large for $Q_C^{\bar{x},\bar{\varphi}}(h) = 0$, defining $g := f/|\nabla\bar{\varphi}|$ and using the density of $C_c(\Omega)$ in $C_0(\Omega)$ now yields the claim. Note that $g \in L^1(\mathcal{Z}, \mathcal{H}^{d-1})$ follows trivially from $h = g\mathcal{H}^{d-1}|_{\mathcal{Z}} \in \mathcal{M}(\Omega)$. \square

Next, we address the reverse estimate to that in [Proposition 6.9](#).

Lemma 6.10. *Let $g \in C_c(\mathcal{Z})$ be given and let $h := g\mathcal{H}^{d-1}|_{\mathcal{Z}} \in \mathcal{M}(\Omega)$. Then, h is an element of the critical cone $\mathcal{K}_C^*(\bar{x}, \bar{\varphi})$, it holds*

$$\frac{1}{2} \int_{\mathcal{Z}} g^2 |\nabla\bar{\varphi}| d\mathcal{H}^{d-1} = Q_C^{\bar{x},\bar{\varphi}}(h),$$

and for every sequence $\{t_k\} \subset \mathbb{R}^+$ with $t_k \searrow 0$ there exists a sequence $\{r_k\} \subset X$ such that $\bar{x} + t_k h + \frac{1}{2}t_k^2 r_k \in C$ holds for all k and such that $t_k r_k \xrightarrow{*} 0$, $\|h + \frac{1}{2}t_k r_k\|_X \rightarrow \|h\|_X$ and $\langle \bar{\varphi}, r_k \rangle \rightarrow Q_C^{\bar{x},\bar{\varphi}}(h)$ holds for $k \rightarrow \infty$.

Proof. We proceed in several steps:

(a): Since $\bar{\varphi} \in C_0(\Omega)$ and $\mathcal{Z} = \{\bar{\varphi} = 0\}$, it trivially holds $\langle \bar{\varphi}, h \rangle = 0$, i.e., $h \in \bar{\varphi}^\perp$.

(b): Assume that a point $p \in \mathcal{Z}$, an open ball $B \subset \mathbb{R}^{d-1}$, an open interval $J := (a, b)$, and a map $\gamma \in C^1(\overline{B})$ are given such that

$$p \in B \times J, \quad \overline{B \times J} \subset \Omega, \quad \mathcal{Z} \cap (B \times J) = \{(z, \gamma(z)) : z \in B\}$$

$$W := \{(z, z') : z \in B, |z' - \gamma(z)| < \varepsilon\} \subset B \times J, \quad \partial_d \bar{\varphi}(p) > 0$$

for some $\varepsilon \in (0, 1)$. Let $0 \leq \psi \in C(W)$ and $g \in C(\mathcal{Z} \cap (B \times J))$ be continuous and bounded functions and let $t_k \in (0, \infty)$ be a sequence with $t_k \searrow 0$. Assume w.l.o.g. that

$$t_k \|g\|_{L^\infty} \|\sqrt{1 + |\nabla\gamma|^2}\|_{L^\infty} \leq \varepsilon$$

for all k (else consider $\{t_k\}_{k \geq K}$, $K \in \mathbb{N}$ sufficiently large) and extend ψ by zero outside of W . With some abuse of notation, we extend the sign of g from $\mathcal{Z} \cap W$ to W by

$$\text{sign}(g)(z, z') := \text{sign } g(z, \gamma(z)).$$

We further define the sets

$$G_k := \left\{ (z, z') \in B \times J \mid t_k \frac{g^-(z, \gamma(z))}{2} \leq \frac{z' - \gamma(z)}{\sqrt{1 + |\nabla \gamma(z)|^2}} \leq t_k \frac{g^+(z, \gamma(z))}{2} \right\} \subset W,$$

where g^+ and g^- are abbreviations for $\max(0, g)$ and $\min(0, g)$, respectively. Now, we set

$$h_k := \frac{2 \operatorname{sign}(g)}{t_k} \mathbb{1}_{G_k},$$

where $\mathbb{1}_A$ denotes the indicator function of a set A . Next, we show the relations

$$\lim_{k \rightarrow \infty} \left(\int_{\Omega} h_k \psi v d\mathcal{L}^d \right) = \int_{\mathcal{Z} \cap (B \times J)} g \psi v d\mathcal{H}^{d-1} \quad \forall v \in C_0(\Omega), \quad (38)$$

$$\lim_{k \rightarrow \infty} \left(\int_{\Omega} |h_k| \psi d\mathcal{L}^d \right) = \int_{\mathcal{Z} \cap (B \times J)} |g| \psi d\mathcal{H}^{d-1}, \quad (39)$$

and

$$\lim_{k \rightarrow \infty} \left(\int_{\Omega} \frac{2h_k}{t_k} \bar{\varphi} \psi d\mathcal{L}^d \right) = \frac{1}{2} \int_{\mathcal{Z} \cap (B \times J)} g^2 \psi |\nabla \bar{\varphi}| d\mathcal{H}^{d-1}. \quad (40)$$

This can be seen as follows: Given a $v \in C_0(\Omega)$, we may calculate (using the abbreviations $dz := d\mathcal{L}^{d-1}(z)$ and $dz' := d\mathcal{L}^1(z')$)

$$\begin{aligned} \int_{\Omega} h_k \psi v d\mathcal{L}^d &= \frac{2}{t_k} \int_B \int_{\gamma(z) + t_k \frac{g^-(z, \gamma(z))}{2}}^{\gamma(z) + t_k \frac{g^+(z, \gamma(z))}{2}} \frac{\sqrt{1 + |\nabla \gamma(z)|^2}}{\sqrt{1 + |\nabla \gamma(z)|^2}} \operatorname{sign}(g) v \psi dz' dz \\ &= 2 \int_B \int_{\frac{g^-(z, \gamma(z))}{2}}^{\frac{g^+(z, \gamma(z))}{2}} \frac{\sqrt{1 + |\nabla \gamma(z)|^2}}{\sqrt{1 + |\nabla \gamma(z)|^2}} (\operatorname{sign}(g) v \psi)|_{(z, \gamma(z) + t_k z')} dz' dz \\ &\rightarrow 2 \int_B \int_{\frac{g^-(z, \gamma(z))}{2}}^{\frac{g^+(z, \gamma(z))}{2}} \frac{\sqrt{1 + |\nabla \gamma(z)|^2}}{\sqrt{1 + |\nabla \gamma(z)|^2}} (\operatorname{sign}(g) v \psi)|_{(z, \gamma(z))} dz' dz \\ &= \int_B \sqrt{1 + |\nabla \gamma(z)|^2} (g v \psi)|_{(z, \gamma(z))} dz = \int_{\mathcal{Z} \cap (B \times J)} g \psi v d\mathcal{H}^{d-1}. \end{aligned}$$

This yields (38). To obtain (39), we can use exactly the same calculation as above (just replace v with $\operatorname{sign}(g)$). It remains to prove (40). To this end, we compute

$$\begin{aligned} \int_{\Omega} \frac{2h_k}{t_k} \bar{\varphi} \psi d\mathcal{L}^d &= \frac{4}{t_k} \int_B \int_{\frac{g^-(z, \gamma(z))}{2}}^{\frac{g^+(z, \gamma(z))}{2}} \frac{\sqrt{1 + |\nabla \gamma(z)|^2}}{\sqrt{1 + |\nabla \gamma(z)|^2}} (\operatorname{sign}(g) \bar{\varphi} \psi)|_{(z, \gamma(z) + t_k z')} dz' dz \\ &= 4 \int_B \int_{\frac{g^-(z, \gamma(z))}{2}}^{\frac{g^+(z, \gamma(z))}{2}} \frac{\sqrt{1 + |\nabla \gamma(z)|^2}}{\sqrt{1 + |\nabla \gamma(z)|^2}} \int_0^1 (\partial_d \bar{\varphi})(z, \gamma(z) + s t_k z') ds (\operatorname{sign}(g) \psi)|_{(z, \gamma(z) + t_k z')} z' dz' dz \\ &\rightarrow 4 \int_B \int_{\frac{g^-(z, \gamma(z))}{2}}^{\frac{g^+(z, \gamma(z))}{2}} \frac{\sqrt{1 + |\nabla \gamma(z)|^2}}{\sqrt{1 + |\nabla \gamma(z)|^2}} z' dz' (\operatorname{sign}(g) \partial_d \bar{\varphi} \psi)|_{(z, \gamma(z))} dz \\ &= \frac{1}{2} \int_B (1 + |\nabla \gamma(z)|^2) (g^2 \partial_d \bar{\varphi} \psi)|_{(z, \gamma(z))} dz. \end{aligned}$$

Differentiating $\bar{\varphi}(z, \gamma(z))$ w.r.t. z_i yields $\partial_d \bar{\varphi}(z, \gamma(z)) \partial_i \gamma(z) = -\partial_i \bar{\varphi}(z, \gamma(z))$ for all $i = 1, \dots, d-1$. Thus,

$$\partial_d \bar{\varphi}(z, \gamma(z)) \sqrt{1 + |\nabla \gamma(z)|^2} = |\nabla \bar{\varphi}(z, \gamma(z))| \quad \forall z \in B.$$

Hence, we arrive at

$$\begin{aligned} \int_{\Omega} \frac{2h_k}{t_k} \bar{\varphi} \psi \, d\mathcal{L}^d &\rightarrow \frac{1}{2} \int_B \sqrt{1 + |\nabla \gamma(z)|^2} (|\nabla \bar{\varphi}| g^2 \psi)|_{(z, \gamma(z))} \, dz \\ &= \frac{1}{2} \int_{\mathcal{Z} \cap (B \times J)} g^2 \psi |\nabla \bar{\varphi}| \, d\mathcal{H}^{d-1}. \end{aligned}$$

This proves (40).

(c): From the implicit function theorem and the definition of \mathcal{Z} , it follows that for every $p \in \mathcal{Z}$ there exist an orthogonal transformation $R \in O(d)$, an open ball $B \subset \mathbb{R}^{d-1}$, an open interval $J := (a, b)$, and a map $\gamma \in C^1(\bar{B})$ with values in J such that

$$\begin{aligned} p \in R(B \times J), \quad \overline{R(B \times J)} \subset \Omega, \quad \mathcal{Z} \cap R(B \times J) &= R(\{(z, \gamma(z)) : z \in B\}), \\ W := \{(z, z') : z \in B, |z' - \gamma(z)| < \varepsilon\} &\subset B \times J, \\ R(\{(z, z') \in B \times J : \gamma(z) < z'\}) &\subset \{\bar{\varphi} > 0\}, \\ R(\{(z, z') \in B \times J : z' < \gamma(z)\}) &\subset \{\bar{\varphi} < 0\} \end{aligned}$$

for some $\varepsilon > 0$. Since $\text{supp}(g)$ is compact, we may find points $p_1, \dots, p_L \in \mathcal{Z}$ with associated R_l, B_l etc. such that the sets $R_l(W_l)$, $l = 1, \dots, L$, cover $\text{supp}(g)$. Define

$$U := \bigcup_{l=1}^L R_l(W_l) \subset \Omega$$

and choose a partition of unity $(\psi_l)_{l=1}^L$ subordinate to the $R_l(W_l)$ -cover of the set U , i.e., a collection of continuous functions $\psi_l \in C(U, [0, 1])$ such that

$$\text{supp}(\psi_l) \subset W_l, \quad \sum_l \psi_l = 1 \text{ on } U.$$

Consider now an arbitrary but fixed sequence $t_k \in (0, \infty)$ with $t_k \searrow 0$, extend the functions ψ_l by zero and define (for k large enough)

$$\begin{aligned} \text{sign}_l(g)(R_l(z, z')) &:= \text{sign } g(R_l(z, \gamma_l(z))) \quad \forall (z, z') \in W_l, \\ G_{l,k} &:= \left\{ (z, z') \in B_l \times J_l \mid t_k \frac{g^-(R_l(z, \gamma_l(z)))}{2} \leq \frac{z' - \gamma_l(z)}{\sqrt{1 + |\nabla \gamma_l(z)|^2}} \leq t_k \frac{g^+(R_l(z, \gamma_l(z)))}{2} \right\}, \\ h_k &:= \sum_{l=1}^L \frac{2 \text{sign}_l(g)}{t_k} \mathbb{1}_{R_l(G_{l,k})} \psi_l. \end{aligned}$$

Then, it holds $\bar{x} + t_k h_k \in C = L^\infty(\Omega, [0, 1])$ (cf. the signs of the involved functions and $\|h_k\|_{L^\infty} \leq 2/t_k$), and we may deduce from (b) that

$$\lim_{k \rightarrow \infty} \left(\int_{\Omega} h_k v \, d\mathcal{L}^d \right) = \sum_{l=1}^L \int_{\mathcal{Z} \cap R_l(B_l \times J_l)} g \psi_l v \, d\mathcal{H}^{d-1} = \langle h, v \rangle \quad \forall v \in C_0(\Omega), \quad (41)$$

$$\lim_{k \rightarrow \infty} \left(\int_{\Omega} |h_k| d\mathcal{L}^d \right) = \sum_{l=1}^L \int_{\mathcal{Z} \cap R_l(B_l \times J_l)} |g| \psi_l d\mathcal{H}^{d-1} = \|h\|_X \quad (42)$$

and

$$\lim_{k \rightarrow \infty} \left(\int_{\Omega} \frac{2h_k}{t_k} \bar{\varphi} d\mathcal{L}^d \right) = \frac{1}{2} \sum_{l=1}^L \int_{\mathcal{Z} \cap R_l(B_l \times J_l)} g^2 \psi_l |\nabla \bar{\varphi}| d\mathcal{H}^{d-1} = \frac{1}{2} \int_{\mathcal{Z}} g^2 |\nabla \bar{\varphi}| d\mathcal{H}^{d-1}. \quad (43)$$

The above proves $h \in \mathcal{T}_C^*(\bar{x})$ and $h \in \mathcal{K}_C^*(\bar{x}, \bar{\varphi})$, see (a). From (41), (42) and (43), we obtain further that $r_k := 2(h_k - h)/t_k$ satisfies $\bar{x} + t_k h + \frac{1}{2} t_k^2 r_k \in C$ for all k and

$$t_k r_k \xrightarrow{*} 0, \quad \|h + \frac{1}{2} t_k r_k\|_X \rightarrow \|h\|_X, \quad \langle \bar{\varphi}, r_k \rangle \rightarrow \frac{1}{2} \int_{\mathcal{Z}} g^2 |\nabla \bar{\varphi}| d\mathcal{H}^{d-1}$$

as $k \rightarrow \infty$. If we combine this with Proposition 6.9 and Definition 3.1, then the claim follows immediately. \square

Using Proposition 6.9 and Lemma 6.10, we finally arrive at an explicit formula for the directional curvature functional in the bang-bang case.

Theorem 6.11. *For every tuple $(\bar{x}, \bar{\varphi}) \in C \times -\mathcal{N}_C^*(\bar{x})$ that satisfies the conditions in Assumption 6.6, it holds*

$$\left\{ h \in \mathcal{K}_C^*(\bar{x}, \bar{\varphi}) \mid Q_C^{\bar{x}, \bar{\varphi}}(h) < \infty \right\} = \left\{ g \mathcal{H}^{d-1}|_{\mathcal{Z}} \mid g \in L^1(\mathcal{Z}, \mathcal{H}^{d-1}) \cap L^2(\mathcal{Z}, |\nabla \bar{\varphi}| \mathcal{H}^{d-1}) \right\}. \quad (44)$$

Moreover, for every element $h = g \mathcal{H}^{d-1}|_{\mathcal{Z}}$ of the above set, it is true that

$$Q_C^{\bar{x}, \bar{\varphi}}(h) = \frac{1}{2} \int_{\mathcal{Z}} g^2 |\nabla \bar{\varphi}| d\mathcal{H}^{d-1}. \quad (45)$$

Proof. Proposition 6.9 yields that “ \subset ” holds in (44) and that “ \supseteq ” holds in (45). To obtain the reverse inclusion/inequality, we consider an arbitrary but fixed $h = g \mathcal{H}^{d-1}|_{\mathcal{Z}} \in \mathcal{M}(\Omega)$ with some $g \in L^1(\mathcal{Z}, \mathcal{H}^{d-1}) \cap L^2(\mathcal{Z}, |\nabla \bar{\varphi}| \mathcal{H}^{d-1})$. Using a compact exhaustion of the domain Ω and mollification, it is easy to see that we can find a sequence $(g_n) \subset C_c(\mathcal{Z})$ with $g_n \rightarrow g$ in $L^1(\mathcal{Z}, \mathcal{H}^{d-1}) \cap L^2(\mathcal{Z}, |\nabla \bar{\varphi}| \mathcal{H}^{d-1})$. The latter implies that $h_n := g_n \mathcal{H}^{d-1}|_{\mathcal{Z}}$ satisfies $h_n \xrightarrow{*} h$ in X and that

$$\frac{1}{2} \int_{\mathcal{Z}} g_n^2 |\nabla \bar{\varphi}| d\mathcal{H}^{d-1} \rightarrow \frac{1}{2} \int_{\mathcal{Z}} g^2 |\nabla \bar{\varphi}| d\mathcal{H}^{d-1}.$$

On the other hand, we obtain from Lemma 6.10 that $h_n \in \mathcal{K}_C^*(\bar{x}, \bar{\varphi})$ holds for all n and that there exist sequences $\{r_{n,k}\} \subset X$ and $\{t_{n,k}\} \subset \mathbb{R}^+$ with $\bar{x} + t_{n,k} h_n + \frac{1}{2} t_{n,k}^2 r_{n,k} \in C$ for all n, k and

$$t_{n,k} \searrow 0, \quad t_{n,k} r_{n,k} \xrightarrow{*} 0, \quad \langle \bar{\varphi}, r_{n,k} \rangle \rightarrow Q_C^{\bar{x}, \bar{\varphi}}(h_n) \quad \text{and} \quad \|h_n + \frac{1}{2} t_{n,k} r_{n,k}\|_X \rightarrow \|h_n\|_X$$

for all n as $k \rightarrow \infty$. Note that, since the sequence $\|h_n\|_X$ is bounded and since the norms $\|h_n + \frac{1}{2} t_{n,k} r_{n,k}\|_X$ converge to $\|h_n\|_X$, in the above situation, we may assume w.l.o.g.

that $\|t_{n,k}r_{n,k}\|_X \leq M$ holds for some constant M that is independent of n and k (just shift the index k appropriately for each n). From [Lemmas 3.3](#) and [6.10](#), we may now deduce that h is an element of the critical cone $\mathcal{K}_C^*(\bar{x}, \bar{\varphi})$ and that

$$Q_C^{\bar{x}, \bar{\varphi}}(h) \leq \liminf_{n \rightarrow \infty} Q_C^{\bar{x}, \bar{\varphi}}(h_n) = \liminf_{n \rightarrow \infty} \frac{1}{2} \int_{\mathcal{Z}} g_n^2 |\nabla \bar{\varphi}| d\mathcal{H}^{d-1} = \frac{1}{2} \int_{\mathcal{Z}} g^2 |\nabla \bar{\varphi}| d\mathcal{H}^{d-1} < \infty.$$

This proves “ \supset ” in [\(44\)](#) and “ \leq ” in [\(45\)](#). \square

We combine [Theorems 6.4](#) and [6.11](#) and arrive at the main result of this section.

Theorem 6.12 (Explicit No-Gap Second-Order Condition for Bang-Bang Problems). *Consider an optimization problem of the form [\(P\)](#) with C , X etc. as in [Assumption 6.2](#). Assume that $\bar{x} \in C$ is fixed, that J satisfies the conditions in [Assumption 4.1](#), that the map $X \ni h \mapsto J''(\bar{x})h^2 \in \mathbb{R}$ is weak- \star continuous, that \bar{x} and $\bar{\varphi}$ satisfy the conditions in [Assumption 6.6](#), and that the constant $K(\bar{\varphi})$ in [\(31\)](#) is non-zero. Then, the condition*

$$\begin{aligned} \frac{1}{2} \int_{\mathcal{Z}} g^2 |\nabla \bar{\varphi}| d\mathcal{H}^{d-1} + J''(\bar{x}) \left(g\mathcal{H}^{d-1}|_{\mathcal{Z}} \right)^2 > 0 \\ \forall g \in L^1(\mathcal{Z}, \mathcal{H}^{d-1}) \cap L^2(\mathcal{Z}, |\nabla \bar{\varphi}| \mathcal{H}^{d-1}) \setminus \{0\} \end{aligned} \quad (46)$$

is equivalent to the quadratic growth condition [\(34\)](#) with constants $c > 0$ and $\varepsilon > 0$.

We conclude this section with some remarks.

Remark 6.13.

- (i) The inequality [\(46\)](#) with “ \geq ” instead of “ $>$ ” is still a necessary optimality condition when $K(\bar{\varphi}) = 0$.
- (ii) [Theorem 6.12](#) is still valid when $\mathcal{Z} = \emptyset$ and [\(46\)](#) is empty. In this case, the only thing that has to be checked to obtain [\(34\)](#) is the level-set condition $K(\bar{\varphi}) > 0$.
- (iii) We expect that the assumptions on $\bar{\varphi}$ in [Theorem 6.12](#) can be weakened (the Taylor expansion in [Lemma 6.7](#), for example, also holds in a far more general setting, see [[Christof and Meyer, 2016, Proposition 5.9](#)]).
- (iv) The techniques used in the proofs of [Lemma 6.8](#) and [Proposition 6.9](#) might be useable in other situations as well (e.g., the idea to exploit the subdifferential structure of the admissible set C , cf. [\(37\)](#)).
- (v) We point out that [\(MRC\)](#) does not hold in the situation of [Theorem 6.12](#) (from $C \subset L^1(\Omega)$ it follows $\mathcal{T}_C(\bar{x}) \cap \bar{\varphi}^\perp = \{0\} \neq \mathcal{K}_C^*(\bar{x}, \bar{\varphi})$ and this is incompatible with the condition [\(MRC\)](#), cf. [Remark 3.5](#)).
- (vi) The first addend on the left-hand side of [\(46\)](#) measures the curvature of the set C in \bar{x} (compare, e.g., with [Theorem 5.11](#)). Note that the surface integral in [\(46\)](#) is only meaningful for a C^1 -function $\bar{\varphi}$. This shows that exploiting the regularity of the gradient $\bar{\varphi} = J'(\bar{x})$ is essential for the derivation of [Theorem 6.12](#).

(vii) *Theorem 6.12* can, e.g., be used to obtain no-gap second-order conditions for bang-bang optimal control problems governed by semilinear PDEs, cf. [Casas et al., 2017b].

7 Conclusion

Given the examples and applications in Sections 5 and 6, we may conclude that the theoretical framework of Section 4 is indeed a handy tool in the derivation of (no-gap) second-order optimality conditions for problems of the type (P). Our results are not only applicable in those situations where the classical assumptions of polyhedricity and second-order regularity are satisfied (see Theorems 5.10 and 5.11) but also allow to study problems with more complicated admissible sets. This is underlined by the no-gap second-order conditions which were derived in the bang-bang setting of Section 6.2. Further investigation is needed for the study of the directional curvature functional in the presence of pointwise state constraints, cf. the comments after Example 5.6. We plan to discuss this topic in a forthcoming paper.

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