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On the sequential normal compactness condition and its restrictiveness in selected function spaces

Patrick Mehlitz*

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Sequential normal compactness is one of the most important properties in terms of modern variational analysis. It is necessary for the derivation of calculus rules for the computation of generalized normals to set intersections or preimages of sets under transformations. While sequential normal compactness is inherent in finite-dimensional Banach spaces, its presence has to be checked in the infinite-dimensional situation. In this paper, we show that broad classes of sets in Lebesgue and Sobolev spaces which are reasonable in the context of optimal control suffer from an intrinsic lack of sequential normal compactness.

Keywords: decomposable set, optimal control, sequential normal compactness
MSC: 28B05, 49J53

1 Introduction

In order to ensure that reasonable calculus rules hold for the tools of modern variational analysis, the presence of a certain compactness condition called sequential normal compactness is necessary. It is indispensable to show that a certain extremal principle w.r.t. limiting normals in Mordukhovich's sense is suitable in order to characterize extremal set systems in Asplund spaces, see Mordukhovich and Shao [1996b]. Based on that, calculus rules for the determination of limiting normals to set intersections and preimages of sets under transformations can be derived, see Mordukhovich [2006].
Recently, the variational calculus of Mordukhovich received some attention in the context of optimal control (e.g. optimal control of variational inequalities or optimal control with

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complementarity constraints on the control function), see Guo and Ye [2016], Harder and Wachsmuth [2017], Hintermüller et al. [2014], Jarušek and Outrata [2007], Mehlitz and Wachsmuth [2016], Outrata et al. [2011], Wachsmuth [2016] and the references therein. To the best of our knowledge, a detailed discussion on whether or not the appearing sets of interest are sequentially normally compact is given in Mehlitz and Wachsmuth [2016] only. However, in order to get a better sensation on the applicability of modern variational analysis in function spaces, it seems to be necessary to study the presence of sequential normal compactness for reasonable classes of sets in Lebesgue and Sobolev spaces.

In this paper, we want to investigate sequential normal compactness in the context of two classes of sets in function spaces. First, we discuss decomposable sets in Lebesgue spaces which are appropriate to characterize the feasible set of optimal control problems with control constraints. The variational geometry of decomposable sets has been studied recently in Mehlitz and Wachsmuth [2016, 2017]. As we will see, such sets suffer from an inherent lack of sequential normal compactness. Afterwards, we focus our attention on standard box constraint sets given by lower and/or upper bounds in Sobolev spaces. Such sets are reasonable in the context of state-constrained optimal control. Our results show that sequential normal compactness is only available if the given Sobolev space is embedded into the Lebesgue space of essentially bounded functions which is a very restrictive condition.

The remaining part of the paper is organized as follows: In Section 2, we comment on the notation used in this manuscript and present some preliminary results. Section 3 is dedicated to the theoretical background of sequential normal compactness. First, we review the most important normal compactness conditions from variational analysis and interrelate them. Particularly, we state a nice characterization of sequential normal compactness for closed, convex sets in reflexive Banach spaces. Afterwards, we present some calculus rules for sequentially normally compact sets. In Section 4, we introduce the notion of decomposable sets and show that broad classes of such sets are nowhere sequentially normally compact. Finally, we deal with the presence of sequential normal compactness for box constraint sets in Sobolev spaces in Section 5.

2 Notation and preliminary results

In this section, we briefly present the notation used in this paper and give some preliminary results we need to exploit later.

2.1 Basic notation

Let $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$ denote the natural numbers (without zero), the rational numbers, and the real numbers, respectively. Furthermore, $\mathbb{R}_0^+$ and $\mathbb{R}^+$ represent the sets of all nonnegative reals and all positive reals, respectively. The sign-function $\operatorname{sgn}\colon \mathbb{R} \to \mathbb{R}$ is defined by

$$\forall \alpha \in \mathbb{R}: \quad \operatorname{sgn} \alpha := \begin{cases} 1 & \text{if } \alpha > 0, \\ 0 & \text{if } \alpha = 0, \\ -1 & \text{if } \alpha < 0. \end{cases}$$

Note that $|\alpha| = (\operatorname{sgn} \alpha)\alpha$ is valid for any $\alpha \in \mathbb{R}$.
For a Banach space $\mathcal{X}$, $\|\cdot\|_{\mathcal{X}}$ is used to express its norm. We denote the open and closed ε-ball around $x \in \mathcal{X}$ w.r.t. the norm $\|\cdot\|_{\mathcal{X}}$ in $\mathcal{X}$ by $\mathbb{U}^{\varepsilon}_{\mathcal{X}}(x)$ and $\mathbb{B}^{\varepsilon}_{\mathcal{X}}(x)$, respectively. Let us define the distance function $\operatorname{dist}\colon \mathcal{X} \times 2^{\mathcal{X}} \to \mathbb{R}^+_0 \cup \{+\infty\}$ of $\mathcal{X}$ as stated below:

$$\forall x \in \mathcal{X}\, \forall A \subset \mathcal{X}: \quad \operatorname{dist}(x, A) := \inf_{a \in A} \|x - a\|_{\mathcal{X}}.$$

Therein, $2^{\mathcal{X}}$ is the power set of $\mathcal{X}$ and $\inf \varnothing := +\infty$ shall hold. The codimension of a subspace L of $\mathcal{X}$ is defined as the dimension of $\mathcal{X}/L := \{\{x\} + L \subset \mathcal{X} \,|\, x \in \mathcal{X}\}$ which is the so-called factor space of $\mathcal{X}$ w.r.t. L. For a nonempty set $A \subset \mathcal{X}$, $\operatorname{int} A$, $\operatorname{ri} A$, $\operatorname{cl} A$, $\operatorname{bd} A$, $\operatorname{cone} A$, $\operatorname{conv} A$, and $\operatorname{lin} A$ denote the interior of A, the relative interior of A, the closure of A, the boundary of A, the conic hull of A (i.e. the smallest cone w.r.t. set inclusion which includes A), the convex hull of A, and the linear hull of A, respectively. In the lemma below, we show a nearby relationship between the operators lin and conv.

Lemma 2.1. *Let $S \subset \mathcal{X}$ be a nonempty subset of a Banach space $\mathcal{X}$. Then the relation $\operatorname{lin} S = \operatorname{conv} \bigcup_{\alpha \in \mathbb{R}} \alpha S$ holds true.*

Proof. Let us prove the inclusion [$\subset$] first. Since we have $S \subset \operatorname{conv} \bigcup_{\alpha \in \mathbb{R}} \alpha S$, it is sufficient to show that $L := \operatorname{conv} \bigcup_{\alpha \in \mathbb{R}} \alpha S$ is a linear space. Therefore, we choose $x, y \in L$ and $\gamma^x, \gamma^y \in \mathbb{R}$ arbitrarily. Then we find integers $n, m \in \mathbb{N}$, vectors $s^x_1, \ldots, s^x_n, s^y_1, \ldots, s^y_m \in S$, scalars $\alpha^x_1, \ldots, \alpha^x_n, \alpha^y_1, \ldots, \alpha^y_m \in \mathbb{R}$, and nonnegative scalars $\mu_1, \ldots, \mu_n, \nu_1, \ldots, \nu_m \in \mathbb{R}^+_0$ such that

$$x = \sum_{i=1}^{n} \mu_i \alpha^x_i s^x_i, \quad y = \sum_{j=1}^{m} \nu_j \alpha^y_j s^y_j, \quad \sum_{i=1}^{n} \mu_i = 1, \quad \sum_{j=1}^{m} \nu_j = 1.$$

Thus, we obtain

$$\gamma^x x + \gamma^y y = \sum_{i=1}^{n} \mu_i \gamma^x \alpha^x_i s^x_i + \sum_{j=1}^{m} \nu_j \gamma^y \alpha^y_j s^y_j = \sum_{i=1}^{n} \frac{\mu_i}{2} 2\gamma^x \alpha^x_i s^x_i + \sum_{j=1}^{m} \frac{\nu_j}{2} 2\gamma^y \alpha^y_j s^y_j \in L$$

since $\frac{\mu_1}{2}, \ldots, \frac{\mu_n}{2}, \frac{\nu_1}{2}, \ldots, \frac{\nu_m}{2} \in \mathbb{R}^+_0$ are nonnegative scalars which satisfy

$$\sum_{i=1}^{n} \frac{\mu_i}{2} + \sum_{j=1}^{m} \frac{\nu_j}{2} = 1.$$

Consequently, L is a linear space which contains S, i.e. $\operatorname{lin} S \subset L$.

The converse inclusion [$\supset$] follows from the nearby observation that $\alpha S \subset \operatorname{lin} S$ holds for all $\alpha \in \mathbb{R}$. This shows $L \subset \operatorname{conv} \operatorname{lin} S = \operatorname{lin} S$ and completes the proof. □

We use $\mathcal{X}^\star$ to denote the topological dual of the Banach space $\mathcal{X}$. The mapping $\langle\cdot,\cdot\rangle_{\mathcal{X}}\colon \mathcal{X}^\star\times\mathcal{X}\to\mathbb{R}_0^+$ represents the dual pairing of $\mathcal{X}$. Recall that the canonical embedding $\mathcal{X}\ni x\mapsto\langle\cdot,x\rangle_{\mathcal{X}}\in\mathcal{X}^{\star\star}$ of $\mathcal{X}$ is an injective isometry. If it is surjective, $\mathcal{X}$ is called reflexive. Thus, any reflexive Banach space is isometrically isomorphic to its bidual space.
Let $\{x_k\}_{k\in\mathbb{N}}\subset\mathcal{X}$ be an arbitrary sequence in $\mathcal{X}$. This sequence is said to converge to $\bar x\in\mathcal{X}$, $x_k\to\bar x$ for short, if the real sequence $\{\|x_k-\bar x\|_{\mathcal{X}}\}_{k\in\mathbb{N}}$ converges to zero. On the other hand, $\{x_k\}_{k\in\mathbb{N}}$ converges weakly to $\bar x$, $x_k\rightharpoonup\bar x$ for short, if $\{\langle x^\star,x_k\rangle_{\mathcal{X}}\}_{k\in\mathbb{N}}$ converges to $\langle x^\star,\bar x\rangle_{\mathcal{X}}$ for any $x^\star\in\mathcal{X}^\star$. We say that a sequence $\{x_k^\star\}_{k\in\mathbb{N}}\subset\mathcal{X}^\star$ converges weakly* to $\bar x^\star\in\mathcal{X}^\star$, $x_k^\star\overset{\star}{\rightharpoonup}\bar x^\star$ for short, if the real sequence $\{\langle x_k^\star,x\rangle_{\mathcal{X}}\}_{k\in\mathbb{N}}$ converges to $\langle\bar x^\star,x\rangle_{\mathcal{X}}$ for any $x\in\mathcal{X}$. Note that whenever the Banach space $\mathcal{X}$ is reflexive, the notions of weak* and weak convergence in the dual space $\mathcal{X}^\star$ are equivalent.
Let $\mathcal{Y}$ be a Banach space as well. Then the product space $\mathcal{X}\times\mathcal{Y}$ becomes a Banach space when equipped, e.g., with the sum norm induced by $\mathcal{X}$ and $\mathcal{Y}$. Particularly, $\mathcal{X}^n$ denotes the n-fold product of $\mathcal{X}$ and is equipped with the norm

$$\forall(x_1,\ldots,x_n)\in\mathcal{X}^n\colon\qquad \|(x_1,\ldots,x_n)\|_{\mathcal{X}^n}:=\sum_{i=1}^n\|x_i\|_{\mathcal{X}}\,.$$

The spaces $(\mathcal{X}\times\mathcal{Y})^\star$ and $\mathcal{X}^\star\times\mathcal{Y}^\star$ are isomorphic and equipped with equivalent norms which is why we are going to identify them in this paper.
The Banach space $\mathcal{X}$ is continuously embedded into the Banach space $\mathcal{Y}$, $\mathcal{X}\hookrightarrow\mathcal{Y}$ for short, if $\mathcal{X}\subset\mathcal{Y}$ holds while there is a constant $\gamma>0$ which satisfies

$$\forall x\in\mathcal{X}\colon\qquad \|x\|_{\mathcal{Y}}\le\gamma\,\|x\|_{\mathcal{X}}\,,$$

i.e. if the identical operator $\mathcal{X}\ni x\mapsto x\in\mathcal{Y}$ is well-defined and continuous.
In this paper, we interpret $\mathbb{R}^n$ as a Hilbert space equipped with the Euclidean inner product denoted by $a\cdot b$ for any two vectors $a,b\in\mathbb{R}^n$. The term $a\perp b$ expresses that a and b are perpendicular to each other, i.e. that $a\cdot b=0$ holds. Note that the Euclidean inner product induces the Euclidean norm in $\mathbb{R}^n$ which will be denoted by $|\cdot|_2$. If necessary, the order relations $a<b$, $a\le b$, and $a\ge b$ have to be interpreted in componentwise fashion. By $\mathtt{e}\in\mathbb{R}^n$, we denote the all-ones vector of appropriate dimension.

2.2 Variational analysis

Let $\mathcal{X}$ be an arbitrary Banach space, fix a set $A\subset\mathcal{X}$, and choose $\bar x\in A$. The tangent and the inner (or adjacent) tangent cone to A at $\bar x$ are defined by

$$\begin{aligned}
\mathcal{T}_A(\bar x)&:=\left\{d\in\mathcal{X}\,\middle|\,\begin{aligned}&\exists\{d_k\}_{k\in\mathbb{N}}\subset\mathcal{X}\,\exists\{t_k\}_{k\in\mathbb{N}}\subset\mathbb{R}^+\colon\\ &\qquad d_k\to d,\ t_k\to0,\ \bar x+t_kd_k\in A\,\forall k\in\mathbb{N}\end{aligned}\right\},\\
\mathcal{T}_A^\flat(\bar x)&:=\left\{d\in\mathcal{X}\,\middle|\,\begin{aligned}&\forall\{t_k\}_{k\in\mathbb{N}}\subset\mathbb{R}^+\colon\ t_k\to0\\ &\qquad\exists\{d_k\}_{k\in\mathbb{N}}\subset\mathcal{X}\colon\ d_k\to d,\ \bar x+t_kd_k\in A\,\forall k\in\mathbb{N}\end{aligned}\right\},
\end{aligned}$$

see e.g. [Aubin and Frankowska, 2009, Section 4.1]. Note that these cones are always closed and satisfy $\mathcal{T}^\flat_A(\bar{x}) \subset \mathcal{T}_A(\bar{x})$. If equality holds, A is said to be derivable at $\bar{x}$. We call A derivable if it is derivable at all of its points. Note that we have

$$\mathcal{T}^\flat_A(\bar{x}) = \mathcal{T}_A(\bar{x}) = \operatorname{cl}\operatorname{cone}(A - \{\bar{x}\})$$

for any convex set A, i.e. any convex set is derivable.
For any $\varepsilon \geq 0$, we define the set of all Fréchet ε-normals to A at $\bar{x}$ as stated below:

$$\widehat{\mathcal{N}}^\varepsilon_A(\bar{x}) := \left\{ x^\star \in \mathcal{X}^\star \,\middle|\, \limsup_{x \to \bar{x},\, x \in A} \frac{\langle x^\star, x - \bar{x} \rangle_{\mathcal{X}}}{\|x - \bar{x}\|_{\mathcal{X}}} \leq \varepsilon \right\}.$$

Note that for $\varepsilon = 0$, $\widehat{\mathcal{N}}_A(\bar{x}) := \widehat{\mathcal{N}}^0_A(\bar{x})$ is a closed, convex cone, the so-called Fréchet (or regular) normal cone to A at $\bar{x}$. Finally, the limiting (or Mordukhovich, basic) normal cone to A at $\bar{x}$ is given as follows:

$$\mathcal{N}_A(\bar{x}) := \left\{ x^\star \in \mathcal{X}^\star \,\middle|\, \begin{array}{c} \exists \{x_k\}_{k\in\mathbb{N}} \subset A,\, \exists \{x^\star_k\}_{k\in\mathbb{N}} \subset \mathcal{X}^\star\, \exists \{\varepsilon_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^+_0 : \\ x_k \to \bar{x},\, x^\star_k \overset{\star}{\rightharpoonup} x^\star,\, \varepsilon_k \to 0,\, x^\star_k \in \widehat{\mathcal{N}}^{\varepsilon_k}_A(x_k) \forall k \in \mathbb{N} \end{array} \right\}.$$

Obviously, $\widehat{\mathcal{N}}_A(\bar{x}) \subset \mathcal{N}_A(\bar{x})$ is valid. If $\mathcal{X}$ is an Asplund space, i.e. if every separable, closed subspace of $\mathcal{X}$ possesses a separable dual space, then the sequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ can be chosen to be identically zero in the above definition of the limiting normal cone, see [Mordukhovich, 2006, Theorem 2.35]. In general, the limiting normal cone is a nonconvex cone which does not need to be closed in the infinite-dimensional situation. On the other hand, for convex sets A, we have

$$\widehat{\mathcal{N}}_A(\bar{x}) = \mathcal{N}_A(\bar{x}) = \{x^\star \in \mathcal{X}^\star \,|\, \forall x \in A\colon\ \langle x^\star, x - \bar{x}\rangle_{\mathcal{X}} \leq 0\},$$

i.e. the above normal cones both coincide with the normal cone in the sense of convex analysis.

2.3 Function spaces

Let $(\Omega, \Sigma, \mathfrak{m})$ be a complete and σ-finite measure space. In order to exclude trivial situations, we assume $\mathfrak{m}(\Omega) > 0$. For any $p \in [1, \infty]$ and any $q \in \mathbb{N}$, $L^p(\mathfrak{m}; \mathbb{R}^q)$ denotes the usual Lebesgue space of (equivalence classes of) measurable, p-integrable functions from Ω to $\mathbb{R}^q$ equipped with the norm as given below:

$$\begin{aligned} \forall p \in [1, \infty)\, \forall u \in L^p(\mathfrak{m}; \mathbb{R}^q)\colon \qquad & \|u\|_{L^p(\mathfrak{m};\mathbb{R}^q)} := \left(\int_\Omega |u(\omega)|_2^p \,\mathrm{d}\mathfrak{m} \right)^{1/p} \\ \forall u \in L^\infty(\mathfrak{m}; \mathbb{R}^q)\colon \qquad & \|u\|_{L^\infty(\mathfrak{m};\mathbb{R}^q)} := \operatorname*{esssup}_{\omega \in \Omega} |u(\omega)|_2 . \end{aligned}$$

For brevity, we set $L^p(\mathfrak{m}) := L^p(\mathfrak{m}; \mathbb{R})$ for any $p \in [1, \infty]$. Note that we have the equivalence $L^p(\mathfrak{m}; \mathbb{R}^q) = L^p(\mathfrak{m})^q$ while the norms $\|\cdot\|_{L^p(\mathfrak{m};\mathbb{R}^q)}$ and $\|\cdot\|_{L^p(\mathfrak{m})^q}$ are equivalent in the sense of norms. If $\Omega \subset \mathbb{R}^d$ is an arbitrary domain, i.e. a nonempty, open set, equipped with the (restricted) d-dimensional Lebesgue measure λ and the corresponding (restricted) Borelean σ-algebra $\mathcal{B}$, we use $L^p(\Omega;\mathbb{R}^q)$ to denote the Lebesgue space which results from the formal completion of the measure space $(\Omega,\mathcal{B},\lambda)$, see [Bogachev, 2007, Section 1.5]. For any $p \in [1,\infty)$, the dual of $L^p(\mathfrak{m};\mathbb{R}^q)$ is isometric to $L^{p'}(\mathfrak{m};\mathbb{R}^q)$ where $p' \in (1,\infty]$ such that $1/p + 1/p' = 1$ holds is the so-called conjugate coefficient of p. The corresponding dual pairing is given as stated below:

$$\forall u \in L^p(\mathfrak{m};\mathbb{R}^q)\,\forall \eta \in L^{p'}(\mathfrak{m};\mathbb{R}^q)\colon \quad \langle \eta, u\rangle_{L^p(\mathfrak{m};\mathbb{R}^q)} := \int_\Omega u(\omega)\cdot\eta(\omega)\mathrm{d}\mathfrak{m}.$$

For any set $A \in \Sigma$, $\chi_A\colon \Omega \to \mathbb{R}$ denotes the characteristic function of A which has value 1 for all $\omega \in A$ and vanishes everywhere on $\Omega \setminus A$. Clearly, we always have $\chi_A \in L^\infty(\mathfrak{m})$. For later use, we need the following consequence of Lebesgue's dominated convergence theorem.

Lemma 2.2. *Fix a complete and σ-finite measure space $(\Omega,\Sigma,\mathfrak{m})$ as well as a parameter $p \in [1,\infty)$. Let $u \in L^p(\mathfrak{m};\mathbb{R}^q)$ be arbitrarily chosen and let $\{\Omega_k\}_{k\in\mathbb{N}} \subset \Sigma$ be a sequence of sets which satisfies $\mathfrak{m}(\Omega_k) \to 0$ as $k \to \infty$. Then we have $\chi_{\Omega_k}u \to 0$ and $(1-\chi_{\Omega_k})u \to u$ in $L^p(\mathfrak{m};\mathbb{R}^q)$ as $k \to \infty$.*

Proof. By construction, the sequences $\{\chi_{\Omega_k}u\}_{k\in\mathbb{N}}$ and $\{(1-\chi_{\Omega_k})u\}_{k\in\mathbb{N}}$ converge pointwise almost everywhere on Ω to 0 and u, respectively. Moreover, $\{|\chi_{\Omega_k}u|_2\}_{k\in\mathbb{N}}$ and $\{|(1-\chi_{\Omega_k})u|_2\}_{k\in\mathbb{N}}$ are both majorized by $|u|_2 \in L^p(\mathfrak{m})$, i.e.

$$\forall k \in \mathbb{N}\,\forall \omega \in \Omega\colon \quad |\chi_{\Omega_k}(\omega)u(\omega)|_2 \le |u(\omega)|_2 \quad |(1-\chi_{\Omega_k}(\omega))u(\omega)|_2 \le |u(\omega)|_2$$

is valid. Thus, the lemma's assertion follows from the dominated convergence theorem, see [Simonnet, 1996, Theorem 5.2.2]. □

Let $\Omega \subset \mathbb{R}^d$ be an arbitrary domain, fix $p \in (1,\infty)$, and choose a function $y \in L^p(\Omega)$. In case of existence, we represent its weak derivative of order 1 w.r.t. ω_j, $j \in \{1,\dots,d\}$, by $\frac{\partial}{\partial\omega_j}y$. By $W^{1,p}(\Omega)$, we denote the common Sobolev space of functions from $L^p(\Omega)$ that possess all weak first order derivatives $\frac{\partial}{\partial\omega_1},\dots,\frac{\partial}{\partial\omega_d}$ which belong to $L^p(\Omega)$ as well. We equip $W^{1,p}(\Omega)$ with

$$\forall y \in W^{1,p}(\Omega)\colon \quad \|y\|_{W^{1,p}(\Omega)} := \left(\|y\|^p_{L^p(\Omega)} + \sum_{j=1}^d \left\|\frac{\partial}{\partial\omega_j}y\right\|^p_{L^p(\Omega)}\right)^{1/p}$$

which is the common Sobolev norm. The following lemma is a straightforward consequence of [Attouch et al., 2006, Corollary 5.8.2] for vector-valued Sobolev spaces.

Lemma 2.3. *Fix $p \in (1,\infty)$. For functions $y, z \in W^{1,p}(\Omega)^q$, the componentwise minimum function $\min\{y;z\}$ belongs to $W^{1,p}(\Omega)^q$ as well. For any $i \in \{1,\dots,q\}$ as well as $j \in \{1,\dots,d\}$, we obtain*

$$\frac{\partial}{\partial\omega_j}\min\{y;z\}_i = \chi_{\{\omega\in\Omega\,|\,y_i(\omega)<z_i(\omega)\}}\frac{\partial}{\partial\omega_j}y_i + \chi_{\{\omega\in\Omega\,|\,y_i(\omega)\ge z_i(\omega)\}}\frac{\partial}{\partial\omega_j}z_i.$$

3 Prelimiaries on sequential normal compactness

3.1 A brief history of normal compactness conditions

In order to perform limiting procedures in Banach spaces, which is essential in the context of variational analysis, the validity of certain compactness conditions is indispensable in order to obtain nontrivial results. Such conditions are often inherently satisfied in the finite-dimensional setting and lead to a reasonable calculus w.r.t. the tools of variational geometry and generalized differentiability which apply to problems in many different mathematical areas, see Aubin and Frankowska [2009], Mordukhovich and Shao [1996a, 1997], Mordukhovich [2006], Rockafellar and Wets [1998] and the references therein.

The first of the aforementioned generalized compactness conditions we want to mention in this paper is the property of a set to be compactly epi-Lipschitzian. It dates back to [Borwein and Strojwas, 1985, Definition 2.1].

Definition 3.1. *A set $A \subset \mathcal{X}$ in a Banach space $\mathcal{X}$ is called compactly epi-Lipschitzian (CEL for short) at $\bar{x} \in A$ if there exist $\varepsilon > 0$, $\delta > 0$, $\kappa > 0$, and a convex, compact set $C \subset \mathcal{X}$, which satisfy*

$$\forall t \in (0,\kappa): \quad A \cap \mathbb{U}^{\varepsilon}_{\mathcal{X}}(\bar{x}) + t\mathbb{U}^{\delta}_{\mathcal{X}}(0) \subset A + tC.$$

We call A CEL if it is CEL at all of its points.

It is mentioned in [Borwein and Strojwas, 1985, Proposition 2.4] that every subset of a finite-dimensional Banach space is CEL. In [Borwein et al., 2000, Theorem 2.5], one can find an explicit characterization of the CEL-property for closed, convex sets in Banach spaces which we are going to reproduce here.

Lemma 3.2. *Let $A \subset \mathcal{X}$ be a closed, convex subset of the Banach space $\mathcal{X}$. Then the following statements are equivalent:*

1. *A is CEL,*
2. *there exists a convex, compact set $C \subset \mathcal{X}$ such that $0 \in \operatorname{int}(A + C)$ holds,*
3. *$\operatorname{lin} A$ is a closed, finite-codimensional space and $\operatorname{ri} A$ is nonempty.*

As a corollary, we obtain the following result.

Corollary 3.3. *Let $A \subset \mathcal{X}$ be a closed, convex subset of a Banach space $\mathcal{X}$. Then the following statements are equivalent:*

1. *A is CEL,*
2. *there exists a point $\bar{x} \in A$ where A is CEL.*

Proof. Clearly, the first statement implies the second one so we only need to prove the converse implication.

Thus, assume that A is CEL at $\bar{x} \in A$. Then we find constants $\varepsilon > 0$, $\delta > 0$, as well as $\kappa > 0$ and a convex, compact set $C \subset \mathcal{X}$ which satisfy

$$A \cap \mathbb{U}_{\mathcal{X}}^{\varepsilon}(\bar{x}) + \tfrac{\kappa}{2}\mathbb{U}_{\mathcal{X}}^{\delta}(0) \subset A + \tfrac{\kappa}{2}C.$$

Particularly, we have

$$\tfrac{\kappa}{2}\mathbb{U}_{\mathcal{X}}^{\delta}(0) \subset A + \tfrac{\kappa}{2}C - \{\bar{x}\}.$$

Thus, for $C' := \frac{\kappa}{2}C - \{\bar{x}\}$, $0 \in \operatorname{int}(A + C')$ is valid. Noting that C' is convex and compact since C is convex and compact, A is CEL due to Lemma 3.2. This completes the proof. □

One of the most important geometrical concepts in variational analysis is the so-called extremal principle. Roughly speaking, it says that two closed sets that share a common point can be locally pushed apart from each other. More precisely, for two closed sets $A_1, A_2 \subset \mathcal{X}$ of a Banach space $\mathcal{X}$ and a point $\bar{x} \in A_1 \cap A_2$, we say that $\bar{x}$ is an extremal point of the system $\{A_1, A_2\}$ if there exist $\varepsilon > 0$ and sequences $\{a_k^1\}_{k\in\mathbb{N}}$ as well as $\{a_k^2\}_{k\in\mathbb{N}}$ converging to 0, such that

$$\forall k \in \mathbb{N}\colon \quad \left(A_1 - \{a_k^1\}\right) \cap \left(A_2 - \{a_k^2\}\right) \cap \mathbb{U}_{\mathcal{X}}^{\varepsilon}(0) = \varnothing$$

is valid, see Mordukhovich and Shao [1996b]. In this case, we call $\{A_1, A_2, \bar{x}\}$ an extremal system. A desireable condition which characterizes extremal systems is

$$\exists x^\star \in \mathcal{X}^\star \setminus \{0\}\colon \quad x^\star \in \mathcal{N}_{A_1}(\bar{x}) \cap (-\mathcal{N}_{A_2}(\bar{x}))\,. \tag{1}$$

In the case where A_1 and A_2 are convex, this condition reduces to

$$\exists x^\star \in \mathcal{X}^\star \setminus \{0\}\, \forall a^1 \in A_1\, \forall a^2 \in A_2\colon \quad \left\langle x^\star, a^2\right\rangle_{\mathcal{X}} \leq \left\langle x^\star, a^1\right\rangle_{\mathcal{X}}$$

which is a classical separability condition. In the finite-dimensional setting, any extremal system $\{A_1, A_2, \bar{x}\}$ where A_1 and A_2 are closed satisfies (1), see [Mordukhovich, 1994, Theorem 3.2]. In the infinite-dimensional setting, an additional assumption is needed in order to guarantee the nontriviality of the dual vector $x^\star$ in (1). In [Mordukhovich and Shao, 1996b, Theorem 3.6], the authors used the so-called normal compactness of sets defined below for that purpose.

Definition 3.4. *A set $A \subset \mathcal{X}$ in a Banach space $\mathcal{X}$ is called normally compact (NC for short) at $\bar{x} \in A$ if there exist constants $\gamma > 0$ as well as $\varepsilon > 0$ and a compact set $C \subset \mathcal{X}$ which satisfy*

$$\forall x \in \mathbb{U}_{\mathcal{X}}^{\varepsilon}(\bar{x}) \cap A\colon \quad \widehat{\mathcal{N}}_A(x) \subset \left\{ x^\star \in \mathcal{X}^\star \,\middle|\, \gamma\,\|x^\star\|_{\mathcal{X}^\star} \leq \max_{c\in C} |\langle x^\star, c\rangle_{\mathcal{X}}| \right\}.$$

We call A NC if it is NC at all of its points.

It is shown in [Loewen, 1992, Proposition 2.7] that whenever $A \subset \mathcal{X}$ is CEL at $\bar{x} \in A$, then it is NC at this point. Moreover, it was proven that for closed sets A, the converse implication is also true, see e.g. [Ioffe, 1990, Theorem 3]. Especially, any subset of a finite-dimensional Banach space $\mathcal{X}$ is NC.
Now, we turn our attention back to the characterization of extremal systems. Let us cite [Mordukhovich and Shao, 1996b, Theorem 3.6] for that purpose.

Lemma 3.5. *Let $\{A_1, A_2, \bar{x}\}$ be an extremal system where $A_1, A_2 \subset \mathcal{X}$ are closed subsets of an Asplund space $\bar{x}$. If at least one of these sets is NC at $\bar{x}$, then (1) is valid.*

As it was remarked in [Mordukhovich and Shao, 1997, Remark 3.5(iii)], the NC-condition in the above result can be replaced by a so-called sequential normal compactness condition defined below.

Definition 3.6. *1. A set $A \subset \mathcal{X}$ in a Banach space $\mathcal{X}$ is said to be sequentially normally compact (SNC for short) at $\bar{x} \in A$ if for any sequences $\{x_k\}_{k\in\mathbb{N}} \subset A$, $\{x_k^\star\}_{k\in\mathbb{N}} \subset \mathcal{X}^\star$, and $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset \mathbb{R}_0^+$ which satisfy $x_k \to \bar{x}$, $x_k^\star \overset{\star}{\rightharpoonup} 0$, $\varepsilon_k \to 0$, and $x_k^\star \in \widehat{\mathcal{N}}_A^{\varepsilon_k}(x_k)$ for all $k \in \mathbb{N}$, we have $x_k^\star \to 0$. We call A SNC if it is SNC at all of its points.*

2. A set $A \subset \mathcal{X}$ in a Banach space $\mathcal{X}$ is said to be topologically normally compact (TNC for short) at $\bar{x} \in A$ if for any nets $\{(x_\tau, x_\tau^\star)\}_{\tau\in T} \subset A \times \mathcal{X}^\star$ and $\{\varepsilon_\tau\}_{\tau\in T} \subset \mathbb{R}_0^+$ such that $\{x_\tau^\star\}_{\tau\in T}$ is bounded and which satisfy $x_\tau \to \bar{x}$, $x_\tau^\star \overset{\star}{\rightharpoonup} 0$, $\varepsilon_\tau \to 0$, as well as $x_\tau^\star \in \widehat{\mathcal{N}}_A^{\varepsilon_\tau}(x_\tau)$ for all $\tau \in T$, we have $x_\tau^\star \to 0$. We call A TNC if it is TNC at all of its points.

The definition of a net and the notion of (Moore-Smith-) net convergence can be found in [Megginson, 1998, Section 2.1]. Noting that any sequence is a net where the underlying directed set T equals the natural numbers equipped with the common less-or-equal relation, a set which is TNC at a fixed point is also SNC there. The converse implication is true whenever $\mathcal{X}$ is reflexive or separable, see [Fabian and Mordukhovich, 2003, Theorem 3.1]. On the other hand, if $A \subset \mathcal{X}$ is a closed subset of an Asplund space $\mathcal{X}$, then A is TNC at $\bar{x} \in A$ if and only if it is CEL at this point, see [Ioffe, 1990, Theorem 3]. Obviously, any subset of a finite-dimensional Banach space is TNC and, thus, SNC.
It is worth to mention that the sequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ and the net $\{\varepsilon_\tau\}_{\tau\in T}$ can be chosen to be identically zero in the above definition of the SNC- and TNC-property, respectively, as long as $\mathcal{X}$ is an Asplund space and $A \subset \mathcal{X}$ is closed (apply [Mordukhovich, 2006, Definition 1.116, Corollary 2.39] to the indicator function δ_A of A).

In the proposition below, we summarize the presented relations between the compactness conditions presented above.

Proposition 3.7. *Let $A \subset \mathcal{X}$ be a closed subset of an Asplund space $\mathcal{X}$. For any $\bar{x} \in \mathcal{X}$, we consider the statements*

(i) A is CEL at $\bar{x}$,

(ii) *A is NC at* $\bar{x}$,

(iii) *A is TNC at* $\bar{x}$,

(iv) *A is SNC at* $\bar{x}$.

Then, (i) to (iii) are equivalent and imply (iv). If, additionally, $\mathcal{X}$ *is reflexive, then (i) to (iv) are equivalent.*

We mentioned earlier that the result of Lemma 3.5 stays valid if the NC-condition is replaced by the weaker SNC-condition.

Lemma 3.8. *Let* $\{A_1, A_2, \bar{x}\}$ *be an extremal system where* $A_1, A_2 \subset \mathcal{X}$ *are closed subsets of an Asplund space* $\bar{x}$. *If at least one of these sets is SNC at* $\bar{x}$, *then (1) is valid.*

In the proposition below, we list the most important consequences of the above lemma, see Mordukhovich and Shao [1996b] and Mordukhovich [2006] for these results and the corresponding proofs. Especially, it visualizes the importance of the SNC-property in modern variational analysis.

Proposition 3.9.

1. *Let* $A \subset \mathcal{X}$ *be a closed subset of an Asplund space* $\mathcal{X}$, *and fix* $\bar{x} \in A$ *where A is SNC. Then* $\bar{x}$ *belongs to* $\operatorname{bd} A$ *if and only if* $\mathcal{N}_A(\bar{x}) \neq \{0\}$ *holds true.*

2. *Let* $A_1, A_2 \subset \mathcal{X}$ *be closed subsets of an Asplund space* $\mathcal{X}$. *Assume that at least one of these sets is SNC at* $\bar{x} \in A_1 \cap A_2$ *and that* $\mathcal{N}_{A_1}(\bar{x}) \cap (-\mathcal{N}_{A_2}(\bar{x})) = \{0\}$ *is valid. Then we have*
$$\mathcal{N}_{A_1 \cap A_2}(\bar{x}) \subset \mathcal{N}_{A_1}(\bar{x}) + \mathcal{N}_{A_2}(\bar{x}).$$

3. *Let* $S \subset \mathcal{Y}$ *be a closed subset of an Asplund space* $\mathcal{Y}$ *and let* $F \colon \mathcal{X} \to \mathcal{Y}$ *be a continuously Fréchet differentiable mapping where* $\mathcal{X}$ *is an Asplund space as well. Let* $\bar{x} \in F^{-1}(S)$ *be fixed and assume that S is SNC at* $F(\bar{x})$. *If the constraint qualification*
$$0 = F'(\bar{x})^\star[y^\star],\ y^\star \in \mathcal{N}_S(F(\bar{x})) \implies y^\star = 0 \tag{2}$$
is valid, then we have
$$\mathcal{N}_{F^{-1}(S)}(\bar{x}) \subset F'(\bar{x})^\star\left[\mathcal{N}_S(F(\bar{x}))\right].$$
Here, $F'(\bar{x})^\star \colon \mathcal{Y}^\star \to \mathcal{X}^\star$ *represents the adjoint operator of the Fréchet derivative* $F'(\bar{x})$ *of F at* $\bar{x}$.

3.2 Calculus rules for sequentially normally compact sets

Combining Lemma 3.2, Corollary 3.3, and Proposition 3.7, it is possible to characterize the presence of sequential normal compactness for closed, convex sets in reflexive Banach spaces explicitly. Note that this result is sharper than [Mordukhovich, 2006, Theorem 1.21] which addresses convex sets in arbitrary Banach spaces.

Lemma 3.10. *Let $\mathcal{X}$ be a reflexive Banach space. For a nonempty, closed, convex set $A \subset \mathcal{X}$, the following statements are equivalent:*

1. *A is SNC,*
2. *A is SNC at some point $\bar{x} \in A$,*
3. *$\operatorname{lin} A$ is a closed, finite-codimensional space and $\operatorname{ri} A$ is nonempty.*

Due to the above lemma, any closed, convex subset of a reflexive Banach space that possesses a nonempty interior is SNC everywhere. However, this result is already clear from [Mordukhovich, 2006, Theorem 1.21] even in the absence of reflexivity.

In the lemma below, we show some nearby consequences resulting from the definition of sequential normal compactness.

Lemma 3.11. *Let $A \subset \mathcal{X}$ be a nonempty subset of a Banach space $\mathcal{X}$ and fix some point $\bar{x} \in A$. Then the following statements hold.*

(i) If $\bar{x} \in \operatorname{int} A$ is valid, then A is SNC at $\bar{x}$.

(ii) Fix $a \in \mathcal{X}$. Then A is SNC at $\bar{x}$ if and only if $A - \{a\}$ is SNC at $\bar{x} - a$.

(iii) Fix $\alpha \in \mathbb{R} \setminus \{0\}$. Then A is SNC at $\bar{x}$ if and only if αA is SNC at $\alpha\bar{x}$.

Proof. Let us start with the proof of the lemma's first assertion. Therefore, we assume that $\bar{x} \in \operatorname{int} A$ is valid. Then we find $\delta > 0$ such that $\mathbb{U}^{\delta}_{\mathcal{X}}(\bar{x}) \subset \operatorname{int} A$ holds true. A simple calculation shows $\widehat{\mathcal{N}}^{\varepsilon}_{A}(x) = \mathbb{B}^{\varepsilon}_{\mathcal{X}^\star}(0)$ for all $x \in \mathbb{U}^{\delta}_{\mathcal{X}}(\bar{x})$ and all $\varepsilon \geq 0$. Obviously, this implies that A is SNC at $\bar{x}$ by definition of the SNC-property.

For the proof of the second statement of the lemma, we choose $a \in \mathcal{X}$ arbitrarily. For any $\varepsilon \geq 0$, it is easy to see that $\widehat{\mathcal{N}}^{\varepsilon}_{A-\{a\}}(\bar{x} - a) = \widehat{\mathcal{N}}^{\varepsilon}_{A}(\bar{x})$ holds true. The definition of the SNC-property shows the claim.

Finally, let us prove the lemma's third assertion. Therefore, we fix $\alpha \in \mathbb{R} \setminus \{0\}$. Then

$$\begin{aligned}
\limsup_{\tilde{x} \xrightarrow{\alpha A} \alpha\bar{x}} \frac{\langle x^\star, \tilde{x} - \alpha\bar{x}\rangle_{\mathcal{X}}}{\|\tilde{x} - \alpha\bar{x}\|_{\mathcal{X}}} \leq \varepsilon \iff & \limsup_{x \xrightarrow{A} \bar{x}} \frac{\langle x^\star, \alpha x - \alpha\bar{x}\rangle_{\mathcal{X}}}{\|\alpha x - \alpha\bar{x}\|_{\mathcal{X}}} \leq \varepsilon \\
\iff & \limsup_{x \xrightarrow{A} \bar{x}} \frac{\alpha}{|\alpha|} \frac{\langle x^\star, x - \bar{x}\rangle_{\mathcal{X}}}{\|x - \bar{x}\|_{\mathcal{X}}} \leq \varepsilon \\
\iff & \limsup_{x \xrightarrow{A} \bar{x}} \frac{\langle (\operatorname{sgn} \alpha) x^\star, x - \bar{x}\rangle_{\mathcal{X}}}{\|x - \bar{x}\|_{\mathcal{X}}} \leq \varepsilon
\end{aligned}$$

is obtained for any $x^\star \in \mathcal{X}^\star$ and any $\varepsilon \geq 0$. Consequently, $\widehat{\mathcal{N}}^{\varepsilon}_{\alpha A}(\alpha\bar{x}) = (\operatorname{sgn} \alpha)\widehat{\mathcal{N}}^{\varepsilon}_{A}(\bar{x})$ holds true.

Assume that A is SNC at $\bar{x}$ and choose sequences $\{\tilde{x}_k\}_{k\in\mathbb{N}} \subset \alpha A$, $\{\tilde{x}^\star_k\}_{k\in\mathbb{N}} \subset \mathcal{X}^\star$, and $\{\varepsilon_k\} \subset \mathbb{R}^+_0$ which satisfy $\tilde{x}_k \to \alpha\bar{x}$, $\tilde{x}^\star_k \xrightharpoonup{\star} 0$, $\varepsilon_k \to 0$, and $\tilde{x}^\star_k \in \widehat{\mathcal{N}}^{\varepsilon_k}_{\alpha A}(\tilde{x}_k)$ for all $k \in \mathbb{N}$. Then we find a sequence $\{x_k\}_{k\in\mathbb{N}} \subset A$ satisfying $x_k = \frac{1}{\alpha}\tilde{x}_k$ for all $k \in \mathbb{N}$. Clearly,

$x_k \to \bar{x}$ is valid while $(\operatorname{sgn}\alpha)\tilde{x}_k^\star \in \widehat{\mathcal{N}}_A^{\varepsilon_k}(x_k)$ holds for all $k \in \mathbb{N}$. From $\tilde{x}_k^\star \overset{\star}{\rightharpoonup} 0$, we have $(\operatorname{sgn}\alpha)\tilde{x}_k^\star \overset{\star}{\rightharpoonup} 0$. Since A is SNC at $\bar{x}$, $(\operatorname{sgn}\alpha)\tilde{x}_k^\star \to 0$ is obtained which yields $\tilde{x}_k^\star \to 0$. As a result, αA is SNC at $\alpha\bar{x}$.
If αA is SNC at $\alpha\bar{x}$, then due to the above arguments, $A = \frac{1}{\alpha}\alpha A$ is SNC at $\bar{x} = \frac{1}{\alpha}\alpha\bar{x}$. This completes the proof. □

In the following lemma, we discuss the SNC-property in the context of certain set operations. Similar results hold for the CEL-property, see [Borwein and Strojwas, 1985, Proposition 2.4].

Lemma 3.12. *1. Let $A_1, A_2 \subset \mathcal{X}$ be nonempty, closed subsets of a Banach space $\mathcal{X}$ and fix $\bar{x} \in A_1 \cup A_2$. Then the following assertions hold:*

a) If $\bar{x} \in A_1 \cap A_2$ holds and if A_1 and A_2 are both SNC at $\bar{x}$, then $A_1 \cup A_2$ is SNC at $\bar{x}$.

b) If $\bar{x} \in A_1 \setminus A_2$ ($\bar{x} \in A_2 \setminus A_1$) holds true and if A_1 (A_2) is SNC at $\bar{x}$, then $A_1 \cup A_2$ is SNC at $\bar{x}$.

2. Let $A_1, A_2 \subset \mathcal{X}$ be nonempty, closed subsets of an Asplund space $\mathcal{X}$. Assume that A_1 and A_2 are SNC at $\bar{x} \in A_1 \cap A_2$ while $\mathcal{N}_{A_1}(\bar{x}) \cap \big(-\mathcal{N}_{A_2}(\bar{x})\big) = \{0\}$ is valid. Then $A_1 \cap A_2$ is SNC at $\bar{x}$.

3. Let $A_i \subset \mathcal{X}_i$, $i = 1, 2$, be a subset of the Banach space $\mathcal{X}_i$ and fix $\bar{x}^i \in A_i$. If A_i is SNC at $\bar{x}^i$ for $i = 1, 2$, then $A_1 \times A_2$ is SNC at $(\bar{x}^1, \bar{x}^2)$.

4. Let $S \subset \mathcal{Y}$ be a closed subset of an Asplund space $\mathcal{Y}$ and let $F\colon \mathcal{X} \to \mathcal{Y}$ be a continuously Fréchet differentiable mapping where $\mathcal{X}$ is an Asplund space as well. Let $\bar{x} \in F^{-1}(S)$ be fixed and assume that S is SNC at $F(\bar{x})$. If the constraint qualification (2) holds, then $F^{-1}(S)$ is SNC at $\bar{x}$.

Proof. Let us start with the proof of the lemma's first assertion. From the definition of Fréchet ε-normals and the closedness of A_1 and A_2, we easily see

$$\forall \varepsilon \in \mathbb{R}_0^+\, \forall x \in A_1 \cup A_2 \colon \quad \widehat{\mathcal{N}}^\varepsilon_{A_1 \cup A_2}(x) \subset \begin{cases} \widehat{\mathcal{N}}^\varepsilon_{A_1}(x) \cap \widehat{\mathcal{N}}^\varepsilon_{A_2}(x) & \text{if } x \in A_1 \cap A_2, \\ \widehat{\mathcal{N}}^\varepsilon_{A_1}(x) & \text{if } x \in A_1 \setminus A_2, \\ \widehat{\mathcal{N}}^\varepsilon_{A_2}(x) & \text{if } x \in A_2 \setminus A_1. \end{cases}$$

Now, fix $\bar{x} \in A_1 \cap A_2$ and assume that A_1 and A_2 are both SNC at $\bar{x}$. Choose sequences $\{x_k\}_{k\in\mathbb{N}} \subset A_1 \cup A_2$, $\{x_k^\star\}_{k\in\mathbb{N}} \subset \mathcal{X}^\star$, and $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset \mathbb{R}_0^+$ which satisfy $x_k \to \bar{x}$, $x_k^\star \overset{\star}{\rightharpoonup} 0$, $\varepsilon_k \to 0$, and $x_k^\star \in \widehat{\mathcal{N}}^{\varepsilon_k}_{A_1 \cup A_2}(x_k)$ for all $k \in \mathbb{N}$.
Note that whenever $\{x_k\}_{k\in\mathbb{N}} \cap (A_1 \cap A_2)$, $\{x_k\}_{k\in\mathbb{N}} \cap (A_1 \setminus A_2)$, or $\{x_k\}_{k\in\mathbb{N}} \cap (A_2 \setminus A_1)$ is of infinite cardinality, then, due to the above upper estimate of $\widehat{\mathcal{N}}^{\varepsilon_k}_{A_1 \cup A_2}(x_k)$ and the property of A_1 and A_2 to be SNC at $\bar{x}$, the associated restriction of $\{x_k^\star\}_{k\in\mathbb{N}}$ converges strongly to 0. Furthermore, we have

$$\Big(\{x_k\}_{k\in\mathbb{N}} \cap (A_1 \cap A_2)\Big) \cup \Big(\{x_k\}_{k\in\mathbb{N}} \cap (A_1 \setminus A_2)\Big) \cup \Big(\{x_k\}_{k\in\mathbb{N}} \cap (A_2 \setminus A_1)\Big) = \{x_k\}_{k\in\mathbb{N}}.$$

Thus, the whole sequence $\{x_k^\star\}_{k\in\mathbb{N}}$ converges strongly to 0, i.e. $A_1 \cup A_2$ is SNC at $\bar{x}$. This shows a).
For the proof of statement b), we only consider the case where $\bar{x} \in A_1 \setminus A_2$ holds and A_1 is SNC at $\bar{x}$. The other assertion follows analogously. Since A_2 is closed, we find $\delta > 0$ such that $\mathbb{U}^\delta_{\mathcal{X}}(\bar{x}) \cap (A_1 \cup A_2) = \mathbb{U}^\delta_{\mathcal{X}}(\bar{x}) \cap A_1$ holds true. Particularly, we have $\widehat{\mathcal{N}}^\varepsilon_{A_1\cup A_2}(x) = \widehat{\mathcal{N}}^\varepsilon_{A_1}(x)$ for all $\varepsilon \in \mathbb{R}^+_0$ and all $x \in \mathbb{U}^\delta_{\mathcal{X}}(\bar{x}) \cap (A_1 \cup A_2)$. Thus, in $\mathbb{U}^\delta_{\mathcal{X}}(\bar{x})$, the variational geometry of $A_1 \cup A_2$ is equivalent to the one of A_1, and since the latter set is SNC at $\bar{x}$, the same holds true for $A_1 \cup A_2$.

The second assertion is taken from [Mordukhovich, 2006, Corollary 3.81].

Let us prove the third assertion. First, we observe that

$$\forall \varepsilon \in \mathbb{R}^+_0\, \forall (x^1, x^2) \in A_1 \times A_2\colon \quad \widehat{\mathcal{N}}^\varepsilon_{A_1\times A_2}(x^1, x^2) \subset \widehat{\mathcal{N}}^\varepsilon_{A_1}(x^1) \times \widehat{\mathcal{N}}^\varepsilon_{A_2}(x^2)$$

holds true. Next, choose sequences $\{(x_k^1, x_k^2)\}_{k\in\mathbb{N}} \subset A_1 \times A_2$, $\{(x_k^{\star,1}, x_k^{\star,2})\}_{k\in\mathbb{N}} \subset \mathcal{X}_1^\star \times \mathcal{X}_2^\star$, and $\{\varepsilon_k\} \subset \mathbb{R}^+_0$ which satisfy $(x_k^1, x_k^2) \to (\bar{x}^1, \bar{x}^2)$, $(x_k^{\star,1}, x_k^{\star,2}) \overset{\star}{\rightharpoonup} (0, 0)$, $\varepsilon_k \to 0$, and $(x_k^{\star,1}, x_k^{\star,2}) \in \widehat{\mathcal{N}}^{\varepsilon_k}_{A_1\times A_2}(x_k^1, x_k^2)$ for all $k \in \mathbb{N}$. We deduce $x_k^i \to \bar{x}^i$, $x_k^{\star,i} \overset{\star}{\rightharpoonup} 0$, as well as $x_k^{\star,i} \in \widehat{\mathcal{N}}^{\varepsilon_k}_{A_i}(x_k^i)$ for all $k \in \mathbb{N}$ and $i = 1, 2$. Exploiting that A_i is SNC at $\bar{x}^i$, $i = 1, 2$, we deduce $x_k^{\star,i} \to 0$, i.e. $(x_k^{\star,1}, x_k^{\star,2}) \to (0, 0)$. This shows that $A_1 \times A_2$ is SNC at $(\bar{x}^1, \bar{x}^2)$.

The forth statement of the lemma follows from [Mordukhovich, 2006, Corollary 1.69, Theorem 3.84]. □

More calculus rules for sequentially normally compact sets can be found in Mordukhovich and Wang [2003], Mordukhovich [2006].

4 Sequential normal compactness of decomposable sets

In optimal control, the set $\mathbb{K}$ of pointwise control constraints might be given as stated below:

$$\mathbb{K} := \{u \in L^p(\mathfrak{m}; \mathbb{R}^q) \,|\, u(\omega) \in K(\omega) \text{ a.e. on } \Omega\}\,. \tag{3}$$

Therein, $(\Omega, \Sigma, \mathfrak{m})$ is a complete as well as σ-finite measure space, $p \in (1, \infty)$ is a fixed parameter, and $K\colon \Omega \rightrightarrows \mathbb{R}^q$ is a set-valued mapping with nonempty, closed images. For the derivation of necessary optimality conditions for optimal control problems whose feasible set is formulated with the aid of $\mathbb{K}$, we need to know more about the variational geometry of this set. In view of Proposition 3.9, the presence of the SNC-property for $\mathbb{K}$ would be benefical in order to ensure a good variational calculus.

Before we can formulate the standing assumptions of this section, we need to present some more definitions for a better understanding of the underlying theory and background material.
Let $(\Omega, \Sigma, \mathfrak{m})$ be a measure space. It is called nonatomic whenever for every set $M \in \Sigma$ satisfying $\mathfrak{m}(M) > 0$, we find $M' \in \Sigma$ such that $0 < \mathfrak{m}(M') < \mathfrak{m}(M)$ holds true. On the other hand, the measure space $(\Omega, \Sigma, \mathfrak{m})$ is called separable, if all the Lebesgue spaces $L^s(\mathfrak{m})$, $s \in [1, \infty)$, are separable. An arbitrary domain $\Omega \subset \mathbb{R}^d$ equipped with Lebesgue's measure and the corresponding Borelean σ-algebra is σ-finite, nonatomic,

and separable, see [Adams and Fournier, 2003, Theorem 2.21]. Its formal completion is complete, additionally.
Fix an arbitrary measure space $(\Omega, \Sigma, \mathfrak{m})$. A set-valued mapping $K\colon \Omega \rightrightarrows \mathbb{R}^q$ is said to be measurable if for every open set $O \subset \mathbb{R}^q$, the preimage $\{\omega \in \Omega \,|\, K(\omega) \cap O \neq \varnothing\}$ is measurable. Whenever K possesses only closed images, there exist many equivalent useful characterizations of the measurability of K, see [Aubin and Frankowska, 2009, Section 8.1] or [Papageorgiou and Kyritsi-Yiallourou, 2009, Section 6.3].
Let $(\Omega, \Sigma, \mathfrak{m})$ be a complete and σ-finite measure space again. For $p \in (1, \infty)$, a set $\mathbb{K} \subset L^p(\mathfrak{m}; \mathbb{R}^q)$ is called decomposable if for any triplet $(A, u_1, u_2) \in \Sigma \times \mathbb{K} \times \mathbb{K}$, we have $\chi_A u_1 + (1 - \chi_A) u_2 \in \mathbb{K}$ as well. Obviously, decomposability seems to be a generalization of convexity. The notion of decomposability dates back to Rockafellar, see Rockafellar [1968]. It is well-known that a nonempty and closed set $\mathbb{K} \in L^p(\mathfrak{m}; \mathbb{R}^q)$ is decomposable if and only there is a measurable set-valued mapping $K\colon \Omega \rightrightarrows \mathbb{R}^q$ with nonempty and closed images such that $\mathbb{K}$ possesses the representation (3), see [Papageorgiou and Kyritsi-Yiallourou, 2009, Theorem 6.4.6]. In Hiai and Umegaki [1977] and [Papageorgiou and Kyritsi-Yiallourou, 2009, Sections 6.2-6.4], the interested reader can find some more properties and calculus rules for decomposable sets. Recently, the variational geometry of decomposable sets has been studied in Mehlitz and Wachsmuth [2016, 2017].

Throughout the section, we postulate the following general assumptions.

Assumption 4.1. *We assume that $(\Omega, \Sigma, \mathfrak{m})$ is a complete, σ-finite, nonatomic, and separable measure space.*

Furthermore, let $\mathbb{K} \subset L^p(\mathfrak{m}; \mathbb{R}^q)$ be a nonempty, closed, and decomposable set with $p \in (1, \infty)$. By $K\colon \Omega \rightrightarrows \mathbb{R}^q$, we denote the closed-valued, measurable set-valued mapping associated to $\mathbb{K}$. We assume that the images of K are derivable almost everywhere on Ω.

Let $p' \in (1, \infty)$ be the conjugate coefficient associated to p.

Assumption 4.1 allows us to state an explicit formula for the Fréchet normal cone to the decomposable set $\mathbb{K}$ of interest. The following result is taken from [Mehlitz and Wachsmuth, 2016, Corollary 3.7].

Lemma 4.2. *For arbitrary $u \in \mathbb{K}$, we have*

$$\widehat{\mathcal{N}}_{\mathbb{K}}(u) = \left\{ \eta \in L^{p'}(\mathfrak{m}; \mathbb{R}^q) \,\middle|\, \eta(\omega) \in \widehat{\mathcal{N}}_{K(\omega)}(u(\omega)) \text{ a.e. on } \Omega \right\}.$$

Next, we present the main result of this section. It shows that all nontrivial decomposable sets which satisfy Assumption 4.1 are nowhere SNC.

Theorem 4.3. *If $\mathbb{K} \neq L^p(\mathfrak{m}; \mathbb{R}^q)$ holds, then $\mathbb{K}$ is nowhere SNC.*

Proof. Fix $\bar{u} \in L^p(\mathfrak{m}; \mathbb{R}^q) \setminus \mathbb{K}$. Then we find a measurable set $\Omega' \in \Sigma$ with finite measure $\mathfrak{m}(\Omega') > 0$ and a constant $\varepsilon > 0$ such that $\operatorname{dist}(\bar{u}(\omega), K(\omega)) \geq \varepsilon$ is valid for all $\omega \in \Omega$.

We define a set-valued mapping $\Psi\colon \Omega \rightrightarrows \mathbb{R}^q$ by

$$\forall \omega \in \Omega\colon \quad \Psi(\omega) := \operatorname{argmin} \left\{ \tfrac{1}{2} |z - \bar{u}(\omega)|_2^2 \,\middle|\, z \in K(\omega) \right\}.$$

The measurability of $\bar{u}$ and K implies that Ψ is measurable as well, see [Aubin and Frankowska, 2009, Theorem 8.2.11]. On the other hand, the images of K are nonempty and closed which is why the images of Ψ possess the same properties. That is why we find a measurable selection of Ψ, i.e. there is a measurable function $\widetilde{u}\colon \Omega \to \mathbb{R}^q$ which satisfies $\widetilde{u}(\omega) \in \Psi(\omega)$ for almost every $\omega \in \Omega$, see [Aubin and Frankowska, 2009, Theorem 8.1.3]. By definition, we have $|\bar{u}(\omega) - \widetilde{u}(\omega)|_2 \geq \varepsilon$ almost everywhere on Ω'. On the other hand, [Mordukhovich, 2006, Proposition 5.1] yields $\bar{u}(\omega) - \widetilde{u}(\omega) \in \widehat{\mathcal{N}}_{K(\omega)}(\widetilde{u}(\omega))$ for almost every $\omega \in \Omega$.

Now, we define a measurable function $\eta\colon \Omega \to \mathbb{R}^q$ by

$$\forall \omega \in \Omega\colon \qquad \eta(\omega) := \chi_{\Omega'}(\omega) \frac{\bar{u}(\omega) - \widetilde{u}(\omega)}{|\bar{u}(\omega) - \widetilde{u}(\omega)|_2}.$$

Due to

$$\|\eta\|^{p'}_{L^{p'}(\mathfrak{m};\mathbb{R}^q)} = \int_{\Omega'} \left| \frac{\bar{u}(\omega) - \widetilde{u}(\omega)}{|\bar{u}(\omega) - \widetilde{u}(\omega)|_2} \right|_2^{p'} \mathrm{d}\mathfrak{m} = \mathfrak{m}(\Omega') < +\infty,$$

we have $\eta \in L^{p'}(\mathfrak{m};\mathbb{R}^q)$. On the other hand, observe that the relation $\eta(\omega) \in \widehat{\mathcal{N}}_{K(\omega)}(\widetilde{u}(\omega))$ holds for almost all $\omega \in \Omega$ by definition. Consequently, Lemma 4.2 yields $\eta \in \widehat{\mathcal{N}}_{\mathbb{K}}(\widetilde{u})$.

Let $\bar{v} \in \mathbb{K}$ be chosen arbitrarily. Since $(\Omega, \Sigma, \mathfrak{m})$ is nonatomic, we find a sequence $\{\Omega_k\}_{k\in\mathbb{N}} \subset \Sigma$ of sets possessing positive measure such that $\mathfrak{m}(\Omega_k) \to 0$ holds true while $\Omega_k \subset \Omega'$ is valid for all $k \in \mathbb{N}$. Let us set $v_k := (1 - \chi_{\Omega_k})\bar{v} + \chi_{\Omega_k}\widetilde{u}$ for all $k \in \mathbb{N}$. Clearly, we have $\{v_k\}_{k\in\mathbb{N}} \subset \mathbb{K}$ and $v_k \to \bar{v}$ in $L^p(\mathfrak{m};\mathbb{R}^q)$ by Lemma 2.2. Next, for all $k \in \mathbb{N}$, we define $\eta_k := \mathfrak{m}(\Omega_k)^{-\frac{1}{p'}} \chi_{\Omega_k}\eta \in L^{p'}(\mathfrak{m};\mathbb{R}^q)$. By construction, $\eta_k(\omega) \in \widehat{\mathcal{N}}_{K(\omega)}(\widetilde{u}(\omega))$ is obtained for almost every $\omega \in \Omega_k$ which leads us to $\eta_k(\omega) \in \widehat{\mathcal{N}}_{K(\omega)}(v_k(\omega))$ for almost all $\omega \in \Omega$ and all $k \in \mathbb{N}$. Once more, we invoke Lemma 4.2 to obtain $\eta_k \in \widehat{\mathcal{N}}_{\mathbb{K}}(v_k)$ for all $k \in \mathbb{N}$. For arbitrary $u \in L^p(\mathfrak{m};\mathbb{R}^q)$, Hölder's inequality yields

$$\begin{aligned}
\left|\langle \eta_k, u\rangle_{L^p(\mathfrak{m};\mathbb{R}^q)}\right| &= \mathfrak{m}(\Omega_k)^{-\frac{1}{p'}} \left| \int_{\Omega_k} \frac{\bar{u}(\omega) - \widetilde{u}(\omega)}{|\bar{u}(\omega) - \widetilde{u}(\omega)|_2} \cdot u(\omega) \mathrm{d}\mathfrak{m} \right| \\
&\leq \mathfrak{m}(\Omega_k)^{-\frac{1}{p'}} \left(\int_{\Omega_k} \left| \frac{\bar{u}(\omega) - \widetilde{u}(\omega)}{|\bar{u}(\omega) - \widetilde{u}(\omega)|_2} \right|_2^{p'} \mathrm{d}\mathfrak{m} \right)^{1/p'} \left(\int_{\Omega_k} |u(\omega)|_2^p \mathrm{d}\mathfrak{m} \right)^{1/p} \\
&= \|\chi_{\Omega_k} u\|_{L^p(\mathfrak{m};\mathbb{R}^q)},
\end{aligned}$$

and by means of Lemma 2.2, the last term falls to zero as $k \to \infty$. Thus, we obtain $\eta_k \rightharpoonup 0$ in $L^{p'}(\mathfrak{m};\mathbb{R}^q)$. On the other hand, we easily see $\|\eta_k\|_{L^{p'}(\mathfrak{m};\mathbb{R}^q)} = 1$ for all $k \in \mathbb{N}$, i.e. $\eta_k \not\to 0$ in $L^{p'}(\mathfrak{m};\mathbb{R}^q)$. Consequently, $\mathbb{K}$ is not SNC at $\bar{v}$. Since the latter point was chosen arbitrarily, $\mathbb{K}$ is nowhere SNC. □

In the following, we present reasonable examples of decomposable sets which are nowhere SNC.

Example 4.4. *Let $(\Omega, \Sigma, \mathfrak{m})$ be a measure space which satisfies Assumption 4.1. Furthermore, for some $p \in (1, \infty)$, choose functions $a, b \in L^p(\mathfrak{m}; \mathbb{R}^q)$ which satisfy $a(\omega) \le b(\omega)$ for almost every $\omega \in \Omega$. We consider the sets*

$$\begin{aligned} \mathbb{L}_a &:= \{u \in L^p(\mathfrak{m}; \mathbb{R}^q) \,|\, a(\omega) \le u(\omega) \text{ a.e. on } \Omega\}, \\ \mathbb{L}_b &:= \{u \in L^p(\mathfrak{m}; \mathbb{R}^q) \,|\, u(\omega) \le b(\omega) \text{ a.e. on } \Omega\}, \\ \mathbb{L}_{a,b} &:= \{u \in L^p(\mathfrak{m}; \mathbb{R}^q) \,|\, a(\omega) \le u(\omega) \le b(\omega) \text{ a.e. on } \Omega\}. \end{aligned}$$

These sets are induced by the set-valued mappings $L_a, L_b, L_{a,b} \colon \Omega \rightrightarrows \mathbb{R}^q$ defined below for all $\omega \in \Omega$:

$$\begin{aligned} L_a(\omega) &:= \{z \in \mathbb{R}^q \,|\, a(\omega) \le z\}, \\ L_b(\omega) &:= \{z \in \mathbb{R}^q \,|\, z \le b(\omega)\}, \\ L_{a,b}(\omega) &:= \{z \in \mathbb{R}^q \,|\, a(\omega) \le z \le b(\omega)\}. \end{aligned}$$

Due to the measurability of a and b, these mappings are measurable as well. Furthermore, the images of L_a, L_b, and $L_{a,b}$ are nonempty, closed, and convex (and, thus, derivable). Consequently, Assumption 4.1 is valid and due to Theorem 4.3, $\mathbb{L}_a$, $\mathbb{L}_b$, and $\mathbb{L}_{a,b}$ are nowhere SNC. Note that these sets represent typical box constraints from control-constrained optimal control.

Example 4.5. *Let $(\Omega, \Sigma, \mathfrak{m})$ be a measure space which satisfies Assumption 4.1. We consider the nonempty, closed, convex cone*

$$L^2(\mathfrak{m}; \mathbb{R}^q)^+_0 := \{u \in L^2(\mathfrak{m}; \mathbb{R}^q) \,|\, 0 \le u(\omega) \text{ a.e. on } \Omega\}.$$

Due to Example 4.4, $L^2(\mathfrak{m}; \mathbb{R}^q)^+_0$ is nowhere SNC.

Let $\mathcal{X}$ be a Banach space and let $G, H \colon \mathcal{X} \to L^2(\mathfrak{m}; \mathbb{R}^q)$ be continuously Fréchet differentiable mappings. The constraint system

$$\begin{aligned} G(x) &\in L^2(\mathfrak{m}; \mathbb{R}^q)^+_0 \\ H(x) &\in L^2(\mathfrak{m}; \mathbb{R}^q)^+_0 \\ \langle H(x), G(x) \rangle_{L^2(\mathfrak{m}; \mathbb{R}^q)} &= 0 \end{aligned}$$

represents a complementarity constraint in the Lebesgue space $L^2(\mathfrak{m}; \mathbb{R}^q)$ which can be used to model optimal control problems with complementarity requirements on the control function, see Guo and Ye [2016], Mehlitz and Wachsmuth [2016]. It is easy to see that the above system is equivalent to

$$(G(x), H(x)) \in \mathbb{K}$$

where $\mathbb{K} \in L^2(\mathfrak{m}; \mathbb{R}^{2q})$ is defined below:

$$\mathbb{K} := \left\{(u, v) \in L^2(\mathfrak{m}; \mathbb{R}^q) \times L^2(\mathfrak{m}; \mathbb{R}^q) \,\middle|\, 0 \le u(\omega) \perp v(\omega) \ge 0 \text{ a.e. on } \Omega\right\}.$$

Thus, $\mathbb{K}$ equals the so-called complementarity set which is induced by the cone $L^2(\mathfrak{m}; \mathbb{R}^q)^+_0$. Defining

$$\Theta := \{(a, b) \in \mathbb{R}^q \times \mathbb{R}^q \,|\, 0 \le a \perp b \ge 0\},$$

we have

$$\mathbb{K} = \left\{(u, v) \in L^2(\mathfrak{m}; \mathbb{R}^q) \times L^2(\mathfrak{m}; \mathbb{R}^q) \,\middle|\, (u(\omega), v(\omega)) \in \Theta \text{ a.e. on } \Omega\right\}.$$

Noting that the closed set Θ can be represented as the finite union of convex sets, it is derivable, see [Mehlitz and Wachsmuth, 2016, Lemma 2.1]. Thus, $\mathbb{K}$ is nowhere SNC due to Theorem 4.3. A direct proof of this result is presented in [Mehlitz and Wachsmuth, 2016, Lemma 4.8].

Due to Theorem 4.3, we cannot rely on the SNC-property when dealing with the variational calculus of decomposable sets. In view of the rich pool of available calculus rules for decomposable sets, see e.g. Hiai and Umegaki [1977], Mehlitz and Wachsmuth [2016, 2017], Papageorgiou and Kyritsi-Yiallourou [2009], it might be possible to show some of the desirable results from Proposition 3.9 even in the absence of the SNC-property. This, however, is beyond the scope of this paper.

5 Sequential normal compactness of convex sets in Sobolev spaces

Below, we postulate the standing assumptions of this section.

Assumption 5.1. *We assume that $\Omega \subset \mathbb{R}^d$ is a bounded domain. Furthermore, we fix functions $a, b \in W^{1,p}(\Omega)^q$ such that $a(\omega) \le b(\omega)$ holds for almost all $\omega \in \Omega$. We consider the sets*

$$\begin{aligned} \mathbb{W}_a &:= \left\{y \in W^{1,p}(\Omega)^q \,\middle|\, a(\omega) \le y(\omega) \text{ a.e. on } \Omega\right\}, \\ \mathbb{W}_b &:= \left\{y \in W^{1,p}(\Omega)^q \,\middle|\, y(\omega) \le b(\omega) \text{ a.e. on } \Omega\right\}, \\ \mathbb{W}_{a,b} &:= \left\{y \in W^{1,p}(\Omega)^q \,\middle|\, a(\omega) \le y(\omega) \le b(\omega) \text{ a.e. on } \Omega\right\} \end{aligned}$$

where $p \in (1, \infty)$ is fixed.

For later use, we define

$$\mathbb{W}_0 := \left\{y \in W^{1,p}(\Omega)^q \,\middle|\, 0 \le y(\omega) \text{ a.e. on } \Omega\right\}.$$

Note that we have $\mathbb{W}_a = \mathbb{W}_0 + \{a\}$, $\mathbb{W}_b = \{b\} - \mathbb{W}_0$, and $\mathbb{W}_{a,b} = (\mathbb{W}_0 + \{a\}) \cap (\{b\} - \mathbb{W}_0)$.

In the following auxiliary lemma, we provide a useful truncation result.

Lemma 5.2. *For some function $y \in \mathbb{W}_0$, we define*

$$\forall k \in \mathbb{N}\colon \quad y_k := \min\{y; k\mathtt{e}\}.$$

Then we have $\{y_k\}_{k\in\mathbb{N}} \subset \mathbb{W}_0$ and $y_k \to y$ in $W^{1,p}(\Omega)^q$ as $k \to \infty$.

Proof. Noting that $k\mathtt{e} \in W^{1,p}(\Omega)^q$ holds due to the boundedness of Ω, $y_k \in W^{1,p}(\Omega)^q$ follows from Lemma 2.3 for any $k \in \mathbb{N}$ and any $y \in \mathbb{W}_0$. Due to the nonnegativity of $k\mathtt{e}$ and y, we already have $y_k \in \mathbb{W}_0$ for any $k \in \mathbb{N}$.

Choose $y \in \mathbb{W}_0$ arbitrarily. Furthermore, fix $i \in \{1, \ldots, q\}$. For any $k \in \mathbb{N}$, we define measurable sets $\Omega_i^1(k)$ and $\Omega_i^2(k)$ as stated below:

$$\begin{aligned} \Omega_i^1(k) &:= \{\omega \in \Omega \,|\, k \leq y_i(\omega)\}, \\ \Omega_i^2(k) &:= \{\omega \in \Omega \,|\, y_i(\omega) < k\}. \end{aligned}$$

Obviously, $\{\Omega_i^1(k), \Omega_i^2(k)\}$ is a disjoint partition of Ω which satisfies

$$\forall \omega \in \Omega\colon \quad y_{k,i}(\omega) = \begin{cases} k & \text{if } \omega \in \Omega_i^1(k), \\ y_i(\omega) & \text{if } \omega \in \Omega_i^2(k). \end{cases}$$

Using Lemma 2.3, we obtain

$$\frac{\partial}{\partial \omega_j} y_{k,i} = \chi_{\Omega_i^2(k)} \frac{\partial}{\partial \omega_j} y_i$$

for any $j \in \{1, \ldots, d\}$. From $y_i \in L^p(\Omega)$, we have $\lambda(\Omega_i^1(k)) \to 0$ as $k \to \infty$. Since

$$y_i - y_{k,i} = \chi_{\Omega_i^1(k)}(y_i - k)$$

is valid for any $k \in \mathbb{N}$, $y_{k,i}$ converges pointwise to y_i almost everywhere on Ω as $k \to \infty$. Moreover, we obtain

$$|y_i(\omega) - y_{k,i}(\omega)| = \left|\chi_{\Omega_i^1(k)}(\omega)(y_i(\omega) - k)\right| \leq |y_i(\omega)|$$

for almost every $\omega \in \Omega$. Since $y_i \in L^p(\Omega)$ is valid, the convergence $y_{k,i} \to y_i$ in $L^p(\Omega)$ as $k \to \infty$ follows from the dominated convergence theorem, see [Simonnet, 1996, Theorem 5.2.2]. For any $j \in \{1, \ldots, d\}$, we deduce

$$\frac{\partial}{\partial \omega_j} y_i - \frac{\partial}{\partial \omega_j} y_{k,i} = \chi_{\Omega_i^1(k)} \frac{\partial}{\partial \omega_j} y_i,$$

i.e. $\frac{\partial}{\partial \omega_j} y_{k,i}$ converges pointwise to $\frac{\partial}{\partial \omega_j} y_i$ almost everywhere on Ω as $k \to \infty$. Furthermore, the estimate

$$\left|\frac{\partial}{\partial \omega_j} y_i(\omega) - \frac{\partial}{\partial \omega_j} y_{k,i}(\omega)\right| \leq \left|\frac{\partial}{\partial \omega_j} y_i(\omega)\right|$$

is obtained from above for almost every $\omega \in \Omega$. Since we have $\frac{\partial}{\partial \omega_j} y_i \in L^p(\Omega)$, the convergence $\frac{\partial}{\partial \omega_j} y_{k,i} \to \frac{\partial}{\partial \omega_j} y_i$ in $L^p(\Omega)$ as $k \to \infty$ follows from the dominated convergence theorem, see [Simonnet, 1996, Theorem 5.2.2]. By definition, $y_{k,i} \to y_i$ in $W^{1,p}(\Omega)$ as $k \to \infty$ is valid for all $i \in \{1, \ldots, q\}$. Thus, we finally derived $y_k \to y$ in $W^{1,p}(\Omega)^q$ which completes the proof. $\square$

Now, we are well-prepared to study the presence of the SNC-property for the closed, convex cone $\mathbb{W}_0$.

In order to ensure that $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ is valid for $p > d$, the underlying domain Ω needs to satisfy the so-called cone condition, see [Adams and Fournier, 2003, Paragraph 4.6] for the definition. It is well-known that any bounded domain with Lipschitz boundary satisfies the cone condition, see [Adams and Fournier, 2003, Paragraph 4.11].

Lemma 5.3. *The following assertions hold true.*

1. *For $p \le d$, the cone $\mathbb{W}_0$ is nowhere SNC.*
2. *Suppose that Ω satisfies the cone condition. For $p > d$, the cone $\mathbb{W}_0$ is SNC.*

Proof. First, let us consider the situation $p \le d$. The main idea of the proof is to show that $\mathbb{W}_0$ possesses an empty relative interior which already shows the lack of the SNC-property by means of Lemma 3.10.
Since we postulated $p \le d$, there exists a function $\bar{y} \in \mathbb{W}_0$ which is not essentially bounded, see [Adams and Fournier, 2003, Examples 4.41, 4.43] where such a function is constructed. Let us set

$$y_k := \min\{y; k\mathsf{e}\}, \qquad w_k := \tfrac{1}{k}\bar{y}, \qquad z_k := y_k - w_k$$

for any $k \in \mathbb{N}$. Clearly, by means of Proposition 5.2, we have $\{y_k\}_{k\in\mathbb{N}} \subset \mathbb{W}_0$ as well as $y_k \to y$ in $W^{1,p}(\Omega)^q$, and $\{w_k\}_{k\in\mathbb{N}} \subset \mathbb{W}_0$ follows from the nonnegativity of $\bar{y}$. From $w_k \to 0$ in $W^{1,p}(\Omega)^q$, we deduce $z_k \to y$ in $W^{1,p}(\Omega)^q$. By definition, y_k is essentially bounded on Ω for any choice of k. On the other hand, w_k is not essentially bounded on Ω for all $k \in \mathbb{N}$ by construction. That is why we have $z_k \notin \mathbb{W}_0$ for all $k \in \mathbb{N}$. On the other hand, the sequence $\{z_k\}_{k\in\mathbb{N}} \subset \operatorname{lin}\mathbb{W}_0$ converges to y which is why the function y does not belong to the relative interior of $\mathbb{W}_0$. Since $y \in \mathbb{W}_0$ was arbitrarily chosen, we deduce $\operatorname{ri}\mathbb{W}_0 = \varnothing$, i.e. $\mathbb{W}_0$ is nowhere SNC by Lemma 3.10.

For the second part of the proof, let Ω satisfy the cone condition and assume $p > d$. Then we have $W^{1,p}(\Omega)^q \hookrightarrow L^\infty(\Omega; \mathbb{R}^q)$, see [Adams and Fournier, 2003, Theorem 4.12], i.e. we find some $\gamma > 0$ such that

$$\forall y \in W^{1,p}(\Omega)^q\colon \qquad \|y\|_{L^\infty(\Omega;\mathbb{R}^q)} \le \gamma \, \|y\|_{W^{1,p}(\Omega)^q}$$

holds. For any $y \in W^{1,p}(\Omega)^q$, $\max\{y; 0\}, -\min\{y; 0\} \in \mathbb{W}_0$ holds true, see Lemma 2.3, and, thus, $y = \max\{y; 0\} + \min\{y; 0\} \in \operatorname{lin}\mathbb{W}_0$ is derived. As a consequence, $\operatorname{lin}\mathbb{W}_0 = W^{1,p}(\Omega)^q$ is valid, i.e. $\operatorname{lin}\mathbb{W}_0$ is closed and possesses finite codimension. Finally, we consider the constant function $\mathsf{e} \in \mathbb{W}_0$. Then for any function $z \in W^{1,p}(\Omega)^q$ which satisfies $\|z - \mathsf{e}\|_{W^{1,p}(\Omega)^q} \le \frac{1}{\gamma}$, $\|z - \mathsf{e}\|_{L^\infty(\Omega;\mathbb{R}^q)} \le 1$ is valid. This yields $|z_i(\omega) - 1| \le 1$ for all $i \in \{1, \ldots, q\}$ and almost every $\omega \in \Omega$. Consequently, we have $z \in \mathbb{W}_0$, i.e. e is an interior point of $\mathbb{W}_0$. By means of Lemma 3.10, $\mathbb{W}_0$ is SNC. $\square$

Now, we consider the sets $\mathbb{W}_a$ and $\mathbb{W}_b$ with only upper or lower constraints, respectively.

Theorem 5.4. *The following assertions hold true.*

1. *For $p \le d$, the sets $\mathbb{W}_a$ and $\mathbb{W}_b$ are nowhere SNC.*
2. *Suppose that Ω satisfies the cone condition. For $p > d$, the sets $\mathbb{W}_a$ and $\mathbb{W}_b$ are SNC.*

Proof. Exploiting $\mathbb{W}_a = \mathbb{W}_0 + \{a\}$ and $\mathbb{W}_b = \{b\} - \mathbb{W}_0$, this result follows from Lemmas 3.11 and 5.3. □

In the upcoming theorem, we analyze the situation for sets $\mathbb{W}_{a,b}$ which are defined via lower and upper bounds. Therefore, an additional assumption is needed.

Assumption 5.5. *In addition to Assumption 5.1, we assume that there exists a subdomain Ω' of Ω such that the restrictions $a|_{\Omega'}$ and $b|_{\Omega'}$ of a and b to Ω' are elements of $L^\infty(\Omega';\mathbb{R}^q)$.*

It is worth to mention that in the case $p > d$ where Ω satisfies the cone condition, we have $W^{1,p}(\Omega)^q \hookrightarrow L^\infty(\Omega;\mathbb{R}^q)$ from [Adams and Fournier, 2003, Theorem 4.12], i.e. Assumption 5.5 is trivially satisfied. In the following example, we show that this property is not inherent whenever $p \le d$ holds true.

Example 5.6. *Let $\Omega \subset \mathbb{R}^d$ be an arbitrary domain. For $p \le d$, we find a nonnegative function $u_0 \in W^{1,p}(\mathbb{R}^d)$ which is not bounded near the origin, see [Adams and Fournier, 2003, Examples 4.41, 4.43]. Now, let $\{\omega_k\}_{k\in\mathbb{N}}$ be a sequence which orders the countable set $\Omega \cap \mathbb{Q}^d$. We define a function $\bar{y}\colon \Omega \to \mathbb{R}^q$ by*

$$\forall \omega \in \Omega\colon \quad \bar{y}(\omega) := \sum_{k=1}^{\infty} 2^{-k}\, \mathsf{e}\, u_0(\omega - \omega_k).$$

Observe that we have

$$\begin{aligned} \|\bar{y}\|_{W^{1,p}(\Omega)^q} &= q \left\| \sum_{k=1}^{\infty} 2^{-k} u_0(\cdot - \omega_k) \right\|_{W^{1,p}(\Omega)} \le q \sum_{k=1}^{\infty} 2^{-k} \|u_0(\cdot - \omega_k)\|_{W^{1,p}(\Omega)} \\ &\le q \sum_{k=1}^{\infty} 2^{-k} \|u_0\|_{W^{1,p}(\mathbb{R}^d)} = q \|u_0\|_{W^{1,p}(\mathbb{R}^d)} < +\infty, \end{aligned}$$

i.e. $\bar{y} \in W^{1,p}(\Omega)^q$ holds true. On the other hand, $\Omega \cap \mathbb{Q}^d$ is dense in $\operatorname{cl}\Omega$, i.e. there cannot exist a subdomain Ω' of Ω where $\bar{y}$ is essentially bounded since $\bar{y}$ is not bounded near the points from $\Omega \cap \mathbb{Q}^d$.

Theorem 5.7. *Let Assumption 5.5 be valid. Then the following assertions hold true.*

1. *For $p \le d$, the set $\mathbb{W}_{a,b}$ is nowhere SNC.*
2. *Suppose that Ω satisfies the cone condition. Furthermore, let $p > d$ be valid and assume that there exists $\varepsilon > 0$ such that $\varepsilon\mathsf{e} \le b(\omega) - a(\omega)$ is satisfied for almost every $\omega \in \Omega$. Then the set $\mathbb{W}_{a,b}$ is SNC.*

Proof. Let us first assume that $p \leq d$ is valid. Assumption 5.5 guarantees the existence of a subdomain Ω' of Ω such that the restrictions $a|_{\Omega'}$ and $b|_{\Omega'}$ of a and b to Ω' belong to $L^\infty(\Omega';\mathbb{R}^q)$. Lemma 2.1 yields

$$\operatorname{lin}\mathbb{W}_{a,b} \subset \left\{ y \in W^{1,p}(\Omega)^q \,\middle|\, y|_{\Omega'} \in L^\infty(\Omega';\mathbb{R}^q) \right\}.$$

As mentioned before, there exists a function $u_0 \in W^{1,p}(\mathbb{R}^d)$ which is unbounded near the origin since $p \leq d$ holds. For some $\bar\omega \in \Omega$, we define $y^{\bar\omega}\colon \Omega \to \mathbb{R}^q$ by

$$\forall \omega \in \Omega\colon \quad y^{\bar\omega}(\omega) := \mathsf{e}\, u_0(\omega - \bar\omega).$$

From $\|y^{\bar\omega}\|_{W^{1,p}(\Omega)^q} \leq q\, \|u_0\|_{W^{1,p}(\mathbb{R}^d)} < +\infty$, we have $y^{\bar\omega} \in W^{1,p}(\Omega)^q$. Let us set

$$\mathfrak{Q} := \{ y^{\bar\omega} \,|\, \bar\omega \in \Omega' \cap \mathbb{Q}^d \}.$$

Clearly, $\mathfrak{Q}$ is a linear independent system of infinitely many functions which are not essentially bounded on Ω'. Thus, we have $\operatorname{lin}\mathbb{W}_{a,b} \cap \mathfrak{Q} = \varnothing$, i.e. $\operatorname{lin}\mathbb{W}_{a,b}$ possesses infinite codimension. Due to Lemma 3.10, $\mathbb{W}_{a,b}$ is nowhere SNC.

Now, we consider the situation where $p > d$ holds true. Then we have the embedding $W^{1,p}(\Omega)^q \hookrightarrow L^\infty(\Omega;\mathbb{R}^q)$. Particularly, a and b are essentially bounded. We show $\operatorname{lin}\mathbb{W}_{a,b} = W^{1,p}(\Omega)^q$. Choose $\hat y \in W^{1,p}(\Omega)^q$ arbitrarily. From $\hat y \in L^\infty(\Omega;\mathbb{R}^q)$, we find constants $\alpha, \beta \in \mathbb{R}$ such that $\alpha\mathsf{e} \leq \hat y(\omega) \leq \beta\mathsf{e}$ holds true for almost all $\omega \in \Omega$. Let us define $\tilde y := a + \frac{1}{2}(b-a) \in \mathbb{W}_{a,b}$. The assumption of the theorem yields

$$\mathbb{W}^{\text{shift}}_{a,b} := \{ y \in W^{1,p}(\Omega)^q \,|\, -\tfrac{\varepsilon}{2}\mathsf{e} \leq y(\omega) \leq \tfrac{\varepsilon}{2}\mathsf{e} \text{ a.e. on } \Omega \} \subset \mathbb{W}_{a,b} - \{\tilde y\} \subset \operatorname{lin}\mathbb{W}_{a,b}.$$

For $M := \max\{|\alpha|;|\beta|\}$, we obtain $\hat y \in (2M/\varepsilon)\mathbb{W}^{\text{shift}}_{a,b} \subset \operatorname{lin}\mathbb{W}_{a,b}$. Since $\hat y$ was arbitrarily chosen, $W^{1,p}(\Omega)^q \subset \operatorname{lin}\mathbb{W}_{a,b}$ follows, which already yields $\operatorname{lin}\mathbb{W}_{a,b} = W^{1,p}(\Omega)^q$. Consequently, $\operatorname{lin}\mathbb{W}_{a,b}$ is a closed, finite-codimensional space. Similar as in the proof of Theorem 5.4, one can show that $\tilde y$ is an interior point of $\mathbb{W}_{a,b}$. This shows the relation $\operatorname{ri}\mathbb{W}_{a,b} = \operatorname{int}\mathbb{W}_{a,b} \neq \varnothing$. Due to Lemma 3.10, $\mathbb{W}_{a,b}$ is SNC. $\square$

Below, we briefly comment on the case where the lower bound a and the upper bound b hit each other on an open subset of Ω.

Remark 5.8. *If there exist a subdomain Ω' of Ω and an index $i_0 \in \{1,\dots,q\}$ such that $a_{i_0}(\omega) = b_{i_0}(\omega)$ holds true for almost every $\omega \in \Omega'$, then we have*

$$\operatorname{lin}\mathbb{W}_{a,b} \subset \left\{ y \in W^{1,p}(\Omega)^q \,\middle|\, \exists \alpha \in \mathbb{R}\colon\ y_{i_0}|_{\Omega'} = \alpha a_{i_0}|_{\Omega'} \right\},$$

see Lemma 2.1. Obviously, the vector space on the right possesses infinite codimension since $W^{1,p}(\Omega')$ is infinite dimensional. Thus, $\operatorname{lin}\mathbb{W}_{a,b}$ possesses infinite codimension. Particularly, $\mathbb{W}_{a,b}$ is nowhere SNC, see Lemma 3.10.

As a result of the above theorem, we obtain that box constraint sets appearing frequently in state-constrained optimal PDE control are nowhere SNC.

Example 5.9. *For fixed vectors $\underline{\mathbf{y}}, \overline{\mathbf{y}} \in \mathbb{R}^q$ satisfying $\underline{\mathbf{y}} < \overline{\mathbf{y}}$, we consider the standard box constraint set*

$$\mathbb{W}^{box} := \left\{ y \in W^{1,p}(\Omega)^q \,\middle|\, \underline{\mathbf{y}} \leq y(\omega) \leq \overline{\mathbf{y}} \text{ a.e. on } \Omega \right\}.$$

For any $p \leq d$, $\mathbb{W}^{box}$ is nowhere SNC. Especially, in the standard setting of state-constrained PDE control, i.e. $p = 2$ and $d \geq 2$, $\mathbb{W}^{box}$ is nowhere SNC.
In contrast, if $\Omega = (0, T)$ equals a real interval (i.e. $d = 1$) while $p \in (1, \infty)$ is arbitrarily chosen, then $\mathbb{W}^{box}$ is everywhere SNC. This covers the setting of state-constrained ODE control.

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