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The limiting normal cone of a complementarity set in Sobolev spaces

Felix Harder* Gerd Wachsmuth†

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We investigate the limiting normal cone of the complementarity set associated with non-negative functions in the Sobolev space $H_0^1(\Omega)$. By using results from homogenization theory, we provide lower estimates for this limiting normal cone and these estimates are unpleasantly large.

Keywords: limiting normal cone, optimality conditions, M-stationarity, obstacle problem

MSC: 49J53, 35B27

1. Introduction

In this paper, we are going to derive lower estimates (w.r.t. set inclusion) for the *limiting normal cone* of the non-convex set

$$\mathbb{K} := \{(v, \mu) \in H_0^1(\Omega) \times H^{-1}(\Omega) : v \geq 0, \mu \leq 0, \langle \mu, v \rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} = 0\},$$

where $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is open and bounded. Here, $v \geq 0$ is to be understood in a pointwise a.e. sense and $\mu \leq 0$ for $\mu \in H^{-1}(\Omega) := H_0^1(\Omega)^\star$ is defined via duality, i.e., $\langle \mu, z \rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} \leq 0$ for all $z \in H_0^1(\Omega)$ with $z \geq 0$. The precise definition of the limiting normal cone is given in Section 2.2, after some notation has been introduced.

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Our research is motivated by the approach of using variational analysis for deriving optimality conditions for the optimal control of the obstacle problem. To highlight this connection, we consider the unilateral obstacle problem

$$\text{Find } y \in H_0^1(\Omega)^+ \quad \text{such that} \quad \langle -\Delta y - u, v - y\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} \geq 0 \; \forall v \in H_0^1(\Omega)^+. \tag{1.1}$$

Here, $H_0^1(\Omega)^+ \subset H_0^1(\Omega)$ is the cone of non-negative functions, $-\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$ is the negative Laplacian and $u \in H^{-1}(\Omega)$ is a given right-hand side. It is well known, that (1.1) has a unique solution and we denote this solution by $\mathcal{S}(u)$. Moreover, by standard arguments, we have the characterization

$$y = \mathcal{S}(u) \quad \Leftrightarrow \quad (y, \Delta y + u) \in \mathbb{K}.$$

Note that $\mathbb{K}$ is the graph of the normal cone mapping of $H_0^1(\Omega)^+$. Next, we consider the optimal control of the obstacle problem by a right-hand side u from the set U_{ad}, i.e.,

$$\begin{aligned} \text{Minimize} \quad & J(y, u) \\ \text{w.r.t.} \quad & (y, u, \lambda) \in H_0^1(\Omega) \times L^2(\Omega) \times H^{-1}(\Omega) \\ \text{such that} \quad & (y, \lambda) \in \mathbb{K}, \\ & -\Delta y + \lambda = u, \\ & u \in U_{\text{ad}}. \end{aligned} \tag{1.2}$$

Here, $J : H_0^1(\Omega) \times L^2(\Omega) \to \mathbb{R}$ is assumed to be continuously Fréchet differentiable, and $U_{\text{ad}} \subset L^2(\Omega)$ is assumed to be closed and convex. Recall that the set $\mathbb{K}$, which appears in the constraints of (1.2), is not convex. The task of providing *necessary* optimality conditions, i.e., conditions which are satisfied for all local minimizers of (1.2), received great interest in the last forty years, we refer exemplarily to [Barbu, 1984; Hintermüller, Kopacka, 2009; Hintermüller, Mordukhovich, Surowiec, 2014; Hintermüller, Surowiec, 2011; Jarušek, Outrata, 2007; Mignot, 1976; Outrata, Jarušek, Stará, 2011; Schiela, D. Wachsmuth, 2013; G. Wachsmuth, 2014; 2016].

Stationarity systems including the limiting normal cone of $\mathbb{K}$ are obtained in [Hintermüller, Mordukhovich, Surowiec, 2014, Section 3] and [G. Wachsmuth, 2016, Proof of Lemma 4.4], see also [Outrata, Jarušek, Stará, 2011, Proof of Theorem 16] in case of controls from $H^{-1}(\Omega)$. Note that in the last two references, the optimality system was not stated explicitly by means of the limiting normal cone, but it can be easily extracted from the referenced proofs. One arrives at the optimality system

$$J_y(\bar{y}, \bar{u}) + \nu - \Delta p = 0, \qquad \gamma \in \mathcal{N}_{U_{\text{ad}}}(\bar{u}), \tag{1.3a}$$

$$J_u(\bar{y}, \bar{u}) + \gamma - p = 0, \qquad (\nu, -p) \in \mathcal{N}_{\mathbb{K}}(\bar{y}, \bar{\lambda}). \tag{1.3b}$$

Here, J_y and J_u denote the partial derivatives of J, and $\mathcal{N}_{U_{\text{ad}}}(\bar{u})$ is the usual normal cone of the convex set U_{ad}. Moreover, $\mathcal{N}_{\mathbb{K}}(\bar{y}, \bar{\lambda}) \subset H^{-1}(\Omega) \times H_0^1(\Omega)$ is the limiting normal cone to $\mathbb{K}$ at $(\bar{y}, \bar{\lambda})$.

It can be shown that this optimality system implies weak stationarity, see (2.8) below. However, there does not exist any stronger upper estimate (w.r.t. set inclusion) for the limiting normal cone of $\mathbb{K}$ in the literature, and it is not clear whether one is able to obtain an optimality system which is stronger than weak stationarity by using this approach.

This is the starting point of our research. Since no improvement on the upper estimate (2.8) seems to be possible, we investigate lower estimates. Indeed, by using results from homogenization theory, we are able to characterize the intersection of the limiting normal cone with $L^p(\Omega) \times H_0^1(\Omega)$, for all values of $p \in (1, 2]$ with $L^p(\Omega) \hookrightarrow H^{-1}(\Omega)$. Unfortunately, these lower estimates are rather big. In the case $(\bar{y}, \bar{\lambda}) = (0, 0)$, i.e., the biactive set coincides with Ω, we obtain that the limiting normal cone contains the set $L^p(\Omega) \times H_0^1(\Omega)$ (for the above mentioned values of p), see Theorem 4.9, and this set is dense in $H^{-1}(\Omega) \times H_0^1(\Omega)$. Similar results are obtained in the case $(\bar{y}, \bar{\lambda}) \neq (0, 0)$.

To our knowledge, there are no characterizations of the limiting normal cone of $\mathbb{K}$ available, only the upper estimate (2.8) from [G. Wachsmuth, 2016] is known. The similar problem of characterizing of the limiting normal cone of sets with pointwise constraints in Lebesgue spaces has been solved only recently, see [Mehlitz, G. Wachsmuth, 2016; 2017].

Let us give a brief outline of the paper. In Section 2 we first introduce some notation. Then we state some facts about capacity theory that are needed in this paper. In Section 2.2 we give the definition of the limiting normal cone, and in Section 2.3 the optimality system (1.3) is compared with known optimality systems from the literature. Afterwards, we provide a generalization of a result from homogenization theory of Cioranescu and Murat (Theorem 2.1) which will play a crucial role for our main results. In Section 3 we characterize the limiting normal cone in the case of $(\bar{y}, \bar{\lambda}) = (0, 0)$ and for multipliers in $L^2(\Omega) \times H_0^1(\Omega)$. These results are generalized in Section 4 where we consider the limiting normal cone at arbitrary points and allow multipliers in $L^p(\Omega) \times H_0^1(\Omega)$, where $p \in (1, 2)$ is chosen such that $L^p(\Omega) \hookrightarrow H^{-1}(\Omega)$. We note that the proof in Section 3 requires significantly less technical considerations and, thus, serves as a motivation for Section 4. Finally, in Section 5 we give an example of an element in the limiting normal cone where one component is not a function, but rather a measure in $H^{-1}(\Omega)$. Appendix A contains some auxiliary results.

2. Preliminaries

We fix some notation. Throughout the paper, $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is assumed to be open and bounded. We do not impose any regularity of Ω.

We use the notation $B_r(x)$ for the open ball with radius r and center $x \in \mathbb{R}^d$. We also denote the d-dimensional Lebesgue measure of a measurable set $M \subset \mathbb{R}^d$ by $\operatorname{vol}(M)$. A frequently appearing constant is the surface measure of the boundary of the d-dimensional unit ball $B_1(0) \subset \mathbb{R}^d$, which will be denoted by S_d. Note that $\operatorname{vol}(B_1(0)) = d^{-1} S_d$.

For convex sets $M \subset H_0^1(\Omega)$ and $\hat{M} \subset H^{-1}(\Omega)$ we use

$$M^\circ := \{\mu \in H^{-1}(\Omega) : \langle \mu, v \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} \leq 0 \ \forall v \in M\},$$

$$\hat{M}^\circ := \{v \in H_0^1(\Omega) : \langle \mu, v\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} \le 0 \ \forall \mu \in \hat{M}\}$$

for the polar cones, and for $\mu \in H^{-1}(\Omega)$ the annihilator is denoted by

$$\mu^\perp := \{v \in H_0^1(\Omega) : \langle \mu, v\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} = 0\}.$$

We define K to be the set of non-negative functions in $H_0^1(\Omega)$, i.e.,

$$K := H_0^1(\Omega)_+ := \{v \in H_0^1(\Omega) : v \ge 0 \text{ a.e. in } \Omega\}.$$

The non-positive functionals in $H^{-1}(\Omega)$ are defined via duality, i.e.,

$$H^{-1}(\Omega)_- := K^\circ = \{\mu \in H^{-1}(\Omega) : \langle \mu, v\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} \le 0 \ \forall v \in K\}.$$

The radial cone and the tangent cone (in the sense of convex analysis) to K at $v \in K$ are defined via

$$\mathcal{R}_K(v) := \bigcup_{\lambda>0} \lambda(K - v) \qquad \text{and} \qquad \mathcal{T}_K(v) := \overline{\mathcal{R}_K(v)},$$

respectively. Recall that the set K is polyhedric, i.e.,

$$\mathcal{T}_K(v) \cap \mu^\perp = \overline{\mathcal{R}_K(v) \cap \mu^\perp}$$

holds for all $v \in K$ and $\mu \in \mathcal{T}_K(v)^\circ$, see [Mignot, 1976, Théorème 3.2]. Note that $v \in K$, $\mu \in \mathcal{T}_K(v)^\circ$ is equivalent to $(v, \mu) \in \mathbb{K}$, i.e., $\mathbb{K}$ is the graph of the normal cone mapping of K. Associated to $(v, \mu) \in \mathbb{K}$, we define the critical cone

$$\mathcal{K}_K(v, \mu) := \mathcal{T}_K(v) \cap \mu^\perp = \{w \in \mathcal{T}_K(v) : \langle \mu, w\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} = 0\}.$$

We mention that we use

$$\|y\|^2_{H_0^1(\Omega)} := \int_\Omega |\nabla y|^2 \, \mathrm{d}x$$

as a norm in $H_0^1(\Omega)$ and the norm in $H^{-1}(\Omega)$ is defined via duality. This implies that $-\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$ is an isometry. For a function $v \in L^1(\Omega)$, we use $v^+ := \max(v, 0)$ and $v^- := \max(-v, 0)$, i.e., $v = v^+ - v^-$. Recall that $v^+, v^- \in H_0^1(\Omega)$ for $v \in H_0^1(\Omega)$.

2.1. Brief introduction to capacity theory

In this section, we recall some facts about capacity theory, which will be needed in the sequel. The $H_0^1(\Omega)$-capacity of a set $O \subset \Omega$ is defined as

$$\operatorname{cap}(O) := \inf\big\{\|v\|^2_{H_0^1(\Omega)} : v \in H_0^1(\Omega) \text{ and } v \ge 1 \text{ a.e. in a neighbourhood of } O\big\},$$

see, e.g., [Attouch, Buttazzo, Michaille, 2006, Section 5.8.2], [Bonnans, Shapiro, 2000, Definition 6.47], and [Delfour, Zolésio, 2001, Section 8.6.1]. We say that a property P

(depending on $x \in \Omega$) holds quasi-everywhere (q.e.) on a subset $S \subset \Omega$, if and only if $\operatorname{cap}(\{x \in S : P(x) \text{ does not hold}\}) = 0$.

A function $v : \Omega \to \mathbb{R}$ is called *quasi-continuous* if for all $\varepsilon > 0$, there exists an open set $G_\varepsilon \subset \Omega$, such that $\operatorname{cap}(G_\varepsilon) < \varepsilon$ and v is continuous on $\Omega \setminus G_\varepsilon$. A set $O \subset \Omega$ is called *quasi-open* if for all $\varepsilon > 0$, there exists an open set $G_\varepsilon \subset \Omega$, such that $\operatorname{cap}(G_\varepsilon) < \varepsilon$ and $O \cup G_\varepsilon$ is open.

It is known, see, e.g., [Bonnans, Shapiro, 2000, Lemma 6.50], [Delfour, Zolésio, 2001, Theorem 8.6.1], that every $v \in H_0^1(\Omega)$ possesses a quasi-continuous representative and this representative is uniquely determined up to sets of zero capacity. When we speak about a function $v \in H_0^1(\Omega)$, we always refer to the quasi-continuous representative. Every sequence which converges in $H_0^1(\Omega)$ possesses a pointwise quasi-everywhere convergent subsequence, see [Bonnans, Shapiro, 2000, Lemma 6.52].

We recall, that a non-negative (or, non-positive) $\mu \in H^{-1}(\Omega)$ can be represented as a regular Borel measure, see, e.g., [Bonnans, Shapiro, 2000, p.564]. Moreover, since μ does not charge sets of capacity zero, it can be extended to finely-open sets and the *fine support*, denoted by $\operatorname{f-supp}(\mu)$, is the complement of the largest finely-open set O with $\mu(O) = 0$. We refer to [G. Wachsmuth, 2014, Appendix A] for details. Due to [G. Wachsmuth, 2014, Lemma A.5], this definition of the fine support is crucial to obtain the characterization

$$\mathcal{K}_K(y, \lambda) = \{w \in H_0^1(\Omega) : w \ge 0 \text{ q.e. on } \{y = 0\} \text{ and } w = 0 \text{ q.e. on } \operatorname{f-supp}(\lambda)\}$$

of the critical cone. The advantage of this representation is that both conditions on w are posed in the q.e.-sense.

By following the proof of [Heinonen, Kilpeläinen, Martio, 1993, Lemma 4.7], we find

$$\operatorname{cap}(O) = \inf\big\{\|\nabla v\|^2_{L^2(\Omega;\mathbb{R}^d)} : v \in H_0^1(\Omega) \text{ and } v \ge 1 \text{ q.e. on } O\big\}. \tag{2.1}$$

We also recall that for all open subsets $\Omega_n \subset \Omega$, we have the characterization

$$u \in H_0^1(\Omega_n) \quad \Leftrightarrow \quad u \in H_0^1(\Omega) \text{ and } u = 0 \text{ q.e. on } \Omega \setminus \Omega_n, \tag{2.2}$$

see [Heinonen, Kilpeläinen, Martio, 1993, Theorem 4.5].

Finally, for any measurable $O \subset \Omega$ we have $\|v\|^2_{H_0^1(\Omega)} \ge C\,\|v\|^2_{L^2(\Omega)} \ge C \operatorname{vol}(O)$ for some $C > 0$ and all functions $v \in H_0^1(\Omega)$ admissible in the definition of $\operatorname{cap}(O)$. Hence,

$$\operatorname{vol}(O) \le \frac{1}{C} \operatorname{cap}(O). \tag{2.3}$$

2.2. Concepts of variational calculus

We mention two basic concepts of variational calculus that will be used in this paper. First, we recall that the Fréchet normal cone $\widehat{\mathcal{N}}_C(\bar{x})$ of a subset $C \subset X$ of a Banach space X is defined via

$$\widehat{\mathcal{N}}_C(\bar{x}) := \left\{\eta \in X^* : \limsup_{x \to \bar{x}, x \in C} \frac{\langle \eta, x - \bar{x}\rangle}{\|x - \bar{x}\|_X} \le 0\right\}.$$

If the Banach space X is reflexive, the limiting normal cone (or Mordukhovich normal cone) to C at a point $x \in C$ can be defined via

$$\mathcal{N}_C(\bar{x}) := \{\eta \in X^* : \exists \{x_n\}_{n\in\mathbb{N}} \subset C, \{\eta_n\}_{n\in\mathbb{N}} \subset X^* : \eta_n \in \widehat{\mathcal{N}}_C(x_n), x_n \to \bar{x}, \eta_n \rightharpoonup \eta\},$$

see [Mordukhovich, 2006, Definition 1.1, Theorem 2.35]. Now, we are going to apply these definitions to the non-convex set $\mathbb{K}$. Due to the polyhedricity of K, we have

$$\widehat{\mathcal{N}}_{\mathbb{K}}(y, \lambda) = \mathcal{K}_K(y, \lambda)^\circ \times \mathcal{K}_K(y, \lambda), \tag{2.4}$$

cf. [Franke, Mehlitz, Pilecka, 2016, Lemma 4.1] and [G. Wachsmuth, 2015, Lemma 5.2]. Hence, $(\nu, w) \in \mathcal{N}_{\mathbb{K}}(y, \lambda)$ if and only if there exist sequences $\{y_n\}_{n\in\mathbb{N}}, \{w_n\}_{n\in\mathbb{N}} \subset H_0^1(\Omega)$, $\{\lambda_n\}_{n\in\mathbb{N}}, \{\nu_n\}_{n\in\mathbb{N}} \subset H^{-1}(\Omega)$ with

$$\begin{aligned} (y_n, \lambda_n) \in \mathbb{K}, \qquad & y_n \to y \text{ in } H_0^1(\Omega), \qquad && w_n \rightharpoonup w \text{ in } H_0^1(\Omega), \qquad && w_n \in \mathcal{K}_K(y_n, \lambda_n), \\ & \lambda_n \to \lambda \text{ in } H^{-1}(\Omega), && \nu_n \rightharpoonup \nu \text{ in } H^{-1}(\Omega), && \nu_n \in \mathcal{K}_K(y_n, \lambda_n)^\circ, \end{aligned}$$

for all $n \in \mathbb{N}$.

2.3. Optimality systems

In this section, we recall two other optimality systems for (1.2), which are of interest for our study of the limiting normal cone. We employ the notions of capacity theory and variational calculus. To this end, let $(\bar{y}, \bar{u}, \bar{\lambda})$ be a locally optimal solution of (1.2). Further, we fix the sets

$$\begin{aligned} \mathcal{A} &:= \{x \in \Omega : \bar{y}(x) = 0\}, & \mathcal{A}_s &:= \text{f-supp}(\bar{\lambda}), \\ \mathcal{I} &:= \{x \in \Omega : \bar{y}(x) > 0\}, & \mathcal{B} &:= \mathcal{A} \setminus \mathcal{A}_s, \end{aligned} \tag{2.5}$$

which are called active set, strictly active set, inactive set, and biactive set, respectively.

The system of weak stationarity is obtained by using

$$\mathcal{N}_{\mathbb{K}}^{\text{weak}}(\bar{y}, \bar{\lambda}) := \{z \in H_0^1(\Omega) : z = 0 \text{ q.e. on } \mathcal{A}\}^\circ \times \{w \in H_0^1(\Omega) : w = 0 \text{ q.e. on } \mathcal{A}_s\} \tag{2.6}$$

instead of $\mathcal{N}_{\mathbb{K}}(\bar{y}, \bar{\lambda})$ in (1.3). This system is satisfied for all local minimizers under very weak assumptions on the data, cf. [G. Wachsmuth, 2016, Lemma 4.4].

Next, we will state the definition of M-stationarity from [G. Wachsmuth, 2016]. Let $\mathcal{B} = \hat{\mathcal{I}} \cup \hat{\mathcal{B}} \cup \hat{\mathcal{A}}_s$ be a disjoint decomposition of the biactive set and we define

$$\hat{\mathcal{K}}(\hat{\mathcal{B}}, \hat{\mathcal{A}}_s) := \{v \in H_0^1(\Omega) : v \geq 0 \text{ q.e. on } \hat{\mathcal{B}} \text{ and } v = 0 \text{ q.e. on } \mathcal{A}_s \cup \hat{\mathcal{A}}_s\}.$$

Note that the critical cone satisfies $\mathcal{K}_K(\bar{y}, \bar{\lambda}) = \hat{\mathcal{K}}(\mathcal{B}, \emptyset)$. Then, the M-stationarity conditions of [G. Wachsmuth, 2016] are obtained by replacing $\mathcal{N}_{\mathbb{K}}(\bar{y}, \bar{\lambda})$ in (1.3) with

$$\mathcal{N}_{\mathbb{K}}^{\text{M}}(\bar{y}, \bar{\lambda}) = \left\{ (\nu, w) \in H^{-1}(\Omega) \times H_0^1(\Omega) : \begin{array}{l} \text{there is a decomposition } \mathcal{B} = \hat{\mathcal{I}} \cup \hat{\mathcal{B}} \cup \hat{\mathcal{A}}_s \\ \text{with } \nu \in \hat{\mathcal{K}}(\hat{\mathcal{B}}, \hat{\mathcal{A}}_s)^\circ, w \in \hat{\mathcal{K}}(\hat{\mathcal{B}}, \hat{\mathcal{A}}_s) \end{array} \right\}.$$

In finite dimensions, a system of M-stationarity can be shown by using the limiting normal cone, cf. [Outrata, 1999, Theorem 3.1]. However, this is not known for the problem (1.2) unless $d = 1$, cf. [Jarušek, Outrata, 2007, Theorem 11] and [G. Wachsmuth, 2016, Lemma 2.3 and Theorem 5.4]. In particular, it is not known whether this system of M-stationarity is a necessary optimality system for (1.2), see also [G. Wachsmuth, 2016, Sections 5, 6].

Finally, we comment on the known relation between the defined normal cones. We trivially have the inclusions

$$\widehat{\mathcal{N}}_{\mathbb{K}}(\bar{y}, \bar{\lambda}) \subset \mathcal{N}_{\mathbb{K}}^{\mathrm{M}}(\bar{y}, \bar{\lambda}) \subset \mathcal{N}_{\mathbb{K}}^{\mathrm{weak}}(\bar{y}, \bar{\lambda}). \tag{2.7}$$

Moreover, the inclusion

$$\mathcal{N}_{\mathbb{K}}(\bar{y}, \bar{\lambda}) \subset \mathcal{N}_{\mathbb{K}}^{\mathrm{weak}}(\bar{y}, \bar{\lambda}) \tag{2.8}$$

can be shown as in the proof of [G. Wachsmuth, 2016, Lemma 4.4] and this implies

$$\widehat{\mathcal{N}}_{\mathbb{K}}(\bar{y}, \bar{\lambda}) \subset \mathcal{N}_{\mathbb{K}}(\bar{y}, \bar{\lambda}) \subset \mathcal{N}_{\mathbb{K}}^{\mathrm{weak}}(\bar{y}, \bar{\lambda}). \tag{2.9}$$

In view of (2.7) and (2.9), we are interested in the relations between $\mathcal{N}_{\mathbb{K}}(\bar{y}, \bar{\lambda})$, $\mathcal{N}_{\mathbb{K}}^{\mathrm{M}}(\bar{y}, \bar{\lambda})$ and $\mathcal{N}^{\mathrm{weak}}(\bar{y}, \bar{\lambda})$. The construction in [G. Wachsmuth, 2016, Section 6] shows that

$$\mathcal{N}_{\mathbb{K}}(0, 0) \not\subset \mathcal{N}_{\mathbb{K}}^{\mathrm{M}}(0, 0)$$

if the dimension d of Ω is at least 2, and in dimension $d = 1$, the inclusion

$$\mathcal{N}_{\mathbb{K}}(\bar{y}, \bar{\lambda}) \subset \mathcal{N}_{\mathbb{K}}^{\mathrm{M}}(\bar{y}, \bar{\lambda})$$

follows from [G. Wachsmuth, 2016, Lemma 2.3 and Section 5].

2.4. A result from homogenization theory

In this section we will repeat a (slightly generalized) result from [Cioranescu, Murat, 1997].

Theorem 2.1 ([Cioranescu, Murat, 1997, Theorem 1.2])**.** Let $\{\Omega_n\}_{n\in\mathbb{N}}$ be a sequence of open subset of Ω. Suppose there exist sequences $\{v_n\}_{n\in\mathbb{N}} \subset H^1(\Omega)$, $\{\gamma_n\}_{n\in\mathbb{N}}, \{\mu_n\}_{n\in\mathbb{N}} \subset H^{-1}(\Omega)$ and a distribution $\mu \in H^{-1}(\Omega)$ such that

$$v_n \in H^1(\Omega) \tag{H.1}$$

$$v_n = 0 \text{ q.e. on } \Omega \setminus \Omega_n \tag{H.2}$$

$$v_n \rightharpoonup 1 \text{ in } H^1(\Omega) \tag{H.3}$$

$$\mu \in \begin{cases} W^{-1,d}(\Omega) & \text{if } d \geq 3, \\ W^{-1,2+\varepsilon}(\Omega) & \text{if } d = 2, \text{ for some } \varepsilon > 0 \end{cases} \tag{H.4'}$$

$$\left.\begin{aligned} \mu_n \to \mu, \gamma_n \rightharpoonup \mu \text{ in } H^{-1}(\Omega),\ -\Delta v_n = \mu_n - \gamma_n, \\ \langle \gamma_n, z_n \rangle = 0 \, \forall z_n \in H_0^1(\Omega_n). \end{aligned}\right\} \tag{H.5'}$$

Let $\xi \in H^{-1}(\Omega)$ be given. We denote by u_n the unique (weak) solution of

$$-\Delta u_n = \xi, \quad u_n \in H_0^1(\Omega_n) \subset H_0^1(\Omega).$$

Then u_n converges weakly in $H_0^1(\Omega)$ towards the unique solution u of

$$-\Delta u + \mu u = \xi, \quad u \in H_0^1(\Omega).$$

Proof. This theorem is only a slight generalization of [Cioranescu, Murat, 1997, Theorem 1.2] and we will not repeat the proof. Instead, we only discuss the differences. First, we use a right-hand side $\xi \in H^{-1}(\Omega)$ instead of $f \in L^2(\Omega)$. It can be seen from the proof, that the right-hand side is only used as a functional over $H_0^1(\Omega)$, so the proof extends to a right-hand side of $\xi \in H^{-1}(\Omega)$.

Next, we generalize the condition $\mu \in W^{-1,\infty}(\Omega)$, which is used in [Cioranescu, Murat, 1997]. Again, inspecting the proof of [Cioranescu, Murat, 1997, Theorem 1.1] and [Cioranescu, Murat, 1997, Theorem 1.2] reveals that we only need the property $\mu z \in H^{-1}(\Omega)$ for all $z \in H_0^1(\Omega)$. Using the Sobolev embedding theorem, this is guaranteed by (H.4').

Finally, instead of condition (H.5) we can use (H.5)' as explained in [Cioranescu, Murat, 1997, Remark 1.6].

Note that the assumptions (H.1)–(H.3) of Theorem 2.1 imply $\Omega_n = \Omega$ for large n in dimension $d = 1$ due to the compact embedding $H^1(\Omega) \hookrightarrow C(\bar{\Omega})$.

Let us explain how Theorem 2.1 is applied later. Suppose that we have a sequence $\{\Omega_n\}_{n\in\mathbb{N}}$ of open subsets of Ω such that the assumptions (H.1)–(H.5') are satisfied for some sequences $\{v_n\}_{n\in\mathbb{N}}$, $\{\gamma_n\}_{n\in\mathbb{N}}$ and $\{\mu_n\}_{n\in\mathbb{N}}$. Further, let $u \in H_0^1(\Omega)$ be arbitrary. We define u_n as the unique weak solution of

$$-\Delta u_n = -\Delta u + \mu u, \quad u_n \in H_0^1(\Omega_n).$$

Then, Theorem 2.1 implies that $u_n \rightharpoonup u$ in $H_0^1(\Omega)$. That is, every $u \in H_0^1(\Omega)$ can be approximated weakly by a sequence $u_n \in H_0^1(\Omega_n)$. In particular, $u_n = 0$ q.e. on $\Omega \setminus \Omega_n$.

3. The biactive case with multipliers in $L^2(\Omega)$

In this section, we are going to calculate the intersection of $\mathcal{N}_{\mathbb{K}}(0,0)$ with $L^2(\Omega) \times H_0^1(\Omega)$. In fact, we proof that every $(\nu, w) \in L^2(\Omega) \times H_0^1(\Omega)$ belongs to the limiting normal cone $\mathcal{N}_{\mathbb{K}}(0,0)$.

The results of this section will be generalized in Section 4. The main purpose of this section is the illustration of the technique of proof, which is much simpler in the $L^2(\Omega)$-case, since we can built upon the results from [Cioranescu, Murat, 1997].

We are going to cover Ω by closed cubes. Therefore, fix a number $n \in \mathbb{N}$ and let $\{x_i^n\}_{i\in\mathbb{N}} = \frac{2}{n}\mathbb{Z}^d$ be a regular grid, and we define the cubes $P_i^n := x_i^n + [-\frac{1}{n}, \frac{1}{n}]^d$. These cubes have edge length $\frac{2}{n}$ and their interiors are pairwise disjoint. Furthermore, each contains a closed hole $T_i^n := \overline{B_{a_n}(x_i^n)}$ at the center, which is a closed ball with radius

$$a_n := \begin{cases} n^{\frac{d}{2-d}} & \text{if } d \geq 3, \\ \exp(-n^2) & \text{if } d = 2. \end{cases}$$

Furthermore we define $B_i^n := B_{1/n}(x_i^n) \subset P_i^n$ to be the (open) ball with radius $\frac{1}{n}$ that is contained in P_i^n. Finally, we define the perforated domain $\Omega_n := \Omega \setminus \bigcup_{i\in I_n} T_i^n$, where I_n is the finite set of indices i such that $P_i^n \cap \Omega \neq \emptyset$. Note that this construction will allow us to apply Theorem 2.1, see [Cioranescu, Murat, 1997, Theorem 2.2].

We briefly describe our approach to show that a fixed $(\nu, w) \in L^2(\Omega) \times H_0^1(\Omega)$ belongs to the limiting normal cone $\mathcal{N}_{\mathbb{K}}(0,0)$. As described after Theorem 2.1, we get a sequence $\{w_n\}_{n\in\mathbb{N}}$ with $w_n \in H_0^1(\Omega_n)$ and $w_n \rightharpoonup w$ in $H_0^1(\Omega)$. The next step is the definition of $\nu_n \in H^{-1}(\Omega)$. In order to obtain the desired inclusions $w_n \in \mathcal{K}_K(y_n, \lambda_n)$ and $\nu_n \in \mathcal{K}_K(y_n, \lambda_n)^\circ$, we propose to define ν_n such that it is only supported on the holes. The construction of y_n and λ_n is then straightforward, see Theorem 3.4.

First, we provide the definition and the boundedness of ν_n in $H^{-1}(\Omega)$. The idea is to distribute the mass of ν on a cell P_i^n to the small hole T_i^n.

Lemma 3.1. Let $\nu \in L^2(\Omega) \subset H^{-1}(\Omega)$ be given. We set $J_n := \{i \in \mathbb{N} : P_i^n \subset \Omega\}$ and define ν_n via

$$\nu_n := \sum_{i\in J_n} \chi_{T_i^n} \frac{1}{\operatorname{vol}(T_i^n)} \int_{P_i^n} \nu \,\mathrm{d}x.$$

Then there is a constant $C > 0$ (which only depends on the domain Ω) such that

$$\|\nu_n\|_{H^{-1}(\Omega)} + \|\nu_n^+\|_{H^{-1}(\Omega)} \leq C\,\|\nu\|_{L^2(\Omega)}$$

holds for all $n \in \mathbb{N}$.

Note that the boundedness of the non-negative part $\nu_n^+ = \max(\nu_n, 0)$ does not simply follow from the boundedness of $\nu_n \in H^{-1}(\Omega)$.

Proof. We show the boundedness of ν_n and the estimate for ν_n^+ follows by similar arguments, see also Lemma 4.2. Let $n \in \mathbb{N}$ be fixed. We define

$$\beta_{i,n} := \frac{1}{\operatorname{vol}(T_i^n)} \int_{P_i^n} \nu \,\mathrm{d}x$$

if $i \in J_n$ and $\beta_{i,n} = 0$ otherwise. For $i \in J_n$, we denote by $u_{i,n}$ the solution of

$$-\Delta u_{i,n} = \chi_{T_i^n} - a_n^d n^d \chi_{B_i^n} \quad \text{in } \Omega, \quad u_{i,n} \in H_0^1(\Omega).$$

It follows that

$$\nu_n = -\Delta\Big(\sum_{i \in J_n} \beta_{i,n} u_{i,n}\Big) + \sum_{i \in J_n} a_n^d n^d \beta_{i,n} \chi_{B_i^n}. \tag{3.1}$$

We can use Lemma A.1 (a) to calculate the norm of $u_{i,n}$, which results in $\|u_{i,n}\|^2_{H_0^1(\Omega)} \le C\, a_n^{2d} n^d$, where we used $L^d(a_n) = n^{-d}$, see (A.1). We also note that $\operatorname{supp} u_{i,n} = \overline{B_i^n} \subset \overline{P_i^n}$, and this implies that $u_{i,n}$ and $u_{j,n}$ are orthogonal w.r.t. the $H_0^1(\Omega)$-inner product for $i \neq j$. Now we continue with the calculation of the $H^{-1}(\Omega)$-norm of ν_n. By using the splitting (3.1) and since $-\Delta$ is an isometry, we obtain

$$\|\nu_n\|_{H^{-1}(\Omega)} \le \Big\|\sum_{i \in J_n} \beta_{i,n} u_{i,n}\Big\|_{H_0^1(\Omega)} + \Big\|\sum_{i \in J_n} a_n^d n^d \beta_{i,n} \chi_{B_i^n}\Big\|_{H^{-1}(\Omega)}.$$

Using the orthogonality of the functions $\{u_{i,n}\}_{i \in J_n}$ we get

$$\begin{aligned}
\Big\|\sum_{i \in J_n} \beta_{i,n} u_{i,n}\Big\|^2_{H_0^1(\Omega)} &= \sum_{i \in J_n} |\beta_{i,n}|^2 \|u_{i,n}\|^2_{H_0^1(\Omega)} \\
&\le C\, a_n^{2d} n^d \sum_{i \in J_n} |\beta_{i,n}|^2 \le C\, a_n^{2d} n^d \sum_{i \in J_n} \frac{1}{\operatorname{vol}(T_i^n)^2} \Big|\int_{P_i^n} \nu \,\mathrm{d}x\Big|^2 \\
&\le C\, a_n^{2d} n^d \sum_{i \in J_n} \frac{\operatorname{vol}(P_i^n)}{\operatorname{vol}(T_i^n)^2} \int_{P_i^n} |\nu|^2 \,\mathrm{d}x \le C\, \|\nu\|^2_{L^2(\Omega)},
\end{aligned}$$

where we used $\operatorname{vol}(P_i^n) = (2/n)^d$ and $\operatorname{vol}(T_i^n) = d^{-1} S_d\, a_n^d$ in the last step. For the other term, we have

$$\begin{aligned}
\Big\|\sum_{i \in J_n} a_n^d n^d \beta_{i,n} \chi_{B_i^n}\Big\|^2_{H^{-1}(\Omega)} &\le C \Big\|\sum_{i \in J_n} a_n^d n^d \beta_{i,n} \chi_{B_i^n}\Big\|^2_{L^2(\Omega)} \\
&\le C\, a_n^{2d} n^d \sum_{i \in J_n} |\beta_{i,n}|^2 \le C\, \|\nu\|^2_{L^2(\Omega)},
\end{aligned}$$

where $\sum_{i \in J_n} |\beta_{i,n}|^2$ is estimated in the same way as above. Now the claim follows from the combination of the above inequalities.

The following counterexample shows that the above statement does not generalize to $L^p(\Omega)$ with $p < 2$.

Example 3.2. Let $1 < p < 2$. We set $\Omega = (0,1)^2 \subset \mathbb{R}^2$ and define $\nu(x,y) = x^\alpha$ for some $\alpha \in (-\frac{1}{p}, -\frac{1}{2})$. We have $\nu \in L^p(\Omega) \setminus L^2(\Omega)$ and due to the Sobolev embedding theorem, $\nu \in H^{-1}(\Omega)$. However, if we define ν_n as in Lemma 3.1, then $\|\nu_n\|_{H^{-1}(\Omega)}$ is not bounded for $n \to \infty$.

Sketch of the proof. We define $\beta_{i,n}$ and $u_{i,n} \in H_0^1(\Omega)$ as in the proof of Lemma 3.1. It can be shown that $\|u_{i,n}\|_{H_0^1(\Omega)}^2 \geq C\, a_n^{2d} n^d$. By calculating the coefficients $\beta_{i,n}$ it follows that $\big\|\sum_{i\in I_n} \beta_{i,n} u_{i,n}\big\|_{H_0^1(\Omega)} \to \infty$. On the other hand, since $L^p(\Omega)$ embeds into $H^{-1}(\Omega)$, $\big\|\sum_{i\in I_n} a_n^d n^d \beta_{i,n} \chi_{B_i^n}\big\|_{H^{-1}(\Omega)} \leq C\,\|\nu\|_{L^p(\Omega)}$ can be shown in the same way as in the previous proof. Then, using the triangle inequality and since $-\Delta$ is an isometry, it follows that

$$\|\nu_n\|_{H^{-1}(\Omega)} \geq \Big\|\sum_{i\in I_n} \beta_{i,n} u_{i,n}\Big\|_{H_0^1(\Omega)} - \Big\|\sum_{i\in I_n} a_n^d n^d \beta_{i,n} \chi_{B_i^n}\Big\|_{H^{-1}(\Omega)} \to \infty \quad (n\to\infty).$$

Using the above boundedness of ν_n in $H^{-1}(\Omega)$ provided in Lemma 3.1, one can show the weak convergence by proving $\langle \nu_n - \nu, f\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} \to 0$ for all $f \in C_0^\infty(\Omega)$. We refer to Lemma 4.3 for a similar proof.

Lemma 3.3. Let $\nu_n, \nu \in H^{-1}(\Omega)$ as in Lemma 3.1. Then $\nu_n \rightharpoonup \nu$.

Now we are in the position to prove the main result of this section, which shows that the limiting normal cone to $\mathbb{K}$ at $(0,0)$ contains a dense subset.

Theorem 3.4. We have $L^2(\Omega) \times H_0^1(\Omega) \subset \mathcal{N}_{\mathbb{K}}(0,0)$.

Proof. Let $w \in H_0^1(\Omega)$ and $\nu \in L^2(\Omega) \subset H^{-1}(\Omega)$ be given. First, we choose $w_n \in H_0^1(\Omega_n) \subset H_0^1(\Omega)$ as the solution of

$$-\Delta w_n = -\Delta w + \mu w \quad \text{in } \Omega_n,\ w_n \in H_0^1(\Omega_n),$$

where $\mu := S_d 2^{-d} \max(1, d-2)$. Recall that S_d is the surface measure of the unit sphere in $\mathbb{R}^d$. Then, according to [Cioranescu, Murat, 1997, Theorem 2.2] and Theorem 2.1 we have $w_n \rightharpoonup w$ in $H_0^1(\Omega)$.

Next, we choose ν_n as in Lemma 3.1. The weak convergence $\nu_n \rightharpoonup \nu$ in $H^{-1}(\Omega)$ follows from in Lemma 3.3. We also choose $y_n := \frac{1}{n} w_n^- \in H_0^1(\Omega)_+$ and $\lambda_n := -\frac{1}{n}\nu_n^+ \in H^{-1}(\Omega)_-$. Since w_n is zero on all holes T_i^n, $i \in J_n$, and ν_n is only defined on these holes, it follows that $(y_n, \lambda_n) \in \mathbb{K}$. The convergences $y_n \to 0, \lambda_n \to 0$ follows since w_n^- and ν_n^+ are bounded in $H_0^1(\Omega)$ and $H^{-1}(\Omega)$, respectively, see also Lemma 3.1.

It remains to show that $w_n \in \mathcal{K}_K(y_n, \lambda_n)$ and $\nu_n \in \mathcal{K}_K(y_n, \lambda_n)^\circ$. The first one follows from $w_n = n(\frac{1}{n} w_n^+ - y_n)$ and the fact that $\{w_n \neq 0\} \subset \Omega_n$. In order to show the condition for ν_n, we use the polyhedricity of K, which implies $\mathcal{K}_K(y_n, \lambda_n)^\circ = ((K - y_n) \cap \lambda_n^\perp)^\circ$. Let $v \in K$ with $v - y_n \in \lambda_n^\perp$ be given. Then

$$\langle \nu_n, v - y_n\rangle = \big\langle -\nu_n^- - n\lambda_n, v - y_n\big\rangle = \big\langle -\nu_n^-, v - w_n^-\big\rangle = -\big\langle \nu_n^-, v\big\rangle \leq 0,$$

where the last inequality follows from $\nu_n^-, v \geq 0$. Because z was chosen arbitrarily, this implies $\nu_n \in \mathcal{K}_K(y_n, \lambda_n)^\circ$. Thus we have shown that (ν, w) lies in $\mathcal{N}_K(0,0)$.

4. The extension to multipliers in $L^p(\Omega)$

In this section, we characterize $\mathcal{N}_{\mathbb{K}}(\bar{y}, \bar{\lambda}) \cap \big(L^p(\Omega) \times H_0^1(\Omega)\big)$ for $(\bar{y}, \bar{\lambda}) \in \mathbb{K}$, where $p < 2$ is given such that $L^p(\Omega) \hookrightarrow H^{-1}(\Omega)$, i.e., $p \in (1, 2)$ if $d = 2$ and $p \in [\frac{2d}{d+2}, 2)$ if $d \ge 3$.

In contrary to our approach in Section 3, we cannot work with holes possessing a uniform radius, cf. Example 3.2. Hence, the holes T_i^n will have different sizes, depending on their location. We denote the radius of T_i^n with $a_{i,n}$, i.e., $T_i^n = \overline{B_{a_{i,n}}(x_i^n)}$. As in Section 3, P_i^n denotes the cube $x_i^n + [-\frac{1}{n}, \frac{1}{n}]^d$, $B_i^n = B_{1/n}(x_i^n)$ is a ball with diameter $1/n$ and I_n is the set of indices i with $P_i^n \cap \Omega \neq \emptyset$. However, the definition of Ω_n will differ slightly from Section 3, see below.

In order to avoid some case distinctions between $d = 2$ and $d \ge 3$, we introduce the auxiliary function L^d via

$$L^d(a) := \begin{cases} -\log(a)^{-1} & \text{for } a \in (0,1) \text{ and } d = 2 \\ a^{d-2} & \text{for } a \in (0, \infty) \text{ and } d \ge 3. \end{cases}$$

In any case, L^d is monotonically increasing and the range is $(0, \infty)$. Throughout this section, $\nu \in L^p(\Omega)$ is chosen arbitrarily but fixed. For technical reasons, we introduce another index set J_n, defined as

$$J_n := \{i \in I_n : P_i^n \subset \Omega,\ 0 < \|\nu\|^p_{L^p(P_i^n)} < n^{1-d}\}.$$

For $i \in J_n$, we define the radius $a_{i,n} > 0$ of the holes T_i^n via

$$L^d(a_{i,n}) = \operatorname{vol}(P_i^n) \operatorname{avg}(P_i^n, |\nu|^p)^{\frac{2}{p}-1}, \tag{4.1}$$

where

$$\operatorname{avg}(P_i^n, |\nu|^p) := \frac{1}{\operatorname{vol}(P_i^n)} \int_{P_i^n} |\nu|^p \,\mathrm{d}x$$

is the average of the function $|\nu|^p$ over the set P_i^n. In the case that $i \in I_n \setminus J_n$ we set $a_{i,n} := 0$. We define

$$\Omega_n := \Omega \setminus \bigcup_{i \in J_n} T_i^n,$$

and this differs from the corresponding definition in Section 3. The next lemma shows that we have $a_{i,n} \le \frac{1}{n}$ for n large enough, i.e., $T_i^n \subset P_i^n$ holds. Afterwards, we will only consider these parameters $n \in \mathbb{N}$ which guarantee $a_{i,n} \le \frac{1}{n}$.

Lemma 4.1. (a) For large $n \in \mathbb{N}$ we have

$$a_{i,n} < \left(\frac{1}{2n}\right)^{1+\varepsilon} \quad \forall i \in J_n,$$

where ε depends on the dimension, but not on i and n. In the case of $d = 2$ we even have

$$a_{i,n} < \exp(-\frac{1}{8}n)\ \forall i \in J_n$$

for large $n \in \mathbb{N}$.

(b) For every large $n \in \mathbb{N}$ there exists a constant $C_n > 1$ such that

$$L^d(a_{i,n}) \le \frac{1}{L^d(a_{i,n})^{-1} - L^d(\frac{1}{n})^{-1}} \le C_n L^d(a_{i,n})$$

holds for all $i \in J_n$. Moreover, $C_n \to 1$ as $n \to \infty$.

(c) It holds that

$$\lim_{n\to\infty} \operatorname{vol}\big(\{\nu \neq 0\} \setminus \bigcup_{i\in J_n} P_i^n\big) = 0.$$

Proof. (a): Using the definition of the index set J_n and $\frac{2}{p} - 1 \in (0, 1)$, we have

$$\begin{aligned} L^d(a_{i,n}) &= \operatorname{vol}(P_i^n) \operatorname{avg}(P_i^n, |\nu|^p)^{\frac{2}{p}-1} \le \operatorname{vol}(P_i^n)(1 + \operatorname{avg}(P_i^n, |\nu|^p)) \\ &= \left(\frac{2}{n}\right)^d + \|\nu\|^p_{L^p(P_i^n)} < \left(\frac{2}{n}\right)^d + n^{1-d} < 2^{d+1} n^{1-d}. \end{aligned} \tag{4.2}$$

For $d \ge 3$ this implies

$$a_{i,n} < 2^{d+1} n^{(1-d)/(d-2)} < \left(\frac{1}{2n}\right)^{1+\varepsilon}$$

for large $n \in \mathbb{N}$ and $\varepsilon := \frac{1}{2(d-2)}$. For $d = 2$ inequality (4.2) yields $a_{i,n} < \exp(-\frac{1}{8}n)$.

(b): By part (a) it follows that $L^d(a_{i,n})/L^d(\frac{1}{n}) \to 0$ as $n \to \infty$, uniformly in $i \in J_n$. This implies the claim.

(c): If $i \in I_n$ does not belong to J_n then there are three possible reasons: $\|\nu\|^p_{L^p(P_i^n)} = 0$, $\|\nu\|^p_{L^p(P_i^n)} \ge n^{1-d}$, or $P_i^n \not\subset \Omega$. Therefore, we have

$$\begin{aligned} \operatorname{vol}\big(\{\nu \neq 0\} \setminus \bigcup_{i\in J_n} P_i^n\big) &= \sum_{i\in I_n \setminus J_n} \operatorname{vol}(\{\nu \neq 0\} \cap P_i^n) \\ &\le \sum_{i:\|\nu\|^p_{L^p(P_i^n)} \ge n^{1-d}} \operatorname{vol}(P_i^n) + \sum_{i:P_i^n \not\subset \Omega} \operatorname{vol}(\Omega \cap P_i^n). \end{aligned}$$

For the first term we have

$$\begin{aligned} \sum_{i:\|\nu\|^p_{L^p(P_i^n)} \ge n^{1-d}} \operatorname{vol}(P_i^n) &= \sum_{i:\|\nu\|^p_{L^p(P_i^n)} \ge n^{1-d}} (2n)^{-d} \\ &\le 2^{-d} n^{-1} \sum_{i:\|\nu\|^p_{L^p(P_i^n)} \ge n^{1-d}} \|\nu\|^p_{L^p(P_i^n)} \\ &\le 2^{-d} n^{-1} \|\nu\|^p_{L^p(\Omega)} \to 0 \quad (n \to \infty). \end{aligned}$$

The convergence of the second term follows from $\Omega = \bigcup_{n\in\mathbb{N}} \bigcup_{i:P_i^n \subset \Omega} P_i^n$. This proves the claim.

The next lemma parallels Lemma 3.1. For this result, the adaptive choice of the radii $a_{i,n}$ in (4.1) is crucial.

Lemma 4.2. We define the measurable function $\tilde{\nu}_n$ via

$$\tilde{\nu}_n := \sum_{i \in J_n} \chi_{T_i^n} \beta_{i,n},$$

where the real-valued coefficients $\beta_{i,n}$ satisfy

$$|\beta_{i,n}| \le \frac{1}{\operatorname{vol}(T_i^n)} \int_{P_i^n} |\nu| \, \mathrm{d}x.$$

Then there is a constant $C > 0$ (depending only on the domain Ω and p) such that

$$\|\tilde{\nu}_n\|_{H^{-1}(\Omega)} \le C \, \|\nu\|_{L^p(\Omega)}^{\frac{p}{2}} + C \, \|\nu\|_{L^p(\Omega)}$$

holds for all $n \in \mathbb{N}$.

Proof. Let $n \in \mathbb{N}$ be fixed. As in the proof of Lemma 3.1 we define $u_{i,n}$ as the solution of

$$-\Delta u_{i,n} = \chi_{T_i^n} - a_{i,n}^d n^d \chi_{B_i^n} \quad \text{in}\, \Omega, \quad u_{i,n} \in H_0^1(\Omega).$$

It follows that

$$\tilde{\nu}_n = -\Delta\Big(\sum_{i \in J_n} \beta_{i,n} u_{i,n}\Big) + \sum_{i \in J_n} a_{i,n}^d n^d \beta_{i,n} \chi_{B_i^n}. \tag{4.3}$$

From Lemma A.1 (a) we know that $\operatorname{supp} u_{i,n} \subset \overline{B_i^n} \subset P_i^n$ and

$$\|u_{i,n}\|_{H_0^1(\Omega)}^2 \le C \, a_{i,n}^{2d} L^d(a_{i,n})^{-1},$$

where the constant $C > 0$ does not depend on n and i. We continue with the boundedness of $\|\tilde{\nu}_n\|_{H^{-1}(\Omega)}$. Using the isometry of $-\Delta$, (4.3) yields

$$\|\tilde{\nu}_n\|_{H^{-1}(\Omega)} \le \Big\|\sum_{i \in J_n} \beta_{i,n} u_{i,n}\Big\|_{H_0^1(\Omega)} + \Big\|\sum_{i \in J_n} a_{i,n}^d n^d \beta_{i,n} \chi_{B_i^n}\Big\|_{H^{-1}(\Omega)}.$$

Since the functions $u_{i,n}$ are orthogonal with respect to the scalar product in $H_0^1(\Omega)$, we have

$$\begin{aligned}\Big\|\sum_{i \in J_n} \beta_{i,n} u_{i,n}\Big\|_{H_0^1(\Omega)}^2 &= \sum_{i \in J_n} |\beta_{i,n}|^2 \|u_{i,n}\|_{H_0^1(\Omega)}^2 \\ &\le C \sum_{i \in J_n} a_{i,n}^{2\,d} \, L^d(a_{i,n})^{-1} \frac{1}{\operatorname{vol}(T_i^n)^2} \Big(\int_{P_i^n} |\nu| \, \mathrm{d}x\Big)^2 \\ &\le C \sum_{i \in J_n} L^d(a_{i,n})^{-1} \operatorname{vol}(P_i^n)^{2-\frac{2}{p}} \, \|\nu\|_{L^p(P_i^n)}^2\end{aligned}$$

$$= C \sum_{i\in J_n} L^d(a_{i,n})^{-1} \operatorname{vol}(P_i^n) \operatorname{avg}(P_i^n, |\nu|^p)^{\frac{2}{p}-1} \|\nu\|^p_{L^p(P_i^n)}$$
$$= C \sum_{i\in J_n} \|\nu\|^p_{L^p(P_i^n)} \le C \|\nu\|^p_{L^p(\Omega)}.$$

For the other term we have

$$\begin{aligned}
\Big\| \sum_{i\in J_n} a_{i,n}^d n^d \beta_{i,n} \chi_{B_i^n} \Big\|^p_{H^{-1}(\Omega)} &\le C \Big\| \sum_{i\in J_n} a_{i,n}^d n^d \beta_{i,n} \chi_{B_i^n} \Big\|^p_{L^p(\Omega)} \\
&= C \sum_{i\in J_n} (a_{i,n} n)^{dp} |\beta_{i,n}|^p \|\chi_{B_i^n}\|^p_{L^p(\Omega)} \\
&\le C \sum_{i\in J_n} n^{dp-d} \Big(\int_{P_i^n} |\nu| \,\mathrm{d}x \Big)^p \\
&\le C \sum_{i\in J_n} n^{dp-d} \operatorname{vol}(P_i^n)^{(1-\frac{1}{p})p} \int_{P_i^n} |\nu|^p \,\mathrm{d}x \\
&\le C \sum_{i\in J_n} \int_{P_i^n} |\nu|^p \,\mathrm{d}x \le C \|\nu\|^p_{L^p(\Omega)}.
\end{aligned}$$

This completes the proof.

After we have established this boundedness, we are in position to prove that every function $\tilde\nu$ which is pointwise bounded by ν can be approximated weakly by functions living on the holes T_i^n.

Lemma 4.3. Let $\tilde\nu$ be a function in $L^p(\Omega) \subset H^{-1}(\Omega)$ such that $|\tilde\nu| \le |\nu|$. If we define

$$\tilde\nu_n := \sum_{i\in J_n} \chi_{T_i^n} \frac{1}{\operatorname{vol}(T_i^n)} \int_{P_i^n} \tilde\nu \,\mathrm{d}x,$$

then $\tilde\nu_n \rightharpoonup \tilde\nu$ in $H^{-1}(\Omega)$.

Proof. Because $\tilde\nu_n$ satisfies the requirements for Lemma 4.2, we know that $\tilde\nu_n$ is bounded in $H^{-1}(\Omega)$. Hence it suffices to show the convergence on the dense subspace $C_0^\infty(\Omega) \subset H_0^1(\Omega)$. Let $f \in C_0^\infty(\Omega)$ be given. We have

$$\begin{aligned}
\Big| \langle \tilde\nu_n - \tilde\nu, f \rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} \Big| &= \Big| \int_\Omega (\tilde\nu_n - \tilde\nu) f \,\mathrm{d}x \Big| \\
&\le \Big| \sum_{i\in J_n} \int_{P_i^n} (\tilde\nu_n - \tilde\nu) f \,\mathrm{d}x \Big| + \Big| \int_{\{\tilde\nu\ne 0\}\setminus \bigcup_{i\in J_n} P_i^n} f \tilde\nu \,\mathrm{d}x \Big|
\end{aligned}$$

The second term converges to 0, because of Lemma 4.1 (c). For the first term we can use that f is uniformly continuous. That mean that for each $\varepsilon > 0$ we can find arbitrarily large $n \in \mathbb{N}$ such that $|f(x) - f(y)| < \varepsilon$ for all $x, y \in P_i^n, i \in J_n$. Thus

$$\begin{aligned}\left|\sum_{i\in J_n}\int_{P_i^n}(\tilde{\nu}_n - \tilde{\nu}) f \,\mathrm{d}x\right| &\leq \sum_{i\in J_n}\left|\frac{1}{\operatorname{vol}(T_i^n)}\int_{P_i^n}\tilde{\nu}\,\mathrm{d}x\int_{T_i^n} f\,\mathrm{d}y - \int_{P_i^n} f\tilde{\nu}\,\mathrm{d}x\right| \\ &\leq \sum_{i\in J_n}\int_{P_i^n}|\tilde{\nu}(x)|\left|\frac{1}{\operatorname{vol}(T_i^n)}\int_{T_i^n}|f(y) - f(x)|\,\mathrm{d}y\right|\mathrm{d}x \\ &\leq \varepsilon\sum_{i\in J_n}\int_{P_i^n}|\tilde{\nu}|\,\mathrm{d}x \leq \varepsilon\,\|\tilde{\nu}\|_{L^1(\Omega)}.\end{aligned}$$

Since ε can be arbitrarily small, this proves that $\langle\tilde{\nu}_n - \tilde{\nu}, f\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} \to 0$ as $n \to \infty$.

The next lemma provides some technical estimates, which will be used to verify the assumptions of Theorem 2.1 in our setting with differently sized holes. Again, the specific choice (4.1) is crucial for these estimates.

Lemma 4.4. Let the size of the holes T_i^n be chosen according to (4.1). Then, there exists a constant $C > 0$, such that

$$\sum_{i\in J_n} L^d(a_{i,n}) \leq C\,\|\nu\|_{L^p(\Omega)}^p, \tag{4.4a}$$

$$n^{d-2}\sum_{i\in J_n} L^d(a_{i,n})^2 \to 0, \tag{4.4b}$$

$$n^{d\,q-d}\sum_{i\in J_n} L^d(a_{i,n})^q \leq C\,\|\nu\|_{L^p(\Omega)}^p, \tag{4.4c}$$

where $q = p/(2-p)$.

Proof. We start with (4.4a). Using the definition (4.1) and $\operatorname{vol}(P_i^n) = (2/n)^d$, we find

$$\begin{aligned}\sum_{i\in J_n} L^d(a_{i,n}) &= \sum_{i\in J_n}\operatorname{vol}(P_i^n)\operatorname{avg}(P_i^n, |\nu|^p)^{\frac{2}{p}-1} = \sum_{i\in J_n}\operatorname{vol}(P_i^n)^{2-\frac{2}{p}}\Big(\int_{P_i^n}|\nu|^p\,\mathrm{d}x\Big)^{\frac{2}{p}-1} \\ &= 2^{2\,d\,(1-\frac{1}{p})}\,n^{d\,(\frac{2}{p}-2)}\sum_{i\in J_n}\Big(\int_{P_i^n}|\nu|^p\,\mathrm{d}x\Big)^{\frac{2}{p}-1}\end{aligned}$$

Since $\frac{2}{p} - 1 \in (0,1)$, we can use Hölder's inequality to obtain

$$\begin{aligned}\sum_{i\in J_n} L^d(a_{i,n}) &\leq 2^{2\,d\,(1-\frac{1}{p})}\,n^{d\,(\frac{2}{p}-2)}\Big(\sum_{i\in J_n}\int_{P_i^n}|\nu|^p\,\mathrm{d}x\Big)^{\frac{2}{p}-1}\Big(\sum_{i\in J_n}1\Big)^{2-\frac{2}{p}} \\ &\leq C\,n^{d\,(\frac{2}{p}-2)+d\,(2-\frac{2}{p})}\,\|\nu\|_{L^p(\Omega)}^p \leq C\,\|\nu\|_{L^p(\Omega)}^p.\end{aligned}$$

This shows (4.4a).

Next, we verify (4.4b). We use (4.2) and (4.4a) and obtain

$$n^{d-2} \sum_{i\in J_n} L^d(a_{i,n})^2 \le n^{d-2} \sum_{i\in J_n} L^d(a_{i,n})\, 2^{d+1}\, n^{1-d} = 2^{d+1}\, n^{-1} \sum_{i\in J_n} L^d(a_{i,n}) \to 0.$$

Finally, we address (4.4c). Using (4.1) and $q\,(2/p-1)=1$ we get

$$n^{d\,q-d} \sum_{i\in J_n} L^d(a_{i,n})^q = n^{d\,q-d} \sum_{i\in J_n} \operatorname{vol}(P_i^n)^{q-1} \int_{P_i^n} |\nu|^p \,\mathrm{d}x \le 2^{d\,(q-1)}\, \|\nu\|^p_{L^p(\Omega)}.$$

As a next step, we verify that the conditions (H.1) to (H.5') of Theorem 2.1 are satisfied for the above choice of the perforated domain Ω_n. We are following the strategy of [Cioranescu, Murat, 1997, Theorem 2.2]. However, due to the variable size of the holes, the analysis is more involved.

We start by defining an appropriate $v_n \in H^1(\Omega)$. For $i \in J_n$ let $v_{i,n} \in H_0^1(\Omega)$ be defined as the solution to

$$\begin{aligned} v_{i,n} &= 1 \quad \text{in } T_i^n \\ -\Delta v_{i,n} &= 0 \quad \text{in } B_i^n \setminus T_i^n \\ v_{i,n} &= 0 \quad \text{in } \Omega \setminus B_i^n. \end{aligned}$$

Functions of this type are discussed in Lemma A.1 (b). Note that the requirements on $a_{i,n}$ in this lemma are satisfied by Lemma 4.1 for $n \in \mathbb{N}$ large enough. We then define

$$v_n := 1 - \sum_{i\in J_n} v_{i,n}.$$

The next two lemmas show that (H.1) to (H.5') are satisfied.

Lemma 4.5. The conditions (H.1)–(H.3) are satisfied by the above choice of $\{v_n\}_{n\in\mathbb{N}}$.

Proof. We start by proving (H.1). Because of $0 \le v_n \le 1$ it suffices to calculate $\|\nabla v_n\|^2_{L^2(\Omega)^d}$. Due to Lemma A.1 (b) and (4.4a) we have

$$\|\nabla v_n\|^2_{L^2(\Omega)^d} \le \sum_{i\in J_n} \|v_{i,n}\|^2_{H_0^1(\Omega)} \le C \sum_{i\in J_n} L^d(a_{i,n}) \le C$$

which shows that the sequence $\{v_n\}_{n\in\mathbb{N}}$ is bounded in $H^1(\Omega)$.

The condition (H.2) follows directly from our choice of v_n.

Finally, we want to show that $\{v_n\}_{n\in\mathbb{N}}$ satisfies (H.3). We check that $v_n \to 1$ in $L^1(\Omega)$. Indeed,

$$\|v_n - 1\|_{L^1(\Omega)} = \sum_{i\in J_n} \int_{B_i^n} |v_{i,n}| \,\mathrm{d}x \le C_d \sum_{i\in J_n} \frac{1}{n} L^d(a_{i,n}) \to 0,$$

where we used (A.5) and (4.4a). Together with the boundedness of $\{v_n\}_{n\in\mathbb{N}}$ in $H^1(\Omega) \hookrightarrow L^1(\Omega)$ and the reflexivity of $H^1(\Omega)$, this implies $v_n \rightharpoonup 1$ in $H^1(\Omega)$. This shows (H.3).

We remark that (4.4a) shows that the capacity of the holes $\bigcup_{i\in J_n} T_i^n$ remains bounded. Indeed, the function $1 - v_n$ from the proof can be used in (2.1) and we obtain

$$\operatorname{cap}\Big(\bigcup_{i\in J_n} T_i^n\Big) \le \|1 - v_n\|^2_{H_0^1(\Omega)} \le C.$$

Lemma 4.6. The conditions (H.4'), (H.5') are satisfied by the above choice of $\{v_n\}_{n\in\mathbb{N}}$ and some sequences $\{\mu_n\}_{n\in\mathbb{N}}$ and $\{\gamma_n\}_{n\in\mathbb{N}}$. In particular, we have $\mu = C_d\,|\nu|^{2-p}$, with $C_d = \max(1, d-2)S_d$.

Proof. First, we prove (H.5'). We note that Δv_n only acts on the boundaries ∂T_i^n and ∂B_i^n. We set $\gamma_n, \mu_n \in H^{-1}(\Omega)$ such that $-\Delta v_n = \mu_n - \gamma_n$ and μ_n only acts on ∂B_i^n whereas γ_n only acts on ∂T_i^n. Then it can be seen that the condition $\langle \gamma_n, z_n\rangle = 0$ is true for all $z_n \in H_0^1(\Omega_n)$. It is possible to explicitly calculate μ_n. We denote by $\delta_{i,n} \in H^{-1}(\Omega)$ the surface measure on ∂B_i^n, i.e.,

$$\langle \delta_{i,n}, f\rangle = \int_{\partial B_i^n} f(s)\,\mathrm{d}s \quad \forall f \in C_0^\infty(\Omega).$$

Then, using integration by parts and (A.6), it turns out that

$$\mu_n = \sum_{i\in J_n} \frac{\partial v_n}{\partial n}\bigg|_{\partial B_i^n} \delta_{i,n} = -\sum_{i\in J_n} \frac{\partial v_{i,n}}{\partial n}\bigg|_{\partial B_i^n} \delta_{i,n} = \sum_{i\in J_n} \frac{1}{n\,d}\alpha_{i,n}\delta_{i,n} \tag{4.5}$$

with real-valued coefficients

$$\alpha_{i,n} := \frac{\max(1, d-2)\, n^d\, d}{L^d(a_{i,n})^{-1} - L^d(\frac{1}{n})^{-1}}. \tag{4.6}$$

For later use we note that Lemma 4.1 (b) implies the existence of a constant C independent of i and n such that

$$0 \le \alpha_{i,n} \le C\, n^d\, L^d(a_{i,n}). \tag{4.7}$$

Now we introduce the function $z_{i,n}$ for $i \in J_n$ as the solution of the equation

$$-\Delta z_{i,n} = \alpha_{i,n} \quad \text{in } B_i^n, \qquad\qquad z_{i,n} = 0 \quad \text{on } \Omega \setminus B_i^n.$$

This function can be calculated explicitly and we find

$$z_{i,n}(x) = \frac{\alpha_{i,n}}{2\,d}\big(n^{-2} - |x - x_i^n|^2\big)\ \forall x \in B_i^n, \qquad -\Delta z_{i,n} = \alpha_{i,n}\chi_{B_i^n} - \frac{1}{n\,d}\alpha_{i,n}\delta_{i,n}. \tag{4.8}$$

For the $H_0^1(\Omega)$-norm of $z_{i,n}$ we have

$$\|z_{i,n}\|^2_{H_0^1(\Omega)} = C\,\alpha_{i,n}^2 n^{-d-2}$$

and due to the orthogonality we have

$$\Big\|\sum_{i\in J_n} z_{i,n}\Big\|^2_{H_0^1(\Omega)} = \sum_{i\in J_n}\|z_{i,n}\|^2_{H_0^1(\Omega)} = C\sum_{i\in J_n}\alpha_{i,n}^2 n^{-d-2} \le C\sum_{i\in J_n} L^d(a_{i,n})^2\, n^{d-2} \to 0,$$

due to (4.7) and (4.4b). Hence, (4.5) and (4.8) imply

$$\mu_n - \sum_{i\in J_n}\alpha_{i,n}\chi_{B_i^n} = \Delta\Big(\sum_{i\in J_n} z_{i,n}\Big) \to 0 \quad \text{in } H^{-1}(\Omega) \quad (n\to\infty).$$

Using Lemma 4.7 below yields $\mu_n \to \mu$ in $H^{-1}(\Omega)$, where $\mu := C_d\,|\nu|^{2-p}$. Finally, $\gamma_n \rightharpoonup \mu$ follows from $-\Delta v_n \rightharpoonup 0$ and $\mu_n \to \mu$, which completes the proof of (H.5').

Now, $\mu = C_d\,|\nu|^{2-p}$, $\nu \in L^p(\Omega)$ and the bounds on p imply

$$\mu \in L^{p/(2-p)}(\Omega) \subset \begin{cases} W^{-1,2+\varepsilon}(\Omega) & \text{if } d = 2,\\ W^{-1,d}(\Omega) & \text{if } d \ge 3\end{cases}$$

for some $\varepsilon > 0$. Thus the remaining condition (H.4') follows.

It remains to check the announced convergence of μ_n towards $\mu = C_d\,|\nu|^{2-p}$.

Lemma 4.7. Let $\alpha_{i,n}$ be defined as in (4.6). Then

$$\sum_{i\in J_n}\alpha_{i,n}\chi_{B_i^n} \to C_d\,|\nu|^{2-p} \quad \text{in } H^{-1}(\Omega),$$

where $C_d = \max(1, d-2) S_d$ is a constant.

Proof. We will proof this by showing the weak convergence in $L^q(\Omega)$, where $q = \frac{p}{2-p}$. Indeed, the boundedness follows from

$$\Big\|\sum_{i\in J_n}\alpha_{i,n}\chi_{B_i^n}\Big\|^q_{L^q(\Omega)} = \sum_{i\in J_n}\alpha_{i,n}^q \operatorname{vol}(B_i^n) \le C\sum_{i\in J_n} n^{q\,d} L^d(a_{i,n})^q\, n^{-d} \le C\,\|\nu\|^p_{L^p(\Omega)},$$

where the last two inequalities are due to (4.7) and (4.4c), respectively.

Now it is sufficient to show the weak convergence on the dense subset $C_0^\infty(\Omega) \subset L^q(\Omega)^\star$. Let $f \in C_0^\infty(\Omega)$ be given.

Due to the definition of $\alpha_{i,n}$ and Lemma 4.1 (b) we have

$$|\alpha_{i,n} - \max(1, d-2)\, d\, n^d\, L^d(a_{i,n})| \le (C_n - 1)\, \max(1, d-2)\, d\, n^d\, L^d(a_{i,n}),$$

where $\{C_n\}_{n\in\mathbb{N}}$ is a sequence of constants such that $C_n \to 1$. It follows that

$$\begin{aligned}\Big\| \sum_{i\in J_n} \alpha_{i,n}\chi_{B_i^n} - \max(1,d-2)d\,n^d \sum_{i\in J_n} L^d(a_{i,n})\chi_{B_i^n}\Big\|_{L^q(\Omega)}^q \\ \le C\,(C_n-1)^q \sum_{i\in J_n} n^{dq-d} L^d(a_{i,n})^q \le C\,(C_n-1)^q \|\nu\|_{L^p(\Omega)}^p \to 0,\end{aligned} \tag{4.9}$$

where we used (4.4c) again. Now using Lemma 4.1 (c) we also have

$$\Big\| \sum_{i\in I_n\setminus J_n} \operatorname{avg}(P_i^n, |\nu|^p)^{\frac{2}{p}-1}\chi_{B_i^n}\Big\|_{L^q(\Omega)}^q \le \int_{\Omega\setminus\bigcup_{i\in J_n} P_i^n} |\nu|^p \,\mathrm{d}x \to 0$$

as $n\to\infty$. By combining this with (4.9) and the definition (4.1) of $a_{i,n}$ we arrive at

$$\sum_{i\in J_n} \alpha_{i,n}\chi_{B_i^n} - \max(1,d-2)d\,2^d \sum_{i\in I_n} \operatorname{avg}(P_i^n, |\nu|^p)^{\frac{2}{p}-1}\chi_{B_i^n} \to 0$$

in $L^q(\Omega)$. Using the uniform continuity of f (similar to the proof of Lemma 4.3) it is possible to replace $\chi_{B_i^n}$ with $\chi_{P_i^n}$, i.e.

$$\Big\langle \sum_{i\in J_n} \alpha_{i,n}\chi_{B_i^n} - \max(1,d-2)S_d \sum_{i\in I_n} \operatorname{avg}(P_i^n, |\nu|^p)^{\frac{2}{p}-1}\chi_{P_i^n},\, f\Big\rangle \to 0,$$

where we used that $2^{-d}d^{-1}S_d = \frac{\operatorname{vol}(B_i^n)}{\operatorname{vol}(P_i^n)}$, and $d^{-1}S_d$ is the volume of the d-dimensional unit ball. Now we apply Lemma A.2 (b) to $g = |\nu|^p$. As a consequence, we have

$$C_d \sum_{i\in I_n} \operatorname{avg}(P_i^n, |\nu|^p)^{\frac{2}{p}-1}\chi_{P_i^n} \to C_d|\nu|^{2-p}$$

in $L^q(\Omega)$ with the constant $C_d = \max(1, d-2)S_d$. Combined with the calculations above, we have

$$\Big\langle \sum_{i\in J_n} \alpha_{i,n}\chi_{B_i^n} - C_d|\nu|^{2-p},\, f\Big\rangle \to 0.$$

The boundedness in $L^q(\Omega)$ of $\sum_{i\in J_n} \alpha_{i,n}\chi_{B_i^n}$ and the compact embedding into $H^{-1}(\Omega)$ (which follows from $q > p$) completes the proof.

We note that the choice

$$\nu \equiv \big(2^d\,C_0\big)^{-1/(2-p)} \text{ if } d = 2, \qquad \nu \equiv \big(2^d\,C_0^{2-d}\big)^{-1/(2-p)} \text{ if } d \ge 3$$

yields the same size of the holes as in [Cioranescu, Murat, 1997, (2.4)] and we obtain the same value of μ, cf. [Cioranescu, Murat, 1997, (2.3)].

Now, the assumptions of Theorem 2.1 are satisfied and, by arguing as in the first lines of the proof Theorem 3.4, we obtain the following corollary.

Corollary 4.8. Let $w \in H_0^1(\Omega)$ be given. Then there exists a sequence $\{w_n\}_{n\in\mathbb{N}}$ with $w_n \in H_0^1(\Omega_n) \subset H_0^1(\Omega)$ such that $w_n \rightharpoonup w$ in $H_0^1(\Omega)$.

By using the same arguments as in the proof of Theorem 3.4, one can show $L^p(\Omega) \times H_0^1(\Omega) \subset \mathcal{N}_{\mathbb{K}}(0,0)$. The next theorem shows a more general result.

Theorem 4.9. Let $p \in (1,2)$ such that $L^p(\Omega) \hookrightarrow H^{-1}(\Omega)$ and let $(\nu, w) \in L^p(\Omega)\times H_0^1(\Omega)$ be given. Then the equivalence

$$(\nu, w) \in \mathcal{N}_{\mathbb{K}}(\bar{y}, \bar{\lambda}) \quad \Longleftrightarrow \quad (\nu, w) \in \mathcal{N}_{\mathbb{K}}^{\text{weak}}(\bar{y}, \bar{\lambda})$$

holds for every $(\bar{y}, \bar{\lambda}) \in \mathbb{K}$.

Proof. The implication "$\Rightarrow$" follows directly from (2.8) and it remains to check "$\Leftarrow$". Therefore, let $(\bar{y}, \bar{\lambda}) \in \mathbb{K}$ be given. As in (2.5) we define the sets $\mathcal{A}_s :=$ f-supp$(\bar{\lambda})$, $\mathcal{I} := \{\bar{y} > 0\}$, and $\mathcal{A} := \{\bar{y} = 0\}$. Due to $\bar{y} = 0$ q.e. on $\mathcal{A}_s$ we can enforce $\bar{y} = 0$ everywhere on $\mathcal{A}_s$. This implies $\mathcal{A}_s \cap \mathcal{I} = \emptyset$. Note that the set $\mathcal{I}$ is quasi-open.

Now, suppose that $(\nu, w) \in \mathcal{N}^{\text{weak}}(\bar{y}, \bar{\lambda})$. As a reminder, $\mathcal{N}_{\mathbb{K}}^{\text{weak}}(\bar{y}, \bar{\lambda})$ was introduced as

$$\mathcal{N}_{\mathbb{K}}^{\text{weak}}(\bar{y}, \bar{\lambda}) := \{z \in H_0^1(\Omega) : z = 0 \text{ q.e. on } \mathcal{A}\}^\circ \times \{z \in H_0^1(\Omega) : z = 0 \text{ q.e. on } \mathcal{A}_s\}$$

in (2.6). It can be shown that $\nu = 0$ a.e. on $\mathcal{I}$. In fact, Lemma A.3 implies $\langle \nu, z\rangle_{L^p(\Omega),L^{p'}(\Omega)} = 0$ for all $z \in L^{p'}(\Omega)$ with $z = 0$ a.e. on $\mathcal{A}$. Here, $p' \in (2, \infty)$ is the exponent conjugate to p, i.e., $1 = 1/p + 1/p'$.

It well be convenient to work with open sets. Therefore, let $\varepsilon > 0$ be given. Because $\mathcal{I}$ is quasi-open, there exists an open set G_ε, such that $\mathcal{I} \cup G_\varepsilon$ is open and $\operatorname{cap}(G_\varepsilon) < \varepsilon$.

The remaining part of the proof is divided into several steps. In steps 1 and 2, we use Corollary 4.8 and Lemma 4.3 to construct approximations w_n to w and $\nu_{n,\varepsilon}$ to ν. The functions w_n will vanish on the holes, whereas $\nu_{n,\varepsilon}$ is supported only on the holes. In step 3, we construct an approximation to $\bar{y}$, which vanishes on the support of $\nu_{n,\varepsilon}$. Afterwards, we find a point in $\mathbb{K}$ such that $(\nu_{n,\varepsilon}, w_n)$ belongs to the Fréchet normal cone in this point, cf. steps 4 and 5. Finally, we pick a diagonal sequence in step 6 and conclude.

Step 1 (Construction of w_n)**:** Applying Corollary 4.8 yields the existence of a sequence $\{\tilde{w}_n\}_{n\in\mathbb{N}}$ with $\tilde{w}_n \in H_0^1(\Omega_n) \subset H_0^1(\Omega)$ and $\tilde{w}_n \rightharpoonup w$ in $H_0^1(\Omega)$. Next, we define $w_n \in H_0^1(\Omega)$ by $w_n := \max(\min(\tilde{w}_n, w^+), -w^-)$. From [G. Wachsmuth, 2016, Lemma 4.1] we know that max and min are weakly sequentially continuous from $H_0^1(\Omega) \times H_0^1(\Omega)$ to $H_0^1(\Omega)$. It follows that $w_n \rightharpoonup w$. Moreover, we have $\{w_n \neq 0\} \subset \{\tilde{w}_n \neq 0\} \subset \Omega_n$ and $\mathcal{A}_s \subset \{w = 0\} \subset \{w_n = 0\}$. By [G. Wachsmuth, 2014, Theorem A.5] it follows that $w = 0$ $\bar{\lambda}$-a.e. This implies $w_n^\pm = 0$ $\bar{\lambda}$-a.e., hence

$$\langle \bar{\lambda}, w_n^\pm \rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} = \int_\Omega w_n^\pm \,\mathrm{d}\bar{\lambda} = 0. \tag{4.10}$$

Step 2 (Construction of $\nu_{n,\varepsilon}$)**:** We define $\nu_\varepsilon := \nu\chi_{\Omega\setminus G_\varepsilon}$ and

$$\nu_{n,\varepsilon} := \sum_{i\in J_n} \chi_{T_i^n} \frac{1}{\operatorname{vol}(T_i^n)} \int_{P_i^n} \nu_\varepsilon \,\mathrm{d}x.$$

According to Lemma 4.3, $\nu_{n,\varepsilon} \rightharpoonup \nu_\varepsilon$ as $n \to \infty$ in $H^{-1}(\Omega)$. Moreover, we have

$$\|\nu_{n,\varepsilon}\|_{H^{-1}(\Omega)} + \|\nu_{n,\varepsilon}^+\|_{H^{-1}(\Omega)} \le C\,\|\nu\|_{L^p(\Omega)}^{\frac{p}{2}} + C\,\|\nu\|_{L^p(\Omega)} \tag{4.11}$$

for a constant $C > 0$ by applying Lemma 4.2 twice.

Step 3 (Construction of $\bar y_{n,\varepsilon}$)**:** Now we will argue that we can choose a sequence $\{\bar y_{n,\varepsilon}\}_{n\in\mathbb{N}} \subset H_0^1(\Omega)$ such that

$$0 \le \bar y_{n,\varepsilon} \le \bar y, \tag{4.12a}$$

$$\lim_{n\to\infty} \bar y_{n,\varepsilon} = \bar y, \tag{4.12b}$$

$$\{\bar y_{n,\varepsilon} > 0\} \subset \bigcup_{i:P_i^n \subset \mathcal{I}\cup G_\varepsilon} P_i^n. \tag{4.12c}$$

Indeed, this is possible: Because of $\bar y \in H_0^1(\mathcal{I}\cup G_\varepsilon)$ and the fact that $C_0^\infty(\mathcal{I}\cup G_\varepsilon)$ is dense in $H_0^1(\mathcal{I}\cup G_\varepsilon)$ there exists a sequence $\{\tilde y_{n,\varepsilon}\}_{n\in\mathbb{N}}$ in $C_0^\infty(\mathcal{I}\cup G_\varepsilon)$ such that $\lim_{n,\to\infty} \tilde y_{n,\varepsilon} = \bar y$ and $\{\tilde y_{n,\varepsilon} > 0\} + B_{\frac{2\sqrt{d}}{n}}(0) \subset \mathcal{I} \cup G_\varepsilon$. The last condition implies

$$\{\tilde y_{n,\varepsilon} > 0\} \subset \bigcup_{i:P_i^n \subset \mathcal{I}\cup G_\varepsilon} P_i^n.$$

Then we define $\bar y_{n,\varepsilon} := \max(\min(\bar y, \tilde y_{n,\varepsilon}), 0)$, and we get (4.12a). Because max and min are continuous in $H_0^1(\Omega)$, we also have $\lim_{n\to\infty} \bar y_{n,\varepsilon} = \bar y$. The remaining condition follows from $\{\bar y_{n,\varepsilon} > 0\} \subset \{\tilde y_{n,\varepsilon} > 0\}$. This yields a sequence $\{\bar y_{n,\varepsilon}\}_{n\in\mathbb{N}}$ satisfying (4.12).

Step 4 (Construction of $(y_{n,\varepsilon}, \lambda_{n,\varepsilon}) \in \mathbb{K}$)**:** In a next step, we define $y_{n,\varepsilon} := \bar y_{n,\varepsilon} + \frac{1}{n} w_n^- \ge 0$ and $\lambda_{n,\varepsilon} := \bar\lambda - \frac{1}{n}\nu_{n,\varepsilon}^+ \le 0$. In order to show that this pair belongs to $\mathbb{K}$, it remains to check

$$\langle \lambda_{n,\varepsilon}, y_{n,\varepsilon}\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} = \langle\bar\lambda,\, \bar y_{n,\varepsilon}\rangle + \frac{1}{n}\langle\bar\lambda,\, w_n^-\rangle - \frac{1}{n}\langle\nu_{n,\varepsilon}^+,\, \bar y_{n,\varepsilon}\rangle - \frac{1}{n^2}\langle\nu_{n,\varepsilon}^+,\, w_n^-\rangle \stackrel{!}{=} 0. \tag{4.13}$$

The first term vanishes due to $0 = \langle\bar\lambda,\, \bar y\rangle \le \langle\bar\lambda,\, \bar y_{n,\varepsilon}\rangle \le 0$, where we used $\bar\lambda \le 0$ and (4.12a). The second terms is zero due to (4.10). The function $\nu_{n,\varepsilon}$ can only be non-zero on holes T_i^n that belong to cubes P_i^n with $P_i^n \cap (\mathcal{A}\setminus G_\varepsilon) \neq \emptyset$. Thus, using that $\bar y_{n,\varepsilon} = 0$ on these P_i^n, cf. (4.12c), the third term vanishes. Finally, the last term disappears since $\nu_{n,\varepsilon}^+$ only lives on the holes and w_n^- vanishes there. This shows (4.13). Together with the signs of $y_{n,\varepsilon}$ and $\lambda_{n,\varepsilon}$, we have $(y_{n,\varepsilon}, \lambda_{n,\varepsilon}) \in \mathbb{K}$.

Step 5 (Verification of $(\nu_{n,\varepsilon}, w_n) \in \widehat{\mathcal{N}}_{\mathbb{K}}(y_{n,\varepsilon}, \lambda_{n,\varepsilon})$)**:** In face of (2.4), we have to show $\nu_{n,\varepsilon} \in \mathcal{K}_K(y_{n,\varepsilon}, \lambda_{n,\varepsilon})^\circ$ and $w_n \in \mathcal{K}_K(y_{n,\varepsilon}, \lambda_{n,\varepsilon})$. By using arguments similar to those that led to (4.13) we find $\langle\lambda_{n,\varepsilon},\, w_n\rangle = 0$. Together with $\bar y_{n,\varepsilon}, w_n^+ \ge 0$ this yields

$$w_n = n\left(\bar y_{n,\varepsilon} + \frac{1}{n} w_n^+ - y_{n,\varepsilon}\right) \in \mathcal{T}_K(y_{n,\varepsilon}) \cap \lambda_{n,\varepsilon}^\perp = \mathcal{K}_K(y_{n,\varepsilon}, \lambda_{n,\varepsilon}).$$

In order to show $\nu_{n,\varepsilon} \in \mathcal{K}_K(y_{n,\varepsilon}, \lambda_{n,\varepsilon})^\circ$, let $z \in K \cap \lambda_{n,\varepsilon}^\perp$ be given. Similar to the derivation of (4.13), we find $\langle \nu_{n,\varepsilon},\, y_{n,\varepsilon} \rangle = 0$. From $z \in K \cap \lambda_{n,\varepsilon}^\perp$, $\lambda_{n,\varepsilon} = \bar\lambda - \frac{1}{n}\, \nu_{n,\varepsilon}^+$, and $\bar\lambda, -\frac{1}{n}\, \nu_{n,\varepsilon}^+ \le 0$ we have $\langle \nu_{n,\varepsilon}^+,\, z \rangle = 0$. Thus,

$$\langle \nu_{n,\varepsilon}, z - y_{n,\varepsilon} \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} = \langle \nu_{n,\varepsilon}, z \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} = \left\langle -\nu_{n,\varepsilon}^-, z \right\rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} \le 0,$$

where we used $z \ge 0$ and $\nu_{n,\varepsilon}^- \ge 0$ in the last step. Since z was arbitrary, we find $\nu_{n,\varepsilon} \in (K \cap \lambda_{n,\varepsilon}^\perp - y_{n,\varepsilon})^\circ = (\mathcal{R}_K(y_{n,\varepsilon}) \cap \lambda_{n,\varepsilon}^\perp)^\circ$. Using the polyhedricity of K, it follows that $\nu_{n,\varepsilon} \in \mathcal{K}_K(y_{n,\varepsilon}, \lambda_{n,\varepsilon})^\circ$.

Step 6 (Choice of a diagonal sequence)**:** Finally, we have to choose a sequence of indices $\{(n_k, \varepsilon_k)\}_{k\in\mathbb{N}}$ such that

$$y_k := y_{n_k,\varepsilon_k} \to \bar y, \qquad \lambda_k := \lambda_{n_k,\varepsilon_k} \to \bar\lambda, \qquad w_k := w_{n_k} \rightharpoonup w, \qquad \nu_k := \nu_{n_k,\varepsilon_k} \rightharpoonup \nu.$$

Let $\{\varepsilon_k\}_{k\in\mathbb{N}}$ be a sequence with $\varepsilon_k > 0$ and $\varepsilon_k \to 0$. Then, we have

$$\left\|\nu - \nu_{\varepsilon_k}\right\|_{H^{-1}(\Omega)} = \left\|\nu \chi_{G_{\varepsilon_k}}\right\|_{H^{-1}(\Omega)} \le C \left\|\nu \chi_{G_{\varepsilon_k}}\right\|_{L^p(\Omega)} = C \left(\int_{G_{\varepsilon_k}} |\nu|^p \,\mathrm{d}x \right)^{1/p},$$

which converges to 0 as $\varepsilon \to 0$ since $\operatorname{vol}(G_{\varepsilon_k}) \to 0$, which follows from $\operatorname{cap}(G_{\varepsilon_k}) \to 0$, see (2.3).

Because $H_0^1(\Omega)$ is separable, we can find a sequence $\{z_m\}_{m\in\mathbb{N}}$ that is dense in $H_0^1(\Omega)$. We have $\nu_{n,\varepsilon_k} \rightharpoonup \nu_{\varepsilon_k}$ and $\bar y_{n,\varepsilon_k} \to \bar y$ as $n \to \infty$ for fixed k by steps 2 and 3. Therefore, we can choose $n_k \ge k$ in such a way that the conditions

$$\|\bar y_{n_k,\varepsilon_k} - \bar y\|_{H_0^1(\Omega)} < \varepsilon_k \quad \text{and} \quad \left|\langle \nu_{n_k,\varepsilon_k} - \nu_{\varepsilon_k}, z_m \rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}\right| < \varepsilon_k \quad \forall m \le k$$

are satisfied. From the boundedness of w_n^-, we conclude $y_{n_k,\varepsilon_k} = \bar y_{n_k,\varepsilon_k} + \frac{1}{n}\, w_{n_k}^- \to \bar y$. Further, it follows that

$$\lim_{k\to\infty} \langle \nu_{n_k,\varepsilon_k} - \nu, z_m \rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} = 0 \qquad \forall m \in \mathbb{N}.$$

Since ν_{n_k,ε_k} is also bounded, cf. (4.11), and $\{z_m\}_{m\in\mathbb{N}}$ is dense in $H_0^1(\Omega)$, it follows that $\nu_{n_k,\varepsilon_k} \rightharpoonup \nu$. The convergence $\lambda_{n_k,\varepsilon_k} \to \bar\lambda$ follows from $n_k \ge k$ and the boundedness of $\|\nu_{n_k,\varepsilon_k}^+\|_{H^{-1}(\Omega)}$, cf. (4.11). Finally, $w_{n_k} \rightharpoonup w$ follows from step 1.

Step 7 (Conclusion)**:** From steps 4 to 6, we find

$$\begin{aligned}
(y_k, \lambda_k) \in \mathbb{K}, \qquad & y_k \to y \text{ in } H_0^1(\Omega), \qquad & w_k \rightharpoonup w \text{ in } H_0^1(\Omega), \qquad & w_k \in \mathcal{K}_K(y_k, \lambda_k), \\
& \lambda_k \to \lambda \text{ in } H^{-1}(\Omega), \qquad & \nu_k \rightharpoonup \nu \text{ in } H^{-1}(\Omega), \qquad & \nu_k \in \mathcal{K}_K(y_k, \lambda_k)^\circ.
\end{aligned}$$

Hence, $(\nu, w) \in \mathcal{N}_{\mathbb{K}}(\bar y, \bar\lambda)$.

5. Measure on a hyperplane

In the following example we want to show that the limiting normal cone in $(0,0)$ contains elements from $H^{-1}(\Omega) \times H_0^1(\Omega)$ where the $H^{-1}(\Omega)$-component is not a function. In particular, the limiting normal cone is strictly larger than $L^p(\Omega) \times H_0^1(\Omega)$, which was our lower estimate from Theorem 4.9.

Example 5.1. We choose $d = 2$ and $\Omega = (-1,1)^2$. Let $\delta \in H^{-1}(\Omega)$ be defined as

$$\langle \delta, f \rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} = \int_{(-1,1)} f(s,0)\,\mathrm{d}s \quad \forall f \in C_0^\infty(\Omega).$$

For every $w \in H_0^1(\Omega)$, we have

$$(\delta, w) \in \mathcal{N}_{\mathbb{K}}(0,0).$$

Note that δ can be written as a derivative of an $L^2(\Omega)$ function, hence, $\delta \in H^{-1}(\Omega)$. We will proceed similarly to the proof of Lemma 4.2.

Proof. First, we cover the line $(-1,1) \times \{0\}$ with squares $P_i^n := [\frac{2i}{n}, \frac{2i+2}{n}] \times [-\frac{1}{n}, \frac{1}{n}]$. Again, at the center of each square there is a hole T_i^n with radius

$$a_{i,n} = \exp(-n), \tag{5.1}$$

and another ball B_i^n with radius $\frac{1}{n}$. We choose $J_n := \{i \in \mathbb{Z} : P_i^n \subset \Omega\}$. As before, we define $\Omega_n := \Omega \setminus \bigcup_{i\in J_n} T_i^n$.

We start by defining a sequence $\nu_n \in H^{-1}(\Omega)$ and showing $\nu_n \rightharpoonup \delta$ in $H^{-1}(\Omega)$. In particular, we set

$$\nu_n := \sum_{i\in J_n} \chi_{T_i^n} \beta_{i,n}$$

with $\beta_{i,n} = \frac{1}{\pi a_{i,n}^2 n}$. First, we show that $\|\nu_n\|_{H^{-1}(\Omega)}$ is bounded. As in the proof of Lemma 4.2 we define functions $u_{i,n}$ as the solution of

$$-\Delta u_{i,n} = \chi_{T_i^n} - a_{i,n}^2 n^2 \chi_{B_i^n} \quad \text{in}\,\Omega, \quad u_{i,n} \in H_0^1(\Omega).$$

By the triangle inequality and since $-\Delta$ is an isometry, we have

$$\|\nu_n\|_{H^{-1}(\Omega)} \le \Big\| \sum_{i\in J_n} \beta_{i,n} u_{i,n} \Big\|_{H_0^1(\Omega)} + \Big\| \sum_{i\in J_n} a_{i,n}^d n^d \beta_{i,n} \chi_{B_i^n} \Big\|_{H^{-1}(\Omega)}.$$

For the first term we have

$$\Big\| \sum_{i\in J_n} \beta_{i,n} u_{i,n} \Big\|_{H_0^1(\Omega)}^2 = \sum_{i\in J_n} |\beta_{i,n}|^2 \|u_{i,n}\|_{H_0^1(\Omega)}^2 \le C \sum_{i\in J_n} a_{i,n}^{2\,d}\, L^d(a_{i,n})^{-1}\, \beta_{i,n}^2$$

$$\leq C \sum_{i\in J_n} L^d(a_{i,n})^{-1} \frac{1}{n^2} \leq C \sum_{i\in J_n} \frac{1}{n} \leq C,$$

where we used $\|u_{i,n}\|^2_{H_0^1(\Omega)} \leq C\, L^d(a_{i,n})^{-1} a_{i,n}^{2d}$ from Lemma A.1 (a). For the other term we have

$$\Big\| \sum_{i\in J_n} a_{i,n}^d n^d \beta_{i,n} \chi_{B_i^n} \Big\|_{H^{-1}(\Omega)} = \Big\| \pi n \sum_{i\in J_n} \chi_{B_i^n} \Big\|_{H^{-1}(\Omega)}.$$

To calculate this norm, let $z \in H_0^1(\Omega)$ with $z \geq 0$. We define the translation $(z \circ \tau_{\frac{2}{n}})(x, y) := z(x, y + \frac{2}{n})$. We have

$$\begin{aligned} 0 \leq \Big\langle n \sum_{i\in J_n} \chi_{B_i^n}, z \Big\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} &= n \sum_{i\in J_n} \int_{B_i^n} z \,\mathrm{d}x \leq n \int_{(-1,1)\times[-\frac{1}{n},\frac{1}{n}]} z \,\mathrm{d}x \\ &= \int_{(-1,1)\times[-\frac{1}{n},1]} n(z - z \circ \tau_{\frac{2}{n}}) \,\mathrm{d}x \\ &\leq 2 \,\|n(z - z \circ \tau_{\frac{2}{n}})\|_{L^2(\Omega)} \leq 4 \,\|z\|_{H_0^1(\Omega)}, \end{aligned}$$

where the last inequality is the characterization of Sobolev spaces by finite differences, cf. [Dobrowolski, 2010, Satz 5.22]. Thus we have shown that ν_n is bounded in $H^{-1}(\Omega)$.

With this boundedness it is easy to prove that $\nu_n \rightharpoonup \delta$. This can be done in the same way as in the proof of Lemma 4.3.

Now let $w \in H_0^1(\Omega)$ be given. We define w_n as the solution of

$$-\Delta w_n = -\Delta w + \pi \delta w \quad \text{in}\, \Omega_n, w_n \in H_0^1(\Omega_n).$$

Then, according to [Cioranescu, Murat, 1997, Theorem 2.10] the conditions for Theorem 2.1 are satisfied, which implies $w_n \rightharpoonup w$ in $H_0^1(\Omega)$.

Finally, we define $y_n := \frac{1}{n} w_n^-$ and $\lambda_n := -\frac{1}{n}\nu_n$ and this implies $y_n \to 0$, $\lambda_n \to 0$, $(y_n, \lambda_n) \in \mathbb{K}$. By using arguments similar to those in the proof Theorem 3.4, we find $\nu_n \in \mathcal{K}_K(y_n, \lambda_n)^\circ$, and $w_n \in \mathcal{K}_K(y_n, \lambda_n)$. Thus we have shown that $(\delta, w) \in \mathcal{N}_{\mathbb{K}}(0, 0)$.

We note that the choice for the size of the holes in (5.1) is equivalent to $L^d(a_{i,n}) = \frac{1}{n}$. The same approach works also in higher dimensions $d \geq 3$, cf. [Cioranescu, Murat, 1997, Theorem 2.10]. In particular, for a constant measure δ that acts on a hyperplane in Ω and $w \in H_0^1(\Omega)$ we have $(\delta, w) \in \mathcal{N}_{\mathbb{K}}(0, 0)$.

6. Conclusion

We have established lower estimates for the limiting normal cone of the set $\mathbb{K}$. In particular, we have characterized the intersection with $L^p(\Omega) \times H_0^1(\Omega)$ for all p with

$L^p(\Omega) \hookrightarrow H^{-1}(\Omega)$. This intersection is unpleasantly large.

Our method of proof does not allow to handle $\nu \in H^{-1}(\Omega) \setminus L^p(\Omega)$. Therefore, we are not able to give a full characterization of the limiting normal cone. Similarly, there is no counterexample available which shows $\mathcal{N}_{\mathbb{K}}(\bar{y}, \bar{\lambda}) \neq \mathcal{N}_{\mathbb{K}}^{\text{weak}}(\bar{y}, \bar{\lambda})$.

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A. Auxiliary results

First, we will provide results for some rotationally invariant solutions of Poisson's equation. As in Section 4 we will use the helper function

$$L^d(a) := \begin{cases} -\log(a)^{-1} & \text{if } d = 2, \\ a^{d-2} & \text{if } d \geq 3. \end{cases} \tag{A.1}$$

As before, we denote the surface measure of the d-dimensional unit sphere by S_d.

Lemma A.1. Let $B := B_b(0) \subset \Omega \subset \mathbb{R}^d$ be an open ball with radius $b \leq 1$ and $T := \overline{B_a(0)} \subset B$ be a closed ball with radius $a \in (0, b)$.

(a) We consider the problem

$$-\Delta u = \chi_T - a^d b^{-d} \chi_B, \qquad u \in H_0^1(\Omega).$$

The solution $u \in H_0^1(\Omega)$ vanishes on $\Omega \setminus B$ and, under the additional requirement $a < \frac{1}{e}$ in the case $d = 2$, we get the estimate

$$\|u\|^2_{H_0^1(\Omega)} \leq 5 S_d a^{2d} L^d(a)^{-1}. \tag{A.2}$$

(b) We consider the problem

$$u = 1 \text{ in } T, \quad -\Delta u = 0 \text{ in } B \setminus T, \quad u = 0 \text{ in } \Omega \setminus B.$$

Then there is a solution $u \in H_0^1(\Omega)$ with $0 \leq u \leq 1$. Under the additional requirements

$$a < b^2 \quad \text{if } d = 2 \qquad \text{and} \qquad a < \frac{b}{2} \quad \text{if } d \geq 3, \tag{A.3}$$

there is a constant C_d depending only on the dimension d, such that

$$\|u\|^2_{H_0^1(\Omega)} \leq C_d \, L^d(a) \tag{A.4}$$

$$\|u\|_{L^1(\Omega)} \le C_d\, b L^d(a). \tag{A.5}$$

Moreover, the (outer) normal derivative of u at ∂B is given by

$$\frac{\partial u}{\partial n}\Big|_{\partial B} = \frac{-\max(1, d-2) b^{1-d}}{L^d(a)^{-1} - L^d(b)^{-1}}. \tag{A.6}$$

Proof. For part (a), we can give an explicit solution of the partial differential equation. It turns out that the solution satisfies $u(x) = \tilde{u}(|x|)$, where

$$\tilde{u}(r) := \begin{cases} c_1 - \frac{1}{2d}(1 - a^d b^{-d}) r^2 & \text{if } 0 \le r < a, \\ c_2 + \frac{a^d b^{-d}}{2d} r^2 + c_3 \frac{1}{L^d(r)} & \text{if } a \le r < b, \\ 0 & \text{if } r \ge b \end{cases}$$

with coefficients $c_1, c_2 \in \mathbb{R}$, and $c_3 = \frac{a^d}{d \max(1, d-2)}$. The constants c_1, c_2 has to be chosen in such a way that $\tilde{u}$ is continuous. Note that our choice of c_3 guarantees that $\tilde{u}$ is continuously differentiable. For the norm of u we have

$$\begin{aligned} \|u\|^2_{H^1_0(\Omega)} &= \int_B |\nabla u|^2 \,\mathrm{d}x = S_d \int_0^b |\tilde{u}'(r)|^2 r^{d-1} \,\mathrm{d}r \\ &= \frac{S_d}{d^2}(1 - a^d b^{-d})^2 \int_0^a r^{d+1} \,\mathrm{d}r + S_d \int_a^b \Big(\frac{a^d b^{-d}}{d} r - c_3 \frac{\max(1, d-2)}{r^{d-1}} \Big)^2 r^{d-1} \,\mathrm{d}r \\ &\le S_d \int_0^a r^{d+1} \,\mathrm{d}r + 2 S_d a^{2d} b^{-2d} \int_0^b r^{d+1} \,\mathrm{d}r + 2 S_d a^{2d} \int_a^\infty r^{1-d} \,\mathrm{d}r \\ &\le S_d a^{d+2} + 2 S_d a^{2d} b^{2-d} + 2 S_d a^{2d} L^d(a)^{-1} \le 5 S_d a^{2d} L^d(a)^{-1}, \end{aligned}$$

where the last inequality uses $a < \frac{1}{e}$ in the case of $d = 2$. Thus we have shown (A.2).

For part (b) we can again give an explicit representation of u. Since u is rotationally invariant, we can write $u(x) = \tilde{u}(|x|)$ and find

$$\tilde{u}(r) := \begin{cases} 1 & \text{if } 0 \le r \le a, \\ \frac{L^d(r)^{-1} - L^d(b)^{-1}}{L^d(a)^{-1} - L^d(b)^{-1}} & \text{if } a < r < b, \\ 0 & \text{if } b \le r, \end{cases} \qquad \tilde{u}'(r) := \begin{cases} 0 & \text{if } 0 \le r \le a, \\ \frac{\max(1, d-2)\, r^{1-d}}{L^d(a)^{-1} - L^d(b)^{-1}} & \text{if } a < r < b, \\ 0 & \text{if } b \le r. \end{cases}$$

Additionally, (A.6) follows from $\frac{\partial u}{\partial n}\big|_{\partial B} = \lim_{r \uparrow b} \tilde{u}'(r)$. By using the above expression for $\tilde{u}'(r)$ and $\int_a^b \tilde{u}'(r) \,\mathrm{d}r = 1$, we find

$$\begin{aligned} \|u\|^2_{H^1_0(\Omega)} &= \int_{B \setminus T} |\nabla u|^2 \,\mathrm{d}x = S_d \int_a^b \tilde{u}'(r)^2 r^{d-1} \,\mathrm{d}r \\ &= \frac{S_d \max(1, d-2)}{L^d(a)^{-1} - L^d(b)^{-1}} \int_a^b \tilde{u}'(r) \,\mathrm{d}r = \frac{S_d \max(1, d-2)}{L^d(a)^{-1} - L^d(b)^{-1}}. \end{aligned}$$

By using (A.1) it can be shown that the requirements (A.3) imply the inequality

$$\frac{1}{L^d(a)^{-1} - L^d(b)^{-1}} \le 2L^d(a). \tag{A.7}$$

The claim (A.4) follows. Next, we calculate $\|u\|_{L^1(\Omega)}$. We have

$$\begin{aligned}\int_\Omega |u| \,\mathrm{d}x &= \int_{B\setminus T} u \,\mathrm{d}x + \mathrm{vol}(T) = S_d \int_a^b \tilde{u}(r) r^{d-1} \,\mathrm{d}r + \mathrm{vol}(T) \\ &\le \frac{S_d}{L^d(a)^{-1} - L^d(b)^{-1}} \int_a^b L^d(r)^{-1} r^{d-1} \,\mathrm{d}r + 2^d a^d.\end{aligned}$$

By calculating the integral for both the cases $d \ge 3$ and $d = 2$, it can be seen that $\int_a^b L^d(r)^{-1} r^{d-1} \,\mathrm{d}r \le b$. Therefore, using (A.7) and $a^d \le a L^d(a)$ results in

$$\int_\Omega |u| \,\mathrm{d}x \le \frac{S_d b}{L^d(a)^{-1} - L^d(b)^{-1}} + 2^d a^d \le C_d \, b \, L^d(a).$$

The next result shows that each function $g \in L^1(\Omega)$ can be approximated by simple functions given by local averages over small cubes.

Lemma A.2. For each $n \in \mathbb{N}$ let $\{P_i^n\}_{i \in I_n}$ be defined as in Section 3, i.e., each P_i^n is a translation of $[-\frac{1}{n}, \frac{1}{n}]^d$, the collection $\{P_i^n\}_{i \in I_n}$ covers Ω and is pairwise disjoint (up to sets of measure zero). We denote by $\mathrm{avg}(P_i^n, g) = \mathrm{vol}(P_i^n)^{-1} \int_{P_i^n} g \,\mathrm{d}x$ the average of $g \in L^1(\Omega)$, which is extended by zero outside of Ω, over P_i^n.

(a) Let $g \in L^1(\Omega)$ be given. Then

$$\sum_{i \in I_n} \mathrm{avg}(P_i^n, g) \chi_{P_i^n} \to g \quad \text{in } L^1(\Omega).$$

(b) Let $q \ge 1$ and $g \in L^1(\Omega)$ with $g \ge 0$ a.e. on Ω be given. Then

$$\sum_{i \in I_n} \mathrm{avg}(P_i^n, g)^{\frac{1}{q}} \chi_{P_i^n} \to g^{\frac{1}{q}} \quad \text{in } L^q(\Omega).$$

Proof. We start with part (a). Since $C_0^\infty(\Omega)$ is dense in $L^1(\Omega)$, we can find a sequence $\{\varphi_m\}_{m \in \mathbb{N}} \subset C_0^\infty(\Omega)$ such that $\varphi_m \to g$ in $L^1(\Omega)$. Because φ_m is uniformly continuous, the convergence

$$\sum_{i \in I_n} \mathrm{avg}(P_i^n, \varphi_m) \chi_{P_i^n} \to \varphi_m \quad (n \to \infty)$$

in $L^\infty(\Omega)$ and therefore in $L^1(\Omega)$ holds for all $m \in \mathbb{N}$. We have

$$\begin{aligned}\Big\|g - \sum_{i\in I_n} \operatorname{avg}(P_i^n, g)\chi_{P_i^n}\Big\|_{L^1(\Omega)} &\le \|g-\varphi_m\|_{L^1(\Omega)} + \Big\|\varphi_m - \sum_{i\in I_n} \operatorname{avg}(P_i^n, \varphi_m)\chi_{P_i^n}\Big\|_{L^1(\Omega)} \\ &\qquad + \Big\|\sum_{i\in I_n} \operatorname{avg}(P_i^n, g-\varphi_m)\chi_{P_i^n}\Big\|_{L^1(\Omega)} \\ &\le 2\,\|g-\varphi_m\|_{L^1(\Omega)} + \Big\|\varphi_m - \sum_{i\in I_n} \operatorname{avg}(P_i^n, \varphi_m)\chi_{P_i^n}\Big\|_{L^1(\Omega)}.\end{aligned}$$

Now, we can choose $m \in \mathbb{N}$ such that the first term becomes small and, afterwards, we can choose $n \in \mathbb{N}$ such that the second term is small. The convergence in $L^1(\Omega)$ follows.

Now we turn to the proof of part (b). For real numbers $a, b \ge 0$ we have the inequality

$$|a-b|^q \le |a^q - b^q|.$$

Indeed, w.l.o.g. $a \ge b$, and after some rearrangement, the inequality is equivalent to the well-known estimate $\|(b, a-b)\|_{\ell^q} \le \|(b, a-b)\|_{\ell^1}$. By applying this inequality, we get

$$\begin{aligned}\Big\|\sum_{i\in I_n} \operatorname{avg}(P_i^n, g)^{\frac1q}\chi_{P_i^n} - g^{\frac1q}\Big\|_{L^q(\Omega)}^q &\le \sum_{i\in I_n}\int_{P_i^n} |\operatorname{avg}(P_i^n, g)^{\frac1q} - g^{\frac1q}|^q \,\mathrm{d}x \\ &\le \sum_{i\in I_n}\int_{P_i^n} |\operatorname{avg}(P_i^n, g) - g| \,\mathrm{d}x \\ &= \Big\|\sum_{i\in I_n} \operatorname{avg}(P_i^n, g)\chi_{P_i^n} - g\Big\|_{L^1(\Omega)},\end{aligned}$$

which yields convergence according to (a).

Finally, we give a density result on quasi-open sets. Note that the result is classical for open sets. The definition of $H_0^1(\mathcal{D})$ below is motivated by (2.2).

Lemma A.3. Let $\mathcal{D} \subset \Omega$ be quasi-open. Then

$$H_0^1(\mathcal{D}) := \{z \in H_0^1(\Omega) : z = 0 \text{ q.e. on } \Omega \setminus \mathcal{D}\}$$

is dense in $L^s(\mathcal{D})$, where $s \in [1, \infty)$ is such that $H_0^1(\Omega) \hookrightarrow L^s(\Omega)$.

Proof. We note that the linear hull of the set

$$\{f \in L^s(\mathcal{D}) : 0 \le f \le 1\}$$

is dense in $L^s(\mathcal{D})$. Hence, it is sufficient to show that $f \in L^s(\mathcal{D})$ with $0 \le f \le 1$ can be approximated by functions from $H_0^1(\mathcal{D})$.

Let $\varepsilon > 0$ be given. Then we can find an open set G_ε such that $\mathcal{D} \cup G_\varepsilon$ is open and $\operatorname{cap}(G_\varepsilon) < \varepsilon$. Since $H_0^1(\mathcal{D} \cup G_\varepsilon)$ is dense in $L^s(\mathcal{D} \cup G_\varepsilon)$, we can find a function $z_\varepsilon \in H_0^1(\mathcal{D} \cup G_\varepsilon)$ such that $0 \le z_\varepsilon \le 1$ and $\|z_\varepsilon - f\|_{L^s(\Omega)} < \varepsilon$. Using (2.1) yields the existence of $y_\varepsilon \in H_0^1(\Omega)$ such that $\|y_\varepsilon\|_{H_0^1(\Omega)} < 2\sqrt{\varepsilon}$, $y_\varepsilon \ge 0$, and $y_\varepsilon \ge 1$ q.e. on G_ε. We define the function

$$\tilde{z}_\varepsilon := \max(0, z_\varepsilon - y_\varepsilon) \in H_0^1(\Omega).$$

From $z_\varepsilon = 0$ q.e. on $\Omega \setminus (\mathcal{D} \cup G_\varepsilon)$, $z_\varepsilon \le 1$, $y_\varepsilon \ge 0$ and $y_\varepsilon \ge 1$ q.e. on G_ε, we find $\tilde{z}_\varepsilon = 0$ q.e. on $\Omega \setminus (\mathcal{D} \cup G_\varepsilon) \cup G_\varepsilon$. This implies $\tilde{z}_\varepsilon \in H_0^1(\mathcal{D})$. Moreover, we have

$$\begin{aligned} \|\tilde{z}_\varepsilon - z_\varepsilon\|_{L^s(\Omega)}^s = \|\max(-z_\varepsilon, -y_\varepsilon)\|_{L^s(\Omega)}^s &= \int_{\{z_\varepsilon \le y_\varepsilon\}} |z_\varepsilon|^s \,\mathrm{d}x + \int_{\{y_\varepsilon < z_\varepsilon\}} |y_\varepsilon|^s \,\mathrm{d}x \\ &\le \|y_\varepsilon\|_{L^s(\Omega)}^s \le C\,\|y_\varepsilon\|_{H_0^1(\Omega)}^s \le C\,\varepsilon^{\frac{s}{2}}. \end{aligned}$$

Using the triangle inequality yields

$$\|f - \tilde{z}_\varepsilon\|_{L^s(\Omega)} \le \|f - z_\varepsilon\|_{L^s(\Omega)} + \|z_\varepsilon - \tilde{z}_\varepsilon\|_{L^s(\Omega)} \le \varepsilon + C\sqrt{\varepsilon}.$$

Thus we can approximate f with functions in $\tilde{z}_\varepsilon \in H_0^1(\mathcal{D})$, and this proves the claim.

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