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The weak sequential closure of decomposable sets in Lebesgue spaces and its application to variational geometry

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We provide a precise characterization of the weak sequential closure of nonempty, closed, decomposable sets in Lebesgue spaces. Therefore, we have to distinguish between the purely atomic and the nonatomic regime. In the latter case, we get a convexification effect which is related to Lyapunov's convexity theorem, and in the former case, the weak sequential closure equals the strong closure. The characterization of the weak sequential closure is utilized to compute the limiting normal cone to nonempty, closed, decomposable sets in Lebesgue spaces. Finally, we give an example for the possible nonclosedness of the limiting normal cone in this setting.

Keywords: decomposable set, Lebesgue spaces, limiting normal cone, measurability, weak sequential closure
MSC: 49J53, 28B05, 90C30

1 Introduction

Pointwise defined sets in function spaces are standard in order to determine the feasible regions of optimal control problems. Typically, control functions are supposed to lie within sets of the form

$$\mathbb{C} := \{v \in L^p(\mathfrak{m}; \mathbb{R}^q) \mid v(\omega) \in C(\omega) \text{ f.a.a. } \omega \in \Omega\},$$

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where $(\Omega, \Sigma, \mathfrak{m})$ is a complete, σ-finite measure space and $C\colon \Omega \rightrightarrows \mathbb{R}^q$ is a set-valued mapping which possesses nonempty, closed images and certain measurability properties. Note that we do not assume the convexity of the images of C. The generality of this setting is necessary in order to deal, e.g., with optimal control problems comprising mixed control-state complementarity constraints, see Guo and Ye [2016], Mehlitz and Wachsmuth [2016].

It is easy to check that the above set $\mathbb{C}$ is decomposable in the sense

$$\forall (A, v_1, v_2) \in \Sigma \times \mathbb{C} \times \mathbb{C}\colon \quad \chi_A\, v_1 + (1 - \chi_A)\, v_2 \in \mathbb{C},$$

where χ_A denotes the characteristic function of the set A. Conversely, it is well known, that every nonempty, closed, and decomposable set $\mathbb{C} \subset L^p(\mathfrak{m}; \mathbb{R}^q)$ can be written in the above form, see [Papageorgiou and Kyritsi-Yiallourou, 2009, Theorem 6.4.6].

If the set $\mathbb{C}$ appears in the constraints of an optimization problem, then corresponding necessary optimality conditions derived via variational analysis are likely to contain different tangent or normal cones to the set $\mathbb{C}$ at some reference point. This motivated our study of the variational geometry of pointwise defined sets $\mathbb{C}$ in Lebesgue spaces, see Mehlitz and Wachsmuth [2016]. In the latter paper, we obtained explicit formulas for the Fréchet, strong limiting, and Clarke normal cone to the set $\mathbb{C}$ under mild assumptions. Moreover, we presented reasonable upper and lower bounds (w.r.t. set inclusion) of the limiting normal cone which led to the observation that in the nonatomic situation, the limiting normal cone to $\mathbb{C}$ is always dense in the corresponding Clarke normal cone. However, we did not derive an explicit formula for the limiting normal cone and the weak tangent cone.

Here, we want to close this gap. It will be shown that the limiting normal cone to $\mathbb{C}$ equals the so-called weak sequential closure, i.e. the set of all weak accumulation points of sequences, of the corresponding strong limiting normal cone, see Proposition 5.4. In contrast to the weak closure, which is the closure w.r.t. the weak topology, the weak sequential closure does not need to be weakly sequentially closed. In fact, it might not even be closed. Thus, it is possible that the limiting normal cone to pointwise defined sets is not closed, see Example 5.6. This strange behavior of the limiting normal cone was already demonstrated by an example in ℓ^2 in [Mordukhovich, 2006, Example 1.7].

Thus, in order to characterize the limiting normal cone to pointwise defined sets, it will be necessary to clarify how the weak sequential closure to a pointwise defined set in a Lebesgue space looks like. This is the aim of Sections 3 and 4 of this paper where we distinguish between the situations of a nonatomic and a purely atomic measure space, respectively. If nonatomic measure spaces are under consideration, we observe a convexification effect when computing weak sequential closures, see Theorem 3.8, which seems to be related to Lyapunov's convexity theorem, see [Aubin and Frankowska, 2009, Theorem 8.7.3]. Especially, we obtain the supplementary result that pointwise defined sets in Lebesgue spaces are weakly closed if and only if they are weakly sequentially closed under mild assumptions, see Corollary 3.10. This is a remarkable property which does not hold for arbitrary sets in infinite-dimensional Banach spaces. In the setting of purely atomic measure spaces, it is shown in Corollary 4.3 that the weak sequential closure and weak closure coincide with the norm closure of the underlying set.

We organized the paper as follows: In Section 2, we state the basic notation which is used throughout the manuscript. Section 3 is dedicated to the characterization of the weak sequential closure to pointwise defined sets in nonatomic measure spaces. We present some necessary and sufficient criteria for the closedness of the weak sequential closure. Some examples are included to visualize the theory. In Section 4, we deal with the computation of the weak sequential closure of pointwise defined sets in purely atomic measure spaces. Finally, in Section 5, the limiting normal cone and the weak tangent cone to a pointwise defined set in a Lebesgue space are characterized using our results on the weak sequential closure. By means of two examples, we show that neither the limiting normal cone nor the weak tangent cone to a pointwise defined set need to be closed.

2 Notation

Basic notation

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{R}^n$ denote the natural numbers, the integers, the real numbers, and the set of all real vectors with $n \in \mathbb{N}$ components, respectively. Furthermore, we use $\underline{\mathbb{Z}} := \mathbb{Z} \cup \{-\infty\}$, $\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$, $\mathbb{R}^+ := \{r \in \mathbb{R} \,|\, r > 0\}$, and $\mathbb{R}^+_0 := \{r \in \mathbb{R} \,|\, r \geq 0\}$. The standard simplex in $\mathbb{R}^n$ is defined via

$$\Delta_n := \left\{ \lambda \in \mathbb{R}^n \,\middle|\, \sum_{i=1}^n \lambda_i = 1,\, \lambda_1, \ldots, \lambda_n \geq 0 \right\}.$$

The Euclidean inner product of two vectors $x, y \in \mathbb{R}^n$ is represented by $x \cdot y$ while $|x| := \sqrt{x \cdot x}$ is the Euclidean norm of x. We use $\mathcal{B} := \{x \in \mathbb{R}^n \,|\, |x| \leq 1\}$ for the closed unit ball in $\mathbb{R}^n$. The mapping $\operatorname{dist}\colon \mathbb{R}^n \times 2^{\mathbb{R}^n} \to \overline{\mathbb{R}}$ defined by

$$\forall x \in \mathbb{R}^n \,\forall A \subset \mathbb{R}^n\colon \quad \operatorname{dist}(x, A) := \inf_{y \in A} |x - y|$$

is called distance function.

Let $A \subset X$ be a subset of a real Banach space X. The sets $\operatorname{cl} A$, $\operatorname{cl}_w^{\text{seq}} A$, and $\operatorname{cl}_w A$ denote the norm closure of A, the weak sequential closure of A (i.e. the set of all weak accumulation points of sequences coming from A), and the weak closure of A (i.e. the closure w.r.t. the weak topology). Due to [Megginson, 1998, Proposition 2.1.18], $\operatorname{cl}_w A$ equals the set of all weak accumulation points of nets coming from A. Note that we have $\operatorname{cl} A \subset \operatorname{cl}_w^{\text{seq}} A \subset \operatorname{cl}_w A$ and these inclusions are, in general, strict. We use $\operatorname{span} A$, $\operatorname{bd} A$, $\operatorname{cone} A$, $\operatorname{conv} A$, and $\overline{\operatorname{conv}}\, A$ to represent the smallest subspace of X containing A, the boundary of A, the smallest (not necessarily convex) cone in X containing A, the smallest convex set in X containing A, and the smallest closed, convex set in X containing A, respectively. Furthermore, we define the polar cone of A as

$$A^\circ := \{x^\star \in X^\star \,|\, \forall x \in A\colon \langle x^\star, x\rangle \leq 0\}\,.$$

Therein, $X^\star$ is the topological dual of X while $\langle \cdot, \cdot \rangle\colon X^\star \times X \to \mathbb{R}$ denotes the associated dual pairing. Recall that the Banach space X is called reflexive whenever the canonical embedding $X \ni x \mapsto \langle \cdot, x\rangle \in X^{\star\star}$ is surjective.

Tangent and normal cones

Let $A \subset X$ be a nonempty, closed subset of the real reflexive Banach space X and fix some $\bar{x} \in A$. The inner (or adjacent) tangent cone, the tangent (or Bouligand) cone, and the weak tangent cone to A at $\bar{x}$ are defined by

$$\mathcal{T}_A^\flat(\bar{x}) := \left\{ d \in X \,\middle|\, \begin{array}{l} \forall \{t_n\} \subset \mathbb{R}^+ \text{ such that } t_n \searrow 0\colon \\ \qquad \exists \{d_n\} \subset X\colon\ d_n \to d,\ \bar{x} + t_n d_n \in A\ \forall n \in \mathbb{N} \end{array} \right\},$$
$$\mathcal{T}_A(\bar{x}) := \left\{ d \in X \,\middle|\, \exists \{t_n\} \subset \mathbb{R}^+\, \exists \{d_n\} \subset X\colon\ t_n \searrow 0,\ d_n \to d,\ \bar{x} + t_n d_n \in A\ \forall n \in \mathbb{N} \right\},$$
$$\mathcal{T}_A^w(\bar{x}) := \left\{ d \in X \,\middle|\, \exists \{t_n\} \subset \mathbb{R}^+\, \exists \{d_n\} \subset X\colon\ t_n \searrow 0,\ d_n \rightharpoonup d,\ \bar{x} + t_n d_n \in A\ \forall n \in \mathbb{N} \right\},$$

respectively, see e.g. [Aubin and Frankowska, 2009, Section 4.1]. Note that the inner tangent cone and the tangent cone are always closed while this property is not inherent for the weak tangent cone. Obviously, we always have the inclusions $\mathcal{T}_A^\flat(\bar{x}) \subset \mathcal{T}_A(\bar{x}) \subset \mathcal{T}_A^w(\bar{x})$ by definition. If A is convex, then all these cones coincide with $\operatorname{cl}\operatorname{cone}(A - \{\bar{x}\})$. We call A derivable at $\bar{x}$ if $\mathcal{T}_A^\flat(\bar{x}) = \mathcal{T}_A(\bar{x})$ holds. Thus, any convex set is derivable everywhere. One may check [Mehlitz and Wachsmuth, 2016, Section 2.2] for more general criteria ensuring the derivability of a set. Amongst others, the finite union of derivable sets is derivable again, see [Mehlitz and Wachsmuth, 2016, Lemma 2.1].

Next, we introduce the Fréchet (or regular) normal cone, the strong (or norm) limiting normal cone, the limiting (or basic, Mordukhovich) normal cone, and the Clarke (or convexified) normal cone to A at $\bar{x}$ (see Mordukhovich [2006], Geremew et al. [2009]) via

$$\widehat{\mathcal{N}}_A(\bar{x}) := \mathcal{T}_A^w(\bar{x})^\circ,$$
$$\mathcal{N}_A^S(\bar{x}) := \left\{ \eta \in X^\star \,\middle|\, \exists \{x_n\} \subset A\, \exists \{\eta_n\} \subset X^\star\colon\ x_n \to \bar{x},\ \eta_n \to \eta,\ \eta_n \in \widehat{\mathcal{N}}_A(x_n)\ \forall n \in \mathbb{N} \right\},$$
$$\mathcal{N}_A(\bar{x}) := \left\{ \eta \in X^\star \,\middle|\, \exists \{x_n\} \subset A\, \exists \{\eta_n\} \subset X^\star\colon\ x_n \to \bar{x},\ \eta_n \rightharpoonup \eta,\ \eta_n \in \widehat{\mathcal{N}}_A(x_n)\ \forall n \in \mathbb{N} \right\},$$
$$\mathcal{N}_A^C(\bar{x}) := \overline{\operatorname{conv}}\, \mathcal{N}_A(\bar{x}),$$

respectively. By definition the Fréchet as well as the Clarke normal cone are closed and convex. A diagonal sequence argument reveals that the strong limiting normal cone is closed as well. On the other hand, the limiting normal cone does not need to be closed if X is infinite dimensional, see [Mordukhovich, 2006, Example 1.7]. We have the inclusions $\widehat{\mathcal{N}}_A(\bar{x}) \subset \mathcal{N}_A^S(\bar{x}) \subset \mathcal{N}_A(\bar{x}) \subset \mathcal{N}_A^C(\bar{x})$ and, in general, these inclusions are strict. If the set A is convex, then all these cones coincide with $(A - \{\bar{x}\})^\circ$. Clearly, if X is finite dimensional, then the strong limiting normal cone and the limiting normal cone coincide.

Measurability, Lebesgue spaces, and decomposable sets

Let (Ω, Σ) be a measurable space and let Y be a separable metric space. A set-valued mapping $C\colon \Omega \rightrightarrows Y$ is called measurable if for any open set $O \subset Y$, the preimage $C^{-1}(O) := \{\omega \in \Omega \,|\, C(\omega) \cap O \neq \varnothing\}$ is measurable. If C is assumed to be closed-valued, then there are equivalent useful characterizations of measurability, see [Papageorgiou and Kyritsi-Yiallourou, 2009, Theorem 6.3.19]. Particularly, the closed-valued mapping

C is measurable if and only if there exists a sequence $\{c_n : \Omega \to Y\}_{n\in\mathbb{N}}$ of measurable functions such that $C(\omega) = \operatorname{cl}\{c_n(\omega) \mid n \in \mathbb{N}\}$ is valid for every $\omega \in \Omega$.

The map C is called graph measurable if its graph $\operatorname{gph} C := \{(\omega, y) \in \Omega \times Y \mid y \in C(\omega)\}$ is measurable w.r.t. the σ-algebra $\Sigma \otimes \mathcal{B}(Y)$. Therein, $\mathcal{B}(Y)$ denotes the Borelean σ-algebra induced by the metric space Y and $\Sigma \otimes \mathcal{B}(Y)$ represents the smallest σ-algebra containing the Cartesian product $\Sigma \times \mathcal{B}(Y)$. Note that whenever C is a measurable set-valued mapping with closed images, then it is graph measurable, see [Papageorgiou and Kyritsi-Yiallourou, 2009, Proposition 6.2.10]. Now, let X be a separable metric space as well. A mapping $\varphi : \Omega \times X \to Y$ is called a Carathéodory function if $\varphi(\cdot, x): \Omega \to Y$ is measurable for all $x \in X$ while $\varphi(\omega, \cdot): X \to Y$ is continuous for all $\omega \in \Omega$. By means of [Papageorgiou and Kyritsi-Yiallourou, 2009, Theorem 6.2.6], any Carathéodory function $\varphi : \Omega \times X \to Y$ is $\Sigma \otimes \mathcal{B}(X)$-measurable.

Now, assume that $(\Omega, \Sigma, \mathfrak{m})$ is a complete, σ-finite measure space. Recall that $(\Omega, \Sigma, \mathfrak{m})$ is nonatomic whenever for every set $M \in \Sigma$ with $\mathfrak{m}(M) > 0$, there is some set $N \in \Sigma$ such that $\mathfrak{m}(M) > \mathfrak{m}(N) > 0$ is valid. On the other hand, $(\Omega, \Sigma, \mathfrak{m})$ is referred to as purely atomic if every set of positive measure in Σ is an atom, i.e. for every $M \in \Sigma$ with $\mathfrak{m}(M) > 0$ and every $N \in \Sigma$ with $N \subset M$, we obtain $\mathfrak{m}(N) = 0$ or $\mathfrak{m}(N) = \mathfrak{m}(M)$.

For any $p \in [1, \infty]$ and any $q \in \mathbb{N}$, we denote by $L^p(\mathfrak{m}; \mathbb{R}^q)$ the usual Lebesgue space of (equivalence classes) of measurable functions mapping from Ω to $\mathbb{R}^q$ equipped with the usual norm. For brevity, we set $L^p(\mathfrak{m}) := L^p(\mathfrak{m}; \mathbb{R})$. Recall that for $p \in [1, \infty)$, the dual space $L^p(\mathfrak{m}; \mathbb{R}^q)^\star$ is isometric to $L^{p'}(\mathfrak{m}; \mathbb{R}^q)$ where $p' \in (1, \infty]$ such that $1/p + 1/p' = 1$ holds denotes the conjugate coefficient of p. The corresponding dual pairing in $L^p(\mathfrak{m}; \mathbb{R}^q)$ is given by

$$\forall v \in L^p(\mathfrak{m}; \mathbb{R}^q)\, \forall \eta \in L^{p'}(\mathfrak{m}; \mathbb{R}^q): \quad \langle \eta, v \rangle := \int_\Omega v(\omega) \cdot \eta(\omega)\, \mathrm{d}\omega.$$

Here, we use $\mathrm{d}\omega$ instead of $\mathrm{d}\mathfrak{m}(\omega)$ since the measure $\mathfrak{m}$ is fixed throughout each section of the paper.

The measurable space $(\Omega, \Sigma, \mathfrak{m})$ is called separable if for any $s \in [1, \infty)$, the Banach space $L^s(\mathfrak{m})$ is separable. Note that an arbitrary domain $\Omega \subset \mathbb{R}^d$ equipped with the corresponding Borelean σ-algebra and Lebesgue's measure forms a σ-finite, nonatomic, and separable measure space, see e.g. [Adams and Fournier, 2003, Theorem 2.21]. Thus, its formal completion, see [Bogachev, 2007, Section 1.5], is a complete, σ-finite, nonatomic, and separable measure space.

Now, let $(\Omega, \Sigma, \mathfrak{m})$ be a complete, σ-finite measure space again. A set $\mathbb{C} \subset L^p(\mathfrak{m}; \mathbb{R}^q)$ is called decomposable if for any triplet $(A, v_1, v_2) \in \Sigma \times \mathbb{C} \times \mathbb{C}$, we have the relation $\chi_A v_1 + (1 - \chi_A) v_2 \in \mathbb{C}$. Therein, $\chi_A \in L^\infty(\mathfrak{m})$ denotes the characteristic function of A which has value 1 for all $\omega \in A$ and vanishes everywhere on $\Omega \setminus A$. Clearly, decomposability is some kind of generalized convexity. It dates back to Rockafellar, see Rockafellar [1968]. It is well known that a nonempty and closed set is decomposable if and only if there exists a measurable set-valued mapping $C : \Omega \rightrightarrows \mathbb{R}^q$ with nonempty and closed images such that $\mathbb{C}$ possesses the representation

$$\mathbb{C} = \{v \in L^p(\mathfrak{m}; \mathbb{R}^q) \mid v(\omega) \in C(\omega) \text{ f.a.a. } \omega \in \Omega\},$$

see [Papageorgiou and Kyritsi-Yiallourou, 2009, Theorem 6.4.6]. One may check Hiai and Umegaki [1977] and [Papageorgiou and Kyritsi-Yiallourou, 2009, Section 6.2–6.4] for properties and calculus rules for decomposable sets.

3 Weak sequential closure in nonatomic measure spaces

In this section, we are going to characterize the weak sequential closure of arbitrary closed, decomposable sets in Lebesgue spaces defined by a nonatomic measure.

To this end, we define an auxiliary function on $\mathbb{R}^q$ in Section 3.1. With the help of this function, we are able to characterize the weak sequential closure in Section 3.2. In Section 3.3, we investigate under which circumstances the weak sequential closure is closed. Finally, we give a couple of counterexamples in Section 3.4.

3.1 An auxiliary function

Let $C \subset \mathbb{R}^q$ be a closed set and let $p \in (1, \infty)$ be fixed. We define the auxiliary function $\gamma_C \colon \mathbb{R}^q \to \overline{\mathbb{R}}$ by

$$\gamma_C(x) := \inf\left\{ \sum_{i=1}^{q+1} \lambda_i \, |v_i|^p \;\middle|\; \lambda \in \Delta_{q+1}, v \in C^{q+1}, \sum_{i=1}^{q+1} \lambda_i \, v_i = x \right\} \tag{1}$$

for $x \in \operatorname{conv} C$ and by $\gamma_C(x) = +\infty$ otherwise. Carathéodory's theorem implies that $\gamma_C(x) < +\infty$ holds if and only if $x \in \operatorname{conv} C$ is valid. By convexity of $v \mapsto |v|^p$, we have

$$\gamma_C(v) = |v|^p \qquad \forall v \in C. \tag{2}$$

The following lemmas show that γ_C is lower semicontinuous and convex.

Lemma 3.1. *Let $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^q$ be a sequence with $x_n \to x$ for some $x \in \mathbb{R}^q$ and $\liminf_{n\to\infty} \gamma_C(x_n) < +\infty$. Then, there exist $\lambda \in \Delta_{q+1}$, $v \in C^{q+1}$ such that $\sum_{i=1}^{q+1} \lambda_i \, v_i = x$ and*

$$\sum_{i=1}^{q+1} \lambda_i \, |v_i|^p \le \liminf_{n\to\infty} \gamma_C(x_n). \tag{3}$$

In particular, γ_C is lower semicontinuous and the infimum in (1) is attained for all $x \in \operatorname{conv} C$.

Proof. First, we extract a subsequence (without relabeling), such that $\gamma_C(x_n)$ is finite for all $n \in \mathbb{N}$ and converges towards its least accumulation point as $n \to \infty$.

By definition of $\gamma_C(x_n)$, there exist sequences $\{\lambda^{(n)}\}_{n\in\mathbb{N}} \subset \Delta_{q+1}$, $\{v^{(n)}\}_{n\in\mathbb{N}} \subset C^{q+1}$ with $x_n = \sum_{i=1}^{q+1} \lambda_i^{(n)} \, v_i^{(n)}$ and $\sum_{i=1}^{q+1} \lambda_i^{(n)} \, |v_i^{(n)}|^p \le \gamma_C(x_n) + 1/n$ for all $n \in \mathbb{N}$.

Since Δ_{q+1} is compact, we can further extract a subsequence (without relabeling) such that $\lambda^{(n)} \to \lambda \in \Delta_{q+1}$. W.l.o.g., there is $j \in \{0, \ldots, q+1\}$ such that $\lambda_i > 0$ for all $i \in \{1, \ldots, q+1\}$ with $i \le j$ and $\lambda_i = 0$ for all $i \in \{1, \ldots, q+1\}$ with $i > j$. This implies

$$|v_i^{(n)}|^p \le \frac{1}{\lambda_i^{(n)}} \sum_{i=1}^{q+1} \lambda_i^{(n)} \, |v_i^{(n)}|^p \le \frac{1}{\lambda_i^{(n)}} \left(\gamma_C(x_n) + \frac{1}{n} \right) \le \beta < +\infty \quad \text{for all } i \le j,$$

where $\beta \in \mathbb{R}$ is an upper bound of the convergent sequences $\{1/\lambda_i^{(n)}\,(\gamma_C(x_n) + \frac{1}{n})\}_{n\in\mathbb{N}}$, $i = 1, \ldots, j$. Hence, we can extract a subsequence such that $v_i^{(n)} \to v_i \in C$ as $n \to \infty$ for $i \leq j$. On the other hand,

$$\lambda_i^{(n)}\,|v_i^{(n)}| = \big(\lambda_i^{(n)}\big)^{1-1/p}\,\big(\lambda_i^{(n)}\,|v_i^{(n)}|^p\big)^{1/p} \leq \big(\lambda_i^{(n)}\big)^{1-1/p}\left(\sum_{i=1}^{q+1}\lambda_i^{(n)}\,|v_i^{(n)}|^p\right)^{1/p} \to 0$$

for all $i > j$. Now, we define $v_i \in C$ arbitrarily for $i > j$. This yields

$$\begin{aligned}\sum_{i=1}^{q+1}\lambda_i\,v_i = \sum_{i=1}^{j}\lambda_i\,v_i + 0 &= \lim_{n\to\infty}\sum_{i=1}^{j}\lambda_i^{(n)}\,v_i^{(n)} + \lim_{n\to\infty}\sum_{i=j+1}^{q+1}\lambda_i^{(n)}\,v_i^{(n)}\\ &= \lim_{n\to\infty}\sum_{i=1}^{q+1}\lambda_i^{(n)}\,v_i^{(n)} = \lim_{n\to\infty} x_n = x.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\sum_{i=1}^{q+1}\lambda_i\,|v_i|^p &= \sum_{i=1}^{j}\lambda_i\,|v_i|^p + 0\\ &\leq \lim_{n\to\infty}\sum_{i=1}^{j}\lambda_i^{(n)}\,|v_i^{(n)}|^p + \lim_{n\to\infty}\sum_{i=j+1}^{q+1}\lambda_i^{(n)}\,|v_i^{(n)}|^p = \lim_{n\to\infty}\sum_{i=1}^{q+1}\lambda_i^{(n)}\,|v_i^{(n)}|^p.\end{aligned}$$

This shows the existence of λ and v satisfying (3). By definition of γ_C, we even have

$$\gamma_C(x) \leq \sum_{i=1}^{q+1}\lambda_i\,|v_i|^p \leq \liminf_{n\to\infty}\gamma_C(x_n)$$

and this yields the lower semicontinuity of γ_C.

Choosing $x_n = x$ for all $n \in \mathbb{N}$, we obtain that the infimum in (1) is attained. □

We use the above lemma in order to come up with the following observation.

Corollary 3.2. *If the condition*

$$\exists c > 0\,\forall x \in \operatorname{conv} C\colon \quad \gamma_C(x) \leq c\,(|x|^p + 1) \tag{4}$$

is satisfied, then $\operatorname{conv} C$ *is closed.*

Proof. Suppose on the contrary that $\operatorname{conv} C$ is not closed. Then we find a sequence $\{x_n\} \subset \operatorname{conv} C$ converging to $x \notin \operatorname{conv} C$. Noting that $\{x_n\}$ is bounded, we obtain

$$\liminf_{n\to\infty}\gamma_C(x_n) \leq \liminf_{n\to\infty} c(|x_n|^p + 1) < +\infty.$$

From Lemma 3.1, we easily see

$$\gamma_C(x) \leq \liminf_{n\to\infty}\gamma_C(x_n) < +\infty.$$

On the other hand, we have $\gamma_C(x) = +\infty$ by definition. This is a contradiction. □

We note that the converse result does not hold, cf. Examples 3.17, 3.18 and 3.20.

Lemma 3.3. *The function* $\gamma_C \colon \mathbb{R}^q \to \overline{\mathbb{R}}$ *is convex.*

Proof. Choose $x, y \in \mathbb{R}^q$ and $\kappa \in (0,1)$ arbitrarily. In the case $x \notin \operatorname{conv} C$ or $y \notin \operatorname{conv} C$, we have $\gamma_C(x) = +\infty$ or $\gamma_C(y) = +\infty$. Thus, there is nothing to show. Let us consider $x, y \in \operatorname{conv} C$. By Lemma 3.1, we find $\lambda^x, \lambda^y \in \Delta_{q+1}$ and $v^x, v^y \in C^{q+1}$ satisfying $x = \sum_{i=1}^{q+1} \lambda_i^x v_i^x$, $y = \sum_{i=1}^{q+1} \lambda_i^y v_i^y$ as well as $\gamma_C(x) = \sum_{i=1}^{q+1} \lambda_i^x |v_i^x|^p$ and $\gamma_C(y) = \sum_{i=1}^{q+1} \lambda_i^y |v_i^y|^p$. Now, we consider the following linear program in standard form:

$$\begin{aligned} \text{Minimize} \quad & \sum_{i=1}^{q+1} \mu_i |v_i^x|^p + \sum_{i=1}^{q+1} \nu_i |v_i^y|^p \\ \text{w.r.t.} \quad & (\mu, \nu) \in \mathbb{R}^{q+1} \times \mathbb{R}^{q+1} \\ \text{s.t.} \quad & \sum_{i=1}^{q+1} \mu_i + \sum_{i=1}^{q+1} \nu_i = 1, \\ & \sum_{i=1}^{q+1} \mu_i v_i^x + \sum_{i=1}^{q+1} \nu_i v_i^y = \kappa x + (1-\kappa) y, \\ & \mu, \nu \geq 0. \end{aligned} \tag{5}$$

Obviously, $(\kappa\lambda^x, (1-\kappa)\lambda^y)$ is a feasible point of this program, and due to the fact that its feasible set is bounded, it possesses at least one optimal basic solution $(\bar{\mu}, \bar{\nu})$. Observe that (5) possesses $q+1$ equality constraints which means that $(\bar{\mu}, \bar{\nu})$ has at most $q+1$ positive components. This shows

$$\begin{aligned} \gamma_C(\kappa x + (1-\kappa) y) &\leq \sum_{i=1}^{q+1} \bar{\mu}_i |v_i^x|^p + \sum_{i=1}^{q+1} \bar{\nu}_i |v_i^y|^p \\ &\leq \sum_{i=1}^{q+1} \kappa \lambda_i^x |v_i^x|^p + \sum_{i=1}^{q+1} (1-\kappa) \lambda_i^y |v_i^y|^p = \kappa \gamma_C(x) + (1-\kappa)\gamma_C(y). \end{aligned}$$

This yields the convexity of γ_C. ∎

Let us provide an auxiliary result, which will be helpful later. We define the function $\psi \colon \mathbb{R}^{q(q+1)} \times \mathbb{R}^{q+1} \times \mathbb{R} \to \mathbb{R}^q \times \mathbb{R}$ via

$$\psi(v_1, \ldots, v_{q+1}, \lambda, \alpha) := \left(\sum_{i=1}^{q+1} \lambda_i \, v_i, \sum_{i=1}^{q+1} \lambda_i \, |v_i|^p + \alpha \right)$$

for all $v_1, \ldots, v_{q+1} \in \mathbb{R}^q$, $\lambda \in \mathbb{R}^{q+1}$, and $\alpha \in \mathbb{R}$. Obviously, ψ is continuous and Lemma 3.1 implies

$$\psi(C^{q+1} \times \Delta_{q+1} \times \mathbb{R}_0^+) = \operatorname{epi} \gamma_C := \{(v, \beta) \in \mathbb{R}^q \times \mathbb{R} \mid \gamma_C(v) \leq \beta\}, \tag{6}$$

where $\operatorname{epi}\gamma_C$ is the epigraph of the function γ_C. Due to the lower semicontinuity of γ_C, its epigraph is closed.

In the last result of this section, we apply the function γ_C to a decomposable set in a pointwise fashion.

Lemma 3.4. *Let $(\Omega, \Sigma, \mathfrak{m})$ be a complete and σ-finite measure space and let $p \in (1, \infty)$ be given as above. Suppose that $C\colon \Omega \rightrightarrows \mathbb{R}^q$ is measurable and closed-valued. Then, the function $\Gamma_C\colon L^p(\mathfrak{m}; \mathbb{R}^q) \to \overline{\mathbb{R}}$ defined via*

$$\Gamma_C(v) := \int_\Omega \gamma_{C(\omega)}(v(\omega))\,\mathrm{d}\omega \in \overline{\mathbb{R}}$$

for all $v \in L^p(\mathfrak{m}; \mathbb{R}^q)$ satisfying $v(\omega) \in \operatorname{conv} C(\omega)$ for a.a. $\omega \in \Omega$ and $\Gamma_C(v) = +\infty$ otherwise is well defined, convex, and lower semicontinuous. In particular, it is weakly lower semicontinuous.

Proof. Fix $v \in L^p(\mathfrak{m}; \mathbb{R}^q)$ with $v(\omega) \in \operatorname{conv} C(\omega)$ for a.a. $\omega \in \Omega$. Then, the set-valued mapping $\Upsilon\colon \Omega \rightrightarrows \mathbb{R}^{q+1} \times \mathbb{R}^{q(q+1)}$ defined by

$$\Upsilon(\omega) := \left\{ (\lambda, w) \in \Delta_{q+1} \times C(\omega)^{q+1} \,\middle|\, \sum_{i=1}^{q+1} \lambda_i\, w_i = v(\omega) \right\}$$

for all $\omega \in \Omega$ is closed-valued, and measurable, see [Aubin and Frankowska, 2009, Theorem 8.2.9]. Furthermore, $\Upsilon(\omega)$ is nonempty for a.a. $\omega \in \Omega$. Since we have

$$\gamma_{C(\omega)}(v(\omega)) = \inf \left\{ \sum_{i=1}^{q+1} \lambda_i\, |w_i|^p \,\middle|\, (\lambda, w) \in \Upsilon(\omega) \right\}$$

for almost all $\omega \in \Omega$, the function $\omega \mapsto \gamma_{C(\omega)}(v(\omega))$ is measurable by means of [Aubin and Frankowska, 2009, Theorem 8.2.11]. Together with $\gamma_{C(\omega)}(v(\omega)) \ge 0$, this shows that Γ_C is well defined.

The convexity of the function Γ_C is a simple consequence of the convexity of $\gamma_{C(\omega)}$ for all $\omega \in \Omega$, which was provided in Lemma 3.3.

Let us prove the lower semicontinuity of Γ_C. Choose a sequence $\{v_n\}_{n\in\mathbb{N}}$ of functions in $L^p(\mathfrak{m}; \mathbb{R}^q)$ converging to v such that $v_n(\omega) \in \operatorname{conv} C(\omega)$ holds true for almost all $\omega \in \Omega$ for sufficiently large $n \in \mathbb{N}$ (otherwise, there is nothing to show). Without relabeling, we extract a subsequence such that $\{\Gamma_C(v_n)\}$ converges towards its least accumulation point. From $v_n \to v$ in $L^p(\mathfrak{m}; \mathbb{R}^q)$ we find a subsequence of $\{v_n\}$ (without relabeling) which converges pointwise almost everywhere on Ω to v. Thus, we can apply Fatou's lemma and Lemma 3.1 in order to obtain

$$\begin{aligned}
\liminf_{n\to\infty} \Gamma_C(v_n) = \lim_{n\to\infty} \Gamma_C(v_n) &= \lim_{n\to\infty} \int_\Omega \gamma_{C(\omega)}(v_n(\omega))\mathrm{d}\omega \\
&\ge \int_\Omega \liminf_{n\to\infty} \gamma_{C(\omega)}(v_n(\omega))\mathrm{d}\omega \ge \int_\Omega \gamma_{C(\omega)}(v(\omega))\mathrm{d}\omega = \Gamma_C(v).
\end{aligned}$$

This completes the proof. $\square$

3.2 Characterization of the weak sequential closure

In this section, we give a characterization of the weak sequential closure of closed, decomposable sets in nonatomic measure spaces.

Throughout this section, we use the following assumption on the measure space.

Assumption 3.5. *We assume that $(\Omega, \Sigma, \mathfrak{m})$ is a σ-finite, complete, separable, and nonatomic measure space. Moreover, let $p \in (1, \infty)$ be given.*

First, we recall the following result which is taken from [Papageorgiou and Kyritsi-Yiallourou, 2009, Proposition 6.4.19] and [Mehlitz and Wachsmuth, 2016, Lemma 3.4].

Lemma 3.6. *Suppose that $(\Omega, \Sigma, \mathfrak{m})$ satisfies Assumption 3.5. Let $\mathbb{C} \subset L^p(\mathfrak{m}; \mathbb{R}^q)$ be nonempty, closed, and decomposable. We denote by $C\colon \Omega \rightrightarrows \mathbb{R}^q$ the associated set-valued map. Then, we have*

$$\operatorname{cl}_w \mathbb{C} = \big\{ v \in L^p(\mathfrak{m}; \mathbb{R}^q) \;\big|\; v(\omega) \in \overline{\operatorname{conv}}\, C(\omega) \text{ f.a.a. } \omega \in \Omega \big\} \tag{7}$$

and

$$\operatorname{conv} \mathbb{C} \subset \operatorname{cl}_w^{\text{seq}} \mathbb{C}.$$

The next lemma of this section demonstrates that a partition of unity $1 = \sum_{i=1}^{q+1} \lambda_i(\omega)$ such that $\lambda\colon \Omega \to \Delta_{q+1}$ is measurable can be approximated in the weak-$\star$ topology of $L^\infty(\mathfrak{m})$ by sequences of characteristic functions. For this result, it is essential that the measure $\mathfrak{m}$ does not contain atoms.

Lemma 3.7. *Suppose that $(\Omega, \Sigma, \mathfrak{m})$ satisfies Assumption 3.5. Let $\lambda \in L^\infty(\mathfrak{m}; \mathbb{R}^{q+1})$ with $\lambda(\omega) \in \Delta_{q+1}$ for a.a. $\omega \in \Omega$ be given. Then, there exists a sequence $\{A^{(n)}\}_{n\in\mathbb{N}} \subset \Sigma^{q+1}$, such that $\{A_i^{(n)}\}_{i=1}^{q+1}$ is a partition of Ω for all $n \in \mathbb{N}$ and*

$$\chi_{A_i^{(n)}} \stackrel{\star}{\rightharpoonup} \lambda_i \quad \text{in } L^\infty(\mathfrak{m}) \qquad \text{as } n \to \infty$$

holds for all $i = 1, \ldots, q+1$.

Moreover, it is possible to choose $\{A^{(n)}\}_{n\in\mathbb{N}}$ such that

$$\forall \iota \in \{1, \ldots, q+1\}\colon \quad \left[\lambda_\iota \in L^1(\mathfrak{m}) \quad \Longrightarrow \quad \forall n \in \mathbb{N}\colon\ \mathfrak{m}\big(A_\iota^{(n)}\big) \le 2 \int_\Omega \lambda_\iota(\omega)\, \mathrm{d}\omega \right].$$

Proof. First, assume $\mathfrak{m}(\Omega) < +\infty$. Let $e_1, \ldots, e_{q+1} \in \mathbb{R}^{q+1}$ denote the $q+1$ unit vectors in $\mathbb{R}^{q+1}$, set $E := \{e_1, \ldots, e_{q+1}\}$, and consider the associated closed, decomposable set

$$\mathbb{E} := \{\xi \in L^p(\mathfrak{m}, \mathbb{R}^{q+1}) \mid \xi(\omega) \in E \text{ f.a.a. } \omega \in \Omega\}.$$

From $\mathfrak{m}(\Omega) < +\infty$, $\mathbb{E}$ is nonempty. Thus, we obtain

$$\begin{aligned} \operatorname{cl}_w \mathbb{E} &= \{\xi \in L^p(\mathfrak{m}, \mathbb{R}^{q+1}) \mid \xi(\omega) \in \overline{\operatorname{conv}}\, E \text{ f.a.a. } \omega \in \Omega\} \\ &= \{\xi \in L^p(\mathfrak{m}, \mathbb{R}^{q+1}) \mid \xi(\omega) \in \Delta_{q+1} \text{ f.a.a. } \omega \in \Omega\}, \end{aligned}$$

i.e. $\lambda \in \mathrm{cl}_w \mathbb{E}$, from Lemma 3.6. Since $L^p(\mathfrak{m})$ is separable and reflexive while $\mathbb{E}$ is bounded, we obtain $\lambda \in \mathrm{cl}_w^{\mathrm{seq}} \mathbb{E}$ from [Megginson, 1998, Corollary 2.6.20]. Thus, there is a sequence $\{\xi_n\}_{n\in\mathbb{N}} \subset \mathbb{E}$ which converges weakly to λ. Noting that $\{\xi_n\}_{n\in\mathbb{N}}$ is bounded in $L^\infty(\mathfrak{m};\mathbb{R}^{q+1})$, we already have $\xi_n \overset{\star}{\rightharpoonup} \lambda$ in $L^\infty(\mathfrak{m};\mathbb{R}^{q+1})$, i.e. $\xi_{n,i} \overset{\star}{\rightharpoonup} \lambda_i$ in $L^\infty(\mathfrak{m})$ for all $i = 1,\dots,q+1$. By definition of $\mathbb{E}$, there exists a partition $\{A_i^{(n)}\}_{i=1}^{q+1} \subset \Sigma$ of Ω with

$$\xi_n = \sum_{i=1}^{q+1} e_i \chi_{A_i^{(n)}}$$

which shows $\chi_{A_i^{(n)}} \overset{\star}{\rightharpoonup} \lambda_i$ for all $i = 1,\dots,q+1$ as $n \to \infty$ whenever the underlying measure space is finite. For the proof in the case $\mathfrak{m}(\Omega) = +\infty$, we choose a partition $\{\Omega_j\}_{j\in\mathbb{N}} \subset \Sigma$ of Ω into sets of positive but finite measure and repeat the above arguments on every segment Ω_j, $j \in \mathbb{N}$.

It remains to verify the last assertion. By $\{A^{(n)}\}_{n\in\mathbb{N}} \subset \Sigma^{q+1}$, we denote the sequence which is given by the first assertion. Let $\iota \in \{1,\dots,q+1\}$ be given such that $\lambda_\iota \in L^1(\mathfrak{m})$ holds. We denote by $\{\Omega_j\}_{j\in\mathbb{N}} \subset \Sigma$ a disjoint partition of Ω into sets of finite measure. Since $\chi_{\Omega_j} \in L^1(\mathfrak{m})$ is valid, we can find a subsequence indexed by $\{n(j,k)\}_{k\in\mathbb{N}}$ which satisfies

$$\int_{\Omega_j} \chi_{A_\iota^{(n(j,k))}} \,\mathrm{d}\omega \to \int_{\Omega_j} \lambda_\iota \,\mathrm{d}\omega \quad \text{as } k \to \infty$$
$$\text{and} \qquad \int_{\Omega_j} \chi_{A_\iota^{(n(j,k))}} \,\mathrm{d}\omega \le 2 \int_{\Omega_j} \lambda_\iota \,\mathrm{d}\omega \quad \forall k \in \mathbb{N}.$$

For $i = 1,\dots,q+1$ and $k \in \mathbb{N}$, we set

$$\tilde{A}_i^{(k)} := \bigcup_{j\in\mathbb{N}} \Omega_j \cap A_i^{(n(j,k))}.$$

Then, $\tilde{A}^{(k)} = \{A_i^{(k)}\}_{i=1}^{q+1}$ is a disjoint partition of Ω for all $k \in \mathbb{N}$ as well. Let us consider the sequence $\{\tilde{A}^{(k)}\}_{k\in\mathbb{N}}$. For arbitrary $v \in L^1(\mathfrak{m})$ and fixed $i \in \{1,\dots,q+1\}$, we obtain

$$\int_\Omega v\chi_{\tilde{A}_i^{(k)}} \mathrm{d}\omega = \sum_{j\in\mathbb{N}} \int_\Omega (v\,\chi_{\Omega_j})\,\chi_{A_i^{(n(j,k))}} \mathrm{d}\omega \to \sum_{j\in\mathbb{N}} \int_\Omega (v\,\chi_{\Omega_j})\,\lambda_i \mathrm{d}\omega = \int_\Omega v\lambda_i \mathrm{d}\omega$$

as $k \to \infty$, i.e., $\chi_{\tilde{A}_i^{(k)}} \overset{\star}{\rightharpoonup} \lambda_i$ in $L^\infty(\mathfrak{m})$ as $k \to \infty$. Note that we have used the dominated convergence theorem with the summable dominating sequence $\|v\,\chi_{\Omega_j}\|_{L^1(\mathfrak{m})}$.

On the other hand,

$$\mathfrak{m}\big(\tilde{A}_\iota^{(k)}\big) = \mathfrak{m}\left(\bigcup_{j\in\mathbb{N}} \Omega_j \cap A_\iota^{(n(j,k))}\right) = \sum_{j\in\mathbb{N}} \mathfrak{m}\big(\Omega_j \cap A_\iota^{(n(j,k))}\big)$$
$$= \sum_{j\in\mathbb{N}} \int_{\Omega_j} \chi_{A_\iota^{(n(j,k))}} \,\mathrm{d}\omega \le 2 \sum_{j\in\mathbb{N}} \int_{\Omega_j} \lambda_\iota \,\mathrm{d}\omega = 2 \int_\Omega \lambda_\iota \,\mathrm{d}\omega$$

is valid. Thus, by replacing $\{A^{(n)}\}_{n\in\mathbb{N}}$ by $\{\tilde{A}^{(k)}\}_{k\in\mathbb{N}}$ and by repeating the argumentation for other indices $\iota \in \{1,\dots,q+1\}$ with $\lambda_\iota \in L^1(\mathfrak{m})$ yields the second claim. □

Now, we are in the position to prove the main result of this section.

Theorem 3.8. *Suppose that* $(\Omega, \Sigma, \mathfrak{m})$ *satisfies Assumption 3.5. Let* $\mathbb{C} \subset L^p(\mathfrak{m};\mathbb{R}^q)$ *be closed and decomposable. We denote by* $C\colon \Omega \rightrightarrows \mathbb{R}^q$ *the associated set-valued map. Then,*

$$\operatorname{cl}_w^{\mathrm{seq}} \mathbb{C} = \{v \in L^p(\mathfrak{m};\mathbb{R}^q) \mid \Gamma_C(v) < +\infty\}.$$

Proof. "$\subset$": Let $v \in \operatorname{cl}_w^{\mathrm{seq}} \mathbb{C}$ be given. Then, there is a sequence $\{v_n\} \subset \mathbb{C}$ satisfying $v_n \rightharpoonup v$ in $L^p(\mathfrak{m};\mathbb{R}^q)$. Since $v_n(\omega) \in C(\omega)$ for a.a. $\omega \in \Omega$, we can use (2) and obtain

$$\Gamma_C(v_n) = \int_\Omega \gamma_{C(\omega)}(v_n(\omega))\,\mathrm{d}\omega = \int_\Omega |v_n(\omega)|^p\,\mathrm{d}\omega = \|v_n\|^p_{L^p(\mathfrak{m};\mathbb{R}^q)}.$$

Since Γ_C is weakly lower semicontinuous by Lemma 3.4, we find

$$\Gamma_C(v) \le \liminf_{n\to\infty} \Gamma_C(v_n) = \liminf_{n\to\infty} \|v_n\|^p_{L^p(\mathfrak{m};\mathbb{R}^q)} < +\infty.$$

"$\supset$": Let $v \in L^p(\mathfrak{m};\mathbb{R}^q)$ with $\Gamma_C(v) < +\infty$ be given. Then we have $\gamma_{C(\omega)}(v(\omega)) < +\infty$ for a.a. $\omega \in \Omega$. Let us introduce a set-valued mapping $\Phi\colon \Omega \rightrightarrows \mathbb{R}^{q+1} \times \mathbb{R}^{q(q+1)}$ by

$$\Phi(\omega) := \left\{(\lambda, w) \in \Delta_{q+1} \times C(\omega)^{q+1} \;\middle|\; \sum_{i=1}^{q+1} \lambda_i\,|w_i|^p = \gamma_{C(\omega)}(v(\omega)) \text{ and } \sum_{i=1}^{q+1} \lambda_i\, w_i = v(\omega)\right\}$$

for all $\omega \in \Omega$. Owing to Lemma 3.1, we find that the images of Φ are not empty and it is straightforward to check that they are closed. It is easy to see from [Aubin and Frankowska, 2009, Theorem 8.2.9] that Φ is measurable. By applying the measurable selection theorem, see [Aubin and Frankowska, 2009, Theorem 8.1.3], we find measurable functions $\lambda\colon \Omega \to \Delta_{q+1}$ and $v_i\colon \Omega \to \mathbb{R}^q$, $i = 1,\dots,q+1$, which satisfy $v_i(\omega) \in C(\omega)$ for all $i = 1,\dots,q+1$, $v(\omega) = \sum_{i=1}^{q+1} \lambda_i(\omega)\, v_i(\omega)$ for a.a. $\omega \in \Omega$, and

$$\int_\Omega \sum_{i=1}^{q+1} \lambda_i(\omega)\,|v_i(\omega)|^p\,\mathrm{d}\omega = \int_\Omega \gamma_{C(\omega)}(v(\omega))\,\mathrm{d}\omega < +\infty.$$

For $i \in \{1,\dots,q+1\}$ and $j \in \underline{\mathbb{Z}}$, we define

$$\Omega(i,j) = \begin{cases} \{\omega \in \Omega \mid 2^j \le |v_i(\omega)|^p < 2^{j+1}\} & \text{if } j > -\infty, \\ \{\omega \in \Omega \mid v_i(\omega) = 0\} & \text{if } j = -\infty. \end{cases}$$

Furthermore, for a multiindex $J \in \underline{\mathbb{Z}}^{q+1}$, we introduce

$$\Omega_J := \bigcap_{i=1}^{q+1} \Omega(i, J_i).$$

This implies that $\{\Omega_J\}_{J\in\underline{\mathbb{Z}}^{q+1}}$ is a disjoint partition of Ω.

Now, we apply Lemma 3.7 on each of these Ω_J, $J \in \underline{\mathbb{Z}}^{q+1}$. This yields sequences $\{A^{(J,n)}\}_{n\in\mathbb{N}}$ such that $\{A_i^{(J,n)}\}_{i=1}^{q+1}$ is a disjoint partition of Ω_J and $\chi_{A_i^{(J,n)}} \overset{\star}{\rightharpoonup} \lambda_i|_{\Omega_J}$ in $L^\infty(\mathfrak{m}|_{\Omega_J})$ as $n \to \infty$. Moreover, we observe that λ_i is integrable on Ω_J if $J_i > -\infty$. In this case, we additionally have $\mathfrak{m}(A_i^{(J,n)}) \le 2 \int_{\Omega_J} \lambda_i \,\mathrm{d}\omega$.

Now, we set

$$v^{(n)} := \sum_{J\in\underline{\mathbb{Z}}^{q+1}} \sum_{i=1}^{q+1} \chi_{A_i^{(J,n)}}\, v_i.$$

This implies

$$\begin{aligned}
\|v^{(n)}\|^p_{L^p(\mathfrak{m};\mathbb{R}^q)} &\le \sum_{J\in\underline{\mathbb{Z}}^{q+1}} \sum_{i=1}^{q+1} \int_{A_i^{(J,n)}} |v_i|^p \,\mathrm{d}\omega \\
&\le \sum_{J\in\underline{\mathbb{Z}}^{q+1}} \sum_{i=1}^{q+1} \begin{cases} \mathfrak{m}(A_i^{(J,n)})\, 2^{(J_i+1)p} & \text{if } J_i > -\infty \\ 0 & \text{if } J_i = -\infty \end{cases} \\
&\le \sum_{J\in\underline{\mathbb{Z}}^{q+1}} \sum_{i=1}^{q+1} \begin{cases} 2\cdot 2^p \int_{\Omega_J} \lambda_i 2^{J_i p} \,\mathrm{d}\omega & \text{if } J_i > -\infty \\ 0 & \text{if } J_i = -\infty \end{cases} \\
&\le \sum_{J\in\underline{\mathbb{Z}}^{q+1}} \sum_{i=1}^{q+1} \begin{cases} 2^{1+p} \int_{\Omega_J} \lambda_i\, |v_i|^p \,\mathrm{d}\omega & \text{if } J_i > -\infty \\ 0 & \text{if } J_i = -\infty \end{cases} \\
&= 2^{1+p} \sum_{i=1}^{q+1} \int_\Omega \lambda_i\, |v_i|^p \,\mathrm{d}\omega = 2^{1+p}\, \Gamma_C(v).
\end{aligned}$$

This yields the boundedness of $\{v^{(n)}\}_{n\in\mathbb{N}}$ in $L^p(\mathfrak{m};\mathbb{R}^q)$, and $v^{(n)} \in \mathbb{C}$ follows by construction.

It remains to show $v^{(n)} \rightharpoonup v$ in $L^p(\mathfrak{m};\mathbb{R}^q)$. Let $J \in \underline{\mathbb{Z}}^{q+1}$ and $i \in \{1,\ldots,q+1\}$ be given. First, we consider the case $J_i > -\infty$. In this case, $\chi_{A_i^{(J,n)}}\, v_i$ is bounded in $L^\infty(\mathfrak{m};\mathbb{R}^q)$ and by Lemma 3.7 it converges weakly-$\star$ towards $\chi_{\Omega_J}\, \lambda_i\, v_i$ in $L^\infty(\mathfrak{m};\mathbb{R}^q)$ as $n \to \infty$. By density of $L^1(\mathfrak{m}) \cap L^{p'}(\mathfrak{m})$ in $L^{p'}(\mathfrak{m})$ and the boundedness of $\chi_{A_i^{(J,n)}}\, v_i$ in $L^p(\mathfrak{m};\mathbb{R}^q)$, this yields $\chi_{A_i^{(J,n)}}\, v_i \rightharpoonup \chi_{\Omega_J}\, \lambda_i\, v_i$ in $L^p(\mathfrak{m};\mathbb{R}^q)$ as $n \to \infty$.

Now, we consider the case $J_i = -\infty$. This yields $v_i|_{\Omega_J} = 0$. Hence, we obviously have $\chi_{A_i^{(J,n)}}\, v_i \rightharpoonup \chi_{\Omega_J}\, \lambda_i\, v_i$ in $L^p(\mathfrak{m};\mathbb{R}^q)$ as $n \to \infty$.

This yields $\langle v^{(n)}, g\rangle \to \langle v, g\rangle$ for all $g \in L^{p'}(\mathfrak{m};\mathbb{R}^q)$ which are supported on the union of a finite subset of $\{\Omega_J\}_{J\in\underline{\mathbb{Z}}^{q+1}}$. Such functions are dense in $L^{p'}(\mathfrak{m};\mathbb{R}^q)$ and together with the boundedness of $v^{(n)}$ in $L^p(\mathfrak{m};\mathbb{R}^q)$ this shows the claim. □

Let $\mathbb{C} \subset L^p(\mathfrak{m};\mathbb{R}^q)$ be nonempty, closed, and decomposable, let $C\colon \Omega \rightrightarrows \mathbb{R}^q$ be the associated measurable set-valued mapping, and let Assumption 3.5 be valid. We want to compare the above result for the weak sequential closure of $\mathbb{C}$ with the expression (7) for its weak closure.

By definition of Γ_C and the representation of $\operatorname{cl}_w \mathbb{C}$ presented in Lemma 3.6, we obtain the inclusions

$$\operatorname{cl}_w^{\text{seq}} \mathbb{C} \subset \{v \in L^p(\mathfrak{m}; \mathbb{R}^q) \mid v(\omega) \in \operatorname{conv} C(\omega) \text{ f.a.a. } \omega \in \Omega\} \subset \operatorname{cl}_w \mathbb{C}. \tag{8}$$

By means of an example, we will see later that these inclusions can be strict at the same time.

In the upcoming theorem, we will demonstrate that the weak closure of $\mathbb{C}$ can be obtained by taking the norm closure of its weak sequential closure.

Theorem 3.9. *Suppose that* $(\Omega, \Sigma, \mathfrak{m})$ *satisfies Assumption 3.5. Let* $\mathbb{C} \subset L^p(\mathfrak{m}; \mathbb{R}^q)$ *be decomposable. Then we have*

$$\operatorname{cl} \operatorname{cl}_w^{\text{seq}} \mathbb{C} = \operatorname{cl}_w \mathbb{C}.$$

Proof. If $\mathbb{C}$ is empty, there is nothing to prove. In the case that $\mathbb{C}$ is closed and not empty, we invoke Lemma 3.6 to obtain $\operatorname{conv} \mathbb{C} \subset \operatorname{cl}_w^{\text{seq}} \mathbb{C} \subset \operatorname{cl}_w \mathbb{C}$. Taking the closure yields $\overline{\operatorname{conv}}\, \mathbb{C} \subset \operatorname{cl} \operatorname{cl}_w^{\text{seq}} \mathbb{C} \subset \operatorname{cl} \operatorname{cl}_w \mathbb{C} = \operatorname{cl}_w \mathbb{C}$. Now, observe that the set on the left is a closed as well as convex superset of $\mathbb{C}$. This yields $\operatorname{cl}_w \mathbb{C} \subset \overline{\operatorname{conv}}\, \mathbb{C} \subset \operatorname{cl} \operatorname{cl}_w^{\text{seq}} \mathbb{C} \subset \operatorname{cl}_w \mathbb{C}$ which completes the proof in case that $\mathbb{C}$ is closed.

It remains to consider the case that $\mathbb{C}$ is not closed. By applying the first part of the proof to $\operatorname{cl} \mathbb{C}$, which is decomposable, we obtain $\operatorname{cl} \operatorname{cl}_w^{\text{seq}} \operatorname{cl} \mathbb{C} = \operatorname{cl}_w \operatorname{cl} \mathbb{C}$. By a diagonal sequence argument, we can show $\operatorname{cl}_w^{\text{seq}} \operatorname{cl} \mathbb{C} = \operatorname{cl}_w^{\text{seq}} \mathbb{C}$ and $\operatorname{cl}_w \operatorname{cl} \mathbb{C} = \operatorname{cl}_w \mathbb{C}$ is immediate. This shows the claim in the case that $\mathbb{C}$ is not closed. □

As a corollary, we obtain an interesting property of decomposable sets.

Corollary 3.10. *Suppose that* $(\Omega, \Sigma, \mathfrak{m})$ *satisfies Assumption 3.5. Let* $\mathbb{C} \subset L^p(\mathfrak{m}; \mathbb{R}^q)$ *be decomposable. Then,* $\mathbb{C}$ *is weakly closed if and only if it is weakly sequentially closed.*

The above property of decomposable sets is very remarkable, since there are examples for sets which are weakly sequentially closed, but not weakly closed. For example, it can be checked that the set $\{\sqrt{n}\, e_n \mid n \in \mathbb{N}\} \subset H$, where $\{e_n \mid n \in \mathbb{N}\}$ is an orthonormal basis in the Hilbert space H, is weakly sequentially closed, but 0 belongs to its weak closure.

3.3 Closedness of the weak sequential closure

In this section, we study necessary and sufficient conditions for the closedness of the weak sequential closure. The main motivation for this investigation is that Theorem 3.9 implies $\operatorname{cl}_w^{\text{seq}} \mathbb{C} = \operatorname{cl}_w \mathbb{C}$ in the case that $\operatorname{cl}_w^{\text{seq}} \mathbb{C}$ is closed, and we can use the more convenient formula (7) for the computation of $\operatorname{cl}_w^{\text{seq}} \mathbb{C}$.

Lemma 3.11. *Suppose that* $(\Omega, \Sigma, \mathfrak{m})$ *satisfies Assumption 3.5. Let* $\mathbb{C} \subset L^p(\mathfrak{m}; \mathbb{R}^q)$ *be nonempty, closed, and decomposable such that* $\operatorname{cl}_w^{\text{seq}} \mathbb{C}$ *is closed. Let* $C \colon \Omega \rightrightarrows \mathbb{R}^q$ *be the associated set-valued mapping. Then,* $\operatorname{conv} C(\omega)$ *is closed for almost every* $\omega \in \Omega$.

Proof. The closedness of $\operatorname{cl}_w^{\text{seq}} \mathbb{C}$ implies

$$\begin{aligned}\operatorname{cl}_w^{\text{seq}} \mathbb{C} &\subset \left\{v \in L^p(\mathfrak{m};\mathbb{R}^q) \;\middle|\; v(\omega) \in \operatorname{conv} C(\omega) \text{ f.a.a. } \omega \in \Omega\right\} \\ &\subset \left\{v \in L^p(\mathfrak{m};\mathbb{R}^q) \;\middle|\; v(\omega) \in \overline{\operatorname{conv}}\, C(\omega) \text{ f.a.a. } \omega \in \Omega\right\} = \operatorname{cl}_w \mathbb{C} = \operatorname{cl}_w^{\text{seq}} \mathbb{C},\end{aligned}$$

see Lemma 3.6 as well as Theorems 3.8 and 3.9. Thus, all these sets have to be equal. If we would already know that the set-valued mappings $\omega \mapsto \operatorname{conv} C(\omega)$ and $\omega \mapsto \overline{\operatorname{conv}}\, C(\omega)$ are graph measurable, we could apply [Papageorgiou and Kyritsi-Yiallourou, 2009, Corollary 6.4.4] to get $\operatorname{conv} C(\omega) = \overline{\operatorname{conv}}\, C(\omega)$ for a.a. $\omega \in \Omega$. Hence, it remains to show these graph measurabilities.

From [Aubin and Frankowska, 2009, Theorem 8.2.2], we get that $\omega \mapsto \overline{\operatorname{conv}}\, C(\omega)$ is measurable. Consequently, it is graph measurable by [Papageorgiou and Kyritsi-Yiallourou, 2009, Theorem 6.3.19].

To show the graph measurability of $\omega \mapsto \operatorname{conv} C(\omega)$, we are going to exploit (6) and $v \in \operatorname{conv} C(\omega)$ if and only if $\gamma_{C(\omega)}(v) < +\infty$. To this end, we introduce the Carathéodory function $\varphi\colon \Omega \times \mathbb{R}^{q(q+1)} \times \mathbb{R}^{q+1} \times \mathbb{R} \to \mathbb{R}^q \times \mathbb{R}$, defined via

$$\varphi(\omega, v_1, \ldots, v_{q+1}, \lambda, \alpha) = \left(\sum_{i=1}^{q+1} \lambda_i\, v_i, \sum_{i=1}^{q+1} \lambda_i\, |v_i|^p + \alpha\right)$$

for all $\omega \in \Omega$, $v_1, \ldots, v_{q+1} \in \mathbb{R}^q$, $\lambda \in \mathbb{R}^{q+1}$, and $\alpha \in \mathbb{R}$. By (6), we find

$$\varphi\big(\omega, C(\omega)^{q+1} \times \Delta_{q+1} \times \mathbb{R}_0^+\big) = \operatorname{epi} \gamma_{C(\omega)}.$$

Now, the set-valued map $\omega \mapsto C(\omega)^{q+1} \times \Delta_{q+1} \times \mathbb{R}_0^+$ is measurable. Hence, [Aubin and Frankowska, 2009, Theorem 8.2.8] implies that

$$\omega \mapsto \operatorname{cl} \varphi\big(\omega, C(\omega)^{q+1} \times \Delta_{q+1} \times \mathbb{R}_0^+\big) = \operatorname{cl} \operatorname{epi} \gamma_{C(\omega)} = \operatorname{epi} \gamma_{C(\omega)}$$

is measurable and possesses nonempty, closed images.

For fixed $K \in \mathbb{N}$, we define a closed set $M_K \subset \mathbb{R}^q \times \mathbb{R}$ by $M_K := \mathbb{R}^q \times [0, K]$. Now, we can invoke [Aubin and Frankowska, 2009, Theorem 8.2.4] to see that $\omega \mapsto M_K \cap \operatorname{epi} \gamma_{C(\omega)}$ is measurable and closed-valued. Thus, [Aubin and Frankowska, 2009, Theorem 8.2.8] and [Papageorgiou and Kyritsi-Yiallourou, 2009, Proposition 6.2.10] can be used to obtain the graph measurability of

$$\omega \mapsto \operatorname{cl} \operatorname{Proj}(M_K \cap \operatorname{epi} \gamma_{C(\omega)}),$$

where $\operatorname{Proj}\colon \mathbb{R}^q \times \mathbb{R} \to \mathbb{R}^q$ is the projection onto the first q arguments. Using the lower semicontinuity of $\gamma_{C(\omega)}$ and the characterization

$$\operatorname{Proj}(M_K \cap \operatorname{epi} \gamma_{C(\omega)}) = \{v \in \mathbb{R}^q \mid \gamma_{C(\omega)}(v) \le K\},$$

we observe that the set $\operatorname{Proj}(M_K \cap \operatorname{epi} \gamma_{C(\omega)})$ is already closed for a.a. $\omega \in \Omega$. Hence,

$$\omega \mapsto \operatorname{Proj}(M_K \cap \operatorname{epi} \gamma_{C(\omega)}).$$

is graph measurable.

Finally, we note

$$\begin{aligned}\operatorname{gph}(\omega \mapsto \operatorname{conv} C(\omega)) &= \{(\omega, v) \mid v \in \operatorname{conv} C(\omega)\} = \big\{(\omega, v) \mid 0 \le \gamma_{C(\omega)}(v) < +\infty\big\} \\ &= \bigcup_{K \in \mathbb{N}} \big\{(\omega, v) \mid 0 \le \gamma_{C(\omega)}(v) \le K\big\} \\ &= \bigcup_{K \in \mathbb{N}} \operatorname{gph}(\omega \mapsto \operatorname{Proj}(M_K \cap \operatorname{epi} \gamma_{C(\omega)})).\end{aligned}$$

Hence, $\omega \mapsto \operatorname{conv} C(\omega)$ is graph measurable.

As already said, [Papageorgiou and Kyritsi-Yiallourou, 2009, Corollary 6.4.4] now implies that $\operatorname{conv} C(\omega) = \overline{\operatorname{conv}}\, C(\omega)$ for a.a. $\omega \in \Omega$. □

In the case where the set-valued mapping which characterizes the pointwise defined set $\mathbb{C}$ of interest is constant, the situation is more comfortable. We study

$$\mathbb{C} = \{v \in L^p(\mathfrak{m}; \mathbb{R}^q) \,|\, v(\omega) \in C \text{ f.a.a. } \omega \in \Omega\} \tag{9}$$

for a nonempty, closed set $C \subset \mathbb{R}^q$ and $p \in (1, \infty)$. In this setting, we find a condition which is necessary and sufficient for the closedness of $\operatorname{cl}_w^{\text{seq}} \mathbb{C}$. However, we have to distinguish the cases where the underlying measure space is finite or not.

Lemma 3.12. *Suppose that $(\Omega, \Sigma, \mathfrak{m})$ satisfies Assumption 3.5. Let $\mathbb{C}$ be nonempty and defined as in (9) with $C \subset \mathbb{R}^q$ being nonempty and closed. We further suppose $\mathfrak{m}(\Omega) < +\infty$. Then, the closedness of $\operatorname{cl}_w^{\text{seq}} \mathbb{C}$ is equivalent to condition (4).*

Proof. Suppose that (4) does not hold. Then there is a sequence $\{x_n\}_{n \in \mathbb{N}} \subset \operatorname{conv} C$ with $\gamma_C(x_n) > n\,(|x_n|^p + 1)$. This implies that the sequence $\{\gamma_C(x_n)/(|x_n|^p + 1)\}_{n \in \mathbb{N}}$ is nonnegative and does not belong to ℓ^∞. Hence, there is a nonnegative sequence $\{\alpha_n\}_{n \in \mathbb{N}} \in \ell^1$ with

$$\sum_{n=1}^{\infty} \alpha_n \frac{\gamma_C(x_n)}{|x_n|^p + 1} = +\infty.$$

We define

$$\beta_n := \frac{\alpha_n}{|x_n|^p + 1}, \qquad B := \sum_{n=1}^{\infty} \beta_n \le \sum_{n=1}^{\infty} \alpha_n < +\infty.$$

Since $\mathfrak{m}$ is nonatomic, there is a countable partition $\{\Omega_n\}_{n \in \mathbb{N}}$ of Ω such that $\mathfrak{m}(\Omega_n) = \beta_n B^{-1} \mathfrak{m}(\Omega)$ holds. Now, we consider the function $v := \sum_{n=1}^{\infty} \chi_{\Omega_n} x_n$. We have

$$\begin{aligned}\|v\|_{L^p(\mathfrak{m};\mathbb{R}^q)}^p &= \sum_{n=1}^{\infty} \mathfrak{m}(\Omega_n)\, |x_n|^p = \frac{\mathfrak{m}(\Omega)}{B} \sum_{n=1}^{\infty} \beta_n\, |x_n|^p \le \frac{\mathfrak{m}(\Omega)}{B} \sum_{n=1}^{\infty} \alpha_n < +\infty, \\ \Gamma_C(v) &= \sum_{n=1}^{\infty} \mathfrak{m}(\Omega_n)\, \gamma_C(x_n) = \frac{\mathfrak{m}(\Omega)}{B} \sum_{n=1}^{\infty} \beta_n\, \gamma_C(x_n) = \frac{\mathfrak{m}(\Omega)}{B} \sum_{n=1}^{\infty} \alpha_n \frac{\gamma_C(x_n)}{|x_n|^p + 1} = +\infty\end{aligned}$$

which shows $v \in \operatorname{cl}_w \mathbb{C} \setminus \operatorname{cl}_w^{\text{seq}} \mathbb{C}$. This, however, implies that $\operatorname{cl}_w^{\text{seq}} \mathbb{C}$ is not closed, see Theorem 3.9. This finishes the first part of the proof.

For the converse direction of the proof, we assume that (4) holds. Let $v \in \operatorname{cl}_w \mathbb{C}$ be given. Since $\operatorname{conv} C$ is closed, see Corollary 3.2, we have $v(\omega) \in \operatorname{conv} C$ for a.a. $\omega \in \Omega$. Then, (4) implies $\gamma_C(v(\omega)) \le c\,(|v(\omega)|^p + 1)$. Hence,

$$\Gamma_C(v) = \int_\Omega \gamma_C(v(\omega))\,\mathrm{d}\omega \le c \int_\Omega (|v(\omega)|^p + 1)\,\mathrm{d}\omega = c\,(\|v\|_{L^p(\mathfrak{m};\mathbb{R}^q)}^p + \mathfrak{m}(\Omega)) < +\infty$$

holds and this shows $v \in \operatorname{cl}_w^{\text{seq}} \mathbb{C}$. This implies the closedness of $\operatorname{cl}_w^{\text{seq}} \mathbb{C}$, see Theorem 3.9. □

Lemma 3.13. *Suppose that $(\Omega, \Sigma, \mathfrak{m})$ satisfies Assumption 3.5. Let $\mathbb{C}$ be nonempty and defined as in (9) with $C \subset \mathbb{R}^q$ being nonempty and closed. Furthermore, we assume that $\mathfrak{m}(\Omega) = +\infty$ is valid. Then, the closedness of $\operatorname{cl}_w^{\text{seq}} \mathbb{C}$ is equivalent to*

$$\exists c > 0\, \forall x \in \operatorname{conv} C\colon \quad \gamma_C(x) \le c\,|x|^p. \tag{10}$$

Proof. First, let us assume that (10) is violated. We will show that $\operatorname{cl}_w^{\text{seq}} \mathbb{C}$ is not closed.

There is a sequence $\{x_n\}_{n\in\mathbb{N}} \subset \operatorname{conv} C$ with $\gamma_C(x_n) > n\,|x_n|^p$. Since $\mathfrak{m}(\Omega) = +\infty$, and since $\mathfrak{m}$ is nonatomic, there is a countable sequence $\{\Omega_n\}_{n\in\mathbb{N}}$ of pairwise disjoint subsets of Ω with $\mathfrak{m}(\Omega_n) = n^{-2}\,|x_n|^{-p}$. We set $v := \sum_{n=1}^\infty \chi_{\Omega_n} x_n$. Then, we obtain

$$\begin{aligned}
\|v\|_{L^p(\mathfrak{m};\mathbb{R}^q)}^p &= \sum_{n=1}^\infty \int_{\Omega_n} |x_n|^p\,\mathrm{d}\omega = \sum_{n=1}^\infty \mathfrak{m}(\Omega_n)\,|x_n|^p = \sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6},\\
\Gamma_C(v) &= \sum_{n=1}^\infty \int_{\Omega_n} \gamma_C(x_n)\,\mathrm{d}\omega \ge \sum_{n=1}^\infty \mathfrak{m}(\Omega_n)\,n\,|x_n|^p = \sum_{n=1}^\infty \frac{1}{n} = +\infty.
\end{aligned}$$

Hence, $v \in \operatorname{cl}_w \mathbb{C} \setminus \operatorname{cl}_w^{\text{seq}} \mathbb{C}$. Invoking Theorem 3.9 shows that $\operatorname{cl}_w^{\text{seq}} \mathbb{C}$ is not closed.

Next, we assume that condition (10) holds. Choose an arbitrary sequence $\{v_n\}_{n\in\mathbb{N}} \subset \operatorname{cl}_w^{\text{seq}} \mathbb{C}$ which converges in $L^p(\mathfrak{m};\mathbb{R}^q)$ to some $\bar v \in L^p(\mathfrak{m};\mathbb{R}^q)$. Exploiting the lower semicontinuity of Γ_C, see Lemma 3.4, and condition (10), we obtain

$$\Gamma_C(\bar v) \le \liminf_{n\to\infty} \Gamma_C(v_n) \le c \liminf_{n\to\infty} \int_\Omega |v_n(\omega)|^p\,\mathrm{d}\omega = c \liminf_{n\to\infty} \|v_n\|_{L^p(\mathfrak{m};\mathbb{R}^q)}^p < +\infty$$

which yields $\bar v \in \operatorname{cl}_w^{\text{seq}} \mathbb{C}$, see Theorem 3.8. This shows the closedness of $\operatorname{cl}_w^{\text{seq}} \mathbb{C}$ and completes the proof. □

In the next lemma, we consider the case that C is a cone. Recall that $\mathcal{B} \subset \mathbb{R}^q$ denotes the closed unit ball.

Lemma 3.14. *Suppose that $(\Omega, \Sigma, \mathfrak{m})$ satisfies Assumption 3.5. Let $C \subset \mathbb{R}^q$ be a nonempty and closed cone and consider the decomposable set $\mathbb{C}$ defined in (9). Then, $\operatorname{cl}_w^{\text{seq}} \mathbb{C}$ is closed if and only if there exists some $r > 0$ such that the condition*

$$(\operatorname{conv} C) \cap \mathcal{B} \subset \operatorname{conv}(C \cap r\mathcal{B}) \tag{11}$$

is valid.

Proof. First, let us assume that there is some $r > 0$ such that $(\operatorname{conv} C) \cap \mathcal{B} \subset \operatorname{conv}(C \cap r\mathcal{B})$ is valid. Choose some $x \in (\operatorname{conv} C) \setminus \{0\}$ arbitrarily. Then $x/|x| \in (\operatorname{conv} C) \cap \mathcal{B}$ is valid and, hence, we find $\bar\lambda \in \Delta_{q+1}$ and $\bar w_1, \ldots, \bar w_{q+1} \in C \cap r\mathcal{B}$ with $x/|x| = \sum_{i=1}^{q+1} \bar\lambda_i \bar w_i$. Since C is a cone, we obtain

$$\gamma_C(x) = \gamma_C\left(|x|\frac{x}{|x|}\right) = |x|^p\,\gamma_C\left(\frac{x}{|x|}\right) \le |x|^p \sum_{i=1}^{q+1} \bar\lambda_i |\bar w_i|^p \le r^p\,|x|^p.$$

Noting that r^p is a constant independent of x, we obtain the closedness of $\operatorname{cl}_w^{\mathrm{seq}} \mathbb{C}$ from Lemmas 3.12 and 3.13 since condition (10) is valid which implies (4).

For the proof of the converse inclusion, we assume that $\operatorname{cl}_w^{\mathrm{seq}} \mathbb{C}$ is closed. Then, by means of Lemmas 3.12 and 3.13, condition (4) holds with some $c > 0$. Now, let $x \in \operatorname{conv} C$ with $|x| \le 1$ be given. Since $\gamma_C(x) \le 2\,c$, Lemma 3.1 implies the existence of $\lambda \in \Delta_{q+1}$, $v \in C^{q+1}$ with $x = \sum_{i=1}^{q+1} \lambda_i\, v_i$ and $\gamma_C(x) = \sum_{i=1}^{q+1} \lambda_i\, |v_i|^p$. Now, we define $K := \sum_{i=1}^{q+1} \lambda_i\, |v_i|$ and $\alpha_i := K/|v_i| \ge 0$, $i = 1, \ldots, q+1$. This implies

$$x = \sum_{i=1}^{q+1} \frac{\lambda_i}{\alpha_i}\,(\alpha_i\, v_i), \qquad \sum_{i=1}^{q+1} \frac{\lambda_i}{\alpha_i} = \sum_{i=1}^{q+1} \frac{\lambda_i\,|v_i|}{K} = 1, \qquad |\alpha_i\, v_i| = K \quad \forall i = 1, \ldots, q+1.$$

Since C is a cone, we have $\alpha_i\, v_i \in C$ for all $i = 1, \ldots, q+1$. Thus, $x \in \operatorname{conv}(C \cap K\,\mathcal{B})$ is valid. It remains to find an upper bound for K which is independent of x. Using Hölders inequality, we find with $1 = 1/p + 1/p'$:

$$K = \sum_{i=1}^{q+1} \lambda_i\,|v_i| = \sum_{i=1}^{q+1} (\lambda_i^{1/p}\,|v_i|)\,\lambda_i^{(p-1)/p} \le \left(\sum_{i=1}^{q+1} (\lambda_i^{1/p}\,|v_i|)^p\right)^{1/p} \left(\sum_{i=1}^{q+1} (\lambda_i^{(p-1)/p})^{p'}\right)^{1/p'}.$$

Using $p' = p/(p-1)$ and $\lambda \in \Delta_{q+1}$, this implies

$$K \le \left(\sum_{i=1}^{q+1} \lambda_i\,|v_i|^p\right)^{1/p} \le \gamma_C(x)^{1/p} = (2\,c)^{1/p}.$$

Hence, (11) is satisfied with $r = (2\,c)^{1/p}$. □

Using the above lemma, we obtain the following result.

Corollary 3.15. *Suppose that $(\Omega, \Sigma, \mathfrak{m})$ satisfies Assumption 3.5. Let $C \subset \mathbb{R}^q$ be a nonempty, closed cone such that $\operatorname{conv} C$ is a closed cone and one of the following conditions is satisfied:*

(i) *$\operatorname{conv} C$ is pointed (i.e. we have $\operatorname{conv} C \cap (-\operatorname{conv} C) = \{0\}$),*

(ii) *$\operatorname{conv} C$ is polyhedral (i.e. it coincides with the conic convex hull of finitely many vectors),*

(iii) $q \le 3$.

We consider the decomposable set $\mathbb{C}$ *defined in* (9)*. Then* $\operatorname{cl}_w^{\text{seq}} \mathbb{C}$ *is closed.*

Proof. For the proof, we only need to show that condition (11) holds, cf. Lemma 3.14.

Case (i): Due to the pointedness of $\operatorname{conv} C$, the polar cone has an interior point, i.e., there is $-a \in \operatorname{int} C^\circ$. This implies

$$\forall c \in \operatorname{conv} C \setminus \{0\}: \quad a \cdot c > 0.$$

Clearly, we have

$$(\operatorname{conv} C) \cap \mathcal{B} \subset \{x \in \mathbb{R}^q \,|\, a \cdot x \le |a|\}$$

and the set $C^a := \{c \in C \,|\, a \cdot c \le |a|\}$ is bounded. Consequently, there is some scalar $r > 0$ which satisfies $C^a \subset C \cap r\mathcal{B}$.

Now, choose $x \in (\operatorname{conv} C) \cap \mathcal{B}$ different from 0 arbitrarily. Then, we find $\lambda \in \Delta_{q+1}$ and $v_1, \ldots, v_{q+1} \in C \setminus \{0\}$ such that $x = \sum_{i=1}^{q+1} \lambda_i v_i$. For any $i = 1, \ldots, q+1$, we set $\beta_i := (a \cdot x)/(a \cdot v_i) > 0$. Then, we have

$$x = \sum_{i=1}^{q+1} \frac{\lambda_i}{\beta_i} (\beta_i v_i), \qquad \sum_{i=1}^{q+1} \frac{\lambda_i}{\beta_i} = \sum_{i=1}^{q+1} \frac{\lambda_i \, (a \cdot v_i)}{a \cdot x} = \frac{a \cdot \left(\sum_{i=1}^{q+1} \lambda_i v_i\right)}{a \cdot x} = 1,$$

and $\beta_i v_i \in C$ for all $i = 1, \ldots, q+1$. Furthermore, $a \cdot (\beta_i v_i) = \beta_i \, (a \cdot v_i) = a \cdot x \le |a|$, i.e. $\beta_i v_i \in C^a$ holds for all $i = 1, \ldots, q+1$. This shows $x \in \operatorname{conv} C^a \subset \operatorname{conv}(C \cap r\mathcal{B})$ by construction and, thus, (11) holds.

Case (ii): By assumption, there are a number $N \in \mathbb{N}$ and a set $\{v_1, \ldots, v_N\} \subset \operatorname{conv} C$ with $\operatorname{conv} C = \operatorname{cone} \operatorname{conv}\{v_1, \ldots, v_N\}$. We define

$$\mathcal{I} := \big\{ I \subset \{1, \ldots, N\} \,\big|\, \text{the family } \{v_i\}_{i \in I} \text{ is linearly independent} \big\}.$$

Further, we set

$$t := \min_{I \in \mathcal{I}} \min_{\lambda \in \Delta_I} \left| \sum_{i \in I} \lambda_i \, v_i \right|.$$

Therein, we used $\Delta_I := \{\lambda \in \Delta_N \mid \lambda_i = 0 \text{ for all } i \notin I\}$. Due to the linear independence of the family $\{v_i\}_{i \in I}$ for $I \in \mathcal{I}$, we have $t > 0$. Further, we choose $s > 0$ such that

$$v_i \in \operatorname{conv}(C \cap s\,\mathcal{B}) \quad \forall i = 1, \ldots, N.$$

Now, let $x \in (\operatorname{conv} C) \cap \mathcal{B}$ be arbitrary. By Carathéodory's theorem, there is an index set $I \in \mathcal{I}$, $\lambda \in \Delta_I$ and $\alpha \ge 0$ with $x = \alpha \sum_{i \in I} \lambda_i \, v_i$. Hence,

$$1 \ge |x| = \alpha \left| \sum_{i \in I} \lambda_i \, v_i \right| \ge \alpha \, t.$$

This shows $\alpha \le t^{-1}$ and from $x = \sum_{i \in I} \lambda_i \, (\alpha \, v_i)$, we find

$$x \in \operatorname{conv}(\alpha \operatorname{conv}(C \cap s\,\mathcal{B})) = \operatorname{conv}(C \cap \alpha\, s\,\mathcal{B}) \subset \operatorname{conv}(C \cap t^{-1} s\,\mathcal{B}).$$

Thus, (11) is verified with $r = t^{-1} s$.

Case (iii): If $\operatorname{conv} C$ is pointed, we can apply case (i). Otherwise, let $L \subset \operatorname{conv} C$ be the largest subspace contained in $\operatorname{conv} C$ and set $K := L^\perp \cap (\operatorname{conv} C)$. Then we obtain $\operatorname{conv} C = L + K$. Since the cone K is at most two-dimensional, it is polyhedral. This shows that $\operatorname{conv} C$ is polyhedral, and we can apply case (ii). $\square$

3.4 Counterexamples

In this section, we provide some counterexamples in which the weak sequential closure is not closed. We give a brief overview of the different features of these counterexamples.

Example 3.16: We use the simple, two-dimensional set $C := (\mathbb{R}_0^+ \times \{0\}) \cup \{(0,1)\}$, which possesses a non-closed convex hull, and $\Omega = (0,1)$ equipped with Lebesgue's measure. The weak sequential closure of the associated decomposable set is strictly smaller than the decomposable set associated with the pointwise convex hull. Hence, this example shows that both inclusions in (8) may be strict at the same time.

Example 3.17: We use $C := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \le \sqrt{|x_1|}\}$. Here, the convex hull $\operatorname{conv} C = \mathbb{R}^2$ is closed. Nevertheless, condition (4) is violated.

Example 3.18: We use $C := \{2^{2^n} \mid n \in \mathbb{N}\}$. Again, the convex hull $\operatorname{conv} C = \mathbb{R}$ is closed, but (4) is violated. Hence, even in $\mathbb{R}$, the closedness of $\operatorname{conv} C$ does not imply (4).

Example 3.19: We consider $C = \{0, 1\} \subset \mathbb{R}$ which has a closed convex hull. However, (10) is not satisfied while (4) holds.

Example 3.20: We use

$$C := \operatorname{cone}\left(\{(1,0,0,0), (-1,0,0,0)\} \cup \left\{\left(\frac{1}{2\pi - t}, 1, \cos(t), \sin(t)\right) \middle| t \in [0, 2\pi)\right\}\right).$$

It is shown that C is a closed cone and $\operatorname{conv} C$ is closed, too. However, condition (11) is violated. Hence, even for closed cones, the closedness of the convex hull does not imply (4). Due to Corollary 3.15 (iii), we cannot construct such a counterexample in less than four dimensions.

Example 3.16. In this example, we demonstrate that the inclusions in (8) might be strict at the same time.

We consider $\Omega = (0,1)$ equipped with the Lebesgue measure, $p \in (1, \infty)$, and

$$C := (\mathbb{R}_0^+ \times \{0\}) \cup \{(0,1)\} \subset \mathbb{R}^2.$$

Obviously, the associated set $\mathbb{C} \subset L^p(\mathfrak{m}; \mathbb{R}^2)$ defined in (9) is nonempty, closed, and decomposable. It can be checked that

$$\operatorname{conv} C = (\{0\} \times [0,1]) \cup (\mathbb{R}^+ \times [0,1))$$

holds. Furthermore, we denote by

$$\mathbb{D} := \{v \in L^p(\mathfrak{m}; \mathbb{R}^2) \mid v(\omega) \in \operatorname{conv} C \text{ f.a.a. } \omega \in \Omega\}$$

the pointwise convex hull associated to $\mathbb{C}$.

Observe that we have

$$\forall (x_1, x_2) \in \mathbb{R}^2\colon \quad \gamma_C(x_1, x_2) = \begin{cases} x_1^p\,(1-x_2)^{1-p} + x_2 & \text{if } (x_1, x_2) \in \operatorname{conv} C, \\ +\infty & \text{if } (x_1, x_2) \notin \operatorname{conv} C. \end{cases}$$

For some $\alpha > 0$, we consider the function $\bar v \in \mathbb{D}$ defined by $\bar v(\omega) = (1, 1 - \omega^\alpha)$ for $\omega \in \Omega$. We obtain

$$\begin{aligned} \Gamma_C(\bar v) &= \int_0^1 \left[(\omega^\alpha)^{1-p} + 1 - \omega^\alpha\right] \mathrm{d}\omega \\ &= \frac{1}{\alpha\,(1-p)+1} + 1 - \frac{1}{\alpha+1} - \frac{1}{\alpha\,(1-p)+1} \lim_{\omega \searrow 0} \omega^{\alpha\,(1-p)+1} \end{aligned}$$

which diverges for $\alpha > 1/(p-1)$. Thus, in this case, we obtain $\bar v \notin \operatorname{cl}_w^{\mathrm{seq}} \mathbb{C}$ from Theorem 3.8.

However, we have

$$\begin{aligned} \operatorname{cl}_w \mathbb{C} &= \{v \in L^p(\mathfrak{m}; \mathbb{R}^2) \mid v(\omega) \in \overline{\operatorname{conv}}\, C \text{ f.a.a. } \omega \in \Omega\} \\ &= \{v \in L^p(\mathfrak{m}; \mathbb{R}^2) \mid v(\omega) \in \mathbb{R}_0^+ \times [0,1] \text{ f.a.a. } \omega \in \Omega\}, \end{aligned}$$

see Lemma 3.6, i.e. $\operatorname{cl}_w^{\mathrm{seq}} \mathbb{C} \subsetneq \mathbb{D} \subsetneq \operatorname{cl}_w \mathbb{C}$. ◊

Example 3.17. We give an example similar to Example 3.16, but in which the pointwise convex hull is already closed. We set

$$C := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \le \sqrt{|x_1|}\}.$$

Obviously, we have $\operatorname{conv} C = \mathbb{R}^2$. For given $p \in (1, \infty)$, one can compute

$$\forall x_2 > 0\colon \quad \gamma_C(0, x_2) = (x_2^2 + x_2^4)^{\frac{p}{2}}.$$

In order to do so, one can proceed in three steps:

First, we show that for $(\lambda, v) \in \Delta_3 \times C^3$ with pairwise disjoint v_i satisfying $(0, x_2) = \sum_{i=1}^3 \lambda_i\, v_i$ and $\lambda_1, \lambda_2, \lambda_3 > 0$, we have $\gamma_C(0, x_2) < \sum_{i=1}^3 \lambda_i |v_i|^p$. Clearly, if v_1 belongs to the left half space while v_2, v_3 belong to the right half space, then there is a point $\tilde v$ on the line segment between v_2 and v_3 such that $(0, x_2)$ belongs to the convex hull of v_1 and $\tilde v$. Obviously, the corresponding convex combination yields a better function value in (1), since $v \mapsto |v|^p$ is strictly convex.

Thus, we can assume that $(\bar\lambda, \bar v) \in \Delta_2 \times C^2$ is a minimizer of (1). Next, we show $\bar v_1, \bar v_2 \in \operatorname{bd} C = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 = \sqrt{|x_1|}\}$. Assuming $\bar v_1 \notin \operatorname{bd} C$ on the contrary, for sufficiently small $\varepsilon > 0$, we have $\tilde v := \bar v_1 + \varepsilon(\bar v_2 - \bar v_1) \in C$. From $(0, x_2) = \bar\lambda_1 \bar v_1 + (1 - \bar\lambda_1)\bar v_2$, we obtain $(0, x_2) = \frac{\bar\lambda_1}{1-\varepsilon}\tilde v + \left(1 - \frac{\bar\lambda_1}{1-\varepsilon}\right)\bar v_2$. Using the strict convexity of $v \mapsto |v|^p$, we obtain

$$\gamma_C(0, x_2) \le \tfrac{\bar\lambda_1}{1-\varepsilon}|\tilde v|^p + \left(1 - \tfrac{\bar\lambda_1}{1-\varepsilon}\right)|\bar v_2|^p < \bar\lambda_1 |\bar v_1|^p + (1 - \bar\lambda_1)|\bar v_2|^p = \bar\lambda_1 |\bar v_1|^p + \bar\lambda_2 |\bar v_2|^p$$

which contradicts the optimality of $(\bar\lambda, \bar v)$.

Finally, we show that $(\bar\lambda, \bar v)$, with $\bar\lambda_1 = \bar\lambda_2 = \frac{1}{2}$, $\bar v_1 = (x_2^2, x_2)$, and $\bar v_2 = (-x_2^2, x_2)$, is actually a minimizer of (1). Therefore, assume that a minimizer $(\hat\lambda, \hat v) \in \Delta_2 \times \operatorname{bd} C^2$ of (1) satisfies $\hat v_1 = (\tilde x_2^2, \tilde x_2)$ and $\hat v_2 = (-x_2^2, x_2)$ for $\tilde x_2, x_2 \geq 0$ and $\tilde x_2 \neq x_2$. Set $\hat v_3 := (x_2^2, x_2)$, $\hat v_4 := (-\tilde x_2^2, \tilde x_2)$ as well as $\mu_1 = \mu_4 := \frac{\hat\lambda_1}{2}$ and $\mu_2 = \mu_3 := \frac{\hat\lambda_2}{2}$. Now, we apply the first of our above arguments to the pairs $\hat v_1$, $\hat v_3$ and $\hat v_2$, $\hat v_4$ in order to find a feasible point of (1) with a better objective value than $(\hat\lambda, \hat v)$. This is a contradiction. Thus, a minimizer of (1) needs to satisfy $\tilde x_2 = x_2$. This shows that $(\bar\lambda, \bar v)$ as presented above is a minimizer to (1).

We fix an arbitrary bounded domain $\Omega \subset \mathbb{R}^d$ and equip it with Lebesgue's measure. Next, we pick a function

$$w \in \left\{ v \in L^p(\mathfrak{m}) \setminus L^{2p}(\mathfrak{m}) \,\middle|\, v(\omega) > 0 \text{ f.a.a. } \omega \in \Omega \right\}$$

and consider $\bar v := (0, w) \in L^p(\mathfrak{m}; \mathbb{R}^2)$. From $\operatorname{conv} C = \mathbb{R}^2$ we find $\operatorname{cl}_w \mathbb{C} = L^p(\mathfrak{m}; \mathbb{R}^2)$, where $\mathbb{C}$ denotes the nonempty, closed, decomposable set associated to the set C, see (9). Hence, $v \in \operatorname{cl}_w \mathbb{C}$. However, we easily see

$$\Gamma_C(\bar v) = \int_\Omega \gamma_C(0, w(\omega)) \,\mathrm{d}\omega = \int_\Omega (w^2(\omega) + w^4(\omega))^{p/2} \,\mathrm{d}\omega \geq \|w\|_{L^{2p}(\mathfrak{m})}^{2p} = +\infty.$$

Thus, $\bar v$ does not belong to $\operatorname{cl}_w^{\mathrm{seq}} \mathbb{C}$. ◊

Example 3.18. Let us consider the set

$$C := \left\{ 2^{2^n} \,\middle|\, n \in \mathbb{N} \right\},$$

$p \in (1, \infty)$, and the associated pointwise defined set $\mathbb{C} \subset L^p(\mathfrak{m})$ given in (9). Here, $\Omega \subset \mathbb{R}^d$ is an arbitrary bounded domain which is equipped with Lebesgue's measure.

For arbitrary $n \in \mathbb{N}$, we define

$$\alpha_n := 2^{-(p+1)\,2^{n-1}} \in (0,1)$$

and

$$w_n := (1 - \alpha_n)\, 2^{2^n} + \alpha_n\, 2^{2^{n+1}}.$$

Then, we can check

$$\gamma_C(w_n) = (1 - \alpha_n)\, 2^{p\,2^n} + \alpha_n\, 2^{p\,2^{n+1}} \geq \alpha_n\, 2^{p\,2^{n+1}} = 2^{(3p-1)\,2^{n-1}} = 2^{(p-1)\,2^{n-1}} 2^{p\,2^n}$$

as well as

$$|w_n|^p \leq 2^{p-1} \left[(1 - \alpha_n)^p\, 2^{p\,2^n} + \alpha_n^p\, 2^{p\,2^{n+1}} \right] \leq 2^{p-1}\, 2^{p\,2^n} \left[1 + \alpha_n^p\, 2^{p\,2^n} \right] \leq 2^p\, 2^{p\,2^n},$$

and taking the limit $n \to \infty$, we obtain

$$\frac{\gamma_C(w_n)}{|w_n|^p} \geq \frac{2^{(p-1)\,2^{n-1}}}{2^p} = 2^{(p-1)\,2^{n-1} - p} \to +\infty.$$

Together with the trivial estimate

$$|w_n|^{(p-1)/8} \le 2^{(p-1)\,2^{n-2}},$$

we even get

$$\frac{\gamma_C(w_n)}{|w_n|^{p+(p-1)/8}} \ge 2^{(p-1)\,2^{n-1}-p-(p-1)\,2^{n-2}} = 2^{(p-1)\,2^{n-2}-p} \ge 1,$$

where the last inequality holds for $n \ge N$ where $N \in \mathbb{N}$ is chosen large enough.

Now, choose a function

$$\bar{v} \in \left\{ v \in L^p(\mathfrak{m}) \setminus L^{p+(p-1)/8}(\mathfrak{m}) \,\middle|\, v(\omega) \in \{w_n \,|\, n \in \mathbb{N},\, n \ge N\} \text{ f.a.a. } \omega \in \Omega \right\}.$$

Since we have $\operatorname{conv} C = [2, \infty)$, $\bar{v}$ is an element of $\operatorname{cl}_w \mathbb{C}$. On the other hand,

$$\Gamma_C(\bar{v}) = \int_\Omega \gamma_C(\bar{v}(\omega))\,\mathrm{d}\omega \ge \int_\Omega |\bar{v}(\omega)|^{p+(p-1)/8}\,\mathrm{d}\omega = \|\bar{v}\|^{p+(p-1)/8}_{L^{p+(p-1)/8}(\mathfrak{m})} = +\infty$$

is obtained. Thus, $\bar{v}$ does not belong to $\operatorname{cl}_w^{\text{seq}} \mathbb{C}$ which, therefore, cannot be closed by Theorem 3.9. ◊

Example 3.19. Let us consider the unbounded domain $\Omega = (1, \infty)$ equipped with Lebesgue's measure, $p \in (1, \infty)$, and the closed set $C := \{0, 1\} \subset \mathbb{R}$.

Clearly, we have $\operatorname{conv} C = \overline{\operatorname{conv}}\, C = [0, 1]$ and $\gamma_C(x) = x$ for all $x \in \operatorname{conv} C$. Consequently, the condition (4) is valid while (10) is violated, i.e. $\operatorname{cl}_w^{\text{seq}} \mathbb{C}$, where $\mathbb{C} \subset L^p(\mathfrak{m})$ denotes the associated decomposable set defined in (9), is not closed. Let us visualize this by means of an example.

We have

$$\operatorname{cl}_w \mathbb{C} = \{v \in L^p(\mathfrak{m}) \,|\, v(\omega) \in [0, 1] \text{ f.a.a. } \omega \in \Omega\}$$

from Lemma 3.6. Let us consider the function $\bar{v} \colon \Omega \to \mathbb{R}$ defined by $\bar{v}(\omega) := \omega^{-1}$ for any $\omega \in \Omega$. Due to

$$\|\bar{v}\|^p_{L^p(\mathfrak{m})} = \int_1^\infty \omega^{-p}\,\mathrm{d}\omega = \frac{1}{1-p}\left(\lim_{\omega\to\infty} \frac{1}{\omega^{p-1}} - 1\right) = \frac{1}{p-1},$$

we have $\bar{v} \in \operatorname{cl}_w \mathbb{C}$. On the other hand,

$$\Gamma_C(\bar{v}) = \int_1^\infty \gamma_C(\bar{v}(\omega))\,\mathrm{d}\omega = \int_1^\infty \omega^{-1}\,\mathrm{d}\omega = \lim_{\omega\to\infty} \ln\omega = +\infty.$$

Following Theorem 3.8, we have $\bar{v} \notin \operatorname{cl}_w^{\text{seq}} \mathbb{C}$ and, thus, $\operatorname{cl}_w^{\text{seq}} \mathbb{C}$ is not closed by Theorem 3.9. ◊

Example 3.20. We define

$$C := \operatorname{cone}\left(\{(1,0,0,0), (-1,0,0,0)\} \cup \left\{ \left(\frac{1}{2\pi - t}, 1, \cos(t), \sin(t)\right) \,\middle|\, t \in [0, 2\,\pi) \right\} \right) \subset \mathbb{R}^4.$$

Let us check that C is closed. It is sufficient to consider convergent sequences $\{x_n\}_{n\in\mathbb{N}}$ of the form

$$x_n = \alpha_n \left(\frac{1}{2\pi - t_n}, 1, \cos(t_n), \sin(t_n) \right)$$

with $\{\alpha_n\}_{n\in\mathbb{N}} \subset (0,\infty)$ and $\{t_n\}_{n\in\mathbb{N}} \subset [0, 2\,\pi)$. By considering the second component of $\{x_n\}$, we obtain the boundedness of α_n, thus, $\alpha_n \to \alpha$ holds along a subsequence. Since $\{t_n\}$ is bounded, we find $t_n \to t$ along a subsequence. If $\alpha = 0$, we find

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \left(\frac{\alpha_n}{(2\pi - t_n)}, 0, 0, 0 \right) \in C.$$

On the other hand, $\alpha > 0$ implies $t \neq 2\,\pi$. This yields

$$\lim_{n\to\infty} x_n = \alpha \left(\frac{1}{2\pi - t}, 1, \cos(t), \sin(t) \right) \in C.$$

This shows the closedness of C.

Next, we compute the convex hull of C. We have

$$\begin{aligned} \operatorname{conv} C &= \operatorname{conv} \operatorname{cone}\big(\big\{(1,0,0,0), (-1,0,0,0)\big\} \cup \{(0,1,\cos(t),\sin(t)) \,|\, t \in [0, 2\,\pi)\}\big) \\ &= \big(\mathbb{R} \times \{0\}^3\big) + \big(\{0\} \times L\big), \end{aligned}$$

where

$$L := \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \;\middle|\; x_1 \geq \sqrt{x_2^2 + x_3^2} \right\}$$

is the Lorentz cone in $\mathbb{R}^3$. Hence, $\operatorname{conv} C$ is closed.

Finally, we consider the points

$$y_n = (0, 1, \cos(t_n), \sin(t_n))$$

with $t_n = 2\pi - 1/n$. It is clear that $y_n \in \operatorname{conv}(C) \cap \sqrt{2}\,\mathcal{B}$ is valid. Now, we are writing y_n as a convex combination of vectors from C, i.e.

$$y_n = \sum_{i=1}^{M_n} \lambda_i^n \, c_i^n,$$

with $\lambda_i^n > 0$, $c_i^n \in C$, and $\sum_{i=1}^{M_n} \lambda_i^n = 1$ for all $i = 1, \ldots, M_n$ and $n \in \mathbb{N}$. It is clear that none of the c_i^n can be of the form

$$\alpha \left(\frac{1}{2\pi - t}, 1, \cos(t), \sin(t) \right)$$

with $t \neq t_n$. Thus, we can write

$$\begin{aligned} c_i^n &= \alpha_i^n \, (1, 0, 0, 0) && \text{for } i = 1, \ldots, M_n^1, \\ c_i^n &= \alpha_i^n \, (-1, 0, 0, 0) && \text{for } i = M_n^1 + 1, \ldots, M_n^2, \\ c_i^n &= \alpha_i^n \, (n, 1, \cos(t_n), \sin(t_n)) && \text{for } i = M_n^2 + 1, \ldots, M_n \end{aligned}$$

for scalars $\alpha_i^n \geq 0$, $i = 1, \ldots, M_n$, and $n \in \mathbb{N}$. We set $\alpha^n := \max\{\alpha_i^n \,|\, i = 1, \ldots, M_n\}$. By considering the second component of y_n, we get

$$1 = \sum_{i=M_n^2+1}^{M_n} \lambda_i^n \, \alpha_i^n.$$

The first component of y_n yields

$$0 = \sum_{i=1}^{M_n^1} \lambda_i^n \, \alpha_i^n - \sum_{i=M_n^1+1}^{M_n^2} \lambda_i^n \, \alpha_i^n + \sum_{i=M_n^2+1}^{M_n} \lambda_i^n \, \alpha_i^n \, n = \sum_{i=1}^{M_n^1} \lambda_i^n \, \alpha_i^n - \sum_{i=M_n^1+1}^{M_n^2} \lambda_i^n \, \alpha_i^n + n.$$

Thus, we obtain

$$\alpha^n \geq \sum_{i=M_n^1+1}^{M_n^2} \lambda_i^n \, \alpha_i^n \geq n.$$

Hence, $y_n \notin \operatorname{conv}(C \cap r\,\mathcal{B})$ is valid for $r < n$. Taking the limit $n \to \infty$ shows that (11) is violated. By means of Lemma 3.14, the weak sequential closure of the associated decomposable set $\mathbb{C} \subset L^p(\mathfrak{m}; \mathbb{R}^4)$ defined in (9) cannot be closed. $\Diamond$

4 Weak sequential closure in purely atomic measure spaces

If $(\Omega, \Sigma, \mathfrak{m})$ is σ-additive and purely atomic, there are at most countably many non-equivalent atoms (w.r.t. the equivalence relation defined by $A \sim B \iff \mathfrak{m}(A \triangle B) = 0$). In the case that the measure space contains only finitely many non-equivalent atoms, the spaces $L^p(\mathfrak{m})$ will be finite dimensional and, thus, are isometric to some $\mathbb{R}^n$ with a weighted p-norm. Hence, we will assume that there are countably many atoms $\{A_k\}_{k \in \mathbb{N}} \subset \Sigma$.

Assumption 4.1. *We assume that $(\Omega, \Sigma, \mathfrak{m})$ is a σ-finite, complete, and purely atomic measure space, and we fix $p \in (1, \infty)$. Moreover, we assume that there are countably many non-equivalent atoms $\{A_k\}_{k \in \mathbb{N}} \subset \Sigma$.*

Recall that a measurable function $u \colon \Omega \to \mathbb{R}$ is constant a.e. on every atom A_k. We denote by $u|_{A_k} \in \mathbb{R}$ the value of u on the atom A_k. This implies that $L^p(\mathfrak{m})$ is isometric to the sequence space $L^p(\mathfrak{n})$, where the measure $\mathfrak{n} \colon 2^{\mathbb{N}} \to \mathbb{R}_0^+$ is given by $\mathfrak{n}(N) = \sum_{k \in N} \mathfrak{m}(A_k)$ for all $N \subset \mathbb{N}$. The isometric isomorphism $\mathcal{I} \colon L^p(\mathfrak{m}) \to L^p(\mathfrak{n})$ is given by

$$(\mathcal{I}(u))(k) = u|_{A_k}.$$

Next, we will demonstrate the well-known fact that weak convergence implies pointwise convergence on purely atomic spaces.

Lemma 4.2. *Assume that Assumption 4.1 is satisfied. Let $\{u_i\}_{i \in \mathbb{I}} \subset L^p(\mathfrak{m})$ be a given net over the directed set $\mathbb{I}$ such that $u_i \rightharpoonup u \in L^p(\mathfrak{m})$. Then, $u_i \to u$ a.e. on Ω.*

Proof. Due to $\mathfrak{m}(A_k) < +\infty$, we have $\chi_{A_k} \in L^{p'}(\mathfrak{m})$. Hence, we obtain

$$u_i|_{A_k}\, \mathfrak{m}(A_k) = \int_\Omega u_i\, \chi_{A_k}\, \mathrm{d}\omega \to \int_\Omega u\, \chi_{A_k}\, \mathrm{d}\omega = u|_{A_k}\, \mathfrak{m}(A_k)$$

for all $k \in \mathbb{N}$. This shows the desired pointwise convergence. □

As a corollary, we find that the weak closure is quite well-behaved for decomposable sets in Lebesgue spaces over a purely atomic measure.

Corollary 4.3. *Suppose that Assumption 4.1 is satisfied and that $\mathbb{C} \subset L^p(\mathfrak{m}; \mathbb{R}^q)$ is nonempty and decomposable. Then, $\mathbb{C}$ is closed if and only if $\mathbb{C}$ is weakly closed. Moreover, we obtain $\operatorname{cl}_w \mathbb{C} = \operatorname{cl}_w^{\mathrm{seq}} \mathbb{C} = \operatorname{cl} \mathbb{C}$.*

Proof. In order to verify the first assertion, we have to show that the closedness of $\mathbb{C}$ implies that $\mathbb{C}$ is weakly closed. Since $\mathbb{C}$ is closed, we know that it possesses the representation

$$\mathbb{C} = \{u \in L^p(\mathfrak{m}; \mathbb{R}^q) \mid u(\omega) \in C(\omega) \text{ f.a.a. } \omega \in \Omega\},$$

where the associated measurable set-valued map $C \colon \Omega \rightrightarrows \mathbb{R}^q$ has nonempty and closed images. If the net $\{u_i\}_{i \in \mathbb{I}}$ converges weakly to u in $L^p(\mathfrak{m}; \mathbb{R}^q)$, we can invoke Lemma 4.2 and obtain

$$u(\omega) = \lim\nolimits_{i \in \mathbb{I}} u_i(\omega) \in C(\omega)$$

by closedness of $C(\omega)$. Hence, $u \in \mathbb{C}$ holds true and, thus, $\mathbb{C}$ is weakly closed.

Let us prove the second assertion. We obviously have $\operatorname{cl} \mathbb{C} \subset \operatorname{cl}_w^{\mathrm{seq}} \mathbb{C} \subset \operatorname{cl}_w \mathbb{C}$. By applying the weak closure to $\mathbb{C} \subset \operatorname{cl} \mathbb{C}$, we find $\operatorname{cl}_w \mathbb{C} \subset \operatorname{cl}_w \operatorname{cl} \mathbb{C} = \operatorname{cl} \mathbb{C}$ since $\operatorname{cl} \mathbb{C}$ is closed and decomposable, thus weakly closed by the first assertion. This proves the equalities $\operatorname{cl} \mathbb{C} = \operatorname{cl}_w^{\mathrm{seq}} \mathbb{C} = \operatorname{cl}_w \mathbb{C}$. □

Again, we emphasize that this result is quite surprising since the weak topology on infinite-dimensional Banach spaces does not cooperate well with sequences. However, Corollary 4.3 demonstrates that this does not happen for decomposable sets.

Finally, we note that Corollary 4.3 is also true if the measure space contains only finitely many atoms, since $L^p(\mathfrak{m})$ is finite dimensional in this case. Hence, it is true for all σ-finite, complete, and purely atomic measure spaces.

5 Variational objects associated with decomposable sets

Throughout this section, we postulate the following standing assumption.

Assumption 5.1. *We assume that $(\Omega, \Sigma, \mathfrak{m})$ is a σ-finite, complete, and separable measure space.*

Furthermore, let $\mathbb{K} \subset L^p(\mathfrak{m}; \mathbb{R}^q)$ be a nonempty, closed, and decomposable set with $p \in (1, \infty)$. By $K \colon \Omega \rightrightarrows \mathbb{R}^q$, we denote the associated measurable and closed-valued multifunction. We assume that the images of K are derivable almost everywhere in Ω. Let $p' \in (1, \infty)$ be the conjugate coefficient associated to p, i.e. the scalar which satisfies $1/p + 1/p' = 1$.

Finally, we fix some point $\bar{u} \in \mathbb{K}$.

We start our considerations by recalling some of the results we obtained in Mehlitz and Wachsmuth [2016] for nonatomic measure spaces. However, the proofs can be adapted to the more general setting which is why we present the basic steps of our argumentation below.

Lemma 5.2. *We have*

$$\mathcal{T}_{\mathbb{K}}(\bar{u}) = \left\{d \in L^p(\mathfrak{m};\mathbb{R}^q) \,\middle|\, d(\omega) \in \mathcal{T}_{K(\omega)}(\bar{u}(\omega)) \text{ f.a.a. } \omega \in \Omega\right\},$$
$$\widehat{\mathcal{N}}_{\mathbb{K}}(\bar{u}) = \left\{\eta \in L^{p'}(\mathfrak{m};\mathbb{R}^q) \,\middle|\, \eta(\omega) \in \widehat{\mathcal{N}}_{K(\omega)}(\bar{u}(\omega)) \text{ f.a.a. } \omega \in \Omega\right\},$$

and

$$\mathcal{N}^S_{\mathbb{K}}(\bar{u}) = \left\{\eta \in L^{p'}(\mathfrak{m};\mathbb{R}^q) \,\middle|\, \eta(\omega) \in \mathcal{N}_{K(\omega)}(\bar{u}(\omega)) \text{ f.a.a. } \omega \in \Omega\right\}.$$

Proof. Exploiting the pointwise derivability of K, the formula for the tangent cone is given in [Aubin and Frankowska, 2009, Corollary 8.5.2]. Next, it is possible to show the estimates

$$\mathcal{T}_{\mathbb{K}}(\bar{u}) \subset \mathcal{T}^w_{\mathbb{K}}(\bar{u}) \subset \overline{\operatorname{conv}}\, \mathcal{T}_{\mathbb{K}}(\bar{u}),$$

for the weak tangent cone, see [Mehlitz and Wachsmuth, 2016, Lemma 3.6]. We mention that this proof does not use that the underlying measure is nonatomic. We polarize this chain of inclusions to obtain

$$\mathcal{T}_{\mathbb{K}}(\bar{u})^\circ = \mathcal{T}^w_{\mathbb{K}}(\bar{x})^\circ = (\overline{\operatorname{conv}}\, \mathcal{T}_{\mathbb{K}}(\bar{u}))^\circ,$$

which leads to $\widehat{\mathcal{N}}_{\mathbb{K}}(\bar{u}) = \mathcal{T}^w_{\mathbb{K}}(\bar{u})^\circ = \mathcal{T}_{\mathbb{K}}(\bar{u})^\circ$. Since $\mathcal{T}_{\mathbb{K}}(\bar{u})$ is a decomposable set containing zero, we deduce

$$\begin{aligned}\widehat{\mathcal{N}}_{\mathbb{K}}(\bar{u}) &= \left\{d \in L^p(\mathfrak{m};\mathbb{R}^q) \,\middle|\, d(\omega) \in \mathcal{T}_{K(\omega)}(\bar{u}(\omega)) \text{ f.a.a. } \omega \in \Omega\right\}^\circ \\ &= \left\{\eta \in L^{p'}(\mathfrak{m};\mathbb{R}^q) \,\middle|\, \eta(\omega) \in \mathcal{T}_{K(\omega)}(\bar{u}(\omega))^\circ \text{ f.a.a. } \omega \in \Omega\right\} \\ &= \left\{\eta \in L^{p'}(\mathfrak{m};\mathbb{R}^q) \,\middle|\, \eta(\omega) \in \widehat{\mathcal{N}}_{K(\omega)}(\bar{u}(\omega)) \text{ f.a.a. } \omega \in \Omega\right\}.\end{aligned}$$

From [Mehlitz and Wachsmuth, 2016, Theorem 3.8] we obtain that the set-valued mapping

$$\omega \mapsto \operatorname{gph} \widehat{\mathcal{N}}_{K(\omega)} := \left\{(u,\eta) \in \mathbb{R}^q \times \mathbb{R}^q \,\middle|\, u \in K(\omega),\ \eta \in \widehat{\mathcal{N}}_{K(\omega)}(u)\right\}$$

is graph measurable. Together with [Papageorgiou and Kyritsi-Yiallourou, 2009, Proposition 6.4.20] and our formula for the Fréchet normal cone, we have

$$\begin{aligned}\operatorname{gph} \mathcal{N}^s_{\mathbb{K}} &= \operatorname{cl} \operatorname{gph} \widehat{\mathcal{N}}_{\mathbb{K}} \\ &= \operatorname{cl} \left\{(u,\eta) \in L^p(\mathfrak{m};\mathbb{R}^q) \times L^{p'}(\mathfrak{m};\mathbb{R}^q) \,\middle|\, (u(\omega),\eta(\omega)) \in \operatorname{gph} \widehat{\mathcal{N}}_{K(\omega)} \text{ f.a.a. } \omega \in \Omega\right\} \\ &= \left\{(u,\eta) \in L^p(\mathfrak{m};\mathbb{R}^q) \times L^{p'}(\mathfrak{m};\mathbb{R}^q) \,\middle|\, (u(\omega),\eta(\omega)) \in \operatorname{cl} \operatorname{gph} \widehat{\mathcal{N}}_{K(\omega)} \text{ f.a.a. } \omega \in \Omega\right\} \\ &= \left\{(u,\eta) \in L^p(\mathfrak{m};\mathbb{R}^q) \times L^{p'}(\mathfrak{m};\mathbb{R}^q) \,\middle|\, (u(\omega),\eta(\omega)) \in \operatorname{gph} \mathcal{N}_{K(\omega)} \text{ f.a.a. } \omega \in \Omega\right\}.\end{aligned}$$

This shows the formula for the strong limiting normal cone. ∎

Furthermore, we obtained the inclusions

$$\operatorname{conv} \mathcal{N}^S_{\mathbb{K}}(\bar u) \subset \operatorname{cl}^{\mathrm{seq}}_w \mathcal{N}^S_{\mathbb{K}}(\bar u) \subset \mathcal{N}_{\mathbb{K}}(\bar u) \subset \mathcal{N}^C_{\mathbb{K}}(\bar u)$$

for nonatomic measure spaces in [Mehlitz and Wachsmuth, 2016, Lemma 3.10, Theorem 3.11], which led to the observation that the limiting normal cone $\mathcal{N}_{\mathbb{K}}(\bar u)$ is dense in $\mathcal{N}^C_{\mathbb{K}}(\bar u)$ as long as the underlying measure space is nonatomic. Moreover, we derived

$$\mathcal{N}^C_{\mathbb{K}}(\bar u) = \left\{\eta \in L^{p'}(\mathfrak{m};\mathbb{R}^q) \,\middle|\, \eta(\omega) \in \mathcal{N}^C_{K(\omega)}(\bar u(\omega)) \text{ f.a.a. } \omega \in \Omega\right\}$$

in this situation. However, we were not able to find an explicit formula for the limiting normal cone. Now, we are going to close this gap.

In order to prepare the proof of Proposition 5.4, we need the following property of the limiting normal cone in finite dimensions.

Lemma 5.3. *Let $L \subset \mathbb{R}^q$ be a closed set and let $\bar u \in L$ be given. Suppose that there are sequences $\{u_n\}_{n\in\mathbb{N}} \subset L$ and $\{\eta_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^q$ such that $u_n \to \bar u$ as $n \to \infty$ and $\eta_n \in \widehat{\mathcal{N}}_L(u_n)$ for all $n \in \mathbb{N}$. Then, the boundedness of $\{\eta_n\}_{n\in\mathbb{N}}$ implies* $\operatorname{dist}(\eta_n, \mathcal{N}_L(\bar u)) \to 0$.

Proof. To the contrary, suppose that there is $\varepsilon > 0$ with $\operatorname{dist}(\eta_n, \mathcal{N}_L(\bar u)) \geq \varepsilon$ along a subsequence (without relabeling). Since $\{\eta_n\}_{n\in\mathbb{N}}$ is bounded, a subsequence converges towards some $\eta \in \mathbb{R}^q$. By definition of $\mathcal{N}_L(\bar u)$, this yields $\eta \in \mathcal{N}_L(\bar u)$, and this is a contradiction to $\operatorname{dist}(\eta_n, \mathcal{N}_L(\bar u)) \geq \varepsilon$. □

The next lemma gives a characterization of the limiting normal cone in terms of the strong limiting normal cone.

Proposition 5.4. *We have*

$$\mathcal{N}_{\mathbb{K}}(\bar u) = \operatorname{cl}^{\mathrm{seq}}_w \mathcal{N}^S_{\mathbb{K}}(\bar u).$$

Proof. "$\supset$": Let $\eta \in \operatorname{cl}^{\mathrm{seq}}_w \mathcal{N}^S_{\mathbb{K}}(\bar u)$ be given. Hence, there is a sequence $\{\eta_k\}_{k\in\mathbb{N}} \subset \mathcal{N}^S_{\mathbb{K}}(\bar u)$ with $\eta_k \rightharpoonup \eta$. By definition of the strong limiting normal cone, we find $u_k \in L^p(\mathfrak{m};\mathbb{R}^q)$ and $\hat\eta_k \in \widehat{\mathcal{N}}_{\mathbb{K}}(u_k)$ with $\|u_k - \bar u\|_{L^p(\mathfrak{m};\mathbb{R}^q)} \leq 1/k$ and $\|\hat\eta_k - \eta_k\|_{L^{p'}(\mathfrak{m};\mathbb{R}^q)} \leq 1/k$. This readily implies $\hat\eta_k \rightharpoonup \eta$ and, together with $u_k \to \bar u$, we obtain $\eta \in \mathcal{N}_{\mathbb{K}}(\bar u)$.

"$\subset$": Next, we show $\mathcal{N}_{\mathbb{K}}(\bar u) \subset \operatorname{cl}^{\mathrm{seq}}_w \mathcal{N}^S_{\mathbb{K}}(\bar u)$. Let $\eta \in \mathcal{N}_{\mathbb{K}}(\bar u)$ be given. By definition, there exist sequences $\{u_i\}_{i\in\mathbb{N}} \subset \mathbb{K}$ and $\{\eta_i\}_{i\in\mathbb{N}} \subset L^{p'}(\mathfrak{m};\mathbb{R}^q)$ which satisfy $u_i \to \bar u$, $\eta_i \rightharpoonup \eta$ and $\eta_i \in \widehat{\mathcal{N}}_{\mathbb{K}}(u_i)$ for all $i \in \mathbb{N}$. W.l.o.g. we suppose that $u_i \to \bar u$ holds pointwise a.e. in Ω. We set $S := \sup_{i\in\mathbb{N}} \|\eta_i\|_{L^{p'}(\mathfrak{m};\mathbb{R}^q)} < +\infty$. For each $M \in \mathbb{N}$, we define

$$\eta_{i,M} := \eta_i \, \chi_{\{\omega\in\Omega | M > |\eta_i(\omega)|\}}.$$

Evidently, $\|\eta_{i,M}\|_{L^{p'}(\mathfrak{m};\mathbb{R}^q)} \leq S$. Moreover, the boundedness in $L^{p'}(\mathfrak{m};\mathbb{R}^q)$ implies

$$\begin{aligned}\|\eta_{i,M} - \eta_i\|_{L^1(\mathfrak{m};\mathbb{R}^q)} &\leq \mathfrak{m}(\{\omega \in \Omega \mid M \leq |\eta_i(\omega)|\})^{1/p} \, \|\eta_i\|_{L^{p'}(\mathfrak{m};\mathbb{R}^q)} \\ &\leq \left(\frac{\|\eta_i\|_{L^{p'}(\mathfrak{m};\mathbb{R}^q)}}{M}\right)^{p'/p} \|\eta_i\|_{L^{p'}(\mathfrak{m};\mathbb{R}^q)} \leq \frac{S^{p'}}{M^{p'/p}}.\end{aligned}$$

Thus $\|\eta_{i,M} - \eta_i\|_{L^1(\mathfrak{m};\mathbb{R}^q)} \to 0$ as $M \to \infty$ uniformly in $i \in \mathbb{N}$.

Next, let $\{\Omega_k\}_{k\in\mathbb{N}} \subset \Sigma$ be a nested sequence of measurable sets with finite measure which satisfies $\Omega = \bigcup_{k\in\mathbb{N}} \Omega_k$. We define

$$\mu_{i,M} := \eta_{i,M}\, \chi_{\Omega_M}.$$

Now, let $\nu_{i,M}$ be a measurable selection of

$$\omega \mapsto \operatorname*{arg\,min}_{\nu\in\mathcal{N}_{K(\omega)}(\bar u(\omega))} |\mu_{i,M}(\omega) - \nu|,$$

see [Rockafellar and Wets, 1998, Theorem 14.26] for the measurability of the mapping $\omega \mapsto \mathcal{N}_{K(\omega)}(\bar u(\omega))$, [Aubin and Frankowska, 2009, Theorem 8.2.11] for the measurability of the arg min-mapping, and [Aubin and Frankowska, 2009, Theorem 8.1.3] for the existence of a measurable selection.

Due to $0 \in \mathcal{N}_{K(\omega)}(\bar u(\omega))$, we have $\|\nu_{i,M}\|_{L^{p'}(\mathfrak{m};\mathbb{R}^q)} \le 2\,S$. The pointwise boundedness of $\mu_{i,M}$ together with $\mu_{i,M}(\omega) \in \widehat{\mathcal{N}}_{K(\omega)}(u_i(\omega))$ for a.e. $\omega \in \Omega$ allows us to invoke Lemma 5.3 with $L = K(\omega)$ for almost all $\omega \in \Omega$. This gives $\nu_{i,M} - \mu_{i,M} \to 0$ as $i \to \infty$ a.e. in Ω. Lebesgue's dominated convergence theorem (using $2\,M\,\chi_{\Omega_M}$ as upper bound) implies $\nu_{i,M} - \mu_{i,M} \to 0$ in $L^{p'}(\mathfrak{m};\mathbb{R}^q)$ as $i \to \infty$.

Now, for each M, choose $i(M) \ge M$ such that $\|\nu_{i(M),M} - \mu_{i(M),M}\|_{L^{p'}(\mathfrak{m};\mathbb{R}^q)} \le 1/M$. We claim that $\nu_{i(M),M} \rightharpoonup \eta$ in $L^{p'}(\mathfrak{m};\mathbb{R}^q)$ as $M \to \infty$. Let $g \in L^\infty(\mathfrak{m};\mathbb{R}^q) \cap L^p(\mathfrak{m};\mathbb{R}^q)$ be given. Then,

$$\begin{aligned}
|\langle g, \nu_{i(M),M} - \eta\rangle| &\le |\langle g, \nu_{i(M),M} - \mu_{i(M),M}\rangle| + |\langle g, \mu_{i(M),M} - \eta_{i(M),M}\rangle| \\
&\qquad + |\langle g, \eta_{i(M),M} - \eta_{i(M)}\rangle| + |\langle g, \eta_{i(M)} - \eta\rangle| \\
&\le \frac{1}{M}\,\|g\|_{L^p(\mathfrak{m};\mathbb{R}^q)} + \|g\,\chi_{\Omega\setminus\Omega_M}\|_{L^p(\mathfrak{m};\mathbb{R}^q)}\, S \\
&\qquad + \frac{S^{p'}}{M^{p'/p}}\,\|g\|_{L^\infty(\mathfrak{m};\mathbb{R}^q)} + |\langle g, \eta_{i(M)} - \eta\rangle|.
\end{aligned}$$

Since we have $\eta_{i(M)} \rightharpoonup \eta$ in $L^{p'}(\mathfrak{m};\mathbb{R}^q)$ as $M \to \infty$ and $\|g\,\chi_{\Omega\setminus\Omega_M}\|_{L^p(\mathfrak{m};\mathbb{R}^q)} \to 0$ as $M \to \infty$, this converges to zero as $M \to \infty$. The boundedness of $\nu_{i(M),M}$ in $L^{p'}(\mathfrak{m};\mathbb{R}^q)$ as $M \to \infty$ together with the density of $L^\infty(\mathfrak{m};\mathbb{R}^q) \cap L^p(\mathfrak{m};\mathbb{R}^q)$ in $L^p(\mathfrak{m};\mathbb{R}^q)$ implies $\nu_{i(M),M} \rightharpoonup \eta$ in $L^{p'}(\mathfrak{m};\mathbb{R}^q)$ as $M \to \infty$.

Finally, we recall $\nu_{i(M),M}(\omega) \in \mathcal{N}_{K(\omega)}(\bar u(\omega))$ and, thus, $\nu_{i(M),M} \in \mathcal{N}^S_{\mathbb{K}}(\bar u)$ follows from Lemma 5.2. Hence, we have shown $\eta \in \operatorname{cl}^{\text{seq}}_w \mathcal{N}^S_{\mathbb{K}}(\bar u)$. □

As a corollary, we obtain the following characterization of the Clarke normal cone to a decomposable set, and this characterization is not restricted to the nonatomic regime anymore, cf. [Mehlitz and Wachsmuth, 2016, Theorem 3.11].

Corollary 5.5. *We have*

$$\mathcal{N}^C_{\mathbb{K}}(\bar u) = \left\{\eta \in L^{p'}(\mathfrak{m};\mathbb{R}^q) \,\middle|\, \eta(\omega) \in \mathcal{N}^C_{K(\omega)}(\bar u(\omega)) \text{ f.a.a. } \omega \in \Omega\right\}.$$

Proof. We proceed by proving both inclusions.

"$\subset$": First, we exploit Lemma 5.2 in order to see

$$\mathcal{N}_{\mathbb{K}}^{S}(\bar{u}) \subset \left\{ \eta \in L^{p'}(\mathfrak{m}; \mathbb{R}^q) \,\middle|\, \eta(\omega) \in \mathcal{N}_{K(\omega)}^{C}(\bar{u}(\omega)) \text{ f.a.a. } \omega \in \Omega \right\}.$$

Due to the closedness and convexity of the set on the right-hand side, we obtain

$$\mathcal{N}_{\mathbb{K}}^{C}(\bar{u}) = \overline{\operatorname{conv}}\, \operatorname{cl}_w^{\text{seq}} \mathcal{N}_{\mathbb{K}}^{S}(\bar{u}) \subset \left\{ \eta \in L^{p'}(\mathfrak{m}; \mathbb{R}^q) \,\middle|\, \eta(\omega) \in \mathcal{N}_{K(\omega)}^{C}(\bar{u}(\omega)) \text{ f.a.a. } \omega \in \Omega \right\}$$

from Proposition 5.4.

"$\supset$": Noting that $\omega \mapsto \mathcal{N}_{K(\omega)}(\bar{u}(\omega))$ is graph measurable and closed-valued by [Rockafellar and Wets, 1998, Theorem 14.26], we obtain that the set-valued mapping $\omega \mapsto \overline{\operatorname{conv}}\, \mathcal{N}_{K(\omega)}(\bar{u}(\omega)) = \mathcal{N}_{K(\omega)}^{C}(\bar{u}(\omega))$ is graph measurable as well, see [Aubin and Frankowska, 2009, Theorem 8.2.2] and [Papageorgiou and Kyritsi-Yiallourou, 2009, Proposition 6.2.10]. Next, we exploit [Papageorgiou and Kyritsi-Yiallourou, 2009, Proposition 6.4.17] and Lemma 5.2 to obtain

$$\overline{\operatorname{conv}}\, \mathcal{N}_{\mathbb{K}}^{S}(\bar{u}) = \left\{ \eta \in L^{p'}(\mathfrak{m}; \mathbb{R}^q) \,\middle|\, \eta(\omega) \in \mathcal{N}_{K(\omega)}^{C}(\bar{u}(\omega)) \text{ f.a.a. } \omega \in \Omega \right\}.$$

Now, the claim follows from

$$\mathcal{N}_{\mathbb{K}}^{C}(\bar{u}) = \overline{\operatorname{conv}}\, \mathcal{N}_{\mathbb{K}}(\bar{u}) \supset \overline{\operatorname{conv}}\, \mathcal{N}_{\mathbb{K}}^{S}(\bar{u}).$$

This completes the proof. □

As we have seen in Section 3.4, the weak sequential closure of a decomposable set does not need to be closed. The upcoming example demonstrates that it is indeed possible that the limiting normal cone to a decomposable set is not closed as well. Especially, we cannot strengthen the result $\operatorname{cl} \mathcal{N}_{\mathbb{K}}(\bar{u}) = \mathcal{N}_{\mathbb{K}}^{C}(\bar{u})$ in the nonatomic case.

Example 5.6. Let us consider the nonempty, closed set $C \subset \mathbb{R}^3$ given by

$$C := \left\{ x \in \mathbb{R}^3 \,\middle|\, x_1 \geq \sqrt{x_2^2 + x_3^2} \right\} \cup \left\{ x \in \mathbb{R}^3 \,\middle|\, x_1 - x_3 = 0 \right\}.$$

Obviously, it is the union of two convex sets and, thus, derivable. Let $\mathbb{C} \subset L^p(\mathfrak{m}; \mathbb{R}^3)$ defined in (9) be the decomposable set associated with C. For the underlying measure space, we fix an arbitrary bounded domain $\Omega \subset \mathbb{R}^d$ which is equipped with the Lebesgue measure. We show the nonclosedness of the limiting normal cone $\mathcal{N}_{\mathbb{C}}(0)$.

Therefore, we compute

$$\mathcal{N}_C(0) = \left\{ \eta \in \mathbb{R}^3 \,\middle|\, -\eta_1 = \sqrt{\eta_2^2 + \eta_3^2} \right\} \cup \operatorname{span}\{(1, 0, -1)\}$$

in order to obtain

$$\mathcal{N}_{\mathbb{C}}^{S}(0) = \left\{ \eta \in L^{p'}(\mathfrak{m}; \mathbb{R}^3) \,\middle|\, \eta(\omega) \in \mathcal{N}_C(0) \text{ f.a.a. } \omega \in \Omega \right\}$$

from Lemma 5.2. It is easy to see

$$\operatorname{conv} \mathcal{N}_C(0) = \{\eta \in \mathbb{R}^3 \,|\, \eta_1 + \eta_3 < 0\} \cup \operatorname{span}\{(1, 0, -1)\}$$

which is not closed. Since we have $\mathcal{N}_{\mathbb{C}}(\bar{u}) = \operatorname{cl}_w^{\text{seq}} \mathcal{N}_{\mathbb{C}}^S(\bar{u})$ from Proposition 5.4, Lemma 3.11 shows that $\mathcal{N}_{\mathbb{C}}(\bar{u})$ is not closed and, thus, does not equal

$$\mathcal{N}_{\mathbb{C}}^C(\bar{u}) = \left\{\eta \in L^{p'}(\mathfrak{m}; \mathbb{R}^3) \,\middle|\, \eta_1(\omega) + \eta_3(\omega) \le 0 \text{ f.a.a. } \omega \in \Omega\right\}$$

which follows from Corollary 5.5. ◊

Now, we provide a statement for the tangent cone in finite dimensions which parallels Lemma 5.3. This result is needed in order to characterize the weak tangent cone to a decomposable set.

Lemma 5.7. *Let $L \subset \mathbb{R}^q$ be a closed set and let $\bar{u} \in L$ be given. Suppose that there are sequences $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ and $\{v_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^q$ such that $t_n \searrow 0$ as $n \to \infty$ and $\bar{u}+t_n v_n \in L$ for all $n \in \mathbb{N}$. Then, the boundedness of $\{v_n\}_{n\in\mathbb{N}}$ implies $\operatorname{dist}(v_n, \mathcal{T}_L(\bar{u})) \to 0$.*

Proof. To the contrary, suppose that there is $\varepsilon > 0$ with $\operatorname{dist}(v_n, \mathcal{T}_L(\bar{u})) \ge \varepsilon$ along a subsequence (without relabeling). Since $\{v_n\}_{n\in\mathbb{N}}$ is bounded, a subsequence converges towards some $v \in \mathbb{R}^q$. By definition of $\mathcal{T}_L(\bar{u})$, this yields $v \in \mathcal{T}_L(\bar{u})$, and this is a contradiction to $\operatorname{dist}(v_n, \mathcal{T}_L(\bar{u})) \ge \varepsilon$. □

The above lemma allows us to prove the following characterization of the weak tangent cone to a decomposable set. The result can be validated by transferring the proof of Proposition 5.4 to the situation at hand doing some obvious changes. Particularly, Lemma 5.3 has to be replaced by Lemma 5.7.

Proposition 5.8. *We have*

$$\mathcal{T}_{\mathbb{K}}^w(\bar{u}) = \operatorname{cl}_w^{\text{seq}} \mathcal{T}_{\mathbb{K}}(\bar{u}).$$

Our final example shows that the weak tangent cone to a nonempty, closed, decomposable set does not need to be closed.

Example 5.9. We set

$$C := \left\{x \in \mathbb{R}^3 \,\middle|\, x_1 \ge \sqrt{x_2^2 + x_3^2}\right\} \cup \operatorname{span}\{(1, 0, 1)\}.$$

which is closed as well as derivable and consider the corresponding nonempty, closed, decomposable set $\mathbb{C} \subset L^p(\mathfrak{m}; \mathbb{R}^3)$ defined as in (9) where $\Omega \subset \mathbb{R}^d$ is some bounded domain equipped with Lebesgue's measure. Obviously, we have $\mathcal{T}_{\mathbb{C}}(0) = \mathbb{C}$, and $\mathcal{T}_{\mathbb{C}}^w(0) = \operatorname{cl}_w^{\text{seq}} \mathbb{C}$ is not closed since

$$\operatorname{conv} C = \{x \in \mathbb{R}^3 \,|\, x_1 - x_3 > 0\} \cup \operatorname{span}\{(1, 0, 1)\}$$

is not closed, see Lemma 3.11. ◊

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