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SENSITIVITY ANALYSIS FOR A CLASS OF $\mathbf{H_0^1}$-ELLIPTIC VARIATIONAL INEQUALITIES OF THE SECOND KIND*

C. CHRISTOF† AND C. MEYER†

Abstract. We study the stability of solutions to H_0^1-elliptic variational inequalities of the second kind that contain a non-differentiable Nemytskii operator. The local Lipschitz continuity of the solution map with respect to perturbations of the right-hand side and perturbations of the coefficient of the Nemytskii operator is proved for a large class of problems, and directional differentiability results are obtained under comparatively mild structural assumptions. It is further shown that the directional derivatives of the solution map are typically characterized by elliptic variational inequalities in weighted Sobolev spaces whose bilinear forms contain surface integrals.

Key words. Elliptic Variational Inequalities of the Second Kind, Directional Differentiability, Sensitivity Analysis, Lipschitz Stability, Optimal Control of Variational Inequalities

AMS subject classifications. 35B30 35J20 35R45 47J20 49J40 49K40 65K15

1. Introduction. This paper is concerned with the stability analysis of a class of H_0^1-elliptic variational inequalities (VIs) of the second kind. The problems that we consider have the form

$$w \in H_0^1(\Omega),\ \ a(w, v-w) + \int_\Omega cj(v)\mathrm{d}\lambda - \int_\Omega cj(w)\mathrm{d}\lambda \geq \langle f, v-w\rangle \ \ \forall\, v \in H_0^1(\Omega), \tag{P}$$

where a is a continuous coercive bilinear form and $j : \mathbb{R} \to [0,\infty)$ is a convex function satisfying $j(0) = 0$. The precise assumptions on the quantities in (P) will be given in Section 2 below. We prove that the solution map $S : (c, f) \mapsto w$ associated with (P) is locally Lipschitz continuous and directionally differentiable under comparatively mild structural assumptions and that the directional derivatives of S are typically characterized by elliptic VIs in weighted Sobolev spaces whose bilinear forms contain surface integrals. See Theorem 2.6 and Theorem 4.14 for our main results.

Let us put our work into perspective. To keep the discussion concise, we focus on contributions dealing with infinite dimensional problems. Concerning the sensitivity analysis of finite dimensional VIs, we refer to [3,25] and the references therein. While the Lipschitz stability of elliptic VIs in general and the differential stability of elliptic VIs of the first kind have been studied in a multitude of papers (cf. [5,6,11,13–15, 17,19,20,23,24,33]), the differential stability of VIs of the second kind is only rarely addressed in the literature. To the best of our knowledge, the only contributions providing differentiability results for VIs of the second kind in infinite dimensions are [7] and [26–28]. In [26–28], where the differential stability of a class of frictional contact problems is considered, the main idea of the sensitivity analysis is to transform the VI of the second kind at hand into a VI of the first kind by means of Fenchel duality and to subsequently apply standard differentiability results as found in [14,20]. We point out that such a dual approach cannot be employed in the situation that we are concerned with. As shown in [29, Example 4.23], the admissible set of the dual

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problem to (P) is in general not polyhedric. The property of polyhedricity, however, is crucial for the application of differentiability results for elliptic VIs of the first kind, see, e.g., [20]. Consequently, dualizing the problem (P) does not simplify the sensitivity analysis in any way. The work which is closest to ours is [7], where the case $c \equiv 1$ and $j(x) = |x|$ is considered. In this paper, the authors prove that the solution mapping $f \mapsto w$ is weakly directionally differentiable under rather restrictive assumptions on the structure of the active set $\{w = 0\}$. To be more precise, they require that w is continuous, that the sets $\{w > 0\}$ and $\{w < 0\}$ have a positive distance to each other, and that the set $\{w = 0\}$ does not have $(d-1)$-dimensional components, see [7, Assumptions 3.1, 3.2]. The main idea in [7] is to reformulate the variational inequality (P) as a variational equality by introducing a slack variable q and to analyze the convergence behavior of the difference quotients associated with q and the solution w to obtain information about the regularity of the solution map. In contrast to this primal-dual approach, in the present paper we solely work with primal quantities. This allows us to substantially weaken the assumptions of [7]. In particular, we can also handle $(d-1)$-dimensional components of the set $\{w = 0\}$ and prove that these components manifest themselves as surface integrals in the elliptic VIs that characterize the directional derivatives of the solution map S. The significant change that appears in the VIs for the directional derivatives when the assumptions on the active set $\{w = 0\}$ are weakened is probably the most remarkable result of our work (cf. also with [26–28]).

The outline of this paper is the following: In Section 2, we clarify the notation, make precise our assumptions, and discuss preliminary results concerning the existence, uniqueness, and regularity of solutions to (P) that are needed for our analysis. Here, we also prove that the solution map $S : (c, f) \mapsto w$ associated with (P) is locally Lipschitz continuous in $H^1(\Omega)$ and $L^\infty(\Omega)$, see Theorem 2.6. Section 3 contains several auxiliary results and an overview of the strategy that we use in the subsequent section to study the (directional) differentiability of the solution operator S. Section 4 is concerned with the differentiability of S for functions j which are twice continuously differentiable away from the origin. We will see here that fine properties of the solution w and the second distributional derivative of j (or its pullback by the solution w to be more precise) are relevant for the differentiability properties of the solution operator S. The main result in this section is Theorem 4.14, which shows that the directional derivatives of S are characterized by elliptic variational inequalities in weighted Sobolev spaces. Lastly, in Section 5, we make some concluding remarks and compare our findings with those in [7].

2. Preliminaries and Notation. In what follows, we use the standard notation $H_0^1(\Omega)$, $W^{k,p}(\Omega)$, $C^{k,\gamma}(\overline{\Omega})$, $k \in \mathbb{N}$, $1 \le p \le \infty$, $\gamma \in (0, 1]$, for the Sobolev and Hölder spaces on a domain $\Omega \subseteq \mathbb{R}^d$, $d \ge 1$. We refer to [2, 9] for details on these spaces. The dual of $H_0^1(\Omega)$ w.r.t. $L^2(\Omega)$ and the associated dual pairing are denoted with $H^{-1}(\Omega)$ and $\langle ., . \rangle$, respectively. With λ^k and $\mathcal{H}^k$ we denote the k-dimensional Lebesgue and Hausdorff measure (where $\mathcal{H}^k$ is assumed to be scaled as in [10, Definition 2.1] such that it coincides with the surface measure on sufficiently regular sets). When the dimension is clear from the context, we drop the index k and simply write λ. Further, we define $x^+ := \max(0, x)$, $x^- := \min(0, x)$, and $L_+^p(\Omega) := \{c \in L^p(\Omega) : c \ge 0 \text{ a.e. in } \Omega\}$, $1 \le p \le \infty$. With C we denote a generic constant. If we want to emphasize that C depends on a quantity α, we write $C = C(\alpha)$. The topological closure and interior of a set S are denoted with $\mathrm{cl}(S)$ and $\mathrm{int}(S)$, respectively. For subsets of the Euclidean space, we also use the notation $\overline{S} := \mathrm{cl}(S)$. Given a function $v : \Omega \to \mathbb{R}$, we abbreviate the set $\{x \in \Omega : v(x) = 0\}$ by $\{v = 0\}$. The sets $\{v \neq 0\}$, $\{v > 0\}$, and $\{v < 0\}$ are defined analogously. Our standing assumptions on the quantities a, j, and Ω appearing in (P) are as follows:

Assumption 2.1 (Standing Assumptions).

a) $\Omega \subset \mathbb{R}^d$, $d \geq 2$, *is a bounded strong Lipschitz domain (as defined in [10, Definition 4.4.],*

b) $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ *is a (not necessarily symmetric) bilinear form defined by* $a(v_1, v_2) := \langle A v_1, v_2 \rangle$, *where* $A : H_0^1(\Omega) \to H^{-1}(\Omega)$ *is a second order partial differential operator of the type*

$$Av := -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j} \left(\alpha_{ij} \frac{\partial}{\partial x_i} v \right) + \beta v$$

with $\alpha_{ij} \in C^1(\overline{\Omega})$ *and* $\beta \in L_+^\infty(\Omega)$ *such that there exists a* $C > 0$ *with*

$$\sum_{i,j=1}^{d} \alpha_{ij}(x) \zeta_i \zeta_j \geq C|\zeta|^2 \quad \forall \zeta \in \mathbb{R}^d \quad \forall x \in \Omega,$$

c) $j : \mathbb{R} \to [0, \infty)$ *is a convex function satisfying* $j(0) = 0$.

Remark 2.2. *With minor obvious modifications the subsequent analysis also works in the case* $d = 1$, *yielding a differentiability result analogous to Theorem 4.14. The inclusion of the one-dimensional case, however, would lead to a significant notational overhead, so we restrict our attention to the case* $d > 1$.

We point out that in Section 4 we tighten the assumptions on j and confine ourselves to the situation where j can be written in the form $j(x) = j_1(x^+) + j_2(-x^-)$ with non-negative, convex functions $j_1, j_2 \in C^2([0, \infty))$ satisfying $j_1(0) = j_2(0) = 0$ and $j_1'(0), j_2'(0) > 0$. Up to then, however, the minimal regularity in Assumption 2.1 is sufficient for our needs. The next lemma collects some properties of j which will be useful in the sequel. They all result from the convexity assumption and can be proven by standard arguments from convex analysis, see, e.g., [8].

Lemma 2.3. *If* $j : \mathbb{R} \to [0, \infty)$ *satisfies the conditions in Assumption 2.1, then:*

a) j *is Lipschitz on bounded sets,*

b) j *is directionally differentiable in every* $x \in \mathbb{R}$ *in all directions* $h \in \mathbb{R}$, *and the directional derivative* $j'(x; h)$ *satisfies*

$$j'(x; h) = \inf_{t > 0} \left(\frac{j(x + th) - j(x)}{t} \right),$$

c) j *is Hadamard differentiable in every* $x \in \mathbb{R}$ *in all directions* $h \in \mathbb{R}$, *i.e., if* $(h_n) \subset \mathbb{R}$, $(t_n) \subset (0, \infty)$ *are sequences satisfying* $h_n \to h$ *and* $t_n \to 0$, *then it holds*

$$j'(x; h) = \lim_{n \to \infty} \left(\frac{j(x + t_n h_n) - j(x)}{t_n} \right),$$

d) $j|_{[0,\infty)}$ *is monotonically increasing and* $j|_{(-\infty,0]}$ *is monotonically decreasing,*

e) for all $x, y \in \mathbb{R}$, *it holds*

$$j'(x; y - x) + j'(y; x - y) \leq 0.$$

We emphasize that we do not impose any conditions on the growth of the function j for $x \to \pm\infty$. Consequently, the functional $H_0^1(\Omega) \ni v \mapsto \int_\Omega cj(v)\mathrm{d}\lambda$ appearing in the VI (P) may take the value $+\infty$ if $c \geq 0$ holds a.e. in Ω. Note that, nevertheless, (P) admits a unique solution for all $(c, f) \in L^1_+(\Omega) \times H^{-1}(\Omega)$. More precisely, we have the following:

Theorem 2.4. *Let Assumption 2.1 hold. Then it is true that:*

a) For all $(c, f) \in L^1_+(\Omega) \times H^{-1}(\Omega)$ there exists a unique solution $w \in H_0^1(\Omega)$ to the variational inequality (P).

b) If $(c, f) \in L^1_+(\Omega) \times L^p(\Omega)$ with $p > d/2$, then there exists a $C > 0$ which depends only on d, p, Ω, and A such that $\|w\|_{L^\infty} \leq C\|f\|_{L^p}$.

c) If $(c, f) \in L^{p_1}_+(\Omega) \times L^{p_2}(\Omega)$ with $p_1 > 2d/(d + 2)$, $p_2 > d/2$, then for all $v \in H_0^1(\Omega)$ it holds

$$a(w, v) + \int_\Omega cj'(w; v)\mathrm{d}\lambda \geq \langle f, v\rangle . \tag{2.1}$$

d) If $(c, f) \in L^p_+(\Omega) \times L^p(\Omega)$, $d/2 < p < \infty$, and if Ω has a $C^{1,1}$-boundary, then w is in $W^{2,p}(\Omega)$. In particular, $w \in C^1(\overline{\Omega})$ for all $(c, f) \in L^p_+(\Omega) \times L^p(\Omega)$ with $p > d$.

Proof. Ad a): The bilinear form a is H_0^1-elliptic and continuous due to Assumption 2.1 and Friedrichs' inequality. Further, we obtain from Fatou's lemma and the properties of j that the functional

$$H_0^1(\Omega) \ni v \mapsto \int_\Omega cj(v)\mathrm{d}\lambda \in \mathbb{R} \cup \{\infty\}$$

is convex, proper and lower semi-continuous for all $c \in L^1_+(\Omega)$. This allows us to use standard results as, e.g., [13, Theorem 4.1]) to obtain the unique solvability of (P).

Ad b): We follow a well-known approach of Stampacchia: Let w be the solution to (P) and let $k \geq 0$ be given. Define $w_k := w - \min(k, \max(w, -k))$. Then w_k is in $H_0^1(\Omega)$ according to [18, Theorem II.A.1] and we may choose the test function $v := w - w_k$ in (P) to obtain (using formula (A.1) in [18, Section II])

$$\int_\Omega \sum_{i,j=1}^d \alpha_{ij}\partial_i w_k \partial_j w_k + \beta w w_k \mathrm{d}\lambda - \int_{\{w>k\}} c\Big(j(k) - j(w)\Big)\mathrm{d}\lambda \\ - \int_{\{w<-k\}} c\Big(j(-k) - j(w)\Big)\mathrm{d}\lambda \leq \int_\Omega f w_k \mathrm{d}\lambda. \tag{2.2}$$

Since j is monotonically increasing on $[0, \infty)$ and monotonically decreasing on $(-\infty, 0]$, and since $\beta \geq 0$ a.e. in Ω, it follows from (2.2) and Friedrichs' inequality that there exists a constant C depending only on d, Ω, and A such that

$$\|w_k\|^2_{H^1} \leq C \int_\Omega f\, w_k \mathrm{d}\lambda \quad \forall\, k \geq 0.$$

The claim now follows from [7, Lemma 3.6], see also [18, Theorem II.B.2].

Ad c): Let $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and $t \in (0, 1)$ be arbitrary. Then we may choose the test function $w + tv$ in (P) to obtain

$$a(w, v) + \int_\Omega c\, \frac{j(w + tv) - j(w)}{t}\mathrm{d}\lambda \geq \langle f, v\rangle .$$

Note that it follows from $f \in L^{p_2}(\Omega)$, $p_2 > d/2$, that w is essentially bounded in Ω (see b)). Further, we obtain from Lemma 2.3 a) that j is Lipschitz on the interval $[-\|w\|_{L^\infty} - \|v\|_{L^\infty}, \|w\|_{L^\infty} + \|v\|_{L^\infty}]$. Thus, there exists a $C > 0$ with

$$\left|\frac{j(w+tv) - j(w)}{t}\right| \leq C|v| \in L^\infty(\Omega) \quad \forall t \in (0,1).$$

Using the dominated convergence theorem, we may now deduce

$$a(w,v) + \int_\Omega cj'(w;v)\mathrm{d}\lambda \geq \langle f, v\rangle . \tag{2.3}$$

This proves c) for all $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$. If v is unbounded, then we can choose the function $v_k := \min(v^+, k) + \max(v^-, -k) \in H_0^1(\Omega) \cap L^\infty(\Omega)$, $k \geq 0$, in (2.3) and exploit the positive homogeneity of the directional derivative to obtain

$$\begin{aligned} a(w, v_k) + \int_\Omega & cj'(w;1)\min(v^+,k)\mathrm{d}\lambda \\ &+ \int_\Omega cj'(w;-1)\min(-v^-,k)\mathrm{d}\lambda \geq \langle f, v_k\rangle . \end{aligned}$$

Letting $k \to \infty$ in the above and using again the dominated convergence theorem (with the majorant $(\|j'(w;1)\|_{L^\infty} + \|j'(w;-1)\|_{L^\infty})c|v| \in L^1(\Omega)$, cf. the Sobolev embeddings, $p_1 > 2d/(d+2)$, and the local Lipschitz continuity of the function j) yields (2.1) in the general case.

Ad d): Let $q := f - Aw \in H^{-1}(\Omega)$. Then it follows from part c), the Sobolev embeddings, and the assumption $p > d/2$ that

$$\langle q, v\rangle \leq \int_\Omega cj'(w;v)\mathrm{d}\lambda \quad \forall\, v \in H_0^1(\Omega) \hookrightarrow L^{p'}(\Omega), \tag{2.4}$$

where p' is the Hölder conjugate of p. Define $\psi : L^{p'}(\Omega) \to \mathbb{R}$, $v \mapsto \int_\Omega cj'(w;v)\mathrm{d}\lambda$. Then ψ is sublinear since c is non-negative and since $\mathbb{R} \ni h \mapsto j'(x;h) \in \mathbb{R}$ is convex and positively homogeneous for all $x \in \mathbb{R}$. Moreover, the functional ψ is globally Lipschitz continuous. To see this, let $v_1, v_2 \in L^{p'}(\Omega)$ be arbitrary but fixed and note that b) implies $w \in L^\infty(\Omega)$. Set $M := \|w\|_{L^\infty(\Omega)} + 1$ and denote with $L(M) > 0$ the Lipschitz constant of j on $[-M, M]$. Then for all $x \in [-\|w\|_{L^\infty(\Omega)}, \|w\|_{L^\infty(\Omega)}]$ the mapping $\mathbb{R} \ni h \mapsto j'(x;h) \in \mathbb{R}$ is globally Lipschitz with constant $L(M)$ so that

$$|j'(w;v_1) - j'(w;v_2)| \leq L(M)\,|v_1 - v_2| \quad \text{ a.e. in } \Omega.$$

The above yields

$$|\psi(v_1) - \psi(v_2)| \leq L(M)\,\|c\|_{L^p(\Omega)}\,\|v_1 - v_2\|_{L^{p'}(\Omega)},$$

which proves the Lipschitz continuity of ψ. From (2.4) and the Hahn-Banach theorem, we may deduce that q can be extended to an element of $L^p(\Omega)$ (still denoted by the same symbol for simplicity). As a consequence,

$$-\sum_{i,j=1}^d \frac{\partial}{\partial x_j}\left(\alpha_{ij}\frac{\partial}{\partial x_i}w\right) + \beta w = -q + f \in L^p(\Omega).$$

The above shows that w is the solution to an elliptic PDE of second order whose right-hand side is in $L^p(\Omega)$. Using standard regularity results as [12, Theorem 9.15], the claim now follows immediately. ∎

Remark 2.5. *It is worth mentioning that the L^∞-bound in part b) is obtained without any further assumptions on c or j (beside those in Assumption 2.1 and $c \in L^1_+(\Omega)$). To the best of our knowledge, this was not known before.*

Having studied the existence, the uniqueness, and the regularity of the solution w to the variational inequality (P), we now turn our attention to the mapping properties of the solution operator $S : (c, f) \mapsto w$.

Theorem 2.6. *Let Assumption 2.1 hold, let $p > d/2$, and let $r > 0$ be arbitrary but fixed. Then there exists a constant C depending only on d, p, Ω, A, r, and j such that the solution operator $S : (c, f) \mapsto w$ associated with (P) satisfies*

$$\|S(c_1, f_1) - S(c_2, f_2)\|_{H^1(\Omega)} + \|S(c_1, f_1) - S(c_2, f_2)\|_{L^\infty(\Omega)} \leq C\Big(\|c_1 - c_2\|_{L^p(\Omega)} + \|f_1 - f_2\|_{L^p(\Omega)}\Big)$$

for all $(c_1, f_1), (c_2, f_2) \in L^p_+(\Omega) \times \{f \in L^p(\Omega) : \|f\|_{L^p} \leq r\}$.

Proof. Let $f_1, f_2 \in L^p(\Omega)$ with $\|f_1\|_{L^p}, \|f_2\|_{L^p} \leq r$ and $c_1, c_2 \in L^p_+(\Omega)$ be arbitrary but fixed. Denote with w_1 the solution $S(c_1, f_1)$ and with w_2 the solution $S(c_2, f_2)$. Then we know from Theorem 2.4 b) and c) that $\|w_1\|_{L^\infty}, \|w_2\|_{L^\infty} \leq C$ holds with a constant C depending only on d, p, Ω, A, and r and that

$$a(w_i, v) + \int_\Omega c_i j'(w_i; v) \mathrm{d}\lambda \geq \langle f_i, v\rangle \quad \forall v \in H^1_0(\Omega), \quad i = 1, 2.$$

The above yields

$$a(w_1 - w_2, v) + \int_\Omega (c_1 - c_2) j'(w_1; v) \mathrm{d}\lambda + \int_\Omega c_2 \Big(j'(w_1; v) + j'(w_2; -v)\Big) \mathrm{d}\lambda \geq \langle f_1 - f_2, v\rangle \quad \forall v \in H^1_0(\Omega). \tag{2.5}$$

Choosing $v = w_2 - w_1$ in (2.5) and using Lemma 2.3 e), Sobolev embeddings, and the local Lipschitz continuity of j, we obtain

$$\begin{aligned} &a(w_1 - w_2, w_1 - w_2) \\ &\quad\leq \langle f_1 - f_2, w_1 - w_2\rangle + \int_\Omega (c_1 - c_2) j'(w_1; \operatorname{sgn}(w_2 - w_1))|w_1 - w_2| \mathrm{d}\lambda \\ &\qquad\qquad + \int_\Omega c_2 \Big(j'(w_1; w_2 - w_1) + j'(w_2; w_1 - w_2)\Big) \mathrm{d}\lambda \\ &\quad\leq C\Big(\|f_1 - f_2\|_{L^p} + \|c_1 - c_2\|_{L^p}\Big)\|w_1 - w_2\|_{H^1} \end{aligned}$$

with a constant C depending only on d, p, Ω, r, A, and j. The coercivity of the bilinear form a now yields the local Lipschitz continuity in $H^1(\Omega)$. It remains to prove the L^∞-estimate. To this end, we use an argument similar to that employed in the proof of Theorem 2.4 b). Let $v := -(w_1 - w_2)_k := -(w_1 - w_2) + \min(k, \max(w_1 - w_2, -k))$,

$k \geq 0$, then (2.5) yields (cf. (2.2))

$$
\begin{aligned}
&a((w_1 - w_2)_k, (w_1 - w_2)_k) \\
&\quad \leq \langle f_1 - f_2, (w_1 - w_2)_k \rangle + \int_\Omega (c_1 - c_2) j'(w_1; -(w_1 - w_2)_k) \mathrm{d}\lambda \\
&\qquad\qquad + \int_\Omega c_2 \Big(j'(w_1; -(w_1 - w_2)_k) + j'(w_2; (w_1 - w_2)_k) \Big) \mathrm{d}\lambda. \qquad (2.6)
\end{aligned}
$$

Further, due to the positive homogeneity of the directional derivative, the definition of $(w_1 - w_2)_k$, and Lemma 2.3 e), we have

$$
\begin{aligned}
&\int_\Omega c_2 \Big(j'(w_1; -(w_1 - w_2)_k) + j'(w_2; (w_1 - w_2)_k) \Big) \mathrm{d}\lambda \\
&\quad = \int_{\{|w_1 - w_2| > k\}} c_2 \Big(j'(w_1; w_2 - w_1) + j'(w_2; w_1 - w_2) \Big) \frac{|w_1 - w_2| - k}{|w_1 - w_2|} \mathrm{d}\lambda \leq 0. \qquad (2.7)
\end{aligned}
$$

Combining (2.6) and (2.7), we obtain that for all $k \geq 0$ it holds

$$
\|(w_1 - w_2)_k\|_{H^1}^2 \leq C(d, p, \Omega, A, r, j) \int_\Omega \Big(|f_1 - f_2| + |c_1 - c_2| \Big) |(w_1 - w_2)_k| \mathrm{d}\lambda.
$$

Using again [7, Lemma 3.6], the claim now follows immediately. ◻

3. Strategy for the Differential Sensitivity Analysis. Theorem 2.6 shows that the solution mapping $S : L^p_+(\Omega) \times L^p(\Omega) \to H^1_0(\Omega), (c, f) \mapsto w$, $p > d/2$, is H^1- and L^∞-Lipschitz on bounded sets. This is not only interesting for its own sake, but also the point of departure for our differential sensitivity analysis: If a tuple $(h, g) \in L^p(\Omega) \times L^p(\Omega)$ satisfying $c + t_0 h \in L^p_+(\Omega)$ for some $t_0 > 0$ is given (with $(c, f) \in L^p_+(\Omega) \times L^p(\Omega)$ arbitrary but fixed), then the local Lipschitz continuity of the solution operator S implies that the difference quotients

$$
\delta_t := \frac{S(c + th, f + tg) - S(c, f)}{t}, \quad 0 < t < t_0,
$$

remain bounded in $H^1(\Omega)$ and $L^\infty(\Omega)$ as t tends to zero. This yields that for every sequence $(t_n) \subset (0, t_0)$ satisfying $t_n \to 0$ we can find a subsequence (unrelabeled for simplicity) such that the associated difference quotients δ_{t_n} converge weakly in $H^1(\Omega)$, strongly in $L^2(\Omega)$, and pointwise a.e. in Ω to a function $\delta \in H^1_0(\Omega)$. In what follows, the main idea is to show that this weak limit δ is unique, i.e., independent of the choice of (sub)sequences and that the difference quotients δ_{t_n} converge even strongly in $H^1(\Omega)$. If this is established, then it follows immediately by contradiction that S is directionally differentiable in (c, f) in the direction (h, g) with $S'((c, f); (h, g)) = \delta$ (cf. also with [7]). So let us consider the following situation:

Assumption 3.1.

a) $p > d/2$,
b) $(c, f) \in L^p_+(\Omega) \times L^p(\Omega)$ *is arbitrary but fixed (with* $w := S(c, f)$*),*
c) $(h, g) \in L^p(\Omega) \times L^p(\Omega)$ *is arbitrary but fixed such that there exists a* $t_0 > 0$ *with* $c + t_0 h \in L^p_+(\Omega)$*,*
d) $0 < t_n < t_0$ *is a sequence tending to zero as* $n \to \infty$*,*
e) the difference quotients $\delta_n := \delta_{t_n}$ *associated with* t_n *satisfy*

$$
\delta_n \rightharpoonup \delta \text{ in } H^1(\Omega), \quad \delta_n \to \delta \text{ in } L^2(\Omega), \quad \delta_n \to \delta \text{ pointwise a.e. in } \Omega
$$

for some $\delta \in H^1_0(\Omega)$*.*

To prove that the weak limit δ is unique, we first note that the definition of δ_n yields $S(c+t_n h, f+t_n g) = w + t_n\delta_n$. As a consequence, for all $v \in H_0^1(\Omega)$ it is true that

$$a(w+t_n\delta_n, v-w-t_n\delta_n) + \int_\Omega (c+t_n h)\, j(v)\mathrm{d}\lambda - \int_\Omega (c+t_n h)\, j(w+t_n\delta_n)\mathrm{d}\lambda \geq \langle f+t_n g, v-w-t_n\delta_n\rangle\,. \quad (3.1)$$

If we choose test functions of the form $v = w + t_n z$, $z \in H_0^1(\Omega)$, in (3.1), then we obtain after some manipulations

$$a(\delta_n, z-\delta_n) + H_n(z) + I_n(z) + J_n(z) - \big(H_n(\delta_n) + I_n(\delta_n) + J_n(\delta_n)\big) \geq \langle g, z-\delta_n\rangle \quad \forall z \in H_0^1(\Omega) \quad (3.2)$$

with

$$H_n(z) := \int_\Omega h\,\frac{j(w+t_n z) - j(w)}{t_n}\mathrm{d}\lambda, \quad I_n(z) := \frac{1}{t_n}\Big(a(w,z) + \int_\Omega c\, j'(w;z)\mathrm{d}\lambda - \langle f, z\rangle\Big),$$

and

$$J_n(z) := \frac{1}{t_n}\left(\int_\Omega c\,\frac{j(w+t_n z) - j(w)}{t_n} - c\, j'(w;z)\mathrm{d}\lambda\right). \quad (3.3)$$

In what follows, our aim is to pass to the limit $n \to \infty$ in (3.2) and to show that the limit δ is itself the solution of an elliptic variational inequality which does not depend on the sequence (t_n) appearing in Assumption 3.1. If this is proved, then δ is clearly unique and the solution map S is indeed directionally differentiable. Note that for the terms $H_n(z)$ and $H_n(\delta_n)$ in (3.2), we have the following:

Lemma 3.2. *Let Assumptions 2.1 and 3.1 hold. Then for all $z \in H_0^1(\Omega) \cap L^\infty(\Omega)$ it is true that*

$$H_n(z) - H_n(\delta_n) \to \int_\Omega h j'(w;z)\mathrm{d}\lambda - \int_\Omega h j'(w;\delta)\mathrm{d}\lambda.$$

Proof. We know from Theorem 2.4 that $\|w\|_{L^\infty} \leq C\|f\|_{L^p}$, and if $z \in L^\infty(\Omega)$, then $\|w + t_n z\|_{L^\infty} \leq C\|f\|_{L^p} + t_0\|z\|_{L^\infty}$. Consequently, there exists a constant $C > 0$ such that $\|w\|_{L^\infty}, \|w+t_n z\|_{L^\infty} \in [0, C]$ holds for all n and we may use the local Lipschitz continuity of j in combination with the dominated convergence theorem to obtain

$$H_n(z) = \int_\Omega h\,\frac{j(w+t_n z) - j(w)}{t_n}\mathrm{d}\lambda \to \int_\Omega h j'(w;z)\mathrm{d}\lambda.$$

For $H_n(\delta_n)$ it follows from $w + t_n\delta_n = S(c+t_n h, f+t_n g)$ and Theorem 2.4 that $\|w+t_n\delta_n\|_{L^\infty} \leq C(\|f\|_{L^p} + t_0\|g\|_{L^p})$. Further, we have a uniform bound on $\|\delta_n\|_{L^\infty}$ due to Theorem 2.6. Thus, we may again use the dominated convergence theorem and the Hadamard differentiability of j to deduce

$$H_n(\delta_n) = \int_\Omega h\,\frac{j(w+t_n\delta_n) - j(w)}{t_n}\mathrm{d}\lambda \to \int_\Omega h j'(w;\delta)\mathrm{d}\lambda.$$

This proves the claim. ∎

Unfortunately, passing to the limit $n \to \infty$ with the I_n- and J_n-terms in (3.2) is not as straightforward. Due to the negative powers of t_n in I_n and J_n and since j is not assumed to be twice continuously differentiable, it is perfectly possible that the sequences $I_n(z)$ and $J_n(z)$ diverge, and the pointwise behavior of the second order difference quotients

$$\frac{1}{t_n}\left(\frac{j(w+t_n\delta_n)-j(w)}{t_n}-j'(w;\delta_n)\right) \tag{3.4}$$

appearing in $J_n(\delta_n)$ is in general hard to determine. To overcome these problems, we note the following:

Lemma 3.3. *Let Assumptions 2.1 and 3.1 hold. Then it is true that*

$$0 \le \limsup_{n\to\infty} I_n(\delta_n) < \infty \quad \textit{and} \quad 0 \le \limsup_{n\to\infty} J_n(\delta_n) < \infty.$$

Proof. Choosing the test function $z = 0$ in (3.2) yields

$$\langle g, \delta_n\rangle - H_n(\delta_n) \ge I_n(\delta_n) + J_n(\delta_n).$$

Further, Lemma 2.3 b) and (2.1) imply $I_n(z) \ge 0$ and $J_n(z) \ge 0$ for all $z \in H_0^1(\Omega)$. Since $\langle g, \delta_n\rangle - H_n(\delta_n)$ remains bounded, the claim now follows immediately. ◻

Lemma 3.3 shows that, although the terms $I_n(z)$ and $J_n(z)$ might diverge for arbitrary test functions $z \in H_0^1(\Omega)$, the expressions $I_n(\delta_n)$ and $J_n(\delta_n)$ in (3.2) have to remain bounded. As we will see, this boundedness property is the key to handling the singular limiting behavior of the functionals I_n and J_n in (3.2). The strategy that we pursue in the remainder of this paper can be summarized in the following steps:

1. We deduce as much information as possible about the weak limit δ from the boundedness properties in Lemma 3.3 (see Lemma 3.5 below and, later on, Propositions 4.7 and 4.11).
2. Using the information obtained in 1., we define the so-called *reduced critical cone* $T^{\text{red}}_{\text{crit}}(c, f)$ (cf. Lemma 3.5 and Corollary 4.12). This cone contains δ according to its construction.
3. We prove that there exists a subset V of $T^{\text{red}}_{\text{crit}}(c, f)$ such that V is dense in the reduced critical cone $T^{\text{red}}_{\text{crit}}(c, f)$ and such that for all $z \in V$ it holds $I_n(z) + J_n(z) \to I(z) + J(z)$ for suitable functionals I and J (see Corollary 3.8 and the proof of Proposition 4.11).
4. We show that $\liminf_{n\to\infty} I_n(\delta_n) + J_n(\delta_n) \ge I(\delta) + J(\delta)$ with I and J as in step 3. and pass to the limit $n \to \infty$ in (3.2) with test functions $z \in V$ (see the proof of Proposition 4.11).
5. Using the density of V in $T^{\text{red}}_{\text{crit}}(c, f)$, we deduce that δ is the solution to an elliptic VI in a weighted Sobolev space whose admissible set is the reduced critical cone $T^{\text{red}}_{\text{crit}}(c, f)$ (see Theorem 4.14). This proves the uniqueness of δ and the weak directional differentiability of S. The coercivity of the bilinear form a finally implies norm convergence of δ_n and thus strong convergence of the difference quotients in $H^1(\Omega)$.

Remark 3.4. *What we prove in steps 3., 4., and 5. of our approach can be interpreted as a special form of Mosco epi-convergence of second order difference quotients. For details on the latter concept, we refer to [19], where a technique similar to ours has been used for the sensitivity analysis of VIs of the first kind in Banach spaces.*

As a first consequence of Lemma 3.3, we obtain:

Lemma 3.5. *Let Assumptions 2.1 and 3.1 hold. Then the weak limit δ of the difference quotients δ_n is an element of the so-called critical cone*

$$T_{\text{crit}}(c,f) := \left\{ z \in H_0^1(\Omega) : a(w,z) + \int_\Omega c j'(w;z) \mathrm{d}\lambda = \langle f, z\rangle \right\}.$$

Proof. From the first estimate in Lemma 3.3, it follows

$$\lim_{n\to\infty} \left(a(w,\delta_n) + \int_\Omega c\, j'(w;\delta_n)\mathrm{d}\lambda - \langle f, \delta_n\rangle \right) = 0,$$

and the weak convergence $\delta_n \rightharpoonup \delta$ yields $a(w,\delta_n) - \langle f,\delta_n\rangle \to a(w,\delta) - \langle f, \delta\rangle$. Further, we obtain from the uniform L^∞-bound on δ_n, the local Lipschitz continuity of the function j, and the dominated convergence theorem

$$\int_\Omega c\, j'(w;\delta_n)\mathrm{d}\lambda = \int_\Omega c\, j'(w;1)\delta_n^+ \mathrm{d}\lambda - \int_\Omega c\, j'(w;-1)\delta_n^- \mathrm{d}\lambda \to \int_\Omega c\, j'(w;\delta)\mathrm{d}\lambda.$$

Combining all of the above proves the claim. ∎

Remark 3.6. *If the bilinear form a is symmetric, then (P) is equivalent to the unconstrained optimization problem*

$$\min_{v \in H_0^1(\Omega)} \psi(v) := \frac{1}{2} a(v,v) + \int_\Omega c j(v) \mathrm{d}\lambda - \langle f, v\rangle$$

and it holds $T_{\text{crit}}(c,f) = \{z \in H_0^1(\Omega) : \psi'(w;z) = 0\}$ (see [13, Lemma 4.1.]). This justifies the use of the notion "critical cone" in Lemma 3.5, cf. [6, Section 3.1.1].

Lemma 3.7. *The set $T_{\text{crit}}(c,f)$ defined in Lemma 3.5 is a closed convex cone.*

Proof. The cone property and the closedness of $T_{\text{crit}}(c,f)$ w.r.t. the H^1-topology are evident. To see that $T_{\text{crit}}(c,f)$ is convex, note that from (2.1) and the convexity of j it follows that for all $z_1, z_2 \in T_{\text{crit}}(c,f)$ and all $s \in [0,1]$ it holds

$$\begin{aligned}
0 &\leq a(w, sz_1 + (1-s)z_2) + \int_\Omega c j'(w; sz_1 + (1-s)z_2)\mathrm{d}\lambda - \langle f, sz_1 + (1-s)z_2\rangle \\
&= a(w, sz_1 + (1-s)z_2) - \langle f, sz_1 + (1-s)z_2\rangle \\
&\qquad + \int_\Omega c \left(\lim_{t\to 0^+} \frac{j(sw + (1-s)w + t(sz_1 + (1-s)z_2)) - j(w)}{t} \right) \mathrm{d}\lambda \\
&\leq s\left(a(w,z_1) + \int_\Omega c j'(w;z_1)\mathrm{d}\lambda - \langle f, z_1\rangle \right) \\
&\qquad + (1-s)\left(a(w,z_2) + \int_\Omega c j'(w;z_2)\mathrm{d}\lambda - \langle f, z_2\rangle \right) = 0.
\end{aligned}$$

This proves the claim. ∎

As an immediate consequence of (2.1) and the definition of the set $T_{\text{crit}}(c,f)$, we obtain the following result which will be sufficient for the limit transition with the I_n-terms in (3.2):

Corollary 3.8. *For all $z \in T_{\text{crit}}(c,f)$ and all $n \in \mathbb{N}$ it holds $I_n(z) - I_n(\delta_n) \leq 0$.*

It remains to study which information about δ is encoded in the boundedness property

$$0 \leq \limsup_{n\to\infty} J_n(\delta_n) < \infty \tag{3.5}$$

and how the terms $J_n(z)$ and $J_n(\delta_n)$ behave when n tends to infinity. Unfortunately, what can be deduced from (3.5) depends heavily on the precise nature of the function j, and, at least to the authors' best knowledge, there is currently no general strategy that can be used to study which implications (3.5) has if j is an arbitrary function satisfying the conditions in Assumption 2.1. As a consequence, in what follows, we have to confine our analysis to a suitable subclass of problems.

4. Sensitivity Analysis for VIs Involving Piecewise Smooth Functions.

Henceforth, we impose the following more restrictive conditions on the function j:

Assumption 4.1. *It holds $j(x) = j_1(x^+) + j_2(-x^-)$ with non-negative and convex functions $j_1, j_2 \in C^2([0,\infty))$ satisfying $j_1(0) = j_2(0) = 0$ and $j_1'(0), j_2'(0) > 0$.*

Remark 4.2. *We point out that our analysis can be extended to cover, e.g., cases where one of the derivatives $j_1'(0)$, $j_2'(0)$ in Assumption 4.1 vanishes and where the function j is non-differentiable at several points. The notational effort, however, increases significantly if this more general setting is considered, so we decided to restrict our attention to the setting in Assumption 4.1.*

As we will see in the following, in the situation of Assumption 4.1, the boundedness property (3.5) yields information about the traces of δ on the boundary of the inactive set $\{w \neq 0\}$. To ensure that these traces are well-defined, we need to impose additional assumptions:

Assumption 4.3 (Structural Assumptions).

a) It holds $f \in L^p(\Omega)$, $p > d/2$, $0 < c \in C(\overline{\Omega})$, and $w \in C^1(\Omega) \cap W^{2,1}(\Omega)$.

b) The set $\partial\{w \neq 0\} \subseteq \overline{\Omega}$ is a λ^d-zero set and there exists a set $\mathcal{C} \subseteq \overline{\Omega}$ such that:

- $\mathcal{C}$ is closed and has H^1-capacity zero, i.e., (cf. [4, Chapter 5.8.2])

$$0 = \mathrm{cap}_2(\mathcal{C}, \mathbb{R}^d) = \inf \Big\{ \|\phi\|_{H^1} : \phi \in C_c(\mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^d), \\ 0 \leq \phi \leq 1,\ \phi \equiv 1 \text{ in a nbhd. of } \mathcal{C} \Big\},$$

- $\partial\{w \neq 0\} \setminus \mathcal{C}$ is a strong $(d-1)$-dimensional Lipschitz submanifold of $\mathbb{R}^d$,

- the sets

$$\mathcal{N}^+ := \{\nabla w = 0\} \cap \partial\{w > 0\} \setminus \mathcal{C}$$
$$\mathcal{N}^- := \{\nabla w = 0\} \cap \partial\{w < 0\} \setminus \mathcal{C}$$

are relatively open in $\partial\{w \neq 0\} \setminus \mathcal{C}$.

Here and in what follows, when using the variable w, we always mean the C^1-representative of the solution $S(c, f)$.

Some remarks are in order regarding Assumption 4.3:

Remark 4.4.

a) The assumption $w \in C^1(\Omega) \cap W^{2,1}(\Omega)$ is automatically fulfilled if Ω has a $C^{1,1}$-boundary and if $f \in L^p(\Omega)$ holds for some $p > d$, see Theorem 2.4 d).

b) *Recall that a set $\mathcal{N} \subset \mathbb{R}^d$ is called a strong $(d-1)$-dimensional Lipschitz submanifold of $\mathbb{R}^d$ if the following holds (cf. [30]): For all $p \in \mathcal{N}$ there exist an orthogonal transformation $R \in O(d)$, an open ball $B \subset \mathbb{R}^{d-1}$, an open interval $J = (a, b)$, and a Lipschitz continuous map $h : B \to J$ such that*

$$p \in R(B \times J) \quad \text{and} \quad \mathcal{N} \cap R(B \times J) = R(\{(x, h(x)) : x \in B\}).$$

c) *Since w is assumed to be continuously differentiable, the implicit function theorem implies that the set*

$$\mathcal{M} := \partial\{w \neq 0\} \cap \{\nabla w \neq 0\} = \{w = 0\} \cap \{\nabla w \neq 0\}$$

is a $(d-1)$-dimensional C^1-submanifold of $\mathbb{R}^d$.

d) *We point out that, according to the notation introduced in Section 2, sets of the form $\{w = 0\}$ or $\{\nabla w \neq 0\}$ are defined as subsets of Ω and not of $\overline{\Omega}$. Therefore, $\mathcal{N}^+$, $\mathcal{N}^-$, and $\mathcal{M}$ are also subsets of Ω and do not include the boundary $\partial\Omega$.*

e) *Since $\mathcal{N}^+$ and $\mathcal{N}^-$ are relatively open subsets of $\partial\{w \neq 0\} \setminus \mathcal{C}$, they are themselves strong $(d-1)$-dimensional Lipschitz submanifolds of $\mathbb{R}^d$.*

f) *As $\mathcal{M}$, $\mathcal{N}^+$, and $\mathcal{N}^-$ are strong $(d-1)$-dimensional Lipschitz submanifolds, traces on these sets are well-defined (cf. [2, 22]).*

g) *Assumption 4.3 substantially weakens the assumptions on the active set in [7], where the set $\{w = 0\}$ is not allowed to have $(d-1)$-dimensional components and where the sets $\{w > 0\}$ and $\{w < 0\}$ have to have positive distance from each other. The geometric setting depicted in Figure 4.1, for example, satisfies the conditions in Assumption 4.3 but violates the assumptions imposed in [7].*

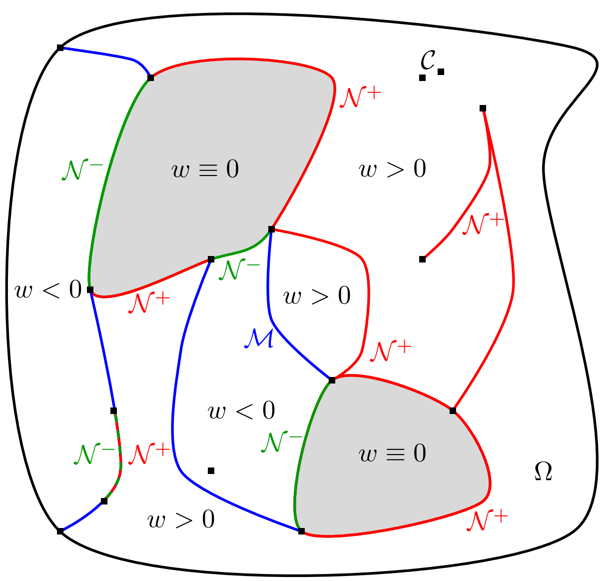

FIG. 4.1. *The geometric situation in Assumption 4.3. All visible lines are part of the boundary $\partial\{w \neq 0\} \subseteq \overline{\Omega}$. In the grey sets, it holds $w \equiv 0$. Points contained in $\mathcal{C}$ are marked by black squares. The sets $\mathcal{M}$, $\mathcal{N}^-$, and $\mathcal{N}^+$ are depicted in blue, green, and red, respectively. We point out that $\mathcal{N}^+$ and $\mathcal{N}^-$ do not necessarily have to be disjoint and that there is always a change of sign along $\mathcal{M}$.*

Note that, in the situation of Assumptions 4.1 and 4.3, the condition in Lemma 3.5 can be rewritten as follows:

LEMMA 4.5. *Suppose that Assumptions 2.1, 4.1, and 4.3 are satisfied. Then it holds* $-cj_2'(0) \le f \le cj_1'(0)$ λ^d*-a.e. in* $\{w = 0\}$ *and it is true that*

$$\begin{aligned} z \in T_{\text{crit}}(c, f) &\iff z^+ \in T_{\text{crit}}(c, f) \text{ and } z^- \in T_{\text{crit}}(c, f) \\ &\iff z^+ = 0 \ \lambda^d\text{-a.e. in } \{w = 0\} \cap \{-cj_2'(0) \le f < cj_1'(0)\} \text{ and} \\ &\qquad z^- = 0 \ \lambda^d\text{-a.e. in } \{w = 0\} \cap \{-cj_2'(0) < f \le cj_1'(0)\}. \end{aligned} \tag{4.1}$$

Proof. Since $w \in W^{2,1}(\Omega)$ and $f \in L^1(\Omega)$, (2.1) implies

$$\begin{aligned} \int_\Omega (Aw) z \mathrm{d}\lambda + \int_{\{w \ne 0\}} cj'(w) z \mathrm{d}\lambda + \int_{\{w=0\}} c\Big(j_1'(0) z^+ - j_2'(0) z^-\Big) \mathrm{d}\lambda& \\ \ge \int_\Omega f z \mathrm{d}\lambda \quad \forall z \in C_c^\infty(\Omega).& \end{aligned} \tag{4.2}$$

If we choose test functions $z \in C_c^\infty(\{w \ne 0\})$ in the above inequality, then we obtain (using that the set $\{w \ne 0\}$ is open)

$$Aw + cj'(w) - f = 0 \text{ a.e. in } \{w \ne 0\}. \tag{4.3}$$

From (4.2) and (4.3), it follows

$$\int_{\{w=0\}} (Aw) z \mathrm{d}\lambda + \int_{\{w=0\}} c\Big(j_1'(0) z^+ - j_2'(0) z^-\Big) \mathrm{d}\lambda \ge \int_{\{w=0\}} f z \mathrm{d}\lambda \tag{4.4}$$

for all $z \in C_c^\infty(\Omega)$. Since $\partial\{w = 0\}$ is a λ^d-zero set, (4.4) yields

$$\int_{\{w=0\}} (cj_1'(0) - f) z^+ - (cj_2'(0) + f) z^- \mathrm{d}\lambda \ge 0 \tag{4.5}$$

for all $z \in C_c^\infty(\Omega)$ and, by approximation, for all $z \in L^\infty(\Omega)$. Choosing the indicator functions $z_1 := \mathbb{1}_{\{f > cj_1'(0)\}}$ and $z_2 := -\mathbb{1}_{\{f < -cj_2'(0)\}}$ in (4.5), we readily obtain $-cj_2'(0) \le f \le cj_1'(0)$ a.e. in $\{w = 0\}$. This proves the first claim of the lemma.

It remains to prove the equivalences in (4.1): If $z^+, z^- \in T_{\text{crit}}(c, f)$, then it follows from the convexity of the cone $T_{\text{crit}}(c, f)$ that $z = z^+ + z^- \in T_{\text{crit}}(c, f)$. If, conversely, $z \in T_{\text{crit}}(c, f)$, then it holds

$$\begin{aligned} 0 &= a(w, z) + \int_{\{w \ne 0\}} cj'(w) z \mathrm{d}\lambda + \int_{\{w=0\}} c\Big(j_1'(0) z^+ - j_2'(0) z^-\Big) \mathrm{d}\lambda - \langle f, z\rangle \\ &= \left(a(w, z^+) + \int_{\{w \ne 0\}} cj'(w) z^+ \mathrm{d}\lambda + \int_{\{w=0\}} cj_1'(0) z^+ \mathrm{d}\lambda - \big\langle f, z^+\big\rangle\right) \\ &\quad + \left(a(w, z^-) + \int_{\{w \ne 0\}} cj'(w) z^- \mathrm{d}\lambda - \int_{\{w=0\}} cj_2'(0) z^- \mathrm{d}\lambda - \big\langle f, z^-\big\rangle\right). \end{aligned}$$

The two bracketed terms on the right-hand side of the last identity are each nonnegative due to (2.1). Thus, they both have to vanish and it follows $z^+, z^- \in T_{\text{crit}}(c, f)$ as claimed. To obtain the second equivalence, we note that, due to the regularity of the functions w and f, the condition in the definition of the set $T_{\text{crit}}(c, f)$ can also be written as (cf. (4.5))

$$\int_{\{w=0\}} (cj_1'(0) - f)z^+ - (cj_2'(0) + f)z^- \mathrm{d}\lambda = 0.$$

If we assume $z^+ \in T_{\text{crit}}(c, f)$, then it follows from the above and $-cj_2'(0) \le f \le cj_1'(0)$ a.e. in $\{w = 0\}$ that

$$0 = \int_{\{w=0\}} (cj_1'(0) - f)z^+ \mathrm{d}\lambda = \int_{\{w=0\}} |(cj_1'(0) - f)z^+| \mathrm{d}\lambda.$$

This yields $z^+ = 0$ a.e. in $\{w = 0\} \cap \{-cj_2'(0) \le f < cj_1'(0)\}$ as claimed. Completely analogously to the above, we obtain that $z^- \in T_{\text{crit}}(c, f)$ implies $z^- = 0$ a.e. in $\{w = 0\} \cap \{-cj_2'(0) < f \le cj_1'(0)\}$. This proves the implication "$\Rightarrow$" in the second equivalence in (4.1). The reverse implication is trivial. □

We now turn our attention back to the boundedness condition (3.5), i.e.,

$$0 \le \limsup_{n\to\infty} J_n(\delta_n) = \limsup_{n\to\infty} \left(\frac{1}{t_n} \int_\Omega c\, \frac{j(w + t_n\delta_n) - j(w)}{t_n} - c\, j'(w; \delta_n) \mathrm{d}\lambda \right) < \infty.$$

To analyze which implications (3.5) has in the situation of Assumptions 4.1 and 4.3 and to pass to the limit with the terms $J_n(z)$ and $J_n(\delta_n)$ in (3.2), we need the following technical result:

Proposition 4.6. *Let $B \subset \mathbb{R}^{d-1}$ be an open ball and let $a > 0$. Suppose that $v, \varphi \in C(\overline{B} \times [0, a])$ are functions satisfying*

$$v = 0 \text{ on } \overline{B} \times \{0\}, \quad v > 0 \text{ in } \overline{B} \times (0, a], \quad \text{and} \quad \varphi \ge 0 \text{ in } \overline{B} \times [0, a].$$

Assume further that $t_n \in (0, \infty)$ and $z_n \in H^1(B \times (0, a))$ are sequences with $t_n \to 0$ and $z_n \rightharpoonup z$ in $H^1(B \times (0, a))$ for some function z. Then the following is true:

a) If it holds $v \in W^{1,\infty}(B \times (0, a))$, $\varphi > 0$ in $\overline{B} \times \{0\}$, and

$$\lim_{t\to 0} \left(\|\nabla v\|_{L^\infty(B\times(0,t))} \right) = 0,$$

and if $(\operatorname{tr} z)^-$ is not identical zero on $B \times \{0\}$, then

$$\liminf_{n\to\infty} \left(\int_{B\times(0,a)} \varphi \frac{(-v - t_n z_n)^+}{t_n^2} \mathrm{d}\lambda \right) = \infty.$$

b) If it holds $v \in C^1(\overline{B} \times [0, a])$ and $\|\nabla v\| \ge \varepsilon > 0$ on $B \times \{0\}$, and if there exists a constant C independent of n with $\|z_n\|_{L^\infty} \le C$, then

$$\int_{B\times(0,a)} \varphi \frac{(-v - t_n z_n)^+}{t_n^2} \mathrm{d}\lambda \to \frac{1}{2} \int_{B\times\{0\}} \varphi \frac{(\operatorname{tr} z^-)^2}{(\partial_d v)} \mathrm{d}\mathcal{H}^{d-1}.$$

Here, $\operatorname{tr} z^- \in L^2(B \times \{0\}, \mathcal{H}^{d-1})$ is the trace of the function z^- on $B \times \{0\}$.

Proof. We may assume w.l.o.g. that $z_n \in C(\overline{B} \times [0, a])$ for all $n \in \mathbb{N}$. If this is not the case, we can simply replace z_n with a sequence $\tilde{z}_n \in C(\overline{B} \times [0, a])$ satisfying $\|z_n - \tilde{z}_n\|_{H^1} \le t_n^2$ (and $\|\tilde{z}_n\|_{L^\infty} \le C$ in b)) since this exchange has no effect on the limiting behavior of the integral expressions under consideration. In the following, we denote the first $d-1$ coordinates of the Euclidean space with $x \in \mathbb{R}^{d-1}$ and the d-th coordinate with $y \in \mathbb{R}$. Further, we introduce the abbreviations $\mathrm{d}x$ and $\mathrm{d}y$ for $\mathrm{d}\lambda^{d-1}(x)$ and $\mathrm{d}\lambda^1(y)$.

Ad a): If $M > 0$ is arbitrary but fixed, then for all large enough n it holds $t_n M < a$ and (since $v \geq 0$ in $B \times (0,a)$)

$$\int_{B\times(0,a)} \varphi \frac{(-v - t_n z_n)^+}{t_n^2} \mathrm{d}\lambda \geq \int_B \int_0^{t_n M} \varphi \frac{(-z_n)^+}{t_n} - \varphi \frac{v}{t_n^2} \mathrm{d}y\mathrm{d}x. \tag{4.6}$$

From $v = 0$ on $\overline{B} \times \{0\}$ and $v \in W^{1,\infty}(B \times (0,a))$, we obtain further

$$\begin{aligned}\left| \int_B \int_0^{t_n M} \varphi(x,y) \frac{v(x,y)}{t_n^2} \mathrm{d}y\mathrm{d}x \right| &\leq \|\varphi\|_{L^\infty} \int_B \int_0^{t_n M} \frac{1}{t_n^2} \int_0^y |(\partial_d v)(x,s)| \mathrm{d}s\mathrm{d}y\mathrm{d}x \\ &\leq \frac{1}{2} \|\varphi\|_{L^\infty} \|\nabla v\|_{L^\infty(B\times(0,t_n M))} \lambda^{d-1}(B) M^2. \end{aligned} \tag{4.7}$$

Similarly, we may calculate that

$$\begin{aligned} &\int_B \int_0^{t_n M} \varphi(x,y) \frac{(-z_n(x,y))^+}{t_n} \mathrm{d}y\mathrm{d}x \\ &\quad = \int_B \int_0^{t_n M} \varphi(x,y) \frac{(-z_n(x,0))^+}{t_n} \mathrm{d}y\mathrm{d}x + R_n \\ &\quad = \int_B (-z_n(x,0))^+ \int_0^M \varphi(x, t_n y) \mathrm{d}y\mathrm{d}x + R_n \end{aligned} \tag{4.8}$$

with

$$\begin{aligned} |R_n| &= \left| \int_B \int_0^{t_n M} \varphi(x,y) \frac{(-z_n(x,y))^+ - (-z_n(x,0))^+}{t_n} \mathrm{d}y\mathrm{d}x \right| \\ &\leq \|\varphi\|_{L^\infty} \int_B \int_0^{t_n M} \frac{1}{t_n} \int_0^y |\partial_d z_n(x,s)| \mathrm{d}s\mathrm{d}y\mathrm{d}x \\ &\leq \|\varphi\|_{L^\infty} \int_B \int_0^{t_n M} \frac{1}{t_n} y^{1/2} \left(\int_0^a |\partial_d z_n(x,s)|^2 \mathrm{d}s \right)^{1/2} \mathrm{d}y\mathrm{d}x \\ &\leq \frac{2}{3} \|\varphi\|_{L^\infty} \|z_n\|_{H^1} \lambda^{d-1}(B)^{1/2} M^{3/2} t_n^{1/2}. \end{aligned} \tag{4.9}$$

Using (4.7), (4.8), (4.9), the boundedness of z_n in $H^1(B \times (0,a))$, our assumptions on v, and the compactness of $\mathrm{tr} : H^1(B \times (0,a)) \to L^2(B, \lambda^{d-1}) \cong L^2(B \times \{0\}, \mathcal{H}^{d-1})$ (cf. [22, Chapter 2, Theorem 6.2]), we can pass to the limit $n \to \infty$ in (4.6) to obtain

$$\begin{aligned} \liminf_{n\to\infty} \left(\int_{B\times(0,a)} \varphi \frac{(-v - t_n z_n)^+}{t_n^2} \mathrm{d}\lambda \right) &\geq M \int_B \varphi(x,0) (-\mathrm{tr}\, z(x))^+ \mathrm{d}x \\ &= M \int_{B\times\{0\}} \varphi |(\mathrm{tr}\, z)^-| \mathrm{d}\mathcal{H}^{d-1}. \end{aligned} \tag{4.10}$$

Since $M > 0$ was arbitrarily large and since $\varphi|(\mathrm{tr}\, z)^-|$ is non-negative and not identical zero on $B \times \{0\}$ (cf. our assumptions), it follows that the limes inferior in (4.10) has to be infinite. This proves part a).

Ad b): The claim in b) is obtained similarly to that in a): Due to the C^1-regularity of v, the function

$$r(x,y) := \frac{v(x,y)}{y} = \int_0^1 \partial_d v(x,sy)\mathrm{d}s$$

is continuous, and from $v > 0$ in $\overline{B} \times (0,a]$ and $\|\nabla v\| = \partial_d v = r \geq \varepsilon > 0$ on $B \times \{0\}$ it follows that r is positive everywhere in $\overline{B} \times [0,a]$. Thus, there exists an $\tilde{\varepsilon} > 0$ such that $r \geq \tilde{\varepsilon}$ holds everywhere in $\overline{B} \times [0,a]$. On the other hand, the integrand in the integral under consideration can only be non-zero if it is true that

$$0 \leq -v(x,y) - t_n z_n(x,y) = -yr(x,y) - t_n z_n(x,y),$$

i.e., if it holds

$$0 \leq y \leq t_n \frac{\|z_n\|_{L^\infty}}{\tilde{\varepsilon}} \leq Ct_n$$

with a constant C independent of n. Note that $\{z_n\}$ is bounded in L^∞ by assumption. Thus, for n large enough, we have

$$\begin{aligned}
&\int_{B\times(0,a)} \varphi \frac{(-v - t_n z_n)^+}{t_n^2}\mathrm{d}\lambda \\
&\quad = \int_B \int_0^{Ct_n} \varphi(x,y) \frac{(-yr(x,y) - t_n z_n(x,y))^+}{t_n^2}\mathrm{d}y\mathrm{d}x \\
&\quad = \int_B \int_0^{C} \varphi(x,t_n y)\Big(- yr(x,t_n y) - z_n(x,t_n y)\Big)^+\mathrm{d}y\mathrm{d}x \\
&\quad = \int_B \int_0^{C} \varphi(x,t_n y)\Big(- yr(x,0) - z_n(x,0)\Big)^+\mathrm{d}y\mathrm{d}x + R_n.
\end{aligned} \tag{4.11}$$

Thanks to the continuity of r and the boundedness of z_n in H^1, we can estimate the remainder R_n by

$$\begin{aligned}
|R_n| &\leq \|\varphi\|_{L^\infty} \int_B \int_0^C |yr(x,t_n y) - yr(x,0)| + |z_n(x,t_n y) - z_n(x,0)|\mathrm{d}y\mathrm{d}x \\
&\leq \|\varphi\|_{L^\infty} \int_B \int_0^C \int_0^{Ct_n} |\partial_d z_n(x,s)|\mathrm{d}s\mathrm{d}y\mathrm{d}x + o(1) \\
&\leq \|\varphi\|_{L^\infty} \int_B \int_0^C (Ct_n)^{1/2} \left(\int_0^a |\partial_d z_n(x,s)|^2\mathrm{d}s\right)^{1/2} \mathrm{d}y\mathrm{d}x + o(1) \\
&\leq \|\varphi\|_{L^\infty} \|z_n\|_{H^1} C^{3/2} \lambda^{d-1}(B)^{1/2}\, t_n^{1/2} + o(1) = o(1).
\end{aligned} \tag{4.12}$$

From (4.11), (4.12), and the compactness of the trace operator, it follows

$$\int_{B\times(0,a)} \varphi \frac{(-v - t_n z_n)^+}{t_n^2}\mathrm{d}\lambda \to \int_B \varphi(x,0) \int_0^C \Big(- y\partial_d v(x,0) - (\operatorname{tr} z)(x)\Big)^+ \mathrm{d}y\mathrm{d}x. \tag{4.13}$$

Using $\|\nabla v\| = \partial_d v \geq \varepsilon > 0$ on $B \times \{0\}$, we can compute the inner integral on the right-hand side of (4.13). This yields

$$\int_{B\times(0,a)} \varphi \frac{(-v - t_n z_n)^+}{t_n^2}\mathrm{d}\lambda \to \frac{1}{2} \int_B \varphi(x,0) \frac{(\operatorname{tr} z^-(x))^2}{(\partial_d v)(x,0)}\mathrm{d}\lambda^{d-1}(x),$$

proving b). ∎

Note that for $j(x) = |x|$, it holds

$$\frac{1}{t_n}\left(\int_\Omega c\,\frac{j(w+t_n\delta_n)-j(w)}{t_n} - c\,j'(w;\delta_n)\mathrm{d}\lambda\right)$$
$$= \int_{\{w>0\}} 2c\,\frac{(-w-t_n\delta_n)^+}{t_n^2}\mathrm{d}\lambda + \int_{\{w<0\}} 2c\,\frac{(w+t_n\delta_n)^+}{t_n^2}\mathrm{d}\lambda,$$

i.e., the integrals studied in Proposition 4.6 are exactly those that appear in the boundedness condition (3.5) when the absolute value function is considered. If j is an arbitrary function satisfying Assumption 4.1, then we can use Taylor expansions and localization arguments to deduce the following from (3.5) and Proposition 4.6 a):

PROPOSITION 4.7. *Let Assumptions 2.1, 4.1, and 4.3 hold. Then the following is true in the situation of Assumption 3.1:*

$$(\operatorname{tr}\delta)^+ = 0\ \mathcal{H}^{d-1}\text{-a.e. on } \mathcal{N}^- \quad \text{and} \quad (\operatorname{tr}\delta)^- = 0\ \mathcal{H}^{d-1}\text{-a.e. on } \mathcal{N}^+.$$

Proof. From (3.5), we obtain that there exists a constant $C > 0$ independent of n with

$$\begin{aligned} C &\geq \frac{1}{t_n}\int_\Omega c\,\frac{j(w+t_n\delta_n)-j(w)}{t_n} - c\,j'(w;\delta_n)\mathrm{d}\lambda \\ &\geq \frac{1}{t_n}\int_{\{w>0\}} c\,\frac{j(w+t_n\delta_n)-j(w)}{t_n} - c\,j'(w)\delta_n\mathrm{d}\lambda \qquad \text{(by Lemma 2.3 b))} \\ &\geq \frac{1}{t_n}\int_{\{w>0\}} c\,\frac{j((w+t_n\delta_n)^+)-j(w^+)}{t_n} - c\,j'(w^+)\delta_n\mathrm{d}\lambda \quad \text{(since } j\geq 0 = j(0)\text{)}. \end{aligned}$$

Since $j = j_1$ on $\mathbb{R}^+$ and $j_1 \in C^2([0,\infty))$ by Assumption 4.1, we may apply a Taylor expansion and can continue the above estimate with

$$\begin{aligned} C \geq \frac{1}{t_n}&\left(\int_{\{w>0\}} c\left(j_1'(w^+)\frac{(w+t_n\delta_n)^+ - w^+}{t_n}\right) - c\,j_1'(w^+)\delta_n\mathrm{d}\lambda\right) \\ &+ \int_{\{w>0\}} c\,\frac{((w+t_n\delta_n)^+ - w^+)^2}{t_n^2}\int_0^1 (1-s)j_1''((1-s)w^+ + s(w+t_n\delta_n)^+)\mathrm{d}s\mathrm{d}\lambda. \end{aligned} \tag{4.14}$$

Further, we obtain from the boundedness of (δ_n) in $L^\infty(\Omega)$ (see Theorem 2.6) that the integrand of the second integral on the right-hand side of (4.14) satisfies

$$\begin{aligned} &\left|c\,\frac{((w+t_n\delta_n)^+ - w^+)^2}{t_n^2}\int_0^1 (1-s)j_1''((1-s)w^+ + s(w+t_n\delta_n)^+)\mathrm{d}s\right| \\ &\qquad \leq \|c\|_{L^\infty}\|\delta_n\|_{L^\infty}^2 \max\left\{j_1''(x) : 0 \leq x \leq \|w\|_{L^\infty} + \|\delta_n\|_{L^\infty}\right\} \leq C \end{aligned}$$

a.e. in Ω for some constant $C > 0$ independent of n. Thus, we may deduce from (4.14) that

$$\begin{aligned} C &\geq \int_{\{w>0\}} cj_1'(w^+)\frac{1}{t_n}\left(\frac{(w+t_n\delta_n)^+ - w^+}{t_n} - \delta_n\right)\mathrm{d}\lambda \\ &= \int_{\{w>0\}} cj_1'(w^+)\frac{(-w-t_n\delta_n)^+}{t_n^2}\mathrm{d}\lambda. \end{aligned} \tag{4.15}$$

As already indicated, we intend to apply Proposition 4.6 a) to (4.15). To do so, we have to localize the problem. For this purpose, consider an arbitrary but fixed point $p \in \mathcal{N}^+ \subseteq \partial\{w \neq 0\} \setminus \mathcal{C}$. Then it follows from Assumption 4.3 b) that (possibly after changing coordinates such that the corresponding orthogonal transformation R becomes the identity) we can find an open ball $B \subset \mathbb{R}^{d-1}$, an open interval J, and a Lipschitz map $h : B \to J$ such that $\partial\{w \neq 0\} \setminus \mathcal{C} \cap (B \times J) = \{(x, h(x)) : x \in B\}$ and $p \in B \times J$. Since $\mathcal{C}$ is closed, since $\mathcal{N}^+ \subset \Omega$, and since $\mathcal{N}^+$ is relatively open in $\partial\{w \neq 0\} \setminus \mathcal{C}$, we can shrink the open sets B and J to obtain

$$p \in B \times J \subset \Omega \setminus \mathcal{C} \quad \text{and} \quad \partial\{w \neq 0\} \cap (B \times J) \subset \mathcal{N}^+. \tag{4.16}$$

Moreover, as J is open and h is Lipschitz, a further reduction of B to a B_0 yields the existence of an $\varepsilon > 0$ such that $p \in B_0 \times J$ and

$$D := \{(x, y) : x \in \overline{B_0},\ |y - h(x)| \leq \varepsilon\} \subset B \times J.$$

Due to (4.16), the set $\partial\{w \neq 0\} \cap D$ is a subset of $\partial\{w > 0\}$. Since $(B \times J) \cap \mathcal{C} = \emptyset$ and thus $(B \times J) \cap \partial\{w \neq 0\} = \{(x, h(x)) : x \in B\}$, we may deduce that w is positive in at least one of the sets

$$D_1 := \{(x, y) \in D : y > h(x)\}, \quad D_2 := \{(x, y) \in D : y < h(x)\}.$$

Let us assume that this is true for D_1 (the other case is analogous). Then the non-negativity of the integrand in (4.15) (cf. $j_1'(0) > 0$ and the convexity of j_1) and the change of variables formula for Lipschitz maps (cf. [10, Theorem 3.9]) imply

$$\begin{aligned} C &\geq \int_{D_1} c j_1'(w^+) \frac{(-w - t_n\delta_n)^+}{t_n^2} \mathrm{d}\lambda \\ &= \int_{B_0} \int_0^\varepsilon \left(c j_1'(w) \frac{(-w - t_n\delta_n)^+}{t_n^2} \right)\Big|_{(x, y+h(x))} \mathrm{d}y\mathrm{d}x. \end{aligned} \tag{4.17}$$

Note that the Jacobian determinant equals one in the above. Defining

$$v(x, y) := w(x, y + h(x)), \quad \varphi(x, y) := c(x, y + h(x)) j_1'(w(x, y + h(x))),$$
$$z_n(x, y) := \delta_n(x, y + h(x)),$$

the right-hand side of (4.17) takes exactly the form of the integral expression studied in Proposition 4.6. Let us verify the assumptions of Proposition 4.6 a) in our setting: First of all, the chain rule for Lipschitz functions (cf. [34, Theorem 2.2.2]) implies

$$v \in W^{1,\infty}(B_0 \times (0, \varepsilon)) \quad \text{with} \quad (\nabla v)(x, y) = \begin{pmatrix} I & 0 \\ \nabla h(x) & 1 \end{pmatrix} (\nabla w)(x, y + h(x)).$$

Since the definition of $\mathcal{N}^+$ yields $(\nabla w)(x, h(x)) = 0$ for all $x \in \overline{B_0}$, the above together with the continuity of ∇w gives $\lim_{t \to 0} \|\nabla v\|_{L^\infty(B_0 \times (0,t))} = 0$, as required in Proposition 4.6 a). In addition, the mapping $B_0 \times (0, \varepsilon) \ni (x, y) \mapsto (x, y + h(x))$ is bi-Lipschitz so that [34, Theorem 2.2.2] together with $\delta_n \in H_0^1(\Omega)$ implies that $z_n \in H^1(B_0 \times (0, \varepsilon))$. Due to the weak convergence of δ_n in $H_0^1(\Omega)$, the sequence z_n is bounded in $H^1(B_0 \times (0, \varepsilon))$. We can thus pass over to weakly convergent subsequences. The pointwise convergence of δ_n a.e. in Ω by Assumption 3.1 moreover implies that z_n converges pointwise a.e. in $B_0 \times (0, \varepsilon)$ to $z(x, y) := \delta(x, y + h(x))$.

Since pointwise and weak limit coincide, we obtain the weak convergence of z_n to z as required in Proposition 4.6. Furthermore, the regularity assumptions on c, w, j_1, and h immediately give $v, \varphi \in C(\overline{B_0} \times [0, \varepsilon])$. Finally, as w is positive in D_1 and since the definition of $\mathcal{N}^+$ implies $w(x, h(x)) = 0$, we also obtain the remaining conditions in Proposition 4.6, i.e.,

$$v = 0 \text{ on } \overline{B_0} \times \{0\}, \quad v > 0 \text{ in } \overline{B_0} \times (0, \varepsilon], \quad \varphi > 0 \text{ in } \overline{B_0} \times [0, \varepsilon].$$

Thus, Proposition 4.6 a) is applicable and we may deduce from (4.17) by contradiction that $(\operatorname{tr}\delta)^- = 0$ $\mathcal{H}^{d-1}$-a.e. on $\mathcal{N}^+ \cap D$. This, together with the arbitrariness of the point $p \in \mathcal{N}^+$, proves the claim for $(\operatorname{tr}\delta)^-$. The result for $(\operatorname{tr}\delta)^+$ is obtained completely analogously. □

We continue the limit analysis of the integrals $J_n(\delta_n)$ and $J_n(z)$ appearing in (3.2) with an auxiliary result that is a globalized version of Proposition 4.6 b):

Proposition 4.8. *Let Ω be a bounded Lipschitz domain, and suppose that functions j, ω, c, and ψ are given such that j satisfies Assumption 4.1 and such that*

$$\begin{aligned} &\omega \in C^1(\Omega), \quad 0 \le c \in C(\Omega), \quad 0 \le \psi \in C_c(\Omega), \\ &\operatorname{supp}(\psi) \cap \partial\{\omega > 0\} \cap \{\nabla\omega = 0\} = \emptyset. \end{aligned} \tag{4.18}$$

Assume further that $t_n \in (0, \infty)$ and $\zeta_n \in H^1(\Omega)$ are sequences satisfying

$$t_n \to 0, \quad \|\zeta_n\|_{L^\infty} \le C, \quad \zeta_n \rightharpoonup \zeta \text{ in } H^1(\Omega), \quad \text{and} \quad \zeta_n \to \zeta \text{ pointwise a.e. in } \Omega$$

for some constant C independent of n and some $\zeta \in H^1(\Omega)$. Then it holds

$$\begin{aligned} \lim_{n\to\infty} &\int_\Omega \psi \frac{c}{t_n}\left(\frac{j(\omega + t_n\zeta_n^-) - j(\omega)}{t_n} - j'(\omega; \zeta_n^-)\right) \mathrm{d}\lambda \\ &= \frac{1}{2}\int_{\{\omega \ne 0\}} \psi c j''(\omega)(\zeta^-)^2 \mathrm{d}\lambda + \frac{1}{2}\int_{\{\omega = 0\}} \psi c j_2''(0)(\zeta^-)^2 \mathrm{d}\lambda \\ &\qquad + \frac{1}{2}\Big(j_1'(0) + j_2'(0)\Big)\int_{\{\omega=0\}\cap\{\nabla\omega\ne 0\}} \psi c \frac{(\operatorname{tr}\zeta^-)^2}{\|\nabla\omega\|} \mathrm{d}\mathcal{H}^{d-1}. \end{aligned} \tag{4.19}$$

Proof. For the left-hand side of (4.19) it follows from the properties of j that

$$\begin{aligned} &\int_\Omega \psi \frac{c}{t_n}\left(\frac{j(\omega + t_n\zeta_n^-) - j(\omega)}{t_n} - j'(\omega; \zeta_n^-)\right) \mathrm{d}\lambda \\ &\quad = \int_{\{\omega \le 0\}} \psi \frac{c}{t_n}\left(\frac{j_2(-\omega - t_n\zeta_n^-) - j_2(-\omega)}{t_n} + j_2'(-\omega)\zeta_n^-\right) \mathrm{d}\lambda \\ &\qquad + \int_{\{\omega > 0\}} \psi \frac{c}{t_n}\left(\frac{j_1((\omega + t_n\zeta_n^-)^+) - j_1(\omega)}{t_n} - j_1'(\omega)\zeta_n^-\right) \mathrm{d}\lambda \\ &\qquad\qquad + \int_{\{\omega > 0\}} \psi \frac{c}{t_n}\left(\frac{j_2(-(\omega + t_n\zeta_n^-)^-)}{t_n}\right) \mathrm{d}\lambda =: I_n^{(1)} + I_n^{(2)} + I_n^{(3)}. \end{aligned}$$

We analyze the limiting behavior of the three integrals $I_n^{(1)}$, $I_n^{(2)}$, and $I_n^{(3)}$ separately:

Ad $I_n^{(1)}$: Using the dominated convergence theorem, the boundedness of ζ_n in $L^\infty(\Omega)$, and Taylor's formula, we obtain

$$I_n^{(1)} = \int_{\{\omega \le 0\}} \psi c (\zeta_n^-)^2 \int_0^1 (1 - s) j_2''(-\omega - s t_n \zeta_n^-) \mathrm{d}s \mathrm{d}\lambda \to \frac{1}{2}\int_{\{\omega \le 0\}} \psi c j_2''(-\omega)(\zeta^-)^2 \mathrm{d}\lambda.$$

Ad $I_n^{(2)}$: Similarly to (4.14), we obtain by Taylor expansion

$$\begin{aligned} I_n^{(2)} &= \frac{1}{t_n}\left(\int_{\{\omega>0\}} \psi c j_1'(\omega)\left(\frac{(\omega+t_n\zeta_n^-)^+-\omega}{t_n}-\zeta_n^-\right)\mathrm{d}\lambda\right) \\ &\quad + \int_{\{\omega>0\}} \psi c\,\frac{((\omega+t_n\zeta_n^-)^+-\omega)^2}{t_n^2}\int_0^1 (1-s)j_1''((1-s)\omega+s(\omega+t_n\zeta_n^-)^+)\mathrm{d}s\mathrm{d}\lambda \\ &=: I_n^{(2a)}+I_n^{(2b)}. \end{aligned}$$

For the integral $I_n^{(2b)}$ it follows from the dominated convergence theorem and the boundedness of $\{\zeta_n\}$ in L^∞ that

$$I_n^{(2b)} \to \frac{1}{2}\int_{\{\omega>0\}} \psi c\, j_1''(\omega)(\zeta^-)^2\mathrm{d}\lambda.$$

For $I_n^{(2a)}$ a distinction of cases yields

$$I_n^{(2a)} = \int_{\{\omega>0\}} \psi c j_1'(\omega)\,\frac{(-\omega-t_n\zeta_n^-)^+}{t_n^2}\mathrm{d}\lambda \tag{4.20}$$

so that we arrive at an expression similar to that in Proposition 4.6. As the limit transition with $I_n^{(2a)}$ is rather involved, let us briefly outline the next steps of our analysis: In what follows, we will split the right-hand side of (4.20) into an integral over a compact subset of $\{\omega>0\}$ and integrals over domains which lie in the immediate vicinity of the boundary $\partial\{\omega>0\}$, see (4.23) below. We will then show that, for large enough n, only the neighborhood of the set $\partial\{\omega>0\}$ contributes to the integral in (4.20), and use localization arguments similar to those in the proof of Proposition 4.7 to pass to the limit $n\to\infty$. As a result, we will obtain the convergence to a surface integral, see (4.25).

To construct an appropriate splitting of the integral in (4.20), we define $K := \operatorname{supp}(\psi)$ and note that (4.18) yields $K\cap\partial\{\omega>0\}\subset\{\nabla\omega\neq 0\}$. The latter implies, together with $\omega\in C^1(\Omega)$ and the compactness of $K\cap\partial\{\omega>0\}$, that there exists a constant $m>0$ with $\|\nabla\omega\|\geq m$ on $K\cap\partial\{\omega>0\}$. Write $\mathcal{M} := \{\omega=0\}\cap\{\nabla\omega\neq 0\}$. Then the implicit function theorem yields that for each point $p\in K\cap\partial\{\omega>0\}\subset\mathcal{M}$ we may find an orthogonal transformation $R_p\in O(d)$, an open ball $B_p\subset\mathbb{R}^{d-1}$, an open interval J_p, and a C^1-function $h_p : B_p\to J_p$ such that

$$\begin{gathered} p\in R_p(B_p\times J_p),\quad R_p(B_p\times J_p)\subset\Omega, \\ \mathcal{M}\cap R_p(B_p\times J_p) = \{\omega=0\}\cap R_p(B_p\times J_p) = R_p(\{(x,h_p(x)) : x\in B_p\}). \end{gathered} \tag{4.21}$$

Note that, since $\nabla\omega$ and h_p are continuous and since we can make the sets B_p and J_p arbitrarily small, we may assume w.l.o.g. that in addition to the above it holds

$$\begin{gathered} \operatorname{cl}(R_p(B_p\times J_p))\subset\Omega,\quad \|\nabla\omega\|\geq m/2 \text{ in } R_p(B_p\times J_p), \\ \omega\neq 0 \text{ in } \operatorname{cl}(R_p(B_p\times J_p))\setminus\operatorname{cl}(R_p(\{(x,h_p(x)) : x\in B_p\})), \\ \text{and}\quad R_p(\{(x,y) : x\in B_p,\ |y-h_p(x)|<\varepsilon_p\})\subseteq R_p(B_p\times J_p) \end{gathered} \tag{4.22}$$

for some $\varepsilon_p>0$. Let us denote the ε_p-tube on the left-hand side of (4.22) with W_p. Then the collection $\{W_p\}$ defines an open cover of $K\cap\partial\{\omega>0\}$ and it follows from the compactness of $K \cap \partial\{\omega > 0\}$ that there exist $p_1, ..., p_L \in K \cap \partial\{\omega > 0\}$, $L \in \mathbb{N}$, with

$$K \cap \partial\{\omega > 0\} \subset \bigcup_{l=1}^{L} W_{p_l} =: U.$$

Further, since U is open, we can find an open set $V \subset \mathbb{R}^d$ such that $U \cup V = \mathbb{R}^d$ and $\overline{V} \cap K \cap \partial\{\omega > 0\} = \emptyset$. Consider now a partition of unity of the Euclidean space $\mathbb{R}^d$ subordinate to the cover $W_{p_1}, ..., W_{p_L}, V$ (cf. [31, Theorem 1.11]), i.e., a collection of smooth functions $\chi_l : \mathbb{R}^d \to [0, 1]$, $l = 1, ..., L + 1$, satisfying

$$\operatorname{supp}(\chi_l) \subset W_{p_l}, \quad l = 1, ..., L, \quad \operatorname{supp}(\chi_{L+1}) \subset V, \quad \text{and} \quad \sum_{l=1}^{L+1} \chi_l \equiv 1.$$

Then we obtain

$$\begin{aligned} I_n^{(2a)} = \sum_{l=1}^{L} \int_{W_{p_l} \cap \{\omega > 0\}} \chi_l \psi c j_1'(\omega) \frac{(-\omega - t_n \zeta_n^-)^+}{t_n^2} \mathrm{d}\lambda \\ + \int_{V \cap K \cap \{\omega > 0\}} \chi_{L+1} \psi c j_1'(\omega) \frac{(-\omega - t_n \zeta_n^-)^+}{t_n^2} \mathrm{d}\lambda. \end{aligned} \tag{4.23}$$

Note that from $\overline{V} \cap K \cap \partial\{\omega > 0\} = \emptyset$ it follows $\overline{V} \cap K \cap \{\omega > 0\} = \overline{V} \cap K \cap \operatorname{cl}(\{\omega > 0\})$. Thus, the set $\overline{V} \cap K \cap \{\omega > 0\}$ is a compact subset of $\{\omega > 0\}$ and the continuity of ω gives the existence of an $\varepsilon > 0$ so that $\omega \geq \varepsilon > 0$ in $\overline{V} \cap K \cap \{\omega > 0\}$. This together with $\|\zeta_n\|_{L^\infty} \leq C$ for all n and $t_n \to 0$ implies that the integral associated with χ_{L+1} in (4.23) is identical zero for n sufficiently large. It remains to analyze the first L integrals on the right-hand side of (4.23), i.e., the contributions to $I_n^{(2a)}$ that come from the vicinity of the boundary $\partial\{\omega > 0\}$. To this end, consider one of the first L integrals in (4.23), drop the index l, and assume w.l.o.g. that $R_p = \mathrm{Id}$. In this prototypical situation, the integral in question satisfies (cf. (4.17)):

$$\begin{aligned} &\int_{W_p \cap \{\omega > 0\}} \chi \psi c j_1'(\omega) \frac{(-\omega - t_n \zeta_n^-)^+}{t_n^2} \mathrm{d}\lambda \\ &\quad = \int_{B_p} \int_{-\varepsilon_p}^{\varepsilon_p} \left[\mathbb{1}_{\{\omega > 0\}} \chi \psi c j_1'(\omega) \frac{(-\omega - t_n \zeta_n^-)^+}{t_n^2} \right] \Bigg|_{(x, y + h_p(x))} \mathrm{d}y \mathrm{d}x. \end{aligned}$$

Note that, since $\{\omega = 0\} \cap W_p = \{(x, h_p(x)) : x \in B_p\} \subset \{\omega = 0\} \cap \{\nabla \omega \neq 0\}$ and due to $\omega \neq 0$ in $\operatorname{cl}(B_p \times J_p) \setminus \operatorname{cl}(\{(x, h_p(x)) : x \in B_p\})$, it has to hold either

$$\begin{aligned} &\omega(x, y + h_p(x)) > 0 \text{ in } \overline{B_p} \times (0, \varepsilon_p], \quad \omega(x, y + h_p(x)) < 0 \text{ in } \overline{B_p} \times [-\varepsilon_p, 0) \\ \text{or} \quad &\omega(x, y + h_p(x)) < 0 \text{ in } \overline{B_p} \times (0, \varepsilon_p], \quad \omega(x, y + h_p(x)) > 0 \text{ in } \overline{B_p} \times [-\varepsilon_p, 0). \end{aligned}$$

If the first case is true (the second one is analogous), then it holds

$$\begin{aligned} &\int_{W_p \cap \{\omega > 0\}} \chi \psi c j_1'(\omega) \frac{(-\omega - t_n \zeta_n^-)^+}{t_n^2} \mathrm{d}\lambda \\ &\quad = \int_{B_p} \int_{0}^{\varepsilon_p} \left[\chi \psi c j_1'(\omega) \frac{(-\omega - t_n \zeta_n^-)^+}{t_n^2} \right] \Bigg|_{(x, y + h_p(x))} \mathrm{d}y \mathrm{d}x. \end{aligned}$$

The integral on the right-hand side of the last equation has exactly the form of that studied in Proposition 4.6 b). Moreover, the functions $v(x,y) := \omega(x, y + h_p(x))$, $\varphi(x,y) := \chi\psi c j_1'(\omega)|_{(x,y+h_p(x))}$, and $z_n(x,y) := \zeta_n^-(x, y+h_p(x))$ satisfy all assumptions of Proposition 4.6 b) as one can easily check using an argumentation similar to that in the proof of Proposition 4.7. Note in this context that $H^1(\Omega) \ni \eta \mapsto \eta^- \in H^1(\Omega)$ is weakly continuous so that $z_n \rightharpoonup z$ in $H^1(B_p \times (0,\varepsilon_p))$ with $z(x,y) := \zeta^-(x, y+h_p(x))$. We thus obtain

$$\begin{aligned}\int_{W_p\cap\{\omega>0\}} \chi\psi c j_1'(\omega)\,\frac{(-\omega - t_n\zeta_n^-)^+}{t_n^2}\mathrm{d}\lambda \\ \to \frac{1}{2}\int_{B_p} j_1'(0)\left[\chi\psi c\frac{(\operatorname{tr}\zeta^-)^2}{\partial_d\omega}\right]\Bigg|_{(x,h_p(x))}\mathrm{d}\lambda^{d-1}(x),\end{aligned} \tag{4.24}$$

where $\operatorname{tr}\zeta^-$ denotes the trace of ζ^- on $\mathcal{M}$. Using the identity

$$\|(\nabla\omega)(x,h_p(x))\| = (\partial_d\omega)(x,h_p(x))\sqrt{1+\|\nabla h_p(x)\|^2} \quad \forall x \in B_p,$$

which follows from the implicit function theorem and the change of variables formula (cf. [10, Theorem 3.9]), we can rewrite (4.24) as follows:

$$\int_{W_p\cap\{\omega>0\}} \chi\psi c j_1'(\omega)\,\frac{(-\omega - t_n\zeta_n^-)^+}{t_n^2}\mathrm{d}\lambda \to \frac{1}{2}\int_{\mathcal{M}} j_1'(0)\chi\psi c\frac{(\operatorname{tr}\zeta^-)^2}{\|\nabla\omega\|}\mathrm{d}\mathcal{H}^{d-1}.$$

Here, we used that $\{(x,h_p(x)) : x \in B_p\} = W_p \cap \mathcal{M}$ according to (4.21) and (4.22). Since W_{p_l}, $l = 1, ..., L$, forms an open cover of $K \cap \partial\{w > 0\}$ and since $\sum_{l=1}^L \chi_l \equiv 1$ on $K \cap \partial\{\omega > 0\} = \operatorname{supp}(\psi) \cap \mathcal{M}$, by summation we finally arrive at

$$I_n^{(2a)} \to \frac{1}{2}j_1'(0)\int_{\mathcal{M}} \psi c\frac{(\operatorname{tr}\zeta^-)^2}{\|\nabla\omega\|}\mathrm{d}\mathcal{H}^{d-1}. \tag{4.25}$$

Ad $I_n^{(3)}$: From Taylor's formula and $j_2(0) = 0$, we obtain

$$\begin{aligned}I_n^{(3)} &= \int_{\{\omega>0\}} \psi\frac{c}{t_n}\left(\frac{j_2(-(\omega+t_n\zeta_n^-)^-)}{t_n}\right)\mathrm{d}\lambda \\ &= \int_{\{\omega>0\}} \psi j_2'(0)c\frac{-(\omega+t_n\zeta_n^-)^-}{t_n^2}\mathrm{d}\lambda \\ &\quad + \int_{\{\omega>0\}} \psi c\frac{((\omega+t_n\zeta_n^-)^-)^2}{t_n^2}\int_0^1 (1-s)j_2''(-s(\omega+t_n\zeta_n^-)^-)\,\mathrm{d}s\mathrm{d}\lambda =: I_n^{(3a)} + I_n^{(3b)}.\end{aligned}$$

Analogously to $I_n^{(2b)}$, the dominated convergence theorem yields $I_n^{(3b)} \to 0$, whereas, because of

$$\int_{\{\omega>0\}} \psi j_2'(0)c\frac{-(\omega+t_n\zeta_n^-)^-}{t_n^2}\mathrm{d}\lambda = \int_{\{\omega>0\}} \psi j_2'(0)c\frac{(-\omega-t_n\zeta_n^-)^+}{t_n^2}\mathrm{d}\lambda,$$

the integral $I_n^{(3a)}$ behaves exactly like $I_n^{(2a)}$. As a consequence,

$$I_n^{(3)} \to \frac{1}{2}j_2'(0)\int_{\mathcal{M}} \psi c\frac{(\operatorname{tr}\zeta^-)^2}{\|\nabla\omega\|}\mathrm{d}\mathcal{H}^{d-1}.$$

Combining all of our results, we arrive at (4.19) as desired. ∎

As a direct consequence of Proposition 4.8, we obtain:

Corollary 4.9. *Let Ω be a bounded Lipschitz domain, and suppose that functions j, ω, c, and ψ are given such that j satisfies Assumption 4.1 and such that*

$$\omega \in C^1(\Omega), \quad 0 \le c \in C(\Omega), \quad 0 \le \psi \in C_c(\Omega),$$
$$\operatorname{supp}(\psi) \cap \partial\{\omega < 0\} \cap \{\nabla\omega = 0\} = \emptyset.$$

Let $t_n \in (0,\infty)$ and $\zeta_n \in H^1(\Omega)$ be sequences as in Proposition 4.8. Then it holds

$$\begin{aligned}\lim_{n\to\infty} \int_\Omega \psi \frac{c}{t_n}\left(\frac{j(\omega + t_n\zeta_n^+) - j(\omega)}{t_n} - j'(\omega;\zeta_n^+)\right)\mathrm{d}\lambda \\ = \frac{1}{2}\int_{\{\omega\neq 0\}} \psi c j''(\omega)(\zeta^+)^2 \mathrm{d}\lambda + \frac{1}{2}\int_{\{\omega=0\}} \psi c j_1''(0)(\zeta^+)^2\mathrm{d}\lambda \\ + \frac{1}{2}\Big(j_1'(0) + j_2'(0)\Big)\int_{\{\omega=0\}\cap\{\nabla\omega\neq 0\}} \psi c \frac{(\operatorname{tr}\zeta^+)^2}{\|\nabla\omega\|}\mathrm{d}\mathcal{H}^{d-1}.\end{aligned} \tag{4.26}$$

Proof. Write $\zeta_n^+ = -(-\zeta_n)^-$, then (4.26) follows straightforwardly from (4.19). ☐

Note that the formulas (4.19) and (4.26) are also interesting for their own sake. They yield, for example, the following second order Taylor expansion for the L^1-norm:

Corollary 4.10. *Let Ω be a bounded Lipschitz domain and let $\omega \in C^1(\Omega)$ be a function with $\{\omega = 0\} \cap \{\nabla\omega = 0\} = \emptyset$. Then for all $\zeta \in C_c(\Omega) \cap H^1(\Omega)$ it holds*

$$\int_\Omega |\omega + t\zeta|\mathrm{d}\lambda = \int_\Omega |\omega|\mathrm{d}\lambda + t\int_\Omega \operatorname{sgn}(\omega)\zeta\mathrm{d}\lambda + t^2\int_{\{\omega=0\}} \frac{(\operatorname{tr}\zeta)^2}{\|\nabla\omega\|}\mathrm{d}\mathcal{H}^{d-1} + o(t^2), \quad t \searrow 0.$$

Proof. Choose a function $0 \le \psi \in C_c(\Omega)$ with $\psi \equiv 1$ in $\operatorname{supp}(\zeta)$, then we obtain from (4.19) and (4.26) that, for every sequence $t_n \subset (0,\infty)$ tending to zero, it holds

$$\begin{aligned}&\int_\Omega \frac{1}{t_n}\left(\frac{|\omega + t_n\zeta| - |\omega|}{t_n} - \operatorname{sgn}(\omega)\zeta\right)\mathrm{d}\lambda \\ &= \int_\Omega \frac{\psi}{t_n}\left(\frac{|\omega + t_n\zeta^-| - |\omega|}{t_n} - \operatorname{sgn}(\omega)\zeta^-\right) + \frac{\psi}{t_n}\left(\frac{|\omega + t_n\zeta^+| - |\omega|}{t_n} - \operatorname{sgn}(\omega)\zeta^+\right)\mathrm{d}\lambda \\ &\to \int_{\{\omega=0\}\cap\{\nabla\omega\neq 0\}} \psi\frac{(\operatorname{tr}\zeta^-)^2}{\|\nabla\omega\|}\mathrm{d}\mathcal{H}^{d-1} + \int_{\{\omega=0\}\cap\{\nabla\omega\neq 0\}} \psi\frac{(\operatorname{tr}\zeta^+)^2}{\|\nabla\omega\|}\mathrm{d}\mathcal{H}^{d-1}.\end{aligned}$$

Reformulating the above yields the claim. ☐

We are now in the position to pass to the limit in the variational inequality (3.2) for the difference quotients δ_n (at least for a particular class of test functions):

Proposition 4.11. *Let Assumptions 2.1, 4.1, and 4.3 hold. Let V denote the set of all functions $z \in L^\infty(\Omega) \cap T_{\mathrm{crit}}(c,f)$ satisfying $z^+ = 0$ a.e. in a neighborhood of $\partial\Omega \cup \mathcal{N}^- \cup \mathcal{C}$ and $z^- = 0$ a.e. in a neighborhood of the set $\partial\Omega \cup \mathcal{N}^+ \cup \mathcal{C}$. Then in the situation of Assumption 3.1 for all $z \in V$ it is true that*

$$a(\delta, z) + \int_\Omega h\Big(j'(w;z) - j'(w;\delta)\Big)\mathrm{d}\lambda - \langle g, z - \delta\rangle + J(z) - J(\delta) \ge \limsup_{n\to\infty} a(\delta_n,\delta_n). \tag{4.27}$$

Here,

$$J(z) := \frac{1}{2}\int_{\{w\neq 0\}} cj''(w)z^2 \mathrm{d}\lambda + \frac{1}{2}\int_{\{w=0\}} cj_1''(0)(z^+)^2 + cj_2''(0)(z^-)^2 \mathrm{d}\lambda + \frac{1}{2}\Big(j_1'(0) + j_2'(0)\Big)\int_{\mathcal{M}} c\frac{(\operatorname{tr} z)^2}{\|\nabla w\|}\mathrm{d}\mathcal{H}^{d-1}. \tag{4.28}$$

In particular, it holds

$$\int_{\mathcal{M}} c\frac{(\operatorname{tr}\delta)^2}{\|\nabla w\|}\mathrm{d}\mathcal{H}^{d-1} < \infty,$$

where $\mathcal{M}$ again denotes the set $\{w = 0\} \cap \{\nabla w \neq 0\}$.

Proof. From (3.2) and Corollary 3.8, it follows that for all $z \in V$ it holds

$$a(\delta_n, z) + H_n(z) - H_n(\delta_n) - \langle g, z - \delta_n\rangle + J_n(z) - J_n(\delta_n) \geq a(\delta_n, \delta_n). \tag{4.29}$$

Due to $\delta_n \rightharpoonup \delta$ in $H_0^1(\Omega)$ and Lemma 3.2, we can pass to the limit with the first four terms on the left-hand side of (4.29). This yields

$$a(\delta, z) + \int_\Omega hj'(w; z)\mathrm{d}\lambda - \int_\Omega hj'(w;\delta)\mathrm{d}\lambda - \langle g, z - \delta\rangle + \limsup_{n\to\infty} J_n(z) - \liminf_{n\to\infty} J_n(\delta_n) \geq \limsup_{n\to\infty} a(\delta_n, \delta_n) \quad \forall\, z \in V. \tag{4.30}$$

It remains to pass to the limit with the J_n-terms. Suppose for this purpose that $z \in V$ is arbitrary but fixed and that $U_1, U_2 \subseteq \mathbb{R}^d$ are open sets with $\partial\Omega \cup \mathcal{N}^- \cup \mathcal{C} \subset U_1$, $\partial\Omega \cup \mathcal{N}^+ \cup \mathcal{C} \subset U_2$, $z^+ = 0$ a.e. in $U_1 \cap \Omega$, and $z^- = 0$ a.e. in $U_2 \cap \Omega$. Then we can find functions $\psi_1, \psi_2 \in C_c(\Omega)$ with

$$0 \leq \psi_1, \psi_2 \leq 1 \text{ in } \Omega, \quad \psi_1 \equiv 1 \text{ in } \Omega \setminus U_1, \quad \psi_2 \equiv 1 \text{ in } \Omega \setminus U_2,$$
$$\operatorname{dist}(\operatorname{supp}(\psi_1), \mathcal{C} \cup \mathcal{N}^-) > 0, \quad \text{and} \quad \operatorname{dist}(\operatorname{supp}(\psi_2), \mathcal{C} \cup \mathcal{N}^+) > 0.$$

Note that the definitions of $\mathcal{N}^-$ and $\mathcal{N}^+$ yield $\partial\{w < 0\} \cap \{\nabla w = 0\} \subseteq \mathcal{C} \cup \mathcal{N}^-$ and $\partial\{w > 0\} \cap \{\nabla w = 0\} \subseteq \mathcal{C} \cup \mathcal{N}^+$. We may thus use Proposition 4.8 and Corollary 4.9 to obtain

$$J_n(z) = \int_\Omega \psi_1 \frac{c}{t_n}\left(\frac{j(w + t_n z^+) - j(w)}{t_n} - j'(w; z^+)\right)\mathrm{d}\lambda + \int_\Omega \psi_2 \frac{c}{t_n}\left(\frac{j(w + t_n z^-) - j(w)}{t_n} - j'(w; z^-)\right)\mathrm{d}\lambda \to J(z). \tag{4.31}$$

To study the limit behavior of $J_n(\delta_n)$, let $\psi_1^k, \psi_2^k \in C_c(\Omega)$ be sequences of bump functions such that $0 \leq \psi_1^k, \psi_2^k \leq 1$ in Ω and

$$\psi_1^k \equiv 1 \text{ in } \{x \in \Omega : \operatorname{dist}(x, \mathcal{C} \cup \mathcal{N}^- \cup \partial\Omega) \geq 1/k\},$$
$$\psi_2^k \equiv 1 \text{ in } \{x \in \Omega : \operatorname{dist}(x, \mathcal{C} \cup \mathcal{N}^+ \cup \partial\Omega) \geq 1/k\},$$
$$\operatorname{dist}(\operatorname{supp}(\psi_1^k), \mathcal{C} \cup \mathcal{N}^-) > 0, \quad \operatorname{dist}(\operatorname{supp}(\psi_2^k), \mathcal{C} \cup \mathcal{N}^+) > 0 \quad \forall\, k \in \mathbb{N}.$$

Then the definition of J_n in (3.3) and the convexity of j imply

$$\begin{aligned} J_n(\delta_n) \geq &\int_\Omega \psi_1^k \frac{c}{t_n}\left(\frac{j(w+t_n\delta_n^+)-j(w)}{t_n} - j'(w;\delta_n^+)\right)\mathrm{d}\lambda \\ &+ \int_\Omega \psi_2^k \frac{c}{t_n}\left(\frac{j(w+t_n\delta_n^-)-j(w)}{t_n} - j'(w;\delta_n^-)\right)\mathrm{d}\lambda. \end{aligned}$$

Due to the weak and pointwise convergence of δ_n to δ and the boundedness of the sequence δ_n in $L^\infty(\Omega)$, we may again use Proposition 4.8 and Corollary 4.9 to pass to the limit with the integrals on the right-hand side of the above estimate. This yields

$$\begin{aligned} \liminf_{n\to\infty} J_n(\delta_n) \geq &\frac{1}{2}\int_{\{w\neq 0\}} \psi_1^k c j''(w)(\delta^+)^2 + \psi_2^k c j''(w)(\delta^-)^2 \mathrm{d}\lambda \\ &+ \frac{1}{2}\int_{\{w=0\}} \psi_1^k c j_1''(0)(\delta^+)^2 + \psi_2^k c j_2''(0)(\delta^-)^2 \mathrm{d}\lambda \\ &+ \frac{1}{2}\big(j_1'(0)+j_2'(0)\big)\int_{\mathcal{M}} c\frac{\psi_1^k(\operatorname{tr}\delta^+)^2 + \psi_2^k(\operatorname{tr}\delta^-)^2}{\|\nabla w\|}\mathrm{d}\mathcal{H}^{d-1} \quad \forall k\in\mathbb{N}. \end{aligned}$$

Note that all integrands in the above are non-negative due to the properties of j and that the bump functions ψ_1^k and ψ_2^k converge pointwise to the indicator function of the domain Ω as $k\to\infty$. Therefore, by taking the limes inferior for $k\to\infty$ and by applying Fatou's lemma, we arrive at

$$\liminf_{n\to\infty} J_n(\delta_n) \geq J(\delta). \tag{4.32}$$

Inserting (4.31) and (4.32) in (4.30) yields the claim. □

Let us summarize what we know about the limit δ at this point (cf. Lemma 3.5, Lemma 4.5, Proposition 4.7, and Proposition 4.11):

Corollary 4.12. *Let Assumptions 2.1, 4.1, and 4.3 hold. Then in the situation of Assumption 3.1 the weak limit δ is an element of the so-called reduced critical cone*

$$\begin{aligned} T_{\text{crit}}^{\text{red}}(c,f) := \Big\{ z\in H_0^1(\Omega) : &\int_{\mathcal{M}} c\frac{(\operatorname{tr} z)^2}{\|\nabla w\|}\mathrm{d}\mathcal{H}^{d-1} < \infty, \\ &z^+ = 0 \ \lambda^d\text{-a.e. in } \{w=0\}\cap\{-cj_2'(0)\leq f < cj_1'(0)\}, \\ &z^- = 0 \ \lambda^d\text{-a.e. in } \{w=0\}\cap\{-cj_2'(0) < f \leq cj_1'(0)\}, \\ &\operatorname{tr}(z^+) = 0 \ \mathcal{H}^{d-1}\text{-a.e. on } \mathcal{N}^-,\ \operatorname{tr}(z^-) = 0 \ \mathcal{H}^{d-1}\text{-a.e. on } \mathcal{N}^+ \Big\}. \end{aligned}$$

To complete the argumentation that we have outlined in Section 3 (with V and $T_{\text{crit}}^{\text{red}}(c,f)$ as in Proposition 4.11 and Corollary 4.12, J as in (4.28), and $I\equiv 0$), it remains to prove that (4.27) holds not only for all functions $z\in V$ but also for all other elements of the reduced critical cone $T_{\text{crit}}^{\text{red}}(c,f)$. The following approximation result turns out to be useful in this context. We point out that similar results using a different notion of capacity can also be found in [1, Theorem 9.1.3].

Lemma 4.13. *Suppose that Assumption 2.1 and Assumption 4.3 hold. Then for every function $z\in H_0^1(\Omega)\cap L^\infty(\Omega)$ satisfying*

$$\operatorname{tr}(z^+) = 0 \ \mathcal{H}^{d-1}\text{-a.e. on } \mathcal{N}^- \quad \text{and} \quad \operatorname{tr}(z^-) = 0 \ \mathcal{H}^{d-1}\text{-a.e. on } \mathcal{N}^+ \tag{4.33}$$

there exists a sequence $z_l \in H_0^1(\Omega) \cap L^\infty(\Omega)$ *such that* z_l *converges to* z *in* $H^1(\Omega)$ *and such that for all* l *it is true that*

$$\begin{aligned} z_l^+ &= 0 \quad \lambda^d\text{-a.e. in a neighborhood of } \partial\Omega \cup \mathcal{N}^- \cup \mathcal{C}, \\ z_l^- &= 0 \quad \lambda^d\text{-a.e. in a neighborhood of } \partial\Omega \cup \mathcal{N}^+ \cup \mathcal{C}. \end{aligned}$$

Proof. Let $z \in H_0^1(\Omega) \cap L^\infty(\Omega)$ be an arbitrary function satisfying (4.33). Then it follows from the hypotheses on $\mathcal{C}$ in Assumption 4.3 b) that there exists a sequence $\phi_m \in C_c(\mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^d)$ with

$$0 \le \phi_m \le 1, \quad \phi_m \equiv 1 \text{ in a nbhd. of } \mathcal{C}, \quad \text{and} \quad \|\phi_m\|_{H^1} \to 0 \text{ as } m \to \infty.$$

Define $z_m := (1 - \phi_m)z \in H_0^1(\Omega)$. Then each z_m vanishes in a neighborhood of $\mathcal{C}$, and it holds (due to the dominated convergence theorem and Friedrichs' inequality)

$$\|z - z_m\|_{H^1} \le C\|\nabla z - \nabla z_m\|_{L^2} \le C\|z\|_{L^\infty} \|\phi_m\|_{H^1} + C\|\phi_m \nabla z\|_{L^2} \to 0.$$

Consequently, we may assume w.l.o.g. that $z = 0$ holds λ^d-a.e. in $\Omega \cap U$, where $U \subseteq \mathbb{R}^d$ is an open neighborhood of $\mathcal{C}$. Further, we have $z = z^+ + z^-$ with $z^+, z^- \in H_0^1(\Omega)$ according to [18, Theorem II.A.1]. This allows us to approximate the positive and the negative part of z separately. Let us consider the positive part z^+ (the argumentation for z^- is analogous). Then from the properties of the function z we obtain that $\operatorname{tr}(z^+) = 0$ $\mathcal{H}^{d-1}$-a.e. on $(\partial\Omega \cup \mathcal{N}^-) \setminus U$ and $z^+ = 0$ λ^d-a.e. in $\Omega \cap U$. Further, Assumption 4.3 implies $\operatorname{cl}(\mathcal{N}^-) \setminus \mathcal{N}^- \subseteq \mathcal{C}$ so that $(\partial\Omega \cup \mathcal{N}^-) \setminus U$ is a compact subset of the Lipschitz submanifold $\partial\{w \neq 0\} \setminus \mathcal{C}$. From the latter, it follows that we may cover $(\partial\Omega \cup \mathcal{N}^-) \setminus U$ by a finite number of rectification neighborhoods $R(B \times J)$ (as appearing in the definition of a strong $(d-1)$-dimensional Lipschitz submanifold, see Remark 4.4). Using this cover, a partition of unity, and classical localization and rectification arguments in the spirit of, e.g., [9, Theorem 5.5.2], [22, Theorem 4.10], it is straightforward to prove that z^+ can be approximated in $H^1(\Omega)$ by continuous functions z_l whose supports have a positive distance to $\partial\Omega \cup \mathcal{N}^- \cup \mathcal{C}$. Due to the continuity of max in $H^1(\Omega)$ (cf. [18, Theorem II.A.1]), we may assume w.l.o.g. that the approximating functions z_l are non-negative everywhere. Combining all of the above, we obtain that z^+ can be approximated in $H^1(\Omega)$ by non-negative functions z_l with the desired properties. Using an analogous argument for the negative part z^- and adding the approximating sequences for z^+ and z^- proves the claim. ∎

We are now finally in the position to prove our main result:

THEOREM 4.14. *Let Assumptions 2.1 and 4.1 hold, let* $p > d/2$, *and let* $(c, f) \in L_+^p(\Omega) \times L^p(\Omega)$ *be a tuple such that* c, f, *and the solution* $w := S(c, f)$ *satisfy the conditions in Assumption 4.3. Then the solution operator* $S : L_+^p(\Omega) \times L^p(\Omega) \to H_0^1(\Omega)$ *associated with problem* (P) *is Hadamard directionally differentiable at* (c, f) *in all directions* $(h, g) \in \mathbb{R}^+(L_+^p(\Omega) - c) \times L^p(\Omega)$ *and the derivative* $\delta := S'((c, f); (h, g))$ *in a direction* (h, g) *is uniquely characterized by the variational inequality*

$$\begin{aligned} &\delta \in T_{\text{crit}}^{\text{red}}(c, f), \\ &a(\delta, z - \delta) + J(z) - J(\delta) \\ &\qquad + \int_\Omega h\, j'(w; z)\, \mathrm{d}\lambda - \int_\Omega h\, j'(w; \delta)\, \mathrm{d}\lambda \ge \langle g, z - \delta \rangle \qquad \forall\, z \in T_{\text{crit}}^{\text{red}}(c, f). \end{aligned} \tag{4.34}$$

Here, J *and* $T_{\text{crit}}^{\text{red}}(c, f)$ *are defined as in* (4.28) *and Corollary 4.12, respectively.*

Proof. Let $(h, g) \in \mathbb{R}^+(L^p_+(\Omega) - c) \times L^p(\Omega)$ be arbitrary but fixed, and let $(t_n) \subset (0, \infty)$ be an arbitrary null sequence. Then $c + t_n\, h \in L^p_+(\Omega)$ for sufficiently large n so that $\delta_n := [S(c + t_n\, h, f + t_n\, g) - S(c, f)]/t_n$ is well-defined. From the local Lipschitz continuity of S (cf. Theorem 2.6), we obtain that the sequence δ_n is bounded in $H^1(\Omega)$ and $L^\infty(\Omega)$. We may thus pass over to a subsequence (unrelabeled for simplicity) to arrive at the situation of Assumption 3.1, where $\delta_n \rightharpoonup \delta$ for some δ in $H^1_0(\Omega)$.

In what follows, we verify that the weak limit δ is indeed the solution to (4.34). To this end, we first note that $\delta \in T^{\text{red}}_{\text{crit}}(c, f)$ due to Corollary 4.12, i.e., δ is admissible for (4.34). Consider now an arbitrary but fixed element z of the reduced critical cone $T^{\text{red}}_{\text{crit}}(c, f)$ and define $z_k := \min(z^+, k) + \max(z^-, -k) \in H^1_0(\Omega) \cap L^\infty(\Omega)$, $k \in \mathbb{N}$. Then z_k is an element of $T^{\text{red}}_{\text{crit}}(c, f)$ for all k and we may employ Lemma 4.13 to obtain that there exist sequences $z_{k,l} \in H^1_0(\Omega) \cap L^\infty(\Omega)$ with $z_{k,l} \to z_k$ as $l \to \infty$ in $H^1(\Omega)$, $z^+_{k,l} = 0$ a.e. in a neighborhood of $\partial\Omega \cup \mathcal{N}^- \cup \mathcal{C}$ for all l, and $z^-_{k,l} = 0$ a.e. in a neighborhood of $\partial\Omega \cup \mathcal{N}^+ \cup \mathcal{C}$ for all l. Define

$$\tilde{z}_{k,l} := \min(z^+_{k,l}, \min(z^+, k)) + \max(z^-_{k,l}, \max(z^-, -k)) \in H^1_0(\Omega) \cap L^\infty(\Omega).$$

Then it holds

$$\begin{aligned}
\tilde{z}^+_{k,l} &= 0 \ \lambda^d\text{-a.e. in a neighborhood of } \partial\Omega \cup \mathcal{N}^- \cup \mathcal{C},\\
\tilde{z}^-_{k,l} &= 0 \ \lambda^d\text{-a.e. in a neighborhood of } \partial\Omega \cup \mathcal{N}^+ \cup \mathcal{C},\\
\tilde{z}^+_{k,l} &= 0 \ \lambda^d\text{-a.e. in } \{w = 0\} \cap \{-c j_2'(0) \le f < c j_1'(0)\},\\
\tilde{z}^-_{k,l} &= 0 \ \lambda^d\text{-a.e. in } \{w = 0\} \cap \{-c j_2'(0) < f \le c j_1'(0)\},
\end{aligned}$$

and we may employ Proposition 4.11 to obtain

$$\begin{aligned}
a(\delta, \tilde{z}_{k,l}) + J(\tilde{z}_{k,l}) - J(\delta) + \int_\Omega h\Big(j'(w; \tilde{z}_{k,l}) - j'(w; \delta)\Big)\mathrm{d}\lambda - \langle g, \tilde{z}_{k,l} - \delta\rangle&\\
\ge \limsup_{n\to\infty} a(\delta_n, \delta_n) \quad \forall k, l.& \qquad (4.35)
\end{aligned}$$

Note that the construction of $\tilde{z}_{k,l}$ and z_k yields

$$\begin{gathered}
|\tilde{z}_{k,l}| \le |z_k| \le |z| \ \ \lambda^d\text{-a.e. in } \Omega \quad \forall k, l,\\
c\frac{(\operatorname{tr} \tilde{z}_{k,l})^2}{\|\nabla w\|} \le c\frac{(\operatorname{tr} z_k)^2}{\|\nabla w\|} \le c\frac{(\operatorname{tr} z)^2}{\|\nabla w\|} \in L^1(\mathcal{M}, \mathcal{H}^{d-1}) \ \ \mathcal{H}^{d-1}\text{-a.e. on } \mathcal{M} \quad \forall k, l.
\end{gathered}$$

Furthermore, we clearly have $z_k \to z$ in $H^1(\Omega)$ as $k \to \infty$ and $\tilde{z}_{k,l} \to z_k$ in $H^1(\Omega)$ as $l \to \infty$ for all $k \in \mathbb{N}$. Thus, using the dominated convergence theorem and the weak lower semi-continuity of $H^1_0(\Omega) \ni z \mapsto a(z, z) \in \mathbb{R}$, we can pass to the limit in (4.35) (first with l then with k) to obtain

$$\begin{aligned}
a(\delta, z) + J(z) - J(\delta) + \int_\Omega h\Big(j'(w; z) - j'(w; \delta)\Big)\mathrm{d}\lambda - \langle g, z - \delta\rangle&\\
\ge \limsup_{n\to\infty} a(\delta_n, \delta_n) \ge \liminf_{n\to\infty} a(\delta_n, \delta_n) \ge a(\delta, \delta).& \qquad (4.36)
\end{aligned}$$

The last estimate has several implications: First of all, it yields that the weak limit δ is indeed a solution to (4.34). Moreover, since (4.34) can only have one solution (as one can easily see using contradiction and the coercivity of the bilinear form a), (4.36) proves that the weak limit δ is unique. This implies that the weak limit of the difference quotients δ_n is independent of the choice of (sub)sequences and that the whole original sequence (δ_n) has to be weakly convergent. As a consequence, we obtain that S is weakly directionally differentiable in (c, f) in the direction (h, g).

To prove that (δ_n) converges even strongly, we note that by choosing $z = \delta \in T^{\text{red}}_{\text{crit}}(c, f)$ in (4.36) we obtain

$$a(\delta, \delta) \geq \limsup_{n\to\infty} a(\delta_n, \delta_n) \geq \liminf_{n\to\infty} a(\delta_n, \delta_n) \geq a(\delta, \delta).$$

Since a is coercive, the above yields $\|\delta_n\|_{H^1} \to \|\delta\|_{H^1}$. This, together with the weak convergence, yields strong convergence $\delta_n \to \delta$ in $H^1(\Omega)$ and proves that S is indeed strongly directionally differentiable. The Hadamard differentiability finally follows from the strong directional differentiability and the Lipschitz continuity of the solution operator S, cf. [6, Proposition 2.49]. ◻

5. Conclusions and Comparison with Known Results. Several points are noteworthy regarding Theorem 4.14 and the variational inequality (4.34):

First of all, we remark that (4.34) is in general neither a variational inequality of the first nor a variational inequality of the second kind (in the sense of [13]). If we consider, e.g., the special case $j(x) = |x|$, $c \equiv 1$, and $h \equiv -1$, then (4.34) becomes

$$\begin{aligned}
&\delta \in T^{\text{red}}_{\text{crit}}(c, f),\\
&a(\delta, z-\delta) + \int_{\mathcal{M}} \frac{(\operatorname{tr} z)^2}{\|\nabla w\|} \mathrm{d}\mathcal{H}^{d-1} - \int_{\{w=0\}} |z| \mathrm{d}\lambda - \int_{\mathcal{M}} \frac{(\operatorname{tr} \delta)^2}{\|\nabla w\|} \mathrm{d}\mathcal{H}^{d-1} + \int_{\{w=0\}} |\delta| \mathrm{d}\lambda\\
&\qquad\qquad \geq \langle g, z-\delta\rangle + \int_{\{w\neq 0\}} \operatorname{sgn}(w)(z-\delta)\mathrm{d}\lambda \quad \forall z \in T^{\text{red}}_{\text{crit}}(c, f),
\end{aligned}$$

and we end up with a variational inequality which involves an in general non-convex functional of the form

$$z \mapsto \int_{\mathcal{M}} \frac{(\operatorname{tr} z)^2}{\|\nabla w\|} \mathrm{d}\mathcal{H}^{d-1} - \int_{\{w=0\}} |z| \mathrm{d}\lambda.$$

Note that, although (4.34) does not fit into the classical setting, the unique solvability of the variational inequality in Theorem 4.14 is still guaranteed. The existence of a solution follows directly from our analysis (since we have proved that the limit of the difference quotients δ_t satisfies (4.34)), and the uniqueness of the solution is a trivial consequence of the ellipticity of the bilinear form a (cf. [13, Theorem 4.1] and the proof of Theorem 4.14).

Secondly, we point out that the natural space for the study of (4.34) is the Hilbert space

$$H := \left\{ z \in H^1_0(\Omega) : \int_{\mathcal{M}} c \frac{(\operatorname{tr} z)^2}{\|\nabla w\|} \mathrm{d}\mathcal{H}^{d-1} < \infty \right\}$$

endowed with the scalar product

$$(z_1, z_2)_H := \int_\Omega \nabla z_1 \cdot \nabla z_2 \mathrm{d}\lambda + \int_{\mathcal{M}} 2c \frac{(\operatorname{tr} z_1)(\operatorname{tr} z_2)}{\|\nabla w\|} \mathrm{d}\mathcal{H}^{d-1}.$$

This can be seen, e.g., in the case $j(x) = |x|$, $h \equiv 0$, and $A = -\Delta$, where (4.34) can be rewritten as

$$\begin{aligned}
&\delta \in T_{\text{crit}}^{\text{red}}(c,f),\\
&\quad \int_\Omega \nabla\delta \cdot \nabla(z-\delta)\mathrm{d}\lambda + \int_{\mathcal{M}} 2c\frac{(\operatorname{tr}\delta)(\operatorname{tr} z - \operatorname{tr}\delta)}{\|\nabla w\|}\mathrm{d}\mathcal{H}^{d-1} \geq \langle g, z-\delta\rangle \quad \forall z \in T_{\text{crit}}^{\text{red}}(c,f)
\end{aligned} \tag{5.1}$$

and the directional derivative $\delta = S'((c,f);(h,g))$ is exactly the $(.,.)_H$-projection of the Riesz representative of g onto the reduced critical cone $T_{\text{crit}}^{\text{red}}(c,f)$. Note that, due to the weight $1/\|\nabla w\|$ in the trace integral of $(.,.)_H$, the space H is typically a proper subspace of $H_0^1(\Omega)$. This has to be taken into account if, e.g., the differentiability result in Theorem 4.14 is used for the derivation of strong stationarity conditions for optimal control problems governed by VIs of the type (P) in the sense of Mignot/Puel (cf. the analysis in [7, 21]). We remark that the emergence of the weighted Sobolev space H is also the reason why we need the rather involved approximation argument in the proof of Theorem 4.14.

It should be noted that the $\mathcal{M}$-integral in the VI (5.1) does not appear in the results of [7], where the case $c \equiv 1$, $h \equiv 0$, and $j(x) = |x|$ was investigated. This can be explained as follows: The analysis in [7] is based on two main assumptions, namely, that the slack variable $q := f - Aw \in H^{-1}(\Omega)$ is a continuous function and that the active set $\{w = 0\}$ can be decomposed into a set of zero capacity and a set with a non-empty interior and a regular boundary (see [7, Assumptions 3.1, 3.2]). It is easy to see that these assumptions imply that the sets $\{w > 0\}$ and $\{w < 0\}$ have a positive distance to each other (see [7, Lemma 3.9]) and that for every $z \in T_{\text{crit}}(f)$ it holds

$$\operatorname{tr}(z^+) = 0 \ \mathcal{H}^{d-1}\text{-a.e. on } \mathcal{N}^- \quad \text{and} \quad \operatorname{tr}(z^-) = 0 \ \mathcal{H}^{d-1}\text{-a.e. on } \mathcal{N}^+.$$

As a consequence, in the situations considered in [7], it is always true that $\mathcal{M} = \emptyset$ and $T_{\text{crit}}^{\text{red}}(c,f) = T_{\text{crit}}(c,f)$ so that the variational inequality for the directional derivatives $\delta = S'((1,f);(0,g))$ in (4.34) simplifies to

$$\delta \in T_{\text{crit}}(c,f), \quad a(\delta, z-\delta) \geq \langle g, z-\delta\rangle \quad \forall\, z \in T_{\text{crit}}(c,f). \tag{5.2}$$

The above is exactly the VI derived in [7, Theorem 3.3]. The differentiability result obtained in [7] is thus a special case of (4.34) under additional structural assumptions on the active set $\{w = 0\}$ and the function j. The drastic change that appears in the structure of the VI for the directional derivatives when the assumptions on the active set in [7] are weakened (compare, e.g., (5.1) with (5.2)) is striking and maybe the most important result of this work (cf. also with [26–28]).

Lastly, we remark that the terms $J(z)$ and $J(\delta)$ appearing in (4.34) are closely related to the pullback $w * j''$ of the second distributional derivative of j by w in the sense of Hörmander [16, Chapter VI]. To be more precise, we have the formal identities

$$J(z) = \frac{1}{2}\left\langle c(w * j''), z^2 \right\rangle \quad \text{and} \quad J(\delta) = \frac{1}{2}\left\langle c(w * j''), \delta^2 \right\rangle,$$

where the brackets $\langle .,. \rangle$ denote the distributional pairing (cf. [16, Example 6.1.5] and [32, Section V.13]). Recall that in the classical theory, the pullback of a distribution by a function $v \in C^1(\Omega)$ is defined by extending the composition map

$$C_c^\infty(\mathbb{R}) \ni \varphi \mapsto \varphi(v) \in C^1(\Omega)$$

continuously to the space $\mathcal{D}'(\mathbb{R})$ and that this extension is only possible if the gradient of the function v under consideration vanishes nowhere in Ω (cf. [16, Theorem 6.1.2]). What can be observed in the situation of Theorem 4.14 is that in the variational inequality (4.34) for the directional derivative $S'((c,f);(h,g))$ the terms known from the classical pullback $w*j''$ appear everywhere where they are well-defined and that on the set $\partial\{w \neq 0\} \cap \{\nabla w = 0\}$, where the classical construction fails, the pullback terms are replaced with the trace conditions $\mathrm{tr}(z^+) = 0$ $\mathcal{H}^{d-1}$-a.e. on $\mathcal{N}^-$ and $\mathrm{tr}(z^-) = 0$ $\mathcal{H}^{d-1}$-a.e. on $\mathcal{N}^+$ in the definition of $T^{\mathrm{red}}_{\mathrm{crit}}(c,f)$. That the quantity $w * j''$ emerges in the above way when the (Mosco epi-) convergence of second order difference quotients of the type (3.4) is considered is remarkable and has, at least to the authors' best knowledge, not been studied systematically so far.

It should be noted that the analysis of the difference quotients in (3.4) becomes even more complicated when the bilinear form a in (P) is assumed to be H^k-elliptic for some $k > 1$. In this situation, also $(d-2)$-, $(d-3)$- etc. dimensional components of the level sets of w are relevant for the sensitivity analysis and traces of derivatives have to be taken into account, too. Finding a systematic approach towards the study of such H^k-elliptic problems seems to be difficult and is subject to further research. The same holds true for the extension of our differentiability results to variational inequalities that do not fit into the setting of Theorem 4.14, e.g., inequalities which involve terms of the form $j(\|\nabla v\|)$, $v \in H^1_0(\Omega)$.

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